21-300 Assignment 1

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August 31, 2011

X1004

(a,b,c) First we prove that the empty formula is not a wff. To do this we prove that the length of a wff is at least 1 using X1000. This holds true for every propositional variable **p** since each propositional variable has length 1. For (2) if $\#\mathbf{A} = n \geq 1$ then $\# \sim \mathbf{A} = n + 1 > 1$ and so the property holds. Finally for (3) if $\#\mathbf{A} = n \geq 1$ and $\#\mathbf{B} = m \geq 1$ then $\#[\mathbf{A} \vee \mathbf{B}] = n + m > 1$. Thus we have that all wff have length at least 1. Since the length of the empty formula is 0, it is not a wff.

Now we introduce the notation that the symbols of a formula are enumerated from left to right. That is $\mathbf{A} = x_0 x_1 x_2 \ldots x_n$.

Next we define a "pair" of square brackets. A "pair" is composed of a left square bracket with a right square bracket to its right. If there are no other brackets between them, they are already a pair. Otherwise, a left bracket is paired with the closest right bracket with a higher index.

First I claim that all brackets are paired. Using X1000 we know that this holds for all propositional variables since they have no brackets. (2) If the property holds for **A** then it also holds for **A** since no new brackets are introduced. (3) If the property holds for **A** and **B** then it also holds for $[\mathbf{A} \vee \mathbf{B}] = \mathbf{C}$. The additional brackets in **C** automatically pair because we've assumed that all of the brackets in **A** and **B** are already paired. By definition, since the new right bracket is the closest right bracket to the new left bracket they are paired. Thus it holds that the brackets in C are paired and so the property holds for all wff by X1000.

Now I claim that every pair of brackets in a wff must have a disjunction with a wff on both sides. Specifically x_l is the left bracket and x_r is the right bracket, then I claim that there exists an x_d with $l < d < r$ that is the disjunction symbol and that the symbols from $[x_{l+1}, x_{d-1}]$ and the symbols from $[x_{d+1}, x_{r-1}]$ both form wffs. Again we use X1000. (1) Every propositional variable does not have a pair of brackets so this is vacuously true. (2) If the property holds for **A** then it also holds for **A** because the constructed wff does not have any new pairs of square brackets. (3) If the property holds for **A** and **B** then $[\mathbf{A} \vee \mathbf{B}] = \mathbf{C}$ also satisfies the property. To show this, we know first know that all brackets in **A** and **B** are already paired as proved by the previous lemma. Thus by definition the new brackets in **C** are paired. By inspection there is an x_d with $l < d < r$. Finally, since have assumed **A** and **B** are wffs the property holds for **C**. Thus by X1000 the property holds for all wffs.

Finally, for each of the formulas in a) $\sim p[\sim q]$, b) $[\,]p$, c) $[\sim p \vee]q$ it is clear that they do not satisfy the properties above so they are not wffs. In particular, the third formula does not have a wff on the right side of the disjunction (since we proved that the empty formula is not a wff).