Intro to QM due: October 22, 2018

Problem Set 7

Positronium is the bound state of an electron and a positron. They are both spin-$\frac{1}{2}$ particles with the same mass, m, and gyromagnetic ratio, g, but with opposite charges: $-e$ for the electron and $+e$ for the positron. Consider the spin states of positronium in a uniform magentic field $\vec{B} = B_0\hat{z}$. Define the spin precession frequency $\omega_0 := geB_0/2mc$ as usual. Also, define the $\widehat{J}_z$ eigenbasis of the Hilbert space of the two particles by $|\pm\pm\rangle := |\pm\hat{z}, \pm\hat{z}\rangle$, where the first sign refers to the electron's spin and the second to the positron's.

Problem 1: If you neglect the interaction between the electron and positron spins, show that the spin Hamiltonian for positronium is

$$\widehat{H} = \omega_0(\widehat{J}_{1z} - \widehat{J}_{2z}), \tag{1}$$

where $\vec{J}_1$ is the spin angular momentum of the electron and $\vec{J}_2$ is that of the positron.
Solution: `The energy of a particle's spin in a magnetic field is` $E = -(qg/2mc)\vec{B}\cdot\vec{J} = -(qgB_0/2mc)J_z$ `where I've used that` $\vec{B} = B_0\hat{z}$. `So the energy of the electron's spin is` $E_1 = \omega_0 J_{1z}$`, and the energy of the positron's spin is` $E_2 = -\omega_0 J_{2z}$`. Thus, since we are neglecting the interaction between the two, the total energy operator is given in` (1)`.`

Problem 2: What are the energy eigenvalues and an energy eigenbasis for (1) in terms of the $|\pm\pm\rangle$ basis? **Solution:** `The` $|\pm\pm\rangle$ `is itself an energy eigenbasis, with eigenvalues`

$$\begin{aligned}
\widehat{H}|++\rangle &= \quad\;\; 0\cdot|++\rangle,\\
\widehat{H}|+-\rangle &= +\hbar\omega_0\cdot|+-\rangle,\\
\widehat{H}|-+\rangle &= -\hbar\omega_0\cdot|-+\rangle,\\
\widehat{H}|--\rangle &= \quad\;\; 0\cdot|--\rangle.
\end{aligned}$$

`So the energy eigenvalues are` $E_{\pm\pm} = (\pm1\mp1)\hbar\omega_0/2$ `where the signs are correlated.`

Problem 3: Write the general state of the system as $|\psi(t)\rangle = a_{++}(t)|++\rangle + a_{+-}(t)|+-\rangle + a_{-+}(t)|-+\rangle + a_{--}(t)|--\rangle$. Find $a_{\pm\pm}(t)$ in terms of their initial values $a_{\pm\pm}(0)$. **Solution:** `From the general solution` $|\psi(t)\rangle = \sum_n e^{-iE_nt/\hbar}|E_n\rangle\langle E_n|\psi(0)\rangle$`, and fromthe last problem where we found that` $|E_n\rangle = |\pm\pm\rangle$`, and since` $\langle\pm\pm|\psi(0)\rangle = a_{\pm\pm}$`, we have`

$$|\psi(t)\rangle = \sum_{\pm\pm} e^{-iE_{\pm\pm}t/\hbar}a_{\pm\pm}(0)|\pm\pm\rangle = \sum_{\pm\pm} e^{-i(\pm1\mp1)\omega_0t/2}a_{\pm\pm}(0)|\pm\pm\rangle,$$

`(pairs of signs correlated). This then gives`

$$a_{\pm\pm}(t) = e^{-i(\pm1\mp1)\omega_0t/2}a_{\pm\pm}(0).$$

Problem 4: If at time $t = 0$ the positronium is in the total spin $j = 0$ state, what are $a_{\pm\pm}(0)$? **Solution:** `The total spin` $j = 0$ `state is` $|j{=}0\rangle = (|+-\rangle - |-+\rangle)/\sqrt{2}$`, so has`

$$a_{++}(0) = a_{--}(0) = 0, \qquad a_{+-}(0) = -a_{-+}(0) = 1/\sqrt{2}.$$

Problem 5: Given the initial conditions found in **problem 4**, show that the state of the system oscillates between the total spin $j = 0$ and spin $j = 1$ states, and determine the frequency of oscillation. **Solution:** `Plug the initial conditions found in` **problem 4** `into the solution found in` **problem 3** `to get`

$$a_{++}(t) = a_{--}(t) = 0, \qquad a_{+-}(t) = +e^{-i\omega_0 t}/\sqrt{2}, \qquad a_{-+}(t) = -e^{+i\omega_0 t}/\sqrt{2},$$

`so the state at time` t `is`

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}\left(e^{-i\omega_0 t}|+-\rangle - e^{+i\omega_0 t}|-+\rangle\right).$$

`Define` $T := \pi/\omega_0$. `At times` $t = nT$ `for` n `an integer,` $e^{-i\omega_0 t} = e^{i\omega_0 t}$`, while at times` $t = (n+\frac{1}{2})T$, $e^{-i\omega_0 t} = -e^{i\omega_0 t}$. `So at times` nT `the particles are in the total spin singlet` $|j,m\rangle = |0,0\rangle$ `state, while at times` $(n+\frac{1}{2})T$ `they are in the total spin` $|j,m\rangle = 1,0$ `state. Thus the system oscillates between a` $j = 0$ `and a` $j = 1$ `state with frequency` $\omega = 2\pi/T = 2\omega_0$.

