Intro to QM due: October 1, 2018

Exam 1

Problem 1: (15 points) Consider a two-dimensional Hilbert space with some orthonormal basis in which three operators $\widehat{A}$, $\widehat{B}$, $\widehat{C}$ have matrix elements

$$\widehat{A} = \begin{pmatrix} 2 & i \\ -i & 0 \end{pmatrix}, \qquad \widehat{B} = \begin{pmatrix} 0 & 2i \\ -2i & 1 \end{pmatrix}, \qquad \widehat{C} = \begin{pmatrix} 1 & 2i \\ -2i & i \end{pmatrix}.$$

It may be useful to recall that $\det \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = ad - bc$.

(a) Which of these operators is hermitian?

Solution: $\widehat{A}$ and $\widehat{B}$.

(b) What are the eigenvalues of $\widehat{A}$?

Solution: The characteristic equation is $0 = \det(\widehat{A} - \lambda) = (2-\lambda)(-\lambda) - (i)(-i) = \lambda^2 - 2\lambda - 1$, giving $\lambda = 1 \pm \sqrt{2}$.

(c) Are $\widehat{A}$ and $\widehat{B}$ simultaneously diagonalizable?

Solution: No, because $[\widehat{A}, \widehat{B}] = \widehat{A}\widehat{B} - \widehat{B}\widehat{A} = \left(\begin{smallmatrix} 2 & 5i \\ 0 & 2 \end{smallmatrix}\right) - \left(\begin{smallmatrix} 2 & 0 \\ -5i & 2 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 5i \\ 5i & 0 \end{smallmatrix}\right) \neq 0$.

Problem 2: (20 points) Consider a three-dimensional Hilbert space with some orthonormal basis in which a hermitian operator $\widehat{A}$ has the following matrix elements, and a (normalized) state $|\psi\rangle$ has components

$$\widehat{A} = \begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \qquad |\psi\rangle = \frac{1}{\sqrt{6}} \begin{pmatrix} \sqrt{3} \\ 1 \\ i\sqrt{2} \end{pmatrix}.$$

(a) What is the probability for measuring $A = \sqrt{2}$ in the state $|\psi\rangle$?

Solution:

$$\mathcal{P}(A{=}\sqrt{2}) = \langle\psi|\widehat{P}_{A=\sqrt{2}}|\psi\rangle = |\langle A{=}\sqrt{2}|\psi\rangle|^2 = \left| \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \frac{1}{\sqrt{6}} \begin{pmatrix} \sqrt{3} \\ 1 \\ i\sqrt{2} \end{pmatrix} \right|^2 = \frac{1}{2}.$$

(b) What is the probability for measuring $A = 3$ in the state $|\psi\rangle$?

Solution:

$$\mathcal{P}(A{=}3) = \langle\psi|\widehat{P}_{A=3}|\psi\rangle = \sum_{j=1,2} |\langle A{=}3_j|\psi\rangle|^2 = \left| \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \frac{1}{\sqrt{6}} \begin{pmatrix} \sqrt{3} \\ 1 \\ i\sqrt{2} \end{pmatrix} \right|^2 + \left| \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \frac{1}{\sqrt{6}} \begin{pmatrix} \sqrt{3} \\ 1 \\ i\sqrt{2} \end{pmatrix} \right|^2 = \frac{1}{2}.$$

(c) What is the probability for measuring $A = \frac{1}{2}(3 + \sqrt{2})$ in the state $|\psi\rangle$?

Solution: The probability is 0, since $(3 + \sqrt{2})/2$ is not an eigenvalue of $\widehat{A}$.

(d) If A is measured on $|\psi\rangle$ and the result $A = 3$ is found, what is the state of the system immediately after the measurement?

