Intro to QM due: November 14, 2018

Problem Set 8

Problem 1: Show that $\delta(x-y)=\delta(y-x)$ by showing that they give the same result when integrated against any pair of functions $f(x)g(y)$ by using the definition of the δ-function that $\int dy\, f(y)\delta(x-y)=f(x)$.

Solution: $\int dxdy\, f(x)g(y)\delta(x-y)=\int dx\, f(x)g(x)$ by the definition. On the other hand, $\int dxdy\, f(x)g(y)\delta(y-x)=\int dy\, f(y)g(y)$, by the definition. But $\int dy\, f(y)g(y)=\int dx\, f(x)g(x)$, since this is just a trivial change of variables in the integral.

Problem 2: Show that $(x-y)\frac{d}{dx}\delta(x-y)=-\delta(x-y)$. **Solution:**

$$\begin{aligned}\int dx\, f(x)\cdot(x-y)\frac{d}{dx}\delta(x-y) &= -\int dx\,\frac{d}{dx}\left[(x-y)f(x)\right]\delta(x-y)\\ &= -\int dx\,\left[f(x)+(x-y)f'(x)\right]\delta(x-y)\\ &= -\left[f(y)+(y-y)f'(y)\right]=-f(y)=-\int dx\, f(x)\delta(x-y).\end{aligned}$$

Since this is true for all $f(x)$, we have shown that $(x-y)\frac{d}{dx}\delta(x-y)=-\delta(x-y)$.

For the rest of the problem set we are considering the position and momentum operators $\widehat{x}$ and $\widehat{p}$ and their associated eigenbases defined by

$$\begin{aligned}\widehat{x}|x\rangle &= x|x\rangle, & \langle x|x'\rangle &= \delta(x-x'),\\ \widehat{p}|p\rangle &= p|p\rangle, & \langle p|p'\rangle &= \delta(p-p'),\end{aligned}$$

where x and p are any real numbers. Recall that

$$\langle p|\widehat{x}|p'\rangle=+i\hbar\frac{d}{dp}\delta(p-p'),\qquad \langle x|\widehat{p}|x'\rangle=-i\hbar\frac{d}{dx}\delta(x-x').$$

Problem 3: Compute $\langle p|\widehat{p}\widehat{x}|p'\rangle$.

Solution: $\langle p|\widehat{p}\widehat{x}|p'\rangle=\langle p|p\widehat{x}|p'\rangle=p\langle p|\widehat{x}|p'\rangle=ip\hbar\frac{d}{dp}\delta(p-p')$.

Problem 4: Compute $\langle x|\widehat{p}e^{2\widehat{x}}|x'\rangle$.

Solution: $\langle x|\widehat{p}e^{2\widehat{x}}|x'\rangle=\langle x|\widehat{p}e^{2x'}|x'\rangle=e^{2x'}\langle x|\widehat{p}|x'\rangle=-i\hbar e^{2x'}\frac{d}{dx}\delta(x-x')$.

Problem 5: Compute $\langle x|[\widehat{x},\widehat{p}]|x'\rangle$.

Solution: $\langle x|[\widehat{x},\widehat{p}]|x'\rangle=\langle x|\widehat{x}\widehat{p}|x'\rangle-\langle x|\widehat{p}\widehat{x}|x'\rangle=\langle x|x\widehat{p}|x'\rangle-\langle x|\widehat{p}x'|x'\rangle=x\langle x|\widehat{p}|x'\rangle-x'\langle x|\widehat{p}|x'\rangle=-i\hbar(x-x')\frac{d}{dx}\delta(x-x')$.

Problem 6: Prove that $\widehat{p}=\widehat{p}^\dagger$, starting from $\langle x|\widehat{p}|\psi\rangle=-i\hbar\frac{d}{dx}\langle x|\psi\rangle$ by showing that $\langle\psi|\widehat{p}|\phi\rangle^*=\langle\phi|\widehat{p}|\psi\rangle$ for all $|\phi\rangle$ and $|\psi\rangle$.

