Problem Set 3

Consider a 3-dimensional Hilbert space with orthonormal basis $\{|n\rangle, n = 1, 2, 3\}$ with respect to which the hermitean operators $\widehat{A}$ and $\widehat{B}$ have matrix elements

$$\widehat{A} = \begin{pmatrix} 17 & -7(1-i)\sqrt{2} & 7i\sqrt{3} \\ -7(1+i)\sqrt{2} & -4 & -7(1-i)\sqrt{6} \\ -7i\sqrt{3} & -7(1+i)\sqrt{6} & 3 \end{pmatrix},$$

$$\widehat{B} = \begin{pmatrix} 25 & 3(1-i)\sqrt{2} & -i\sqrt{3} \\ 3(1+i)\sqrt{2} & 28 & 3(1-i)\sqrt{6} \\ i\sqrt{3} & 3(1+i)\sqrt{6} & 27 \end{pmatrix}.$$

For problems 1 - 6, compute or verify the following "by hand" (ie, do not use *Mathematica* or some other automated system), showing your work:

Problem 1: $\det \widehat{A}$. **Solution:** `I find` $\det \widehat{A} = -18,432$. `(I don't have to show my work because I'm the professor.)`

Problem 2: The eigenvalues, λ_n, of $\widehat{A}$. **Solution:** `I find two eigenvalues` $\lambda_1 = -32$ `and` $\lambda_2 = 24$.

Problem 3: *All* the corresponding eigenvectors, $|v_n\rangle$, of $\widehat{A}$. **Solution:** `The eigenvectors` $|v_1\rangle$ `corresponding to` λ_1 `all have the form` $|v_1\rangle = c(-i, (1-i)\sqrt{2}, \sqrt{3})^T$ `where` c `is any complex constant. The eigenvectors` $|v_2\rangle$ `corresponding to` λ_2 `all have the form` $|v_2\rangle = (-(1-i)\sqrt{2}d + i\sqrt{3}e, d, e)^T$ `where` d `and` e `are any complex constants.`

Problem 4: An orthonormal basis, $\{|f_n\rangle\}$, of eigenvectors corresponding to the eigenvalues λ_n of $\widehat{A}$. **Solution:** `Normalizing` $|v_1\rangle$ `(ie, setting its length to 1), I find` $|c| = 1/\sqrt{8}$. `Arbitrarily choosing the phase to be 1, I thus have the normalized eigenvector` $|f_1\rangle = \frac{1}{2\sqrt{2}}(-i, (1-i)\sqrt{2}, \sqrt{3})^T$. `The` λ_2 `eigenspace is 2-dimensional, so we need to find a pair of orthonormal vectors from this space. Let me arbitrarily choose the first one to be` $|v_2\rangle$ `with` $e = 0$`; to be normalized I then have to pick` $|d| = 1/\sqrt{5}$`, and I arbitrarily choose the phase` $|f_2\rangle = \frac{1}{\sqrt{5}}(-(1-i)\sqrt{2}, 1, 0)^T$. `The second eigenbasis vector in this eigenspace has to be chosen orthogonal to the first, so we need to find a` $|v_2\rangle$ `so that` $0 = \langle f_2||v_2\rangle \propto (-(1-i)\sqrt{2}, 1, 0)^*(-(1-i)\sqrt{2}d + i\sqrt{3}e, d, e)^T = -(1+i)\sqrt{2}\cdot(-(1-i)\sqrt{2}d + i\sqrt{3}e) + 1\cdot d + 0\cdot e = 5d + (1-i)\sqrt{6}e$. `Therefore we must have` $d = -(1-i)\sqrt{6}e/5$`, so that` $|f_3\rangle = \frac{e}{5}(i\sqrt{3}, -(1-i)\sqrt{6}, 5)^T$. `Normalizing this gives` $|e| = \sqrt{5}/(2\sqrt{2})$. `Choosing the phase to be 1 I then get` $|f_3\rangle = \frac{1}{2\sqrt{10}}(i\sqrt{3}, -(1-i)\sqrt{6}, 5)^T$`, completing the orthonormal eigenbasis. It should be clear that (infinitely) many different such bases could have been chosen.`

