Problem Set 11

Spin-1/2 particle moving in one dimension

The position and momentum observables of a particle in one dimension are $\widehat{X}$ and $\widehat{P}$ satisfying $[\widehat{X}, \widehat{P}] = i\hbar$. The spin angular momentum observables for such a particle are $\widehat{J}_a$, for $a = x, y, z$ and satisfy $[\widehat{J}_x, \widehat{J}_y] = i\hbar\widehat{J}_z$ and two other relations coming from cyclicly permuting the (x, y, z) indices. We assume that the position and momentum operators commute with the spin operators: $[\widehat{X}, \widehat{J}_a] = [\widehat{P}, \widehat{J}_a] = 0$.

This implies that the Hilbert space of a spin-1/2 particle is then the tensor product, $\mathcal{H} = \mathcal{H}_x \otimes \mathcal{H}_{j=1/2}$, of the infinite-dimensional position/momentum space $\mathcal{H}_x$ of the particle with the 2-dimensional spin $j = 1/2$ space $\mathcal{H}_{j=1/2}$ of the spin degrees of freedom of the particle. In particular, we can take an orthonormal basis of $\mathcal{H}$ to be the set of states

$$|x, m\rangle := |x\rangle \otimes |j = \tfrac{1}{2}, m\rangle \qquad \text{for } -\infty < x < \infty \quad \text{and} \quad m \in \{-\tfrac{1}{2}, +\tfrac{1}{2}\},$$

where $|x\rangle$ are the usual position eigenbasis of $\mathcal{H}_x$ and $|j, m\rangle$ are the usual $\widehat{J}^2$ and $\widehat{J}_z$ eigenbasis of $\mathcal{H}_{j=1/2}$. So, in particular,

$$\begin{aligned}
\widehat{X}|x, m\rangle &= x|x, m\rangle & \widehat{P}|x, m\rangle &= i\hbar\frac{d}{dx}|x, m\rangle \\
\widehat{J}_z|x, m\rangle &= \hbar m|x, m\rangle & \widehat{J}^2|x, m\rangle &= \frac{3}{4}\hbar^2|x, m\rangle \\
\langle x, m|x', m'\rangle &= \delta(x - x')\,\delta_{m,m'} & 1 &= \sum_m \int dx\, |x, m\rangle\langle x, m|.
\end{aligned}$$

Problem 1: Show that any state $|\psi\rangle$ can be written as

$$|\psi\rangle = \int dx \left[\psi_+(x)|x, +\tfrac{1}{2}\rangle + \psi_-(x)|x, -\tfrac{1}{2}\rangle\right],$$

where we define

$$\psi_\pm := \langle x, \pm\tfrac{1}{2}|\psi\rangle.$$

This shows that we can describe any state of our spin-1/2 one-dimensional particle by a 2-component vector of wave functions,

$$|\psi\rangle \leftrightarrow \begin{pmatrix}\psi_+(x) \\ \psi_-(x)\end{pmatrix}.$$

Solution: `Insert the identity operator using the completeness relation:` $|\psi\rangle = 1{\cdot}|\psi\rangle = \sum_m \int dx\, |x,m\rangle\langle x,m|\psi\rangle$, `which gives the result.`

Problem 2: If $|\psi\rangle$ is normalized, i.e., $\langle\psi|\psi\rangle = 1$, what does this imply about the $\psi_\pm(x)$ wave functions? **Solution:** `Again, insert the completeness relation:` $1 = \langle\psi|\psi\rangle = \sum_m \int dx\, \langle\psi|x,m\rangle\langle x,m|\psi\rangle = \int dx\, \left(|\psi_+|^2 + |\psi_-|^2\right)$.