Solution: $\langle\phi|\widehat{p}|\psi\rangle = \int dx\langle\phi|x\rangle\langle x|\widehat{p}|\psi\rangle = -i\hbar\int dx\phi^*(x)\psi'(x)$ (where the $'$ means derivative with respect to x). By the same reasoning, $\langle\psi|\widehat{p}|\phi\rangle = -i\hbar\int dx\psi^*(x)\phi'(x)$, so $\langle\psi|\widehat{p}|\phi\rangle^* = i\hbar\int dx\psi(x)\phi'^*(x) = -i\hbar\int dx\psi'(x)\phi^*(x)$, where I used integration by parts in the last step. This is the same as the result we found for $\langle\phi|\widehat{p}|\psi\rangle$, so we have shown that $\langle\psi|\widehat{p}|\phi\rangle^* = \langle\phi|\widehat{p}|\psi\rangle$. Finally, the left side is $\langle\psi|\widehat{p}|\phi\rangle^* = \langle\phi|\widehat{p}^\dagger|\psi\rangle$, so we have shown that $\langle\phi|\widehat{p}^\dagger|\psi\rangle = \langle\phi|\widehat{p}|\psi\rangle$ for all ϕ and ψ, and so $\widehat{p} = \widehat{p}^\dagger$.

Problem 7: If a state $|\psi\rangle$ is described by a *real* wavefunction $\psi(x) = \psi^*(x)$, show that $\langle\widehat{p}\rangle = 0$ in this state.
Solution:

$$\begin{aligned}
\langle\widehat{p}\rangle = \langle\psi|\widehat{p}|\psi\rangle &= \int_{-\infty}^{\infty} dx\langle\psi|x\rangle\langle x|\widehat{p}|\psi\rangle = \int_{-\infty}^{\infty} dx\,\psi^*(x)\left(-i\hbar\frac{d}{dx}\right)\psi(x) \\
&= -i\hbar\int_{-\infty}^{\infty} dx\,\psi(x)\frac{d\psi(x)}{dx} = -\frac{i\hbar}{2}\int_{-\infty}^{\infty} dx\,\frac{d}{dx}\left(\psi(x)^2\right) = -\frac{i\hbar}{2}\,\psi^2\Big|_{-\infty}^{\infty} = 0,
\end{aligned}$$

since $|\psi| \to 0$ as $|x| \to \infty$.

Problem 8: Show that the expectation value, $\langle\widehat{p}\rangle_\phi$ of the momentum in a state $|\phi\rangle$ with wavefunction $\phi(x) = e^{ip_0x/\hbar}\psi(x)$ for some real constant p_0 satisfies $\langle\widehat{p}\rangle_\phi = p_0 + \langle\widehat{p}\rangle_\psi$, where $\langle\widehat{p}\rangle_\psi$ is the expectation value of the momentum in a state $|\psi\rangle$.
Solution:

$$\begin{aligned}
\langle\widehat{p}\rangle_\phi &= \langle\phi|\widehat{p}|\phi\rangle = \langle e^{ip_0x/\hbar}\psi|\widehat{p}|e^{ip_0x/\hbar}\psi\rangle = \int_{-\infty}^{\infty} dx\,\langle e^{ip_0x/\hbar}\psi|x\rangle\langle x|\widehat{p}|e^{ip_0x/\hbar}\psi\rangle \\
&= \int_{-\infty}^{\infty} dx\,\left(e^{ip_0x/\hbar}\psi(x)\right)^*(-i\hbar)\frac{d}{dx}\left(e^{ip_0x/\hbar}\psi(x)\right) \\
&= -i\hbar\int_{-\infty}^{\infty} dx\,\psi^*(x)e^{-ip_0x/\hbar}\left[\frac{ip_0}{\hbar}e^{ip_0x/\hbar}\psi(x) + e^{ip_0x/\hbar}\frac{d\psi}{dx}\right] \\
&= \int_{-\infty}^{\infty} dx\,\psi^*(x)\,p_0\,\psi(x) - i\hbar\int_{-\infty}^{\infty} dx\,\psi^*(x)\frac{d\psi}{dx} = p_0\left[\int_{-\infty}^{\infty} dx\,\langle\psi|x\rangle\langle x|\psi\rangle\right] + \left[\int_{-\infty}^{\infty} dx\,\langle\psi|x\rangle\langle x|\widehat{p}|\psi\rangle\right] \\
&= p_0\langle\psi|\psi\rangle + \langle\psi|\widehat{p}|\psi\rangle = p_0 + \langle\widehat{p}\rangle_\psi.
\end{aligned}$$

For **problems 9-12** consider the operator $\widehat{H} = \widehat{p}^2 + \widehat{x}^2$.

