Intro to QM due: February 13, 2019

Problem Set 13

Problem 1: A hydrogen atom is in the state

$$|\psi(0)\rangle = \frac{1}{\sqrt{3}}\left(\sqrt{2}|1,0,0\rangle + i|2,1,1\rangle\right)$$

at time $t=0$, where $|n,\ell,m\rangle$ are the usual hydrogen energy eigenstates. Calculate $|\psi(t)\rangle$, $\langle\widehat{H}\rangle(t)$, $\langle\widehat{L}^2\rangle(t)$, and $\langle\widehat{L}_z\rangle(t)$. **Solution:**

$$|\psi(t)\rangle = \sum_{n,\ell,m} e^{-iE_n t/\hbar}|n,\ell,m\rangle\langle n,\ell,m|\psi(0)\rangle = \frac{1}{\sqrt{3}}\left(e^{-i\omega t}\sqrt{2}|1,0,0\rangle + e^{-i\omega t/4}i|2,1,1\rangle\right),$$

where I have defined $\omega := -\mu c^2\alpha^2/(2\hbar)$ so that $E_n = \omega\hbar/n^2$, and in the second step I used the orthonormality of the energy eigenstates. Then

$$\begin{aligned}
\langle\widehat{H}\rangle(t) &= \langle\psi(t)|\widehat{H}|\psi(t)\rangle = \frac{1}{3}\left(e^{i\omega t}\sqrt{2}\langle 1,0,0| - e^{i\omega t/4}i\langle 2,1,1|\right)\widehat{H}\left(e^{-i\omega t}\sqrt{2}|1,0,0\rangle + e^{-i\omega t/4}i|2,1,1\rangle\right)\\
&= \frac{1}{3}\left(e^{i\omega t}\sqrt{2}\langle 1,0,0| - e^{i\omega t/4}i\langle 2,1,1|\right)\left(e^{-i\omega t}\omega\hbar\sqrt{2}|1,0,0\rangle + e^{-i\omega t/4}\omega\hbar\frac{i}{4}|2,1,1\rangle\right)\\
&= \frac{\omega\hbar}{3}\left(2+\frac{1}{4}\right) = \frac{3\omega\hbar}{4},\\
\langle\widehat{L}^2\rangle(t) &= \langle\psi(t)|\widehat{L}^2|\psi(t)\rangle = \frac{1}{3}\left(e^{i\omega t}\sqrt{2}\langle 1,0,0| - e^{i\omega t/4}i\langle 2,1,1|\right)\widehat{L}^2\left(e^{-i\omega t}\sqrt{2}|1,0,0\rangle + e^{-i\omega t/4}i|2,1,1\rangle\right)\\
&= \frac{1}{3}\left(e^{i\omega t}\sqrt{2}\langle 1,0,0| - e^{i\omega t/4}i\langle 2,1,1|\right)\left(e^{-i\omega t}\hbar^2 0(0+1)\sqrt{2}|1,0,0\rangle + e^{-i\omega t/4}\hbar^2 1(1+1)i|2,1,1\rangle\right)\\
&= \frac{2\hbar^2}{3},\\
\langle\widehat{L}_z\rangle(t) &= \langle\psi(t)|\widehat{L}_z|\psi(t)\rangle = \frac{1}{3}\left(e^{i\omega t}\sqrt{2}\langle 1,0,0| - e^{i\omega t/4}i\langle 2,1,1|\right)\widehat{L}_z\left(e^{-i\omega t}\sqrt{2}|1,0,0\rangle + e^{-i\omega t/4}i|2,1,1\rangle\right)\\
&= \frac{1}{3}\left(e^{i\omega t}\sqrt{2}\langle 1,0,0| - e^{i\omega t/4}i\langle 2,1,1|\right)\left(e^{-i\omega t}\hbar\cdot 0\sqrt{2}|1,0,0\rangle + e^{-i\omega t/4}\hbar\cdot 1\cdot i|2,1,1\rangle\right) = \frac{\hbar}{3}.
\end{aligned}$$

None of these have any time dependence. This follows from first principles since all three are conserved quantities.