Problem 5: The change of basis matrix $\widehat{R}_{mn}$ from the $\{|n\rangle\}$ to the $\{|f_n\rangle\}$ basis (defined by $|f_n\rangle = \sum_m \widehat{R}_{mn}|m\rangle$). **Solution:** $\widehat{R}_{mn} = \langle m|f_n\rangle$ which are just the components of $|f_n\rangle$. Thus, from the choice of eigenbasis in problem 4 we simply read off

$$\widehat{R} = \frac{1}{2\sqrt{10}} \begin{pmatrix} -i\sqrt{5} & -4(1-i) & i\sqrt{3} \\ (1-i)\sqrt{10} & 2\sqrt{2} & -(1-i)\sqrt{6} \\ \sqrt{15} & 0 & 5 \end{pmatrix}.$$

ie, its three columns are simply the components of $|f_1\rangle$, $|f_2\rangle$, and $|f_3\rangle$.

Problem 6: Verify that $\widehat{R}$ is unitary, ie, that $\widehat{R}^\dagger \widehat{R} = I$. **Solution:**

$$\widehat{R}^\dagger \widehat{R} = \frac{1}{40} \begin{pmatrix} i\sqrt{5} & (1+i)\sqrt{10} & \sqrt{15} \\ -4(1+i) & 2\sqrt{2} & 0 \\ -i\sqrt{3} & -(1+i)\sqrt{6} & 5 \end{pmatrix} \begin{pmatrix} -i\sqrt{5} & -4(1-i) & i\sqrt{3} \\ (1-i)\sqrt{10} & 2\sqrt{2} & -(1-i)\sqrt{6} \\ \sqrt{15} & 0 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Problems 7 - 13 refer to the same operators, $\widehat{A}$ and $\widehat{B}$, defined above. Hint: these problems require very little or no computation.

Problem 7: Compute the matrix elements of $\widehat{A}$ in the $\{|f_n\rangle\}$ basis. **Solution:** $\widehat{A}$ is just diagonal with the eigenvalues as the diagonal enries in an orthonormal eigenbasis. So

$$\widehat{A} = \begin{pmatrix} -32 & 0 & 0 \\ 0 & 24 & 0 \\ 0 & 0 & 24 \end{pmatrix}.$$

Problem 8: Write a general expression for $\widehat{A}$ in terms of the eigenvalues λ_i and the projectors, $\widehat{P}_{\lambda_i}$, onto the λ_i eigenspace. (Don't compute the $\widehat{P}_{\lambda_i}$ explicitly yet.) **Solution:** As shown in class, generally $\widehat{A} = \sum_i \lambda_i \widehat{P}_{\lambda_i} = -32P_{\lambda_1} + 24P_{\lambda_2}$.

Problem 9: Write expressions for the $\widehat{P}_{\lambda_i}$ in terms of the $|f_n\rangle$. **Solution:** The general expression (derived in class) for a projector onto a subspace with orthonormal basis $\{|v_i\rangle\}$ is $\widehat{P} = \sum_i |v_i\rangle\langle v_i|$. Thus, in our case $\widehat{P}_{\lambda_1} = |f_1\rangle\langle f_1|$ and $\widehat{P}_{\lambda_2} = |f_2\rangle\langle f_2| + |f_3\rangle\langle f_3|$.

Problem 10: Compute the matrix elements of the $\widehat{P}_{\lambda_n}$ in the $\{|f_n\rangle\}$ basis. **Solution:** Since, in the $\{|f_n\rangle\}$ basis, $|f_1\rangle = (1,0,0)^T$, etc, we simply have

$$\widehat{P}_{\lambda_1} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\widehat{P}_{\lambda_2} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

in the $\{|f_n\rangle\}$ basis.

