Problem Set 4

All problems in this problem set will be worth 3 points instead of the usual 1 point.

For all these problems we are considering the spin states of a spin-1 particle. The Hilbert space of these states form an $\widehat{J}^2$ eigenspace with eigenvalue $2\hbar^2$, and in the $\widehat{J}_z$ eigenbasis it is spanned by the 3 orthonormal states $|j,m\rangle$ with $j=1$ and $m \in \{-1,0,+1\}$.

Problem 1: Starting from the formulas in the text for the matrix elements of the $\widehat{J}_z$ and $\widehat{J}_\pm$ operators, show that in the $\widehat{J}_z$ eigenbasis that

$$\widehat{J}_x = \frac{\hbar}{\sqrt{2}}\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}, \qquad \widehat{J}_y = \frac{\hbar}{\sqrt{2}}\begin{pmatrix}0 & -i & 0\\ i & 0 & -i\\ 0 & i & 0\end{pmatrix}, \qquad \widehat{J}_z = \hbar\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -1\end{pmatrix}.$$

Solution: Order the $\widehat{J}_z$ eigenbasis so that $|1,1\rangle$ is the first row/column, $|1,0\rangle$ is the second, and $|1,-1\rangle$ is the third. From the definitions of $\widehat{J}_\pm$, $\widehat{J}_x = \frac{1}{2}(\widehat{J}_+ + \widehat{J}_-)$ and $\widehat{J}_y = \frac{-i}{2}(\widehat{J}_+ - \widehat{J}_-)$. From the text, $\widehat{J}_\pm|1,m\rangle = \hbar\sqrt{2-m^2 \mp m}|1,m\pm 1\rangle$, so the $\widehat{J}_x$ matrix elements follow:

$$\begin{aligned}
\langle 1,n|\widehat{J}_x|1,m\rangle &= \langle 1,n|\frac{1}{2}\left(\hbar\sqrt{2-m^2-m}\,|1,m+1\rangle + \hbar\sqrt{2-m^2+m}\,|1,m-1\rangle\right)\\
&= \frac{\hbar}{2}\left(\hbar\sqrt{2-m^2-m}\,\langle 1,n|1,m+1\rangle + \hbar\sqrt{2-m^2+m}\,\langle 1,n|1,m-1\rangle\right)\\
&= \frac{\hbar}{2}\left(\hbar\sqrt{2-m^2-m}\,\delta_{n,m+1} + \hbar\sqrt{2-m^2+m}\,\delta_{n,m-1}\right).
\end{aligned}$$

Plugging in $n,m = 1,0,-1$ then gives the nine matrix elements of the $\widehat{J}_x$. A similiar calculation goes for $\widehat{J}_y$. $\widehat{J}_z$ is even easier since $|1,m\rangle$ is, by definition, the eigenbasis of $\widehat{J}_z$, so in this basis $\widehat{J}_z$ is diagonal with its eigenvalues on the diagonal.

Problem 2: What are the possible values you could find if you measured $\widehat{J}_z$? **Solution:** The possible outcomes are the eigenvalues of $\widehat{J}_z$ which are $J_z = \{\hbar, 0, -\hbar\}$.

Problem 3: Consider the state in which $J_z = \hbar$. In this state what are $\langle J_x\rangle$, $\langle J_x^2\rangle$, and ΔJ_x? **Solution:** $\widehat{J}_z|\psi\rangle = \hbar\cdot|\psi\rangle$ implies

$$|\psi\rangle = \begin{pmatrix}1\\0\\0\end{pmatrix}$$

in the $\widehat{J}_z$ eigenbasis. (Note that I have normalized $|\psi\rangle$!) Then

$$\langle J_x\rangle = \langle\psi|\widehat{J}_x|\psi\rangle = \begin{pmatrix}1 & 0 & 0\end{pmatrix}\frac{\hbar}{\sqrt{2}}\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix} = \frac{\hbar}{\sqrt{2}}\begin{pmatrix}1 & 0 & 0\end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix} = 0.$$

$$\langle J_x^2\rangle = \langle\psi|\widehat{J}_x^2|\psi\rangle = \begin{pmatrix}1 & 0 & 0\end{pmatrix}\frac{\hbar^2}{2}\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix} = \frac{\hbar^2}{2}\begin{pmatrix}0 & 1 & 0\end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix} = \frac{\hbar^2}{2}.$$

$$\Delta J_x = \sqrt{\langle J_x^2\rangle - \langle J_x\rangle^2} = \sqrt{\left(\frac{\hbar^2}{2}\right) - 0^2} = \frac{\hbar}{\sqrt{2}}.$$

