Intro to QM due: September 5, 2018

Problem Set 1

Let $\widehat{n}$ be a unit vector in 3 dimensions with polar and azimuthal angles (θ, ϕ) in a spherical coordinates (see figure 1.11 of the text). Consider the following vectors in the state space of a spin-$\frac{1}{2}$ particle:

$$
\begin{aligned}
|+\widehat{n}\rangle &:= \cos\frac{\theta}{2}|+\widehat{z}\rangle + e^{i\phi}\sin\frac{\theta}{2}|-\widehat{z}\rangle, \\
|-\widehat{n}\rangle &:= \sin\frac{\theta}{2}|+\widehat{z}\rangle - e^{i\phi}\cos\frac{\theta}{2}|-\widehat{z}\rangle.
\end{aligned}
$$

Problem 1: Show that $\{|+\widehat{n}\rangle, |-\widehat{n}\rangle\}$ form an orthonormal basis of the spin-$\frac{1}{2}$ state space. **Solution:** `To show they form an orthonormal basis, we need to show that` $\langle \pm\widehat{n}|\pm\widehat{n}\rangle = 1$ `and` $\langle \pm\widehat{n}|\mp\widehat{n}\rangle = 0$.

$$
\begin{aligned}
\langle +\widehat{n}|+\widehat{n}\rangle &= \left(\cos\frac{\theta}{2}\langle +\widehat{z}| + e^{-i\phi}\sin\frac{\theta}{2}\langle -\widehat{z}|\right)\left(\cos\frac{\theta}{2}|+\widehat{z}\rangle + e^{i\phi}\sin\frac{\theta}{2}|-\widehat{z}\rangle\right) \\
&= \cos^2\frac{\theta}{2}\langle +\widehat{z}|+\widehat{z}\rangle + e^{i\phi}\sin\frac{\theta}{2}\cos\frac{\theta}{2}\langle +\widehat{z}|-\widehat{z}\rangle + e^{-i\phi}\sin\frac{\theta}{2}\cos\frac{\theta}{2}\langle -\widehat{z}|+\widehat{z}\rangle + \sin^2\frac{\theta}{2}\langle -\widehat{z}|-\widehat{z}\rangle \\
&= \cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2} = 1,
\end{aligned}
$$

`using the orthonormality of the` $|\pm\widehat{z}\rangle$ `basis. An almost identitical calculation shows that` $\langle -\widehat{n}|-\widehat{n}\rangle = 1$.

$$
\begin{aligned}
\langle +\widehat{n}|-\widehat{n}\rangle &= \left(\cos\frac{\theta}{2}\langle +\widehat{z}| + e^{-i\phi}\sin\frac{\theta}{2}\langle -\widehat{z}|\right)\left(\sin\frac{\theta}{2}|+\widehat{z}\rangle - e^{i\phi}\cos\frac{\theta}{2}|-\widehat{z}\rangle\right) \\
&= \sin\frac{\theta}{2}\cos\frac{\theta}{2}\langle +\widehat{z}|+\widehat{z}\rangle - e^{i\phi}\cos^2\frac{\theta}{2}\langle +\widehat{z}|-\widehat{z}\rangle + e^{-i\phi}\sin^2\frac{\theta}{2}\langle -\widehat{z}|+\widehat{z}\rangle - \sin\frac{\theta}{2}\cos\frac{\theta}{2}\langle -\widehat{z}|-\widehat{z}\rangle \\
&= \sin\frac{\theta}{2}\cos\frac{\theta}{2} - \sin\frac{\theta}{2}\cos\frac{\theta}{2} = 0,
\end{aligned}
$$

`which also implies that` $\langle -\widehat{n}|+\widehat{n}\rangle = 0$.

Problem 2: Compute the probability $\mathcal{P}(|+\widehat{n}\rangle \Rightarrow |+\widehat{n}'\rangle)$ of measuring a particle prepared in state $|+\widehat{n}\rangle$ to be in state $|+\widehat{n}'\rangle$, where $\widehat{n}'$ is the unit vector with angles (θ', ϕ'). **Solution:** `First calculate`

$$
\begin{aligned}
\langle +\widehat{n}'|+\widehat{n}\rangle &= \left(\cos\tfrac{\theta'}{2}\langle +\widehat{z}| + e^{-i\phi'}\sin\tfrac{\theta'}{2}\langle -\widehat{z}|\right)\left(\cos\tfrac{\theta}{2}|+\widehat{z}\rangle + e^{i\phi}\sin\tfrac{\theta}{2}|-\widehat{z}\rangle\right) \\
&= \cos\tfrac{\theta}{2}\cos\tfrac{\theta'}{2} + e^{i(\phi-\phi')}\sin\tfrac{\theta}{2}\sin\tfrac{\theta'}{2}.
\end{aligned}
$$

