21-329 Assignment 2

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2.6

Assume for the sake of contradiction that there is a set V such that $x \in V$ for every set x. Consider the powerset $\mathcal{P}(V)$. First note that every $x \in V$ is also a member of $\mathcal{P}(V)$. This is because every $y \in x$ is a set so $y \in V$ and so $x \subseteq V \Rightarrow x \in \mathcal{P}(V)$. Now consider the function $f(x) = x$ where $f : V \to \mathcal{P}(V)$. Using the Comprehension Scheme we know there is a set B such that $x \in B \Leftrightarrow (x \in V \text{ and } x \notin f(x))$. Note that $B \in \mathcal{P}(V)$ because it is a subset of V. However, we can also see that there is no $f(x)$ such that $f(x) = B$. Consider any $x \in V$. If $x \in f(x)$ then $x \notin B$. Otherwise $x \notin f(x)$ so $x \in B$. Thus for every $x \in V$, it is in $f(x)$ iff it's not in B so they are not equal. Since this holds for every $f(x)$ and $f(x) = x$ then $B \neq x$ for any $x \in V$. But then $B \notin V$ which contradicts the assumption that V contains all sets. $\lightning$

2.7

(Claim 1)

First we describe the LHS.

$$\begin{aligned} \bigcap \mathcal{F} &= \{a \mid \forall X \in \mathcal{F}\ (a \in X)\} && \text{(by definition)} \\ \Rightarrow S - \bigcap \mathcal{F} &= \{a \mid a \in S \text{ and } \exists X \in \mathcal{F}\ (a \notin X)\} \end{aligned}$$

The latter line is justified by definition of difference. We want all elements of S such that they aren't in the intersection of $\mathcal{F}$. Every element that isn't in $\bigcap \mathcal{F}$ must not be in some set $X \in \mathcal{F}$.

Now we decribe the RHS.

$$\begin{aligned} \bigcup \{S - X \mid X \in \mathcal{F}\} &= \bigcup \{D \mid X \in \mathcal{F},\ d \in D \Leftrightarrow (d \in S \text{ and } d \notin X)\} && \text{(by definition of set difference)} \\ &= \{d \mid d \in S \text{ and } \exists X \in \mathcal{F}\ (d \notin X)\} && \text{(by definition of union)} \end{aligned}$$

For the first line we are constructing sets D for each $X \in \mathcal{F}$. Given an $X \in \mathcal{F}$ we want all $d \in S$ that aren't in X. For the second line, every member of the union must be a member of some set D in the previous line (by definition of union). If d is a member of some D then by definition of D we know $d \in S$ and $d \notin X$ where $X \in \mathcal{F}$ is the chosen set for that D.

(Claim 2)

LHS:

$$\begin{aligned} \bigcup \mathcal{F} &= \{a \mid \exists X \in \mathcal{F}\ (a \in X)\} && \text{(by definition of union)} \\ \Rightarrow S - \bigcup \mathcal{F} &= \{a \mid a \in S \text{ and } \forall X \in \mathcal{F}\ (a \notin X) \end{aligned}$$

In the second line we don't want any $a \in \bigcup \mathcal{F}$. By definition, this means that for every $X \in \mathcal{F}$, $a \notin X$.

RHS:

$$\begin{aligned} \bigcap \{S - X \mid X \in \mathcal{F}\} &= \bigcap \{D \mid X \in \mathcal{F},\ d \in D \Leftrightarrow (d \in S \text{ and } d \notin X)\} && \text{(by definition of set difference)} \\ &= \{d \mid d \in S \text{ and } \forall X \in \mathcal{F}\ (d \notin X)\} \end{aligned}$$

The first line is justified as in claim 1. For the second line d must be a member of every D by definition of intersection. This is true iff $d \in S$ and for every $X \in \mathcal{F}$ (because we have a D for each X) we have $d \notin X$.

2.8

(1) For shorthand let the LHS be the *set* on the left hand side of the equation. Similarly let RHS be the *set* on the right hand side of the equation. Consider some $x \in LHS$. Then it must be the case (by definition) that for every $a \in A$ there is some $b \in B$ such that $x \in S_{(a,b)}$. We construct *some* function (there may be more than one) $f \in {}^AB$ such that if $(a, b) \in f$ then $x \in S_{(a,b)}$. To do this we invoke the axiom of choice. In particular, for some $a \in A$ we consider the set $\{a\} \times B$. We use comprehension to select only the pairs such that $x \in S_{(a,b)}$. We can use this construction for any $a \in A$. Now consider the family of sets $\mathcal{F}$ which contains all of these constructed sets (i.e. for each $a \in A$ construct the set of pairs such that $x \in S_{(a,b)}$). Each of the constructed sets is nonempty because of the definition of LHS. Furthermore, each set can contain infinite members and so we can't construct the function explicitly. Instead, we just claim such a function exists by the axiom of choice on $\mathcal{F}$ to pick one pair for each $a \in A$. This is by definition a function $f \in {}^AB$. Now since we have an $f \in {}^AB$ such that for every $a \in A,\ x \in S(a, f(a))$ then we know that $x \in RHS$. Thus we have $LHS \subseteq RHS$.