Problem 9: Show that the matrix elements of $\widehat{H}$ in the position are

$$\langle x|\widehat{H}|x'\rangle = -\hbar^2\frac{d^2\delta(x-x')}{dx^2} + x^2\delta(x-x').$$

Solution:

$$\begin{aligned}
\langle x|\widehat{H}|x'\rangle &= \langle x|\widehat{p}^2|x'\rangle + \langle x|\widehat{x}^2|x'\rangle = \int dy \langle x|\widehat{p}|y\rangle\langle y|\widehat{p}|x'\rangle + \langle x|x^2|x'\rangle \\
&= \int dy \left(-i\hbar\frac{d\delta(x-y)}{dx}\right)\left(-i\hbar\frac{d\delta(y-x')}{dy}\right) + x^2\langle x|x'\rangle \\
&= -\hbar^2\frac{d}{dx}\int dy\,\delta(x-y)\frac{d\delta(y-x')}{dy} + x^2\delta(x-x') \\
&= -\hbar^2\frac{d}{dx}\frac{d\delta(x-x')}{dx} + x^2\delta(x-x') = -\hbar^2\frac{d^2\delta(x-x')}{dx^2} + x^2\delta(x-x').
\end{aligned}$$

Problem 10: Find the matrix elements of $\widehat{H}$ in the momentum basis.

Solution: `This is almost identical to the last problem:`

$$\begin{aligned}
\langle p|\widehat{H}|p'\rangle &= \langle p|\widehat{p}^2|p'\rangle + \langle p|\widehat{x}^2|p'\rangle = \langle p|p^2|p'\rangle + \int dq \langle p|\widehat{x}|q\rangle\langle q|\widehat{x}|p'\rangle \\
&= p^2\langle p|p'\rangle + \int dq \left(i\hbar\frac{d\delta(p-q)}{dp}\right)\left(i\hbar\frac{d\delta(q-p')}{dq}\right) \\
&= p^2\delta(p-p') - \hbar^2\frac{d}{dp}\int dq\,\delta(p-q)\frac{d\delta(q-p')}{dq} \\
&= p^2\delta(p-p') - \hbar^2\frac{d}{dp}\frac{d\delta(p-p')}{dp} = p^2\delta(p-p') - \hbar^2\frac{d^2\delta(p-p')}{dp^2}.
\end{aligned}$$

Problem 11: The eigenvalue equation for $\widehat{H}$ is $\widehat{H}|\phi\rangle = \lambda|\phi\rangle$. Show that in the position basis this is the differential equation for the wavefunction $\phi(x)$:

$$0 = -\hbar^2\frac{d^2}{dx^2}\phi(x) + x^2\,\phi(x) - \lambda\phi(x).$$

Solution: `The eigenvalue equation is` $0 = \widehat{H}|\phi\rangle - \lambda|\phi\rangle$. `In the` x`-basis this becomes`

$$\begin{aligned}
0 &= \langle x|\widehat{H}|\phi\rangle - \lambda\langle x|\phi\rangle = \int dy\,\langle x|\widehat{H}|y\rangle\langle y|\phi\rangle - \lambda\phi(x) \\
&= \int dy\,\left(-\hbar^2\frac{d^2\delta(x-y)}{dx^2} + y^2\delta(x-y)\right)\phi(y) - \lambda\phi(x) \\
&= -\hbar^2\frac{d^2}{dx^2}\left(\int dy\,\delta(x-y)\phi(y)\right) + \int dy\,y^2\delta(x-y)\phi(y) - \lambda\phi(x) \\
&= -\hbar^2\frac{d^2}{dx^2}\phi(x) + x^2\,\phi(x) - \lambda\phi(x).
\end{aligned}$$

Problem 12: Show that the state $|\phi\rangle$, with position-basis wavefunction given by $\phi(x) = Ce^{-\alpha x^2}$ is an eigenvector of $\widehat{H}$ for a certain value of α, and determine α, the eigenvalue, and a positive real value of C for which $|\phi\rangle$ is normalized.

Solution: We just have to check that $\phi(x)$ satisfies the differential equation in the previous problem for some λ:

$$-\hbar^2 \frac{d^2}{dx^2}\phi(x) + x^2\,\phi(x) = -\hbar^2 C \frac{d}{dx}\left(-2\alpha x e^{-\alpha x^2}\right) + Cx^2 e^{-\alpha x^2} = -\hbar^2 C(4\alpha^2 x^2 - 2\alpha)e^{-\alpha x^2} + Cx^2 e^{-\alpha x^2}$$
$$= 2\alpha\hbar^2 C e^{-\alpha x^2} + (1-4\alpha^2\hbar^2)x^2 C e^{-\alpha x^2} = 2\alpha\hbar^2\phi(x) + (1-4\alpha^2\hbar^2)x^2\phi(x).$$

For this to be proportional to $\phi(x)$, we need the term proportional to x^2 to vanish, requiring $\alpha = \pm 1/(2\hbar)$. We must choose the plus sign for $\phi(x)$ to vanish at $x = \pm\infty$. In this case we have found that

$$-\frac{d^2}{dx^2}\phi(x) + x^2\,\phi(x) = 2\alpha\hbar^2\phi(x) = \hbar\phi(x)$$

so $|\phi\rangle$ is an eigenvector with eigenvalue $\lambda = \hbar$.