Then

$$
\begin{aligned}
\mathcal{P}(|+\widehat{n}\rangle \Rightarrow |+\widehat{n}'\rangle) &= |\langle +\widehat{n}'|+\widehat{n}\rangle|^2 \\
&= |\cos\tfrac{\theta'}{2}\cos\tfrac{\theta}{2} + e^{i(\phi-\phi')}\sin\tfrac{\theta}{2}\sin\tfrac{\theta'}{2}|^2 \\
&= \left(\cos\tfrac{\theta'}{2}\cos\tfrac{\theta}{2} + e^{i(\phi-\phi')}\sin\tfrac{\theta}{2}\sin\tfrac{\theta'}{2}\right)\left(\cos\tfrac{\theta'}{2}\cos\tfrac{\theta}{2} + e^{-i(\phi-\phi')}\sin\tfrac{\theta}{2}\sin\tfrac{\theta'}{2}\right) \\
&= \tfrac{1}{4}[1+\cos\theta][1+\cos\theta'] + \tfrac{1}{4}e^{i(\phi-\phi')}\sin\theta\sin\theta' + \tfrac{1}{4}e^{-i(\phi-\phi')}\sin\theta\sin\theta' + \tfrac{1}{4}[1-\cos\theta][1-\cos\theta'] \\
&= \tfrac{1}{2}[1+\cos\theta\cos\theta' + \cos(\phi-\phi')\sin\theta\sin\theta'],
\end{aligned}
$$

where in the 4th line I used the identities $\sin\alpha\cos\alpha = \frac{1}{2}\sin(2\alpha)$, $\cos^2\alpha = \frac{1}{2}[1+\cos(2\alpha)]$, $\sin^2\alpha = \frac{1}{2}[1-\cos(2\alpha)]$, and in the last line I used $\cos\alpha = \frac{1}{2}(e^{i\alpha}+e^{-i\alpha})$.

A spin-$\frac{3}{2}$ particle is described by a 4-dimensional state space with an orthonormal basis of states $\{|m\rangle\}$, $m \in \{-\frac{3}{2}, -\frac{1}{2}, +\frac{1}{2}, +\frac{3}{2}\}$. These basis vectors are states with $S_z = m\hbar$. Consider the following two states,

$$
|\psi_1\rangle := N_1 \sum_{m\in\{\pm\frac{3}{2},\pm\frac{1}{2}\}} \frac{1}{m} e^{im}|m\rangle, \qquad \text{and} \qquad |\psi_2\rangle := N_2 \sum_{m\in\{\pm\frac{3}{2},\pm\frac{1}{2}\}} \frac{1}{m}|m\rangle,
$$

where N_1 and N_2 are some positive numbers.

Problem 3: Compute $\langle m|n\rangle$ for all $m, n \in \{-\frac{3}{2}, -\frac{1}{2}, +\frac{1}{2}, +\frac{3}{2}\}$. (Ie, write a simple formula that gives the answer for all values of m and n.) **Solution:** Since the problem stated that $|m\rangle$ form an orthonormal basis, $\langle m|n\rangle = \delta_{m,n}$ by the definition of ''orthonormal''.

Problem 4: Compute N_1 and N_2 so that $|\psi_1\rangle$ and $|\psi_2\rangle$ are normalized. **Solution:** $|\psi_1\rangle$ normalized means that

$$
\begin{aligned}
1 = \langle\psi_1|\psi_1\rangle &= \left(N_1 \sum_{n\in\{\pm\frac{3}{2},\pm\frac{1}{2}\}} \frac{1}{n}e^{-in}\langle n|\right)\left(N_1 \sum_{m\in\{\pm\frac{3}{2},\pm\frac{1}{2}\}} \frac{1}{m}e^{im}|m\rangle\right) \\
&= N_1^2 \sum_{m,n\in\{\pm\frac{3}{2},\pm\frac{1}{2}\}} \frac{1}{nm}e^{i(m-n)}\langle n|m\rangle = N_1^2 \sum_{m,n\in\{\pm\frac{3}{2},\pm\frac{1}{2}\}} \frac{1}{nm}e^{i(m-n)}\delta_{n,m} \\
&= N_1^2 \sum_{m\in\{\pm\frac{3}{2},\pm\frac{1}{2}\}} \frac{1}{m^2} = N_1^2\frac{80}{9} \quad\Rightarrow\quad N_1 = \frac{3}{4\sqrt{5}}.
\end{aligned}
$$