Problem 6: Given the initial conditions found in **problem 4**, measure $\widehat{J}_{1x}$ and $\widehat{J}_{2x}$ at time t. What is the probability that *both* measurements give $+\hbar/2$? **Solution:** `The probability of measuring` $\widehat{J}_{1x} = \hbar/2$ `is`

$$\begin{aligned}\mathcal{P}(J_{1x}{=}J_{2x}{=}\tfrac{\hbar}{2}) &= \langle\psi(t)|\widehat{P}_{J_{x1}=J_{2x}=\frac{\hbar}{2}}|\psi(t)\rangle = \langle\psi(t)|\Big(|+\widehat{x}\rangle_1\langle+\widehat{x}|_1\Big)\otimes\Big(|+\widehat{x}\rangle_2\langle+\widehat{x}|_2\Big)|\psi(t)\rangle \\ &= \frac{1}{4}\langle\psi(t)|\Big(|+\rangle_1+|-\rangle_1\Big)\Big(\langle+|_1+\langle-|_1\Big)\otimes\Big(|+\rangle_2+|-\rangle_2\Big)\Big(\langle+|_2+\langle-|_2\Big)|\psi(t)\rangle \\ &= \frac{1}{4}\langle\psi(t)|\Big(|++\rangle+|+-\rangle+|-+\rangle+|--\rangle\Big)\Big(\langle++|+\langle+-|+\langle-+|+\langle--|\Big)|\psi(t)\rangle.\end{aligned}$$

`But` $\Big(\langle++|+\langle+-|+\langle-+|+\langle--|\Big)|\psi(t)\rangle = \sum_{\pm\pm} a_{\pm\pm}(t) = i\sqrt{2}\sin(\omega_0 t)$`, where in the last step we used the solution we found in` **problem 5**. `Plugging this in gives`

$$\mathcal{P}(J_{1x}{=}J_{2x}{=}\tfrac{\hbar}{2}) = \frac{1}{4}\Big[-i\sqrt{2}\sin(\omega_0 t)\Big]\cdot\Big[i\sqrt{2}\sin(\omega_0 t)\Big] = \tfrac{1}{2}\sin^2(\omega_0 t).$$

Problem 7: If we also include the spin-spin interaction between the electron and positron, the spin Hamiltonian becomes

$$\widehat{H} = \omega_0(\widehat{J}_{1z} - \widehat{J}_{2z}) + \frac{2A}{\hbar^2}\vec{\widehat{J}}_1\cdot\vec{\widehat{J}}_2, \tag{2}$$

where A is a real constant with dimensions of energy. Using that the total spin angular momentum is $\vec{J} = \vec{J}_1 + \vec{J}_2$, show that $\widehat{H}$ can be rewritten as

$$\widehat{H} = \omega_0(\widehat{J}_{1z} - \widehat{J}_{2z}) + A\left(\frac{1}{\hbar^2}\widehat{J}^2 - \frac{3}{2}\right). \tag{3}$$

Solution: The basic trick is to notice that $J^2 = \vec{J}\cdot\vec{J} = (\vec{J}_1+\vec{J}_2)\cdot(\vec{J}_1+\vec{J}_2) = \vec{J}_1\cdot\vec{J}_1+\vec{J}_1\cdot\vec{J}_2+\vec{J}_2\cdot\vec{J}_1+\vec{J}_2\cdot\vec{J}_2 = J_1^2+J_2^2+2\vec{J}_1\cdot\vec{J}_2$, where we used in the last step that $\vec{J}_1\cdot\vec{J}_2 = \vec{J}_2\cdot\vec{J}_1$ since $\vec{J}_1$ and $\vec{J}_2$ act on different factors of the tensor product. Thus $2\vec{J}_1\cdot\vec{J}_2 = J^2 - J_1^2 - J_2^2$. Plugging this into (2) then gives

$$\begin{aligned}
\widehat{H} &= \omega_0(\widehat{J}_{1z} - \widehat{J}_{2z}) + \frac{A}{\hbar^2}\left(\widehat{J}^2 - \widehat{J}_1^2 - \widehat{J}_2^2\right) \\
&= \omega_0(\widehat{J}_{1z} - \widehat{J}_{2z}) + \frac{A}{\hbar^2}\left(\widehat{J}^2 - \frac{3}{4}\hbar^2 - \frac{3}{4}\hbar^2\right) \\
&= \omega_0(\widehat{J}_{1z} - \widehat{J}_{2z}) + A\left(\frac{1}{\hbar^2}\widehat{J}^2 - \frac{3}{2}\right),
\end{aligned}$$

where we used in the second step that $\widehat{J}_1^2 = \hbar^2 j_1(j_1+1) = \frac{3}{4}\hbar^2$ since $j_1 = \frac{1}{2}$, and similarly for $\widehat{J}_2^2$.