Problem 2: Calculate the probability that an electron in the ground state of hydrogen is outside the classically allowed region. **Solution:** The hydrogen ground state energy is $E_1 = -\mu c^2\alpha^2/2$. The classical turning point in the Coulomb potential $V(r) = -e^2/r = -\hbar c\alpha/r$ at this energy is at the radius r_c such that $V(r_c) = E_1$, or

$$r_c = 2\hbar/(\mu c\alpha) = 2a_0,$$

where I used the definition of the Bohr radius. The ground state wavefunction is

$$\psi_{1,0,0} = R_{1,0}(r)Y_{0,0}(\theta,\phi) = \frac{2}{\sqrt{4\pi a_0^3}}e^{-r/a_0},$$

so the probability of finding the electron in the classically forbidden region is

$$\text{Prob}(r > r_c) = \int_{r_c}^{\infty} r^2 dr \int d\Omega \ |\psi_{1,0,0}(r,\theta,\phi)|^2 = \frac{1}{\pi a_0^3} 4\pi \cdot \int_{2a_0}^{\infty} r^2 dr e^{-2r/a_0} = \frac{4}{a_0^3} \cdot \frac{13 a_0^3}{4e^4} = \frac{13}{e^4} \approx .24$$

Calculate (i) the ground state energy, (ii) the Bohr radius, and (iii) the wavelength of the radiation emitted in the transition from the $n = 2$ to $n = 1$ state for each of the following 2-particle systems (problems 3-5):

Problem 3: H^2, which is a bound state of a deuteron and an electron. **Solution:** The ground state energy, Bohr radius, and $2 \to 1$ transition wavelength of a Coulombic bound state of two particles with charges $\pm e$ are

$$E_1 = \mu c^2 \alpha^2 / 2, \qquad a_0 = \hbar / (\mu c \alpha), \qquad \lambda_{2\to 1} = \frac{2h}{\mu c \alpha^2} \left(1^{-2} - 2^{-2}\right)^{-1} = \frac{8h}{3\mu c \alpha^2}.$$

$\hbar$, c, and α do not change for this problem (nor for problems 4 and 5) relative to the hydrogen atom. All that changes is the reduced mass. Writing μ_{H^2} for the H^2 reduced mass, and μ_H for the hydrogen reduced mass, we get

$$\frac{\mu_{H^2}}{\mu_H} = \frac{m_e m_d}{m_e + m_d} \cdot \frac{m_e + m_p}{m_e m_p} = \frac{2m_p(m_e + m_p)}{m_p(m_e + 2m_p)} = \frac{1 + (m_e/m_p)}{1 + \frac{1}{2}(m_e/m_p)} \approx 1 + \tfrac{1}{4000}. \tag{1}$$

Here I used that $m_d \approx 2m_p$, and that $m_p \approx 2000 m_e$. So, relative to hydrogen, the ground state energy of H^2 is reduced by about one part in 2000, while the Bohr radius and transition wavelength are increased by about one part in 2000.

Problem 4: Positronium. **Solution:** Just as in **problem 3**, since

$$\frac{\mu_{e^+e^-}}{\mu_H} = \frac{m_e m_e}{m_e + m_e} \cdot \frac{m_e + m_p}{m_e m_p} = \frac{m_e + m_p}{2m_p} = \frac{1}{2}(1 + \tfrac{m_e}{m_p}) \approx \frac{1}{2},$$

(where I used that the mass of an anti-electron is the same as the mass of an electron) it follows from (1) that, relative to hydrogen, the ground state energy of H^2 is reduced by a factor of 2, while the Bohr radius and transition wavelength are both increased by a factor of 2.

Problem 5: A bound state of a proton and a negative muon. **Solution:** Just as in **problem 3**,

$$\frac{\mu_{p\mu^-}}{\mu_H} = \frac{m_p m_\mu}{m_p + m_\mu} \cdot \frac{m_e + m_p}{m_e m_p} = \frac{m_\mu(m_e + m_p)}{m_e(m_p + m_\mu)} = \frac{m_\mu}{m_e} \frac{1 + \frac{m_e}{m_p}}{1 + \frac{m_\mu}{m_p}} \approx 200 \frac{1 + \frac{1}{2000}}{1 + \frac{1}{10}} \approx 180,$$

where I used that $m_\mu / m_e \approx 200$, $m_\mu / m_p \approx 1/10$, and $m_e / m_p \approx 1/2000$. It follows from (1) that, relative to hydrogen, the ground state energy of $p\mu^-$ is increased by a factor of 180, while the Bohr radius and transition wavelength are both decreased by a factor of 180.