Intro to QM March 27, 2019

Exam 5

Consider two identical non-interacting fermions of mass m in a common 1-dimensional harmonic oscillator. Thus they have the hamiltonian

$$\widehat{H}^{(0)} = \frac{1}{2m}\left(\widehat{p}_1^2 + \widehat{p}_1^2\right) + \frac{m\omega^2}{2}\left(\widehat{x}_1^2 + \widehat{x}_2^2\right).$$

Denote by $|n\rangle$ ($n = 0, 1, 2, \ldots$) the usual normalized one-particle energy eigenstates of the harmonic oscillator.

Recall that for the harmonic oscillator $\widehat{x} = \sqrt{\hbar/(2m\omega)}(\widehat{a} + \widehat{a}^\dagger)$, where $\widehat{a}^\dagger$ and $\widehat{a}$ are the raising and lowering operators which satisfy $[\widehat{a}, \widehat{a}^\dagger] = 1$, $\widehat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$, and $\widehat{a}|n\rangle = \sqrt{n}|n-1\rangle$. So for this 2-particle problem, we will have $\widehat{a}_i$ and $\widehat{a}_i^\dagger$, $i = 1, 2$, for the 2 particles, satisfying the commutation relations

$$[\widehat{a}_i, \widehat{a}_j] = 0, \qquad [\widehat{a}_i, \widehat{a}_j^\dagger] = \delta_{ij}, \qquad [\widehat{a}_i^\dagger, \widehat{a}_j^\dagger] = 0. \tag{1}$$

Problem 1: (10 points)
Suppose that one fermion is in the state $|\psi_1\rangle$ and a second identical fermion is in the state $|\psi_2\rangle$ with

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right), \qquad |\psi_2\rangle = \frac{1}{\sqrt{2}}\left(|1\rangle + |2\rangle\right).$$

What is the correctly normalized 2-particle state, $|\psi_{12}\rangle$?

Solution: `Since they are identical fermions, the 2-particle state must be antisymmetrized:`

$$\begin{aligned}|\psi_{12}\rangle &\sim |\psi_1\rangle|\psi_2\rangle - |\psi_2\rangle|\psi_1\rangle \sim \left(|0\rangle + |1\rangle\right)\left(|1\rangle + |2\rangle\right) - \left(|1\rangle + |2\rangle\right)\left(|0\rangle + |1\rangle\right) \\ &\sim |0,1\rangle + |0,2\rangle + |1,1\rangle + |1,2\rangle - |1,0\rangle - |2,0\rangle - |1,1\rangle - |2,1\rangle \\ &\sim |0,1\rangle - |1,0\rangle + |1,2\rangle - |2,1\rangle + |0,2\rangle - |2,0\rangle.\end{aligned}$$

`Since the norm`^{2} `of this state is` $1^2 + 1^2 + 1^2 + 1^2 + 1^2 + 1^2 = 6$`, the normalized state is`

$$|\psi_{12}\rangle = \frac{1}{\sqrt{6}}\left(|0,1\rangle - |1,0\rangle + |1,2\rangle - |2,1\rangle + |2,0\rangle - |0,2\rangle\right).$$

Problem 2: (10 points)
The ground state and first excited state (lowest and next-to-lowest energy eigenstates) of the two-fermion system are

$$|E_1^{(0)}\rangle := \frac{1}{\sqrt{2}}\left(|0,1\rangle - |1,0\rangle\right), \qquad |E_2^{(0)}\rangle := \frac{1}{\sqrt{2}}\left(|0,2\rangle - |2,0\rangle\right). \tag{2}$$

What are the energy eigenvalues, $E_1^{(0)}$ and $E_2^{(0)}$, of these states? (The "(0)" superscript is put in because these will be the unperturbed energy eigenvalues once we perturb the potential in **problems 4 and 5** below.)

Solution: The 2-particle Hamiltonian is just the sum of two 1-particle ones so acting on the tensor product state $|n,m\rangle$ it just gives $\widehat{H}|n,m\rangle = \left[(n+\frac{1}{2})\hbar\omega + (m+\frac{1}{2})\hbar\omega\right]|n,m\rangle$, the sum of the one-particle energies. Then the energy of $|E_n^{(0)}\rangle$ is $E_n^{(0)} = \frac{1}{2}\hbar\omega + (n+\frac{1}{2})\hbar\omega = (n+1)\hbar\omega$ for $n=1,2$.

Problem 3: (10 points)

What is the value of the next energy eigenvalue, $E_3^{(0)}$, above the two shown in (2)? What is the degeneracy of this energy level? Write down an orthonormal basis of the energy eigenstates for this eigenvalue.