Problem 11: Plug the result of problem 10 into that of problem 8 and verify that you get back the matrix of $\widehat{A}$ in the $\{|f_n\rangle\}$ basis. **Solution:**

$$A = -32\widehat{P}_{\lambda_1} + 24\widehat{P}_{\lambda_2} = -32\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix} + 24\begin{pmatrix}0&0&0\\0&1&0\\0&0&1\end{pmatrix} = \begin{pmatrix}-32&0&0\\0&24&0\\0&0&24\end{pmatrix}.$$

Problem 12: Compute the matrix elements of the $\widehat{P}_{\lambda_i}$ in the $\{|n\rangle\}$ basis. **Solution:** `Since the` $|f_n\rangle$ `are given in the` $\{|n\rangle\}$ `basis in problem 4, we get`

$$\widehat{P}_{\lambda_1} = \frac{1}{8}\begin{pmatrix}-i\\(1-i)\sqrt{2}\\\sqrt{3}\end{pmatrix}\begin{pmatrix}+i & (1+i)\sqrt{2} & \sqrt{3}\end{pmatrix} = \frac{1}{8}\begin{pmatrix}1 & (1-i)\sqrt{2} & -i\sqrt{3}\\(1+i)\sqrt{2} & 4 & (1-i)\sqrt{6}\\+i\sqrt{3} & (1+i)\sqrt{6} & 3\end{pmatrix},$$

$$\widehat{P}_{\lambda_2} = \frac{1}{5}\begin{pmatrix}-(1-i)\sqrt{2}\\1\\0\end{pmatrix}\begin{pmatrix}-(1+i)\sqrt{2} & 1 & 0\end{pmatrix} + \frac{1}{40}\begin{pmatrix}i\sqrt{3}\\-(1-i)\sqrt{6}\\5\end{pmatrix}\begin{pmatrix}-i\sqrt{3} & -(1+i)\sqrt{6} & 5\end{pmatrix}$$

$$= \frac{1}{8}\begin{pmatrix}7 & -(1-i)\sqrt{2} & i\sqrt{3}\\-(1+i)\sqrt{2} & 4 & -(1-i)\sqrt{6}\\-i\sqrt{3} & -(1+i)\sqrt{6} & 5\end{pmatrix}$$

Problem 13: Plug the result of problem 12 into that of problem 8 and verify that you get back the matrix of $\widehat{A}$ in the $\{|n\rangle\}$ basis. **Solution:**

$$\widehat{A} = -32\widehat{P}_{\lambda_1} + 24\widehat{P}_{\lambda_2} = -4\begin{pmatrix}1 & (1-i)\sqrt{2} & -i\sqrt{3}\\(1+i)\sqrt{2} & 4 & (1-i)\sqrt{6}\\+i\sqrt{3} & (1+i)\sqrt{6} & 3\end{pmatrix} + 3\begin{pmatrix}7 & -(1-i)\sqrt{2} & i\sqrt{3}\\-(1+i)\sqrt{2} & 4 & -(1-i)\sqrt{6}\\-i\sqrt{3} & -(1+i)\sqrt{6} & 5\end{pmatrix}$$

$$= \begin{pmatrix}17 & -7(1-i)\sqrt{2} & 7i\sqrt{3}\\-7(1+i)\sqrt{2} & -4 & -7(1-i)\sqrt{6}\\-7i\sqrt{3} & -7(1+i)\sqrt{6} & 3\end{pmatrix}.$$

Problems 14 - 16 still refer to the same operators, $\widehat{A}$ and $\widehat{B}$, defined above. But for these problems I suggest you use *Mathematica* or some other computer program.

Problem 14: Compute $[\widehat{A}, \widehat{B}]$. **Solution:** $[\widehat{A}, \widehat{B}] = 0$.

Problem 15: Compute the eigenvalues, κ_i, and a corresponding orthonormal basis of eigenvectors, $|w_n\rangle$, of $\widehat{B}$. **Solution:** `Mathematica gives the eigenvalues` $\kappa_1 = 40$, $\kappa_2 = 24$, `and` $\kappa_3 = 16$, `with corresponding (non-normalized) eigenvectors (I cleared the denominators)` $|\widetilde{w}_1\rangle = (-i, (1-i)\sqrt{2}, \sqrt{3})^T$, $|\widetilde{w}_2\rangle = (i\sqrt{3}, 0, 1)^T$, `and` $|\widetilde{w}_3\rangle = (-i, (-(1-i)\sqrt{2}, \sqrt{3})^T$. `To make them orthonormal, we only have to normalize them (divide by their lengths). I get` $|w_1\rangle = |\widetilde{w}_1\rangle/\sqrt{8}$, $|w_2\rangle = |\widetilde{w}_2\rangle/\sqrt{3}$, `and` $|w_3\rangle = |\widetilde{w}_3\rangle/\sqrt{8}$.