The computation for N_2 gives the same result since $|\psi_1\rangle$ and $|\psi_2\rangle$ only differ in the phases of their terms and those cancelled inthe above computation. So we have

$$
N_1 = N_2 = \frac{3}{4\sqrt{5}}.
$$

Problem 5: Compute $\langle S_z\rangle$ in the state $|\psi_1\rangle$ and in the state $|\psi_2\rangle$. **Solution:** From the text, in a state $|\psi\rangle = \sum_m c_m|m\rangle$, $\langle S_z\rangle = \sum_m |c_m|^2 m\hbar$. Applying this to $|\psi_1\rangle$ gives

$$\langle S_z\rangle = N_1^2 \sum_m \left|\frac{1}{m}e^{im}\right|^2 m\hbar = \hbar N_1^2 \sum_{m\in\{\pm\frac{3}{2},\pm\frac{1}{2}\}} \frac{1}{m} = \hbar N_1^2\left(-\frac{2}{3}-\frac{2}{1}+\frac{2}{1}+\frac{3}{2}\right) = 0.$$

The answer is the same for the expectation value in the state $|\psi_2\rangle$ since, again, the phases all cancelled.

Problem 6: Compute ΔS_z in the state $|\psi_1\rangle$ and in the state $|\psi_2\rangle$. **Solution:** From the text, $(\Delta S_z)^2 = \langle (S_z)^2\rangle - \langle S_z\rangle^2$. From the last problem $\langle S_z\rangle = 0$, so we need only compute $\langle (S_z)^2\rangle = \sum_m |c_m|^2(\hbar m)^2$, giving in state $|\psi_1\rangle$

$$(\Delta S_z)^2 = N_1^2 \sum_m \left|\frac{1}{m}e^{im}\right|^2 (\hbar m)^2 = \frac{9}{80}\hbar^2 \sum_m 1 = \frac{9}{20}\hbar^2 \qquad \Rightarrow \qquad \Delta S_z = \frac{3\hbar}{2\sqrt{5}}.$$

Again, the answer is the same for the state $|\psi_2\rangle$, for the same reason as in the last two problems.

Problem 7: Compute the probabilities that a measurement gives $S_z = -\frac{\hbar}{2}$ in the state $|\psi_1\rangle$ and in the state $|\psi_2\rangle$. **Solution:**

$$\mathcal{P}(|\psi_1\rangle \Rightarrow |-\tfrac{1}{2}\rangle) = |\langle -\tfrac{1}{2}|\psi_1\rangle|^2 = \left|\langle -\tfrac{1}{2}|N_1\left(\sum_m \frac{1}{m}e^{im}|m\rangle\right)\right|^2 = N_1^2\left|\sum_m \frac{1}{m}e^{im}\langle -\tfrac{1}{2}|m\rangle\right|^2$$
$$= N_1^2\left|\sum_m \frac{1}{m}e^{im}\delta_{m,-1/2}\right|^2 = N_1^2\left|-\frac{2}{1}e^{-i/2}\right|^2 = \frac{9}{80}\cdot 4 = \frac{9}{20}.$$

The answer is the same for the state $|\psi_2\rangle$, for the same reason as in the last three problems.

Problem 8: Compute the probability $\mathcal{P}(|\psi_1\rangle \Rightarrow |\psi_2\rangle)$ of measuring a particle prepared in state $|\psi_1\rangle$ to be in state $|\psi_2\rangle$. **Solution:**

$$\mathcal{P}(|\psi_1\rangle \Rightarrow |\psi_2\rangle) = |\langle\psi_2|\psi_1\rangle|^2 = \left|N_2\left(\sum_n \frac{1}{n}\langle n|\right)N_1\left(\sum_m \frac{1}{m}e^{im}|m\rangle\right)\right|^2 = N_1^2N_2^2\left|\sum_{n,m}\frac{e^{im}}{nm}\langle n|m\rangle\right|^2$$
$$= N_1^2N_2^2\left|\sum_{n,m}\frac{e^{im}}{nm}\delta_{n,m}\right|^2 = N_1^2N_2^2\left|\sum_m \frac{e^{im}}{m^2}\right|^2 = \frac{81}{6400}\left|\frac{4e^{-3i/2}}{9}+4e^{-i/2}+4e^{+i/2}+\frac{4e^{+3i/2}}{9}\right|^2$$
$$= \frac{81}{6400}\left[\frac{8\cos(\frac{3}{2})}{9}+8\cos(\tfrac{1}{2})\right]^2 = \frac{1}{100}\left[\cos(\tfrac{3}{2})+9\cos(\tfrac{1}{2})\right]^2 \approx 0.64.$$