Recall that a particle with spin $\vec{J}$ in the presence of a magnetic field $\vec{B}$ has energy $\gamma\vec{B}\cdot\vec{J}$ where $\gamma := qg/2mc$ is a constant depending on the particle's properties. Let's say that the magnetic field depends only on x but points in the $+\widehat{z}$ direction, and call $\gamma\vec{B}(x) = U(x)\widehat{z}$. Also suppose the particle experiences a force in the $\widehat{x}$-direction due to a potential energy $V(x)$. Then the energy observable (Hamiltonian) for the particle is

$$\widehat{H} = \frac{1}{2m}\widehat{P}^2 + V(\widehat{X}) + U(\widehat{X})\,\widehat{J}_z. \tag{1}$$

Problem 3: Show that the energy eigenvalue equation, $\widehat{H}|\psi\rangle = E|\psi\rangle$, in the $|x,m\rangle$ basis is the pair of equations

$$\begin{aligned}
-\frac{\hbar^2}{2m}\psi_+'' + \left[V(x) + \tfrac{\hbar}{2}U(x)\right]\psi_+ &= E\psi_+ \\
-\frac{\hbar^2}{2m}\psi_-'' + \left[V(x) - \tfrac{\hbar}{2}U(x)\right]\psi_- &= E\psi_-
\end{aligned}$$

where the primes denote derivatives with respect to x. **Solution:** `Take the bracket of the equation with` $\langle x,m|$,

$$\begin{aligned}
0 &= \langle x,m|\widehat{H}|\psi\rangle - E\langle x,m|\psi\rangle \\
&= \frac{1}{2m}\langle x,m|\widehat{P}^2|\psi\rangle + \langle x,m|V(\widehat{X})|\psi\rangle + \langle x,m|U(\widehat{X})\,\widehat{J}_z|\psi\rangle - E\psi_m(x) \\
&= \frac{1}{2m}\left(-i\hbar\frac{d}{dx}\right)^2\langle x,m|\psi\rangle + V(x)\langle x,m|\psi\rangle + U(x)(\hbar m)\langle x,m|\psi\rangle - E\psi_m(x) \\
&= -\frac{\hbar^2}{2m}\psi_m'' + V(x)\psi_m + \hbar m U(x)\psi_m - E\psi_m
\end{aligned}$$

`which gives the result for` $m = \pm\frac{1}{2}$.

Now assume that the potentials $V(x)$ and $U(x)$ are related by a "prepotential" $W(x)$ such that

$$V(x) = [W'(x)]^2, \qquad \text{and} \qquad U(x) = -\sqrt{\frac{2}{m}}\,W''(x). \tag{2}$$

Also, recall that the angular momentum raising and lowering operators, $\widehat{J}_\pm := \widehat{J}_x \pm i\widehat{J}_y$, act on the $|m\rangle$ basis as

$$\widehat{J}_\pm|\pm\tfrac{1}{2}\rangle = 0, \qquad \widehat{J}_\pm|\mp\tfrac{1}{2}\rangle = \hbar|\pm\tfrac{1}{2}\rangle \tag{3}$$

where the $\pm$ signs are correlated in each equation. Also, define the operators $\widehat{Q}_\pm$ by

$$\widehat{Q}_\pm := \frac{1}{\hbar}\left[\frac{1}{\sqrt{2m}}\widehat{P} \mp iW'(\widehat{X})\right]\widehat{J}_\pm, \tag{4}$$

where, as usual, the signs are correlated. Finally, recall that the anti-commutator of two operators is defined by $\{\widehat{A},\widehat{B}\} := \widehat{A}\widehat{B}+\widehat{B}\widehat{A}$.

Problem 4: Show that:

$$\{\widehat{Q}_+,\widehat{Q}_-\} = \widehat{H}, \qquad \{\widehat{Q}_\pm,\widehat{Q}_\pm\} = 0, \qquad [\widehat{Q}_\pm,\widehat{H}] = 0, \qquad \text{and} \qquad \widehat{Q}_\pm^\dagger = \widehat{Q}_\mp,$$

where the $\pm$ signs are correlated within each equation. **Solution:** Write $\widehat{Q}_\pm := \widehat{A}_\pm \otimes \widehat{J}_\pm$ where $\widehat{A}_\pm$ is the factor in square brackets in (4) which acts on the $\mathcal{H}_x$ factor of the Hilbert space. Then $\{\widehat{Q}_+,\widehat{Q}_-\} = \widehat{Q}_+\widehat{Q}_- + \widehat{Q}_-\widehat{Q}_+ = \widehat{A}_+\widehat{A}_-\otimes\widehat{J}_+\widehat{J}_- + \widehat{A}_-\widehat{A}_+\otimes\widehat{J}_-\widehat{J}_+$. Compute