Solution: The state can be of the form $|n_1,n_2\rangle_{\text{antisymmetrized}}$ which has energy eigenvalue $(n_1+n_2+1)\hbar\omega$. The next value above $E_1^{(0)} = 2\hbar\omega$ and $E_2^{(0)} = 3\hbar\omega$ is thus $E_3^{(0)} = 4\hbar\omega$. Thus we must have $n_1+n_2=3$. There are two possible ways of having this: $(n_1,n_2) = (0,3)$ or $(n_1,n_2) = (1,2)$ since, by antisymmetrization, no two of the states can be the same. So the degeneracy of the $E_3^{(0)}$ level is 2. An orthonormal eigenbasis of this level is given by the 2 states

$$|E_3^{(0)},1\rangle := \frac{1}{\sqrt{2}}\left(|0,3\rangle - |3,0\rangle\right), \qquad |E_3^{(0)},2\rangle := \frac{1}{\sqrt{2}}\left(|1,2\rangle - |2,1\rangle\right). \tag{3}$$

This choice of orthonormal basis is not unique.

Now suppose the 2-fermion system is perturbed, so that the hamiltonian is now

$$\widehat{H} = \widehat{H}^{(0)} + \widehat{H}^{(1)}, \qquad \text{with} \qquad \widehat{H}^{(1)} = \lambda\frac{2m\omega}{\hbar}(\widehat{x}_1 - \widehat{x}_2)^2,$$

where λ is some small real constant.

Problem 4: (10 points)

What is the first order perturbative correction to the second energy level, $E_2^{(0)}$?

Solution: Using that $\widehat{x}_i = \sqrt{\hbar/(2m\omega)}(\widehat{a}_i + \widehat{a}_i^\dagger)$, we have

$$\begin{aligned}
E_2^{(1)} = \langle E_2^{(0)}|\widehat{H}^1|E_2^{(0)}\rangle &= \lambda\frac{2m\omega}{\hbar}\langle E_2^{(0)}|(\widehat{x}_1 - \widehat{x}_2)^2|E_2^{(0)}\rangle = \lambda\langle E_2^{(0)}|(\widehat{a}_1 + \widehat{a}_1^\dagger - \widehat{a}_2 - \widehat{a}_2^\dagger)^2|E_2^{(0)}\rangle \\
&= \lambda\langle E_2^{(0)}|(\widehat{a}_1\widehat{a}_1^\dagger + \widehat{a}_1^\dagger\widehat{a}_1 - \widehat{a}_1\widehat{a}_2^\dagger - \widehat{a}_2^\dagger\widehat{a}_1 - \widehat{a}_2\widehat{a}_1^\dagger - \widehat{a}_1^\dagger\widehat{a}_2 + \widehat{a}_2\widehat{a}_2^\dagger + \widehat{a}_2^\dagger\widehat{a}_2)|E_2^{(0)}\rangle \\
&= 2\lambda\langle E_2^{(0)}|(1 + \widehat{a}_1^\dagger\widehat{a}_1 - \widehat{a}_2^\dagger\widehat{a}_1 - \widehat{a}_1^\dagger\widehat{a}_2 + \widehat{a}_2^\dagger\widehat{a}_2)|E_2^{(0)}\rangle
\end{aligned}$$

where in the second line we kept only terms with equal numbers of $\widehat{a}$'s and $\widehat{a}^\dagger$'s, and in the third line we used the commutation relations (1). From the form of $|E_2^{(0)}\rangle$ in (2) it follows that $\langle E_2^{(0)}|\widehat{a}_i^\dagger\widehat{a}_j|E_2^{(0)}\rangle = \delta_{ij}\frac{1}{2}\left(\langle 0,2| - \langle 2,0|\right)\widehat{a}_1^\dagger\widehat{a}_1\left(|0,2\rangle - |2,0\rangle\right) = \delta_{ij}\frac{1}{2}\left(-\sqrt{2}\langle 1,0|\right)\left(-\sqrt{2}|1,0\rangle\right) = \delta_{ij}$. Using this, we then get

$$E_2^{(1)} = 2\lambda(1 + \delta_{11} - \delta_{12} - \delta_{21} + \delta_{22}) = 6\lambda.$$

Problem 5: (10 bonus points)
What is the first order correction to the energies of the next, $E_3^{(0)}$, energy level? Note that this is a degenerate energy level, so you need to do degenerate perturbation theory to determine how this level splits.

Solution: For degenerate perturbation theory, we must find the eigenvalues of $\widehat{H}_1$ when restricted to the degenerate eigenspace. In the basis (3) found in problem 3, this means we want to find the eigenvalues of the matrix

$$M := \begin{pmatrix} \langle E_3^{(0)}, 1|\widehat{H}^{(1)}|E_3^{(0)}, 1\rangle & \langle E_3^{(0)}, 1|\widehat{H}^{(1)}|E_3^{(0)}, 2\rangle \\ \langle E_3^{(0)}, 2|\widehat{H}^{(1)}|E_3^{(0)}, 1\rangle & \langle E_3^{(0)}, 2|\widehat{H}^{(1)}|E_3^{(0)}, 2\rangle \end{pmatrix}.$$