Now consider an $x \in RHS$. Then there is a function $f \in {}^AB$ such that for every $a \in A,\ x \in S(a, f(a))$. Since $f \in {}^AB$ then $f(a) \in B$. Thus for every $a \in A$ there exists a $f(a) \in B$ such that $x \in S_{(a,b)}$. By definition we then have $x \in LHS$ and so $RHS \subseteq LHS$.

Combining our two claims with the extensionality axiom we have $LHS = RHS$ as desired.

(2) The argument from the previous part holds except we do not need to invoke the axiom of choice to construct the function. Because $S_{(a,b)} \cap S_{(a,b')} = \emptyset$ then for some $x \in LHS$ and some $a \in A$ there is precisely one $b \in B$ such that $x \in S_{(a,b)}$. To be explicit for some $a \in A$ consider the set $\{a\} \times B$. Using comprehension we can pick the subset of pairs such that $x \in S_{(a,b)}$. With the extra constraint, this new set has precisely one member. Taking the union of all of these sets (one for each $a \in A$) we have a function $f \in {}^AB$ such that for some $a \in A$, $x \in S_{(a,f(a))}$. The rest of the proof follows as above.

2.11

(1) We need replacement to show there exists a set S containing every member of $\{B_n | n < \omega\}$. We then use comprehension to generate the subset of $\omega \times S$ such that $P(x, y)$ is satisfied. That property can be anything that holds only for $(n, B_n),\ n < \omega$. We can then use union and comprehension multiple times to grab the second item in each pair. So in summary, we need replacement, comprehension, and union to generate C.

(2) Consider $x \in C$. Then there must exist an $n < \omega$ and corresponding B_n such that $x \in B_n$. By definition, if $y \in x$ then $y \in B_{n+1}$. Furthermore, if $y \in B_{n+1}$ then $y \in C$. Thus $x \subseteq C$ and so C is transitive.

(3) First, note that if $x \in C$ then there is some $n < \omega$ and a corresponding B_n such that $x \in B_n$. Thus if we prove that $B_n \subseteq D$ for every $n < \omega$ we will have shown that $C \subseteq D$. We do so by induction on $n < \omega$.

- Base Case

  Since $B_0 = A$ this is true since we've assumed $A \subseteq D$.

- Induction Step Assume the claim holds for every $n < k$ for some $0 < k < \omega$.

  Consider $x \in B_k$. We know there exists a $Y \in B_{k-1}$ such that $x \in Y$. By the induction hypothesis $B_{k-1} \subseteq D$ so $Y \in D$. Since D is transitive, we have $Y \subseteq D$. Finally we have $x \in D$ (because $x \in Y$). Thus for every $x \in B_k$ we have $x \in D$ so $B_k \subseteq D$.

By induction we've shown that every $B_n \subseteq D$ where $n < \omega$. As explained before, this implies that $C \subseteq D$ as desired.

2.13

A note to the grader: I incorrectly applied the definition of $\vee$ and $\wedge$ so that the rules of associativity and distributivity were incorrect. I then attempted to prove these invalid statements for 7 hours. I didn't have time to rigorize most of the proofs below

(1) First we show the operations are well-defined. We assume xEx' and yEy' for the following proofs. First, we have a lemma that the symmetric difference $a \triangle b$ is finite iff $a - b$ is finite and $b - a$ is finite. If the RHS is finite, each member of the LHS is a member of either $a - b$ or $b - a$ so the LHS is finite. Similarly if $a - b$ is not finite, then there are an infinite number of members on the LHS by definition. A symmetric argument holds if $b - a$ is not finite.

As a result of this lemma we know that $(x - x'), (x' - x), (y - y'), (y' - y)$ are all finite sets.

Now we prove that $(x \cup y)E(x' \cup y')$. By the above lemma we just need to show that $(x \cup y) - (x' \cup y') = S$ is finite as well as $(x' \cup y') - (x \cup y) = T$. Consider the set $a \in S$ and $a \in x$. Since $a \in S$ we know $a \notin x'$ and because we have $(x - x')$ is finite then we know the number of members of this set is finite (since removing members that are in y' will only reduce the size of the set). Similarly the number of elements $b \in S$ and $b \in y$ is finite. Since S is the union of these two finite sets (by definition), it too must be finite. A symmetric argument holds for T being finite and so we have $(x \cup y)E(x' \cup y')$.