$$\begin{aligned}
\widehat{A}_\pm\widehat{A}_\mp &= \frac{1}{\hbar^2}\left\{\frac{1}{2m}\widehat{P}^2 + \left[W'(\widehat{X})\right]^2 \pm \frac{i}{\sqrt{2m}}[\widehat{P},W'(\widehat{X})]\right\}\\
&= \frac{1}{\hbar^2}\left\{\frac{1}{2m}\widehat{P}^2 + \left[W'(\widehat{X})\right]^2 \mp \frac{\hbar}{\sqrt{2m}}W''(\widehat{X})\right\}\\
&= \frac{1}{\hbar^2}\left\{\frac{1}{2m}\widehat{P}^2 + V(\widehat{X}) \pm \frac{\hbar}{2}U(\widehat{X})\right\}
\end{aligned}$$

where in the second line I used that $[\widehat{P},\widehat{X}] = -i\hbar$, and in the last line I used (2). Then

$$\begin{aligned}
\{\widehat{Q}_+,\widehat{Q}_-\} &= \frac{1}{\hbar^2}\left\{\frac{1}{2m}\widehat{P}^2 + V(\widehat{X}) + \frac{\hbar}{2}U(\widehat{X})\right\}\otimes\widehat{J}_+\widehat{J}_- + \frac{1}{\hbar^2}\left\{\frac{1}{2m}\widehat{P}^2 + V(\widehat{X}) - \frac{\hbar}{2}U(\widehat{X})\right\}\otimes\widehat{J}_-\widehat{J}_+\\
&= \left[\frac{1}{2m}\widehat{P}^2 + V(\widehat{X})\right]\otimes\frac{1}{\hbar^2}\{\widehat{J}_+,\widehat{J}_-\} + U(\widehat{X})\otimes\frac{1}{2\hbar}[\widehat{J}_+,\widehat{J}_-].
\end{aligned}$$

Now, from (3) we see that $\{\widehat{J}_+,\widehat{J}_-\}|m\rangle = \hbar^2|m\rangle$ and $[\widehat{J}_+,\widehat{J}_-]|m\rangle = 2m\hbar^2|m\rangle = 2\hbar\widehat{J}_z|m\rangle$. Plugging these into the last equation gives

$$\{\widehat{Q}_+,\widehat{Q}_-\} = \left[\frac{1}{2m}\widehat{P}^2 + V(\widehat{X})\right]\otimes 1 + U(\widehat{X})\otimes\widehat{J}_z = \widehat{H},$$

giving the first result. The other relations are easier. $\{\widehat{Q}_\pm,\widehat{Q}_\pm\} = 2\widehat{Q}_\pm^2 = 2\widehat{A}_\pm^2 \otimes \widehat{J}_\pm^2$. But from (3) it follows that $\widehat{J}_\pm^2 = 0$, giving the second result. $[\widehat{Q}_+,\widehat{H}] = [\widehat{Q}_+,\{\widehat{Q}_+,\widehat{Q}_-\}] = \widehat{Q}_+(\widehat{Q}_+\widehat{Q}_- + \widehat{Q}_-\widehat{Q}_+) - (\widehat{Q}_+\widehat{Q}_- + \widehat{Q}_-\widehat{Q}_+)\widehat{Q}_+ = \widehat{Q}_+^2\widehat{Q}_- + \widehat{Q}_+\widehat{Q}_-\widehat{Q}_+ - \widehat{Q}_+\widehat{Q}_-\widehat{Q}_+ - \widehat{Q}_-\widehat{Q}_+^2 = 0$ where in the last step I used that $\widehat{Q}_\pm^2 = 0$ from the previous result. A similar calculation holds for $[\widehat{Q}_-,\widehat{H}]$. Finally, $\widehat{Q}_\pm^\dagger = \widehat{A}_\pm^\dagger\otimes\widehat{J}_\pm^\dagger = \widehat{A}_\mp\otimes\widehat{J}_\mp = \widehat{Q}_\mp$.