Problem 16: Do $\widehat{A}$ and $\widehat{B}$ have a common orthonormal basis of eigenvectors? If not, why not? If they do, what is it? **Solution:** `Yes, they do. It is simply` $\{|w_n\rangle\}$, the eigenbasis of $\widehat{B}$. You can check that $\widehat{A}|w_1\rangle = -32|w_1\rangle$, $\widehat{A}|w_2\rangle = 24|w_2\rangle$, and $\widehat{A}|w_3\rangle = 24|w_3\rangle$. Or, we know this must be true since $[\widehat{A},\widehat{B}] = 0$ (see problem 18 below).

The remaining three problems refer to arbitrary operators in an unspecified Hilbert space.

Problem 17: Show that any eigenvalue, λ, of a hermitean operator, $\widehat{A}$, must be real. **Solution:** Say an eigenvector corresponding to eigenvalue λ is $|v\rangle$, so $\widehat{A}|v\rangle = \lambda|v\rangle$. Consider $\langle v|\widehat{A}|v\rangle$. On the one hand, $\langle v|(\widehat{A}|v\rangle) = \langle v|(\lambda|v\rangle) = \mu\langle v|v\rangle$. On the other hand, $(\langle v|\widehat{A}|v\rangle)^* = \langle v|A^\dagger|v\rangle = \langle v|A|v\rangle = \lambda\langle v|v\rangle$, where we have used that A is hermitean. Comparing the two we learn that $\lambda\langle v|v\rangle = \lambda^*\langle v|v\rangle^* = \lambda^*\langle v|v\rangle$, and so $\lambda = \lambda^*$.

Problem 18: Show that if a hermitean operator $\widehat{A}$ has eigenvectors $|v\rangle$ and $|w\rangle$ corresponding to two different eigenvalues λ and μ, $\lambda \neq \mu$, then $\langle v|w\rangle = 0$. **Solution:** Consider $\langle v|\widehat{A}|w\rangle$. On the one hand, $\langle v|(\widehat{A}|w\rangle) = \langle v|(\mu|w\rangle) = \mu\langle v|w\rangle$. On the other hand, $(\langle v|\widehat{A}|w\rangle)^* = \langle w|A^\dagger|v\rangle = \langle w|A|v\rangle = \lambda\langle w|v\rangle =$, where we have used that A is hermitean. So, we have shown that $\mu^*\langle v|w\rangle^* = \lambda\langle w|v\rangle$, or, since $\langle v|w\rangle^* = \langle w|v\rangle$, that $(\mu^* - \lambda)\langle w|v\rangle = 0$. But from the previous problem we know that $\mu^* = \mu$, so this becomes $0 = (\mu - \lambda)\langle w|v\rangle$. Since by assumption $\mu - \lambda \neq 0$, this implies $\langle w|v\rangle = 0$.

Problem 19: If an operator, $\widehat{A}$, has an eigenvalue λ such that all its corresponding eigenvectors are of the form $c|v\rangle$ for some $|v\rangle$ where c is any complex number, and $\widehat{A}$ commutes with another operator, $\widehat{B}$, then show that $|v\rangle$ is an eigenvector of $\widehat{B}$. **Solution:** Consider $\widehat{A}(\widehat{B}|v\rangle) = \widehat{B}\widehat{A}|v\rangle = \widehat{B}(\lambda|v\rangle) = \lambda(\widehat{B}|v\rangle)$. This shows that $\widehat{B}|v\rangle$ is an eigenvector of $\widehat{A}$ with eigenvalue λ. By assumption all such eigenvectors have the form $c|v\rangle$ for some c. Therefore, we must have $\widehat{B}|v\rangle = c|v\rangle$ for some c, showing that $|v\rangle$ is also an eigenvector of $\widehat{B}$.