Problem 4: Find the normalized eigenstates and the eigenvalues of $\widehat{J}_x$ in the $\widehat{J}_z$ eigenbasis. **Solution:** The characteristic equation for $\widehat{J}_x$ is

$$0 = \det(\widehat{J}_x - \lambda) = \det\begin{pmatrix}-\lambda & \frac{\hbar}{\sqrt{2}} & 0\\ \frac{\hbar}{\sqrt{2}} & -\lambda & \frac{\hbar}{\sqrt{2}}\\ 0 & \frac{\hbar}{\sqrt{2}} & -\lambda\end{pmatrix} = \hbar^2\lambda - \lambda^3, \qquad \Rightarrow \qquad \lambda \in \{\hbar, 0, -\hbar\}.$$

The corresponding eigenvectors, $|\lambda\rangle$, then satisfy

$$0 = (\widehat{J}_x - \lambda)|\lambda\rangle = \begin{pmatrix}-\lambda & \frac{\hbar}{\sqrt{2}} & 0\\ \frac{\hbar}{\sqrt{2}} & -\lambda & \frac{\hbar}{\sqrt{2}}\\ 0 & \frac{\hbar}{\sqrt{2}} & -\lambda\end{pmatrix}\begin{pmatrix}a\\b\\c\end{pmatrix} = \begin{pmatrix}-\lambda a + \frac{\hbar}{\sqrt{2}}b\\ \frac{\hbar}{\sqrt{2}}a - \lambda b + \frac{\hbar}{\sqrt{2}}c\\ \frac{\hbar}{\sqrt{2}}b - \lambda a\end{pmatrix}$$

where we have parameterized the components of $|\lambda\rangle$ by $(a\ \ b\ \ c)$. For $\lambda = \hbar$, we can solve for b and c in terms of a, giving $b = \sqrt{2}a$ and $c = a$. We then determine a by normalizing $|\lambda = \hbar\rangle$:

$$|\lambda = \hbar\rangle = \begin{pmatrix}a\\ \sqrt{2}a\\ a\end{pmatrix}, \qquad \Rightarrow \qquad 1 = \langle\lambda = \hbar|\lambda = \hbar\rangle = \begin{pmatrix}a^* & \sqrt{2}a^* & a^*\end{pmatrix}\begin{pmatrix}a\\ \sqrt{2}a\\ a\end{pmatrix} = 4|a|^2, \qquad \Rightarrow \qquad a = \frac{1}{2}$$

(where I have chosen the arbitrary phase to be 1). Thus, and doing the same thing for $\lambda = 0$ and $\lambda = -\hbar$, gives

$$|J_x = \hbar\rangle = \frac{1}{2}\begin{pmatrix}1\\ \sqrt{2}\\ 1\end{pmatrix}, \qquad |J_x = 0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 0\\ -1\end{pmatrix}, \qquad |J_x = -\hbar\rangle = \frac{1}{2}\begin{pmatrix}1\\ -\sqrt{2}\\ 1\end{pmatrix}.$$

Problem 5: If the particle is in the state with $J_z = -\hbar$, and $\widehat{J}_x$ is measured, what are the possible outcomes and their probabilities? **Solution:** The possible outcomes are $J_x = \{\hbar, 0, -\hbar\}$, which are the eigenvalues of $\widehat{J}_x$. $|\psi\rangle$ is the normalized eigenstate of $\widehat{J}_z$ with eigenvalue $J_z = -\hbar$, which is

$$|\psi\rangle = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

in the $\widehat{J}_z$ eigenbasis. So (here $\mathcal{P}$ stands for "probability of"):

$$\mathcal{P}(J_x = \hbar) = |\langle J_x = \hbar|\psi\rangle|^2 = \left| \frac{1}{2}\begin{pmatrix}1 & \sqrt{2} & 1\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix}\right|^2 = \frac{1}{4},$$

$$\mathcal{P}(J_x = 0) = |\langle J_x = 0|\psi\rangle|^2 = \left| \frac{1}{2}\begin{pmatrix}1 & 0 & -1\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix}\right|^2 = \frac{1}{2},$$

$$\mathcal{P}(J_x = -\hbar) = |\langle J_x = -\hbar|\psi\rangle|^2 = \left| \frac{1}{2}\begin{pmatrix}1 & -\sqrt{2} & 1\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix}\right|^2 = \frac{1}{4}.$$