Problem 8: Use (3) to find the energy eigenvalues. **Solution:** In the total spin $|j,m\rangle$ basis, we know from the addition of two spin-$\frac{1}{2}$'s that a basis is $|1,1\rangle = |++\rangle$, $|1,-1\rangle = |--\rangle$, $|1,0\rangle = (|+-\rangle + |-+\rangle)/\sqrt{2}$, and $|0,0\rangle = (|+-\rangle - |-+\rangle)/\sqrt{2}$. This implies that

$$\begin{aligned}
(\widehat{J}_{1z} - \widehat{J}_{2z})|1,0\rangle &= (\widehat{J}_{1z} - \widehat{J}_{2z})(|+-\rangle + |-+\rangle)/\sqrt{2} \\
&= \frac{1}{\sqrt{2}}\left[\widehat{J}_{1z}|+-\rangle + \widehat{J}_{1z}|-+\rangle - \widehat{J}_{2z}|+-\rangle - \widehat{J}_{2z}|-+\rangle\right] \\
&= \frac{1}{\sqrt{2}}\left[(+\tfrac{\hbar}{2})|+-\rangle + (-\tfrac{\hbar}{2})|-+\rangle - (-\tfrac{\hbar}{2})|+-\rangle - (+\tfrac{\hbar}{2})|-+\rangle\right] \\
&= \frac{\hbar}{\sqrt{2}}\left[|+-\rangle - |-+\rangle\right] = \hbar|0,0\rangle,
\end{aligned}$$

and similarly

$$(\widehat{J}_{1z} - \widehat{J}_{2z})|0,0\rangle = \hbar|1,0\rangle.$$

Also, recall that $\widehat{J}^2|j,m\rangle = \hbar^2 j(j+1)|j,m\rangle$. Putting these all together gives

$$\begin{aligned}
\widehat{H}|1,1\rangle &= \left[\omega_0\cdot 0 + A\left(\frac{1}{\hbar^2}\cdot\hbar^2 1\cdot(1+1) - \frac{3}{2}\right)\right]|1,1\rangle = \frac{A}{2}|1,1\rangle, \\
\widehat{H}|1,-1\rangle &= \left[\omega_0\cdot 0 + A\left(\frac{1}{\hbar^2}\cdot\hbar^2 1\cdot(1+1) - \frac{3}{2}\right)\right]|1,-1\rangle = \frac{A}{2}|1,-1\rangle, \\
\widehat{H}|1,0\rangle &= \omega_0\cdot\hbar|0,0\rangle + A\left(\frac{1}{\hbar^2}\cdot\hbar^2 1\cdot(1+1) - \frac{3}{2}\right)|1,-1\rangle = \hbar\omega_0|0,0\rangle + \frac{A}{2}|1,0\rangle, \\
\widehat{H}|0,0\rangle &= \omega_0\cdot\hbar|1,0\rangle + A\left(\frac{1}{\hbar^2}\cdot\hbar^2 0\cdot(0+1) - \frac{3}{2}\right)|0,0\rangle = \hbar\omega_0|1,0\rangle - \frac{3A}{2}|0,0\rangle.
\end{aligned}$$

From the first two lines we see there is a double-degenerate eigenvalue $A/2$. Writing the last two lines in matrix form gives

$$\widehat{H}\begin{pmatrix}|1,0\rangle \\ |0,0\rangle\end{pmatrix} = \begin{pmatrix}A/2 & \hbar\omega_0 \\ \hbar\omega_0 & -3A/2\end{pmatrix}\begin{pmatrix}|1,0\rangle \\ |0,0\rangle\end{pmatrix}.$$

The eigenvalues of this matrix are then given by the roots of

$$0 = \det\begin{pmatrix}-\lambda + A/2 & \hbar\omega_0 \\ \hbar\omega_0 & -\lambda - 3A/2\end{pmatrix} = (\lambda - \tfrac{1}{2}A)(\lambda + \tfrac{3}{2}A) - (\hbar\omega_0)^2 = \lambda^2 + A\lambda - \left[\tfrac{3}{4}A^2 + (\hbar\omega_0)^2\right],$$

which are $\lambda = -\frac{1}{2}A \pm \sqrt{A^2 + (\hbar\omega_0)^2}$. Thus the eigenvalues of $\widehat{H}$ are

$$-\tfrac{1}{2}A - \sqrt{A^2 + (\hbar\omega_0)^2} \ , \ +\tfrac{1}{2}A \ , \ +\tfrac{1}{2}A \ , \ -\tfrac{1}{2}A + \sqrt{A^2 + (\hbar\omega_0)^2} \ .$$