Similarly we can prove $(x \cap y)E(x' \cap y')$ by showing $(x \cap y) - (x' \cap y') = S$ is finite and $(x' \cap y') - (x \cap y) = T$ is finite. We use the claim that intersection distributes over symmetric difference as proved in Exercise 2.9. In particular, we note that $((x \cup x') \cap y) \triangle ((x \cup x') \cap y') \supseteq (x \cap y) \triangle (x' \cap y')$. This is true by definition. Then we use distributivity $((x \cup x') \cap y) \triangle ((x \cup x') \cap y') = (x \cup x') \cap (y \triangle y')$. Since $(y \triangle y')$ is finite by assumption, then $(x \cup x') \cap (y \triangle y')$ is finite. Since S is a subset of the previous set, then S is finite. A symmetric argument holds for T being finite and so the claim is true.

Finally we wish to prove $(\omega - x)E(\omega - x')$. Again we show $((\omega - x) - (\omega - x')) = S$ is finite and $((\omega - x') - (\omega - x)) = T$ is finite. First we show S is finite. We know $(\omega - x) = \{a \mid a \in \omega \text{ and } a \notin x\}$ and $(\omega - x') = \{a \mid a \in \omega \text{ and } a \notin x'$. Thus the difference between these sets is $\{a \mid a \in \omega,\ a \notin x,\ a \in x'\}$. Since we know $(x' - x)$ is finite, then the subset that is also in ω (which is S) is also finite. A symmetric proof holds for T so the claim holds.

Next we also note that $[\emptyset]_E \neq [\omega]_E$. For instance, $3 \in \mathcal{P}(\omega)$, $3 \in [\emptyset]_E$, but $3 \notin [\omega]_E$.

Now we prove the ten laws of Boolean algebra for $\mathbb{B}$.

- Associativity
  - $[x \cup (y \cup z)]_E = [(x \cup y) \cup z]_E$
    Since union is associative we know that these sets are equal
  - $[x \cap (y \cap z)]_E = [(x \cap y) \cap z]_E$
    True because intersection is associative.
- Commutativity
  - $[x \cup y]_E = [y \cup x]_E$
    By definition of union we have $x \cup y = y \cup x$ and so the equivalence classes must be the same
  - $[x \cap y]_E = [y \cap x]_E$
    By definition of intersection we have $x \cap y = y \cap x$ and so the equivalence classes must be the same
- Distributivity

  - $[x \cup (y \cap z)]_E = [(x \cup y) \cap (x \cup z)]_E$
    We know that $x \cup (y \cap z) = \{a \mid a \in x$ or $(a \in y$ and $a \in z\}$. On the right hand side we have $\{a \mid (a \in x$ or $a \in y)$ and $(a \in x$ or $a \in z)\}$. In both cases a member is either in x or it must be in both y and z. Since both of the sets are equal, their equivalence classes must be the same
  - $[x \cap (y \cup z)]_E = [(x \cap y) \cup (x \cap z)]_E$
    On the LHS we have $\{a \mid a \in x$ and $(x \in y$ or $x \in z)\}$. On the right hand side $\{b \mid (b \in x$ and $b \in y)$ or $(b \in x$ and $b \in z)\}$. In both cases b is in the set iff it is in x and it is in either y or z. Since the sets are equal, their equivalence classes are equal.

- Identity
  - $[x \cup \emptyset]_E = [x]_E$
    We have $x \cup \emptyset = x$ because there are no members in the empty set.
  - $[x \cap \omega]_E = [x]_E$
    Since $x \in \mathcal{P}(\omega)$ we have $x \subseteq \omega$. Since every member of x is also in ω then $x \cap \omega = x$.

- Complementation
  - $[x \cup (\omega - x)]_E = [\omega]_E$
    Since $x \subseteq \omega$ then we have $x \cup (\omega - x) = \omega$ by definition.
  - $[x \cap (\omega - x)]_E = [\emptyset]_E$
    By definition of difference we know $(\omega - x)$ does not contain any elements in x. Thus $x \cap (\omega - x) = \emptyset$.

(2) First, note that if a is finite that $[a]_E = [\emptyset]_E$ because every finite subset is a member of both sets and no infinite subsets are in either set. Now, assume for the sake of contradiction there is a non-finite $a \in \mathcal{P}(\omega)$ that is an atom of $\mathbb{B}$. Using ZFC (in particular, we need comprehension and union) we can construct the a sequence corresponding to some indexing a and the set b such that b is the even-indexed members of this sequence. Clearly, b is not finite in size. Now we know $[b]_E = [b \cap a]_E$ because $b \cap a = b$. Furthermore $[b]_E \neq [\emptyset]_E$ because b is not finite. Finally we can show that $[b]_E \neq [a]_E$. But $a - b$ is not finite because it consists of all the odd-indexed members of the indexed a. Thus the symmetric difference of b and a is not finite and so $[b]_E \neq [a]_E$.