Problem 6: Consider the state

$$|\psi\rangle = \tfrac{1}{2}\begin{pmatrix}1\\1\\\sqrt{2}\end{pmatrix}$$

in the $\widehat{J}_z$ eigenbasis. If $\widehat{J}_z^2$ is meansured in this state and the result $+\hbar^2$ is obtained, what is the state after the measurement? How probable was this result? If, instead, we had measured $\widehat{J}_z$, what are the outcomes and their probabilities? **Solution:**

$$\widehat{J}_z^2 = \hbar^2 \begin{pmatrix}1 & & \\ & 0 & \\ & & 1\end{pmatrix}, \qquad \Rightarrow \qquad \text{the possible outcomes are } J_z^2 = \{0, \hbar^2\},$$

since those are its eigenvalues. An eigenbasis of the $J_z^2 = \hbar^2$ eigenspace is $\{|a\rangle, |b\rangle\}$ with

$$|a\rangle = \begin{pmatrix}1\\0\\0\end{pmatrix}, \qquad |b\rangle = \begin{pmatrix}0\\0\\1\end{pmatrix}.$$

Therefore, upon measuring $\widehat{J}_z^2 = \hbar^2$, the state collapses to

$$|\psi\rangle \longrightarrow |\psi'\rangle = \frac{(|a\rangle\langle a| + |b\rangle\langle b|)|\psi\rangle}{|(|a\rangle\langle a| + |b\rangle\langle b|)|\psi\rangle|}.$$

But

$$[|a\rangle\langle a| + |b\rangle\langle b|]\,|\psi\rangle = \left[\begin{pmatrix}1\\0\\0\end{pmatrix}\begin{pmatrix}1 & 0 & 0\end{pmatrix} + \begin{pmatrix}0\\0\\1\end{pmatrix}\begin{pmatrix}0 & 0 & 1\end{pmatrix}\right]\frac{1}{2}\begin{pmatrix}1\\1\\\sqrt{2}\end{pmatrix} = \begin{pmatrix}1\\0\\0\end{pmatrix}\frac{1}{2} + \begin{pmatrix}0\\0\\1\end{pmatrix}\frac{1}{\sqrt{2}} = \frac{1}{2}\begin{pmatrix}1\\0\\\sqrt{2}\end{pmatrix},$$

has norm

$$\sqrt{\frac{1}{2}\begin{pmatrix}1 & 0 & \sqrt{2}\end{pmatrix}\frac{1}{2}\begin{pmatrix}1\\0\\\sqrt{2}\end{pmatrix}} = \frac{\sqrt{3}}{2},$$

so

$$|\psi'\rangle = \frac{2}{\sqrt{3}}\frac{1}{2}\begin{pmatrix}1\\0\\\sqrt{2}\end{pmatrix} = \frac{1}{\sqrt{3}}\begin{pmatrix}1\\0\\\sqrt{2}\end{pmatrix}.$$

The probability of $J_z^2 = +\hbar^2$ is

$$\mathcal{P}(J_z^2 = \hbar^2) = \langle\psi|\,(|a\rangle\langle a| + |b\rangle\langle b|)\,|\psi\rangle = |\langle a|\psi\rangle|^2 + |\langle b|\psi\rangle|^2$$
$$= \left|\frac{1}{2}\begin{pmatrix}1 & 0 & 0\end{pmatrix}\begin{pmatrix}1\\ 1\\ \sqrt{2}\end{pmatrix}\right|^2 + \left|\frac{1}{2}\begin{pmatrix}0 & 0 & 1\end{pmatrix}\begin{pmatrix}1\\ 1\\ \sqrt{2}\end{pmatrix}\right|^2 = \frac{1}{4} + \frac{1}{2} = \frac{3}{4}.$$