Consider first

$$\begin{aligned} \widehat{H}^{(1)}|E_3^{(0)}, 1\rangle &= \tfrac{\lambda}{\sqrt{2}}(\widehat{a}_1 + \widehat{a}_1^\dagger - \widehat{a}_2 - \widehat{a}_2^\dagger)^2(|0,3\rangle - |3,0\rangle) = \lambda\Big((\widehat{a}_1 + \widehat{a}_1^\dagger - \widehat{a}_2 - \widehat{a}_2^\dagger)^2|0,3\rangle\Big)_A \\ &= \lambda\Big([(\widehat{a}_1 + \widehat{a}_1^\dagger)^2 - 2(\widehat{a}_1 + \widehat{a}_1^\dagger)(\widehat{a}_2 + \widehat{a}_2^\dagger) + (\widehat{a}_2 + \widehat{a}_2^\dagger)^2]|0,3\rangle\Big)_A, \end{aligned}$$

where the A subscript means antisymmetrize. Now

$$\begin{aligned} (\widehat{a}_1 + \widehat{a}_1^\dagger)^2|0,3\rangle &\sim |0 \text{ or } 2,\ 3\rangle \\ (\widehat{a}_1 + \widehat{a}_1^\dagger)(\widehat{a}_2 + \widehat{a}_2^\dagger)|0,3\rangle &\sim |1,\ 2 \text{ or } 4\rangle \\ (\widehat{a}_2 + \widehat{a}_2^\dagger)^2|0,3\rangle &\sim |0,\ 1 \text{ or } 3 \text{ or } 5\rangle. \end{aligned}$$

But we are only interested in states $|0,3\rangle$ or $|1,2\rangle$, so we only need to keep the terms

$$\begin{aligned} \widehat{H}^{(1)}|E_3^{(0)}, 1\rangle &\supset \lambda\Big([(\widehat{a}_1\widehat{a}_1^\dagger + \widehat{a}_1^\dagger\widehat{a}_1) - 2\widehat{a}_1^\dagger\widehat{a}_2 + (\widehat{a}_2\widehat{a}_2^\dagger + \widehat{a}_2^\dagger\widehat{a}_2)]|0,3\rangle\Big)_A \\ &= \lambda\Big(|0,3\rangle - 2\sqrt{3}|1,2\rangle + 7|0,3\rangle\Big)_A = \lambda\Big(8|0,3\rangle_A - 2\sqrt{3}|1,2\rangle_A\Big). \end{aligned}$$

A similar argument gives

$$\begin{aligned} \widehat{H}^{(1)}|E_3^{(0)}, 2\rangle &= \tfrac{\lambda}{\sqrt{2}}(\widehat{a}_1 + \widehat{a}_1^\dagger - \widehat{a}_2 - \widehat{a}_2^\dagger)^2(|1,2\rangle - |2,1\rangle) = \lambda\Big((\widehat{a}_1 + \widehat{a}_1^\dagger - \widehat{a}_2 - \widehat{a}_2^\dagger)^2|1,2\rangle\Big)_A \\ &\supset \lambda\Big([(\widehat{a}_1\widehat{a}_1^\dagger + \widehat{a}_1^\dagger\widehat{a}_1) - 2(\widehat{a}_1\widehat{a}_2^\dagger + \widehat{a}_1^\dagger\widehat{a}_2) + (\widehat{a}_2\widehat{a}_2^\dagger + \widehat{a}_2^\dagger\widehat{a}_2)]|1,2\rangle\Big)_A \\ &= \lambda\Big(2|1,2\rangle - 2(\sqrt{3}|0,3\rangle + 2|2,1\rangle) + 3|1,2\rangle\Big)_A \\ &= \lambda\Big(5|1,2\rangle_A - 2\sqrt{3}|0,3\rangle_A - 4|2,1\rangle_A\Big) = \lambda\Big(9|1,2\rangle_A - 2\sqrt{3}|0,3\rangle_A\Big). \end{aligned}$$

Plugging these into the matrix M gives

$$M = \lambda\begin{pmatrix} {}_A\langle 0,3|\Big(8|0,3\rangle_A - 2\sqrt{3}|1,2\rangle_A\Big) & {}_A\langle 0,3|\Big(9|1,2\rangle_A - 2\sqrt{3}|0,3\rangle_A\Big) \\ {}_A\langle 1,2|\Big(8|0,3\rangle_A - 2\sqrt{3}|1,2\rangle_A\Big) & {}_A\langle 1,2|\Big(9|1,2\rangle_A - 2\sqrt{3}|0,3\rangle_A\Big) \end{pmatrix} = \lambda\begin{pmatrix} 8 & -2\sqrt{3} \\ -2\sqrt{3} & 9 \end{pmatrix}.$$

The eigenvalues of M are then the roots of

$$0 = \det\begin{pmatrix} 8\lambda - \mu & -2\sqrt{3}\lambda \\ -2\sqrt{3}\lambda & 9\lambda - \mu \end{pmatrix} = \mu^2 - 17\lambda\mu + 72\lambda^2 - 12\lambda^2 = (\mu - 12\lambda)(\mu - 5\lambda).$$

Therefore, the first order correction to the $E_3^{(0)}$ level is to split it into two distinct energy levels, $E_{3,j} = E_3^{(0)} + E_{3,j}^{(1)} + \mathcal{O}(\lambda^2)$ for $j = 1, 2$ with

$$E_{3,1}^{(1)} = 12\lambda \qquad \text{and} \qquad E_{3,2}^{(1)} = 5\lambda.$$