Problem 5: Show that if $|\psi\rangle$ is an energy eigenstate of energy E, that $\widehat{Q}_\pm|\psi\rangle$ are also energy eigenstates of the same energy. **Solution:** If $\widehat{H}|\psi\rangle = E|\psi\rangle$, then $\widehat{H}\widehat{Q}_\pm|\psi\rangle = \widehat{Q}_\pm\widehat{H}|\psi\rangle = E\widehat{Q}_\pm|\psi\rangle$, where in the second step I used that $[\widehat{Q}_\pm,\widehat{H}] = 0$ from the previous problem.

Problem 6: Use the results of the last two problems to show that the energy eigenspace with given positive eigenvalue $E > 0$ has multiplicity 2. **Solution:** doesn't exist!

(Sorry!) The correct statement is only that positive energy eigenstates have even multiplicity.

Problem 7: Show that there are no energy eigenstates of negative energy $E < 0$. **Solution:** $E = E\langle\psi|\psi\rangle = \langle\psi|\widehat{H}|\psi\rangle = \langle\psi|\{\widehat{Q}_+, \widehat{Q}_-\}|\psi\rangle = \langle\psi|\widehat{Q}_+\widehat{Q}_-|\psi\rangle + \langle\psi|\widehat{Q}_-\widehat{Q}_+|\psi\rangle = \langle\psi|\widehat{Q}_-^\dagger\widehat{Q}_-|\psi\rangle + \langle\psi|\widehat{Q}_+^\dagger\widehat{Q}_+|\psi\rangle = ||\widehat{Q}_-|\psi\rangle||^2 + ||\widehat{Q}_+|\psi\rangle||^2 \geq 0$.

Problem 8: Show that states $|\psi\rangle$, $|\phi\rangle$, with wavefunctions

$$\begin{pmatrix} \psi_+(x) \\ \psi_-(x) \end{pmatrix} = \begin{pmatrix} e^{-\sqrt{2m}W(x)/\hbar} \\ 0 \end{pmatrix}$$

and

$$\begin{pmatrix} \phi_+(x) \\ \phi_-(x) \end{pmatrix} = \begin{pmatrix} 0 \\ e^{+\sqrt{2m}W(x)/\hbar} \end{pmatrix}$$

are energy eigenstates with eigenvalue $E = 0$. What are the conditions on the pre-potential $W(x)$ at $x = \pm\infty$ for $|\psi\rangle$ and for $|\phi\rangle$ to be normalizable? **Solution:** From **problem 3** we have $\langle x, -\frac{1}{2}|\widehat{H}|\psi\rangle = 0$ and

$$\begin{aligned}\langle x, \tfrac{1}{2}|\widehat{H}|\psi\rangle &= -\frac{\hbar^2}{2m}\psi_+'' + \left[V + \frac{\hbar}{2}U\right]\psi_+ \\ &= -\frac{\hbar^2}{2m}\left[\frac{2m}{\hbar^2}(W')^2 - \frac{\sqrt{2m}}{\hbar}W''\right]e^{-\sqrt{2m}W/\hbar} + \left[(W')^2 - \frac{\hbar}{\sqrt{2m}}W''\right]e^{-\sqrt{2m}W/\hbar} = 0.\end{aligned}$$

Thus $\widehat{H}|\psi\rangle = 0$. A similar calculation holds for $|\phi\rangle$. From **problem 2**, $|\psi\rangle$ is normalizable if $1 = \int dx(|\psi_+|^2 + |\psi_-|^2) = \int dx \exp\{-2\sqrt{2m}W(x)/\hbar\}$. This integral converges only if the integrand vanishes as $|x| \to \infty$, which implies that we must have $W \to +\infty$ as $|x| \to \infty$. Similarly, $|\phi\rangle$ is normalizable only if $W \to -\infty$ as $|x| \to \infty$. So only one of these two states can exist. Also, note that if $|W| \not\to \infty$ as $|x| \to \infty$ or if $W \to \pm\infty$ as $x \to \mp\infty$, then no $E = 0$ state can occur.