Solution: `Upon measuring` $A = 3$, $|\psi\rangle$ `is projected onto the` $A = 3$ `eigenspace and normalized:`

$$|\psi\rangle \to |\psi'\rangle = \frac{\widehat{P}_{A=3}|\psi\rangle}{\|\widehat{P}_{A=3}|\psi\rangle\|},$$

`where`

$$\widehat{P}_{A=3} = \sum_{j=1,2} |A{=}3_j\rangle\langle A{=}3_j| = \begin{pmatrix}0\\1\\0\end{pmatrix}\begin{pmatrix}0 & 1 & 0\end{pmatrix} + \begin{pmatrix}0\\0\\1\end{pmatrix}\begin{pmatrix}0 & 0 & 1\end{pmatrix} = \begin{pmatrix}0 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}.$$

`Then`

$$\widehat{P}_{A=3}|\psi\rangle = \frac{1}{\sqrt{6}}\begin{pmatrix}0 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}\sqrt{3}\\1\\i\sqrt{2}\end{pmatrix} = \frac{1}{\sqrt{6}}\begin{pmatrix}0\\1\\i\sqrt{2}\end{pmatrix}$$

`and` $\|\widehat{P}_{A=3}|\psi\rangle\| = 1/\sqrt{2}$, `so`

$$|\psi'\rangle = \frac{1}{\sqrt{3}}\begin{pmatrix}0\\1\\i\sqrt{2}\end{pmatrix}.$$

Problem 3: (15 points) A deuteron is a kind of hydrogen nucleus which is a bound state of a neutron and a proton. It turns out that it can occur[1] in states with total spin quantum numbers $j = 0$ or $j = 1$. Thus it is described by a 4-dimensional Hilbert space with orthonormal basis given by the simultaneous eigenvectors $|j, m\rangle$ of the $\widehat{J}^2$ and $\widehat{J}_z$ operators: $\{|1,1\rangle, |1,0\rangle, |1,-1\rangle, |0,0\rangle\}$. In this basis in this order (ie, so that $|1,1\rangle$ corresponds to the first row/column, $|1,0\rangle$ corresponds to the second row/column, etc.) the matrix elements of the $\widehat{J}_x$ operator are:

$$\widehat{J}_x = \frac{\hbar}{\sqrt{2}}\begin{pmatrix}0 & 1 & 0 & 0\\1 & 0 & 1 & 0\\0 & 1 & 0 & 0\\0 & 0 & 0 & 0\end{pmatrix}.$$

Recall also that $\widehat{J}^2|j,m\rangle = \hbar^2 j(j+1)|j,m\rangle$ and $\widehat{J}_z|j,m\rangle = \hbar m|j,m\rangle$. Say we have prepared a deuteron in the (normalized) state

$$|\psi\rangle = \frac{1}{\sqrt{3}}\left(i\sqrt{2}|1,0\rangle + |0,0\rangle\right).$$

(a) What are the possible values you can find if you measure $\widehat{J}_z$ on this state and what are their probabilities? **Solution:** `The possible outcomes are` $J_z = 0$ `with probability` $\mathcal{P}(J_z{=}0) = 1$.

[1]Actually, the $j = 0$ state is very short-lived compared to the $j = 1$ states; but we will ignore this fact.

(b) What are the possible values you can find if you measure $\widehat{J}^2$ on this state and what are their probabilities? **Solution:** `The possible outcomes are` $J^2 = \hbar^2$ `with probability` $\mathcal{P}(J^2{=}\hbar^2) = 2/3$ `and` $J^2 = 0$ `with probability` $\mathcal{P}(J^2{=}0) = 1/3$`.`

(c) What is the expectation value of $\widehat{J}_x$ in the state $|\psi\rangle$? **Solution:**

$$\langle J_x\rangle = \langle\psi|\widehat{J}_x|\psi\rangle = \frac{\hbar}{3\sqrt{2}}\begin{pmatrix}0 & -i\sqrt{2} & 0 & 1\end{pmatrix}\begin{pmatrix}0 & 1 & 0 & 0\\ 1 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}\begin{pmatrix}0\\ i\sqrt{2}\\ 0\\ 1\end{pmatrix}$$

$$= \frac{\hbar}{3\sqrt{2}}\begin{pmatrix}-i\sqrt{2} & 0 & -i\sqrt{2} & 0\end{pmatrix}\begin{pmatrix}0\\ i\sqrt{2}\\ 0\\ 1\end{pmatrix} = 0.$$