For $|\phi\rangle$ to be normalized we must have $1 = \langle\phi|\phi\rangle = \int dx\,\langle\phi|x\rangle\langle x|\phi\rangle = \int dx\,\phi^*(x)\phi(x) = |C|^2\int dx\, e^{-\hbar x^2/2}e^{-\hbar x^2/2} = |C|^2\int_{-\infty}^{\infty} dx\, e^{-\hbar x^2} = \sqrt{\pi/\hbar}\,|C|^2$. so $C = (\hbar/\pi)^{1/4}$.

For **problems 13-15**, consider a state with momentum-space wave function

$$\langle p|\psi\rangle = \begin{cases} 0 & p<0 \\ C & 0<p<p_0 \\ 0 & p_0<p \end{cases}, \tag{1}$$

where C and p_0 are some positive real constants.

Problem 13: Determine the value for C such that $|\psi\rangle$ is normalized.

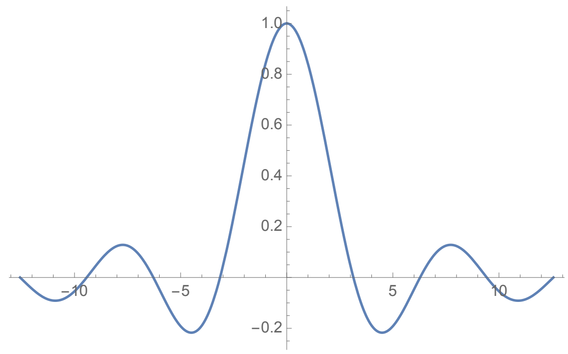
Solution: $|\psi\rangle$ normalized means $1 = \langle\psi|\psi\rangle = \int_{-\infty}^{\infty} dp\langle\psi|p\rangle\langle p|\psi\rangle = \int_{-\infty}^{\infty} dp|\langle p|\psi\rangle|^2 = \int_0^{p_0} dp|C|^2 = |C|^2 p_0$. Thus $C = 1/\sqrt{p_0}$ (where we have set the arbitrary phase to 1).

Problem 14: Determine $\psi(x) = \langle x|\psi\rangle$.

Solution: $\langle x|\psi\rangle = \int dp\langle x|p\rangle\langle p|\psi\rangle = (2\pi\hbar)^{-1/2}\int dp e^{ipx/\hbar}\langle p|\psi\rangle = C(2\pi\hbar)^{-1/2}\int_0^{p_0} dp e^{ipx/\hbar} = C(2\pi\hbar)^{-1/2}\int_0^{p_0} dp e^{ipx/\hbar} = -i(\hbar C/x)(2\pi\hbar)^{-1/2}\, e^{ipx/\hbar}\big|_0^{p_0} = (-i/x)\sqrt{\hbar/(2\pi p_0)}\left(e^{ip_0x/\hbar}-1\right)$, where I put in the value of C from the last problem.

Problem 15: Sketch $|\langle p|\psi\rangle|$ and $|\langle x|\psi\rangle|$. Estimate Δp from $\langle p|\psi\rangle$ and Δx from $\langle x|\psi\rangle$, to estimate $\Delta p\Delta x$. (Simply estimate rather than calculate the uncertainties.) Are there any values of p_0 for which the Heisenberg uncertainty relation is violated?

Solution: $|\langle p|\psi\rangle|$ is just a square bump of height $C = 1/\sqrt{p_0}$ from $p=0$ to $p=p_0$. So its width is about $\Delta p \approx p_0/2$. From the last problem, $|\langle x|\psi\rangle| = (1/|x|)\sqrt{\hbar/(2\pi p_0)}\left|e^{ip_0x/2\hbar}(e^{ip_0x/2\hbar} - e^{-ip_0x/2\hbar})\right| = (1/|x|)\sqrt{\hbar/(2\pi p_0)}2|\sin(p_0x/2\hbar)| \propto \sin(y)/y$, where $y := p_0x/(2\hbar)$. A plot of $\sin(y)/y$ is



which has width $\Delta y \approx \pi$, so I estimate $\Delta x \approx 2\pi\hbar/p_0$. Thus

$$\Delta x \Delta p \approx (2\pi\hbar/p_0) \cdot (p_0/2) = \pi\hbar.$$

The value is independent of p_0, and seems to be approximately a factor of 2π above the Heisenberg uncertainty relation bound.