If instead we measured $\widehat{J}_z$ the possible outcomes are the eigenvalues of $\widehat{J}_z$, $\{0, \pm\hbar\}$, with probabilities

$$\mathcal{P}(J_z = \hbar) = \left|\begin{pmatrix}1 & 0 & 0\end{pmatrix}|\psi'\rangle\right|^2 = \left|\frac{1}{2}\begin{pmatrix}1 & 0 & 0\end{pmatrix}\begin{pmatrix}1\\ 1\\ \sqrt{2}\end{pmatrix}\right|^2 = \frac{1}{4}.$$
$$\mathcal{P}(J_z = 0) = \left|\begin{pmatrix}0 & 1 & 0\end{pmatrix}|\psi'\rangle\right|^2 = \left|\frac{1}{2}\begin{pmatrix}0 & 1 & 0\end{pmatrix}\begin{pmatrix}1\\ 1\\ \sqrt{2}\end{pmatrix}\right|^2 = \frac{1}{4}$$
$$\mathcal{P}(J_z = -\hbar) = \left|\begin{pmatrix}0 & 0 & 1\end{pmatrix}|\psi'\rangle\right|^2 = \left|\frac{1}{2}\begin{pmatrix}0 & 0 & 1\end{pmatrix}\begin{pmatrix}1\\ 1\\ \sqrt{2}\end{pmatrix}\right|^2 = \frac{1}{2}.$$

Problem 7: A particle is in a state for which the probabilities $\mathcal{P}(J_z = \hbar) = 1/4$, $\mathcal{P}(J_z = 0) = 1/2$, and $\mathcal{P}(J_z = -\hbar) = 1/4$. Show that the most general normalized state with this property is

$$|\psi\rangle = \frac{e^{i\delta_1}}{2}|J_z = +\hbar\rangle + \frac{e^{i\delta_2}}{\sqrt{2}}|J_z = 0\rangle + \frac{e^{i\delta_3}}{2}|J_z = -\hbar\rangle. \tag{1}$$

It was stated in the text (and by me in class) that if $|\psi\rangle$ is a normalized state then the state $e^{i\theta}|\psi\rangle$ is a physically equivalent normalized state. Does this mean that the factors of $e^{i\delta_j}$ multiplying the $\widehat{J}_z$ eigenstates are irrelevant? To test this, calculate, for example, $\mathcal{P}(J_x = 0)$. **Solution:** In the $\widehat{J}_z$ eigenbasis,

$$|J_z = +\hbar\rangle = \begin{pmatrix}1\\ 0\\ 0\end{pmatrix}, \qquad |J_z = 0\rangle = \begin{pmatrix}0\\ 1\\ 0\end{pmatrix}, \qquad |J_z = -\hbar\rangle = \begin{pmatrix}0\\ 0\\ 1\end{pmatrix},$$

write the unknown state as

$$|\psi\rangle = \begin{pmatrix}a\\ b\\ c\end{pmatrix}.$$

Then

$$\mathcal{P}(J_z = +\hbar) = \frac{1}{4} = |\langle J_z = +\hbar|\psi\rangle|^2 = \left| \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \right|^2 = |a|^2,$$

$$\mathcal{P}(J_z = 0) = \frac{1}{2} = |\langle J_z = 0|\psi\rangle|^2 = \left| \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \right|^2 = |b|^2,$$

$$\mathcal{P}(J_z = -\hbar) = \frac{1}{4} = |\langle J_z = -\hbar|\psi\rangle|^2 = \left| \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \right|^2 = |c|^2.$$

The most general solution to these three equations is then

$$a = \frac{1}{2}e^{i\delta_1}, \qquad b = \frac{1}{\sqrt{2}}e^{i\delta_2}, \qquad c = \frac{1}{2}e^{i\delta_3},$$

for some arbitrary phases δ_i, which gives the desired answer.

The δ_i phase factors are *not* irrelevant. For example

$$\begin{aligned} \mathcal{P}(J_x = 0) = |\langle J_x = 0|\psi\rangle|^2 &= \left| \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & -1 \end{pmatrix} \frac{1}{2} \begin{pmatrix} e^{i\delta_1} \\ \sqrt{2}e^{i\delta_2} \\ e^{i\delta_3} \end{pmatrix} \right|^2 = \left| \frac{e^{i\delta_1}}{2\sqrt{2}} - \frac{e^{i\delta_3}}{2\sqrt{2}} \right|^2 \\ &= \frac{1}{8}\left(e^{i\delta_1} - e^{i\delta_3}\right)\left(e^{-i\delta_1} - e^{-i\delta_3}\right) = \frac{1}{8}\left(1 - e^{i(\delta_3-\delta_1)} - e^{-i(\delta_3-\delta_1)} + 1\right) \\ &= \frac{1}{4}\left(1 - \cos(\delta_3 - \delta_1)\right), \end{aligned}$$

so something measurable (a probability) depends on the difference of the phases.