Problem Set 2

Let $\{|n\rangle, n = 1, \ldots, d\}$ be an orthonormal basis of a Hilbert space, and consider the two operators

$$\widehat{P}_\ell := \sum_{n=1}^{\ell} |n\rangle\langle n|, \qquad \widehat{P}_{d-\ell} := \sum_{n=\ell+1}^{d} |n\rangle\langle n|,$$

where $1 \leq \ell \leq d$ is some given integer. The following three problems, combined, ask you to show that $\widehat{P}_\ell$ and $\widehat{P}_{d-\ell}$ are orthogonal projection operators which sum to the identity. Please use only the properties of kets, bras, and the inner product to show the following:

Problem 1: $(\widehat{P}_\ell)^\dagger = \widehat{P}_\ell$ and $(\widehat{P}_{d-\ell})^\dagger = \widehat{P}_{d-\ell}$. **Solution:** $(\widehat{P}_\ell)^\dagger = (\sum_{n=1}^{\ell} |n\rangle\langle n|)^\dagger = \sum_{n=1}^{\ell}(|n\rangle\langle n|)^\dagger = \sum_{n=1}^{\ell}(\langle n|)^\dagger(|n\rangle)^\dagger = \sum_{n=1}^{\ell} |n\rangle\langle n| = \widehat{P}_\ell$, and similarly for $\widehat{P}_{d-\ell}$.

Problem 2: $(\widehat{P}_\ell)^2 = \widehat{P}_\ell$, $(\widehat{P}_{d-\ell})^2 = \widehat{P}_{d-\ell}$, and $\widehat{P}_\ell \widehat{P}_{d-\ell} = \widehat{P}_{d-\ell}\widehat{P}_\ell = 0$. **Solution:** $(\widehat{P}_\ell)^2 = (\sum_{n=1}^{\ell} |n\rangle\langle n|)^2 = (\sum_{n=1}^{\ell} |n\rangle\langle n|)(\sum_{m=1}^{\ell} |m\rangle\langle m|) = \sum_{n,m=1}^{\ell}(|n\rangle\langle n|)(|m\rangle\langle m|) = \sum_{n,m=1}^{\ell} |n\rangle\langle n|m\rangle\langle m| = \sum_{n,m=1}^{\ell} |n\rangle\delta_{n,m}\langle m| = \sum_{n=1}^{\ell} |n\rangle\langle n| = \widehat{P}_\ell$, and similarly for $\widehat{P}_{d-\ell}$. $\widehat{P}_\ell\widehat{P}_{d-\ell} = (\sum_{n=1}^{\ell} |n\rangle\langle n|)(\sum_{m=\ell+1}^{d} |m\rangle\langle m|) = \sum_{n=1}^{\ell}\sum_{m=\ell+1}^{d} |n\rangle\langle n|m\rangle\langle m| = \sum_{n=1}^{\ell}\sum_{m=\ell+1}^{d} |n\rangle\delta_{n,m}\langle m| = 0$, where in the last step we used that since n never equals m in the sum so $\delta_{n,m} = 0$ for every term. A similar calculation shows $\widehat{P}_{d-\ell}\widehat{P}_\ell = 0$.

Problem 3: $\widehat{P}_\ell + \widehat{P}_{d-\ell} = 1$. **Solution:** $\widehat{P}_\ell + \widehat{P}_{d-\ell} = (\sum_{n=1}^{\ell} |n\rangle\langle n|) + (\sum_{n=\ell+1}^{d} |n\rangle\langle n|) = \sum_{n=1}^{d} |n\rangle\langle n| = 1$, where in the last step we used the completeness relation for the orthomormal basis.

The 2-dimensional Hilbert space for a spin-$\frac{1}{2}$ particle has an orthonormal basis of states $\{|\pm z\rangle\}$. We define another orthonormal basis, $\{|\pm x\rangle\}$, for this Hilbert space by

$$|\pm x\rangle := \frac{1}{\sqrt{2}}\left(|+z\rangle \pm |-z\rangle\right) \qquad \text{(signs correlated).}$$

The (spin) angular momentum operators $\widehat{S}_z$ and $\widehat{S}_x$ are defined by their actions on these orthonormal bases:

$$\widehat{S}_z|\pm z\rangle := \pm\frac{\hbar}{2}|\pm z\rangle, \qquad \widehat{S}_x|\pm x\rangle := \pm\frac{\hbar}{2}|\pm x\rangle.$$

Finally, the rotation operators $\widehat{R}(\theta\widehat{z})$ and $\widehat{R}(\theta\widehat{x})$ implementing the effects of rotations in space by an angle θ around the $\widehat{z}$ and $\widehat{x}$ axes, respectively, on states in the Hilbert space are defined by

$$\widehat{R}(\theta\widehat{z}) := \exp\left\{-\frac{i}{\hbar}\theta\widehat{S}_z\right\}, \qquad \widehat{R}(\theta\widehat{x}) := \exp\left\{-\frac{i}{\hbar}\theta\widehat{S}_x\right\}.$$

Problem 4: Compute $\widehat{R}(\theta\widehat{z})|+z\rangle$, writing your answer in the $|\pm z\rangle$ basis. **Solution:**
Since $\widehat{S}_z|+z\rangle = (\hbar/2)|+z\rangle$, it follows by successive application that $(\widehat{S}_z)^n|+z\rangle = (\hbar/2)^n|+z\rangle$.
So, using the series definition of the exponential, it follows that $\widehat{R}(\theta\widehat{z})|+z\rangle = \exp\{-i\theta\widehat{S}_z/\hbar\}|+z\rangle = \exp\{-i\theta(\hbar/2)/\hbar\}|+z\rangle = e^{-i\theta/2}|+z\rangle = (\cos\frac{\theta}{2} - i\sin\frac{\theta}{2})|+z\rangle$.

Problem 5: Compute $\widehat{R}(\theta\widehat{x})|+z\rangle$, writing your answer in the $|\pm z\rangle$ basis. **Solution:**
From the definition of $|\pm x\rangle$ given above, it follows that $|+z\rangle = (1/\sqrt{2})(|+x\rangle + |-x\rangle)$.
Then, by the same kind of calculation as in the last problem, we get $\widehat{R}(\theta\widehat{x})|+z\rangle = \exp\{-i\theta\widehat{S}_x/\hbar\}(|+x\rangle + |-x\rangle)/\sqrt{2} = (\exp\{-i\theta\widehat{S}_x/\hbar\}|+x\rangle + \exp\{-i\theta\widehat{S}_x/\hbar\}|-x\rangle)/\sqrt{2} = (\exp\{-i\theta(+\hbar/2)/\hbar\}|+x\rangle + \exp\{-i\theta(-\hbar/2)/\hbar\}|-x\rangle)/\sqrt{2} = (e^{-i\theta/2}|+x\rangle + e^{+i\theta/2}|-x\rangle)/\sqrt{2}$. Now use the definition of $|\pm x\rangle$ again to rewrite this back in terms of the $|\pm z\rangle$ basis: $\widehat{R}(\theta\widehat{x})|+z\rangle = (e^{-i\theta/2}|+x\rangle + e^{+i\theta/2}|-x\rangle)/\sqrt{2} = \frac{1}{2}e^{-i\theta/2}(|+z\rangle + |-z\rangle) + \frac{1}{2}e^{+i\theta/2}(|+z\rangle - |-z\rangle) = \frac{1}{2}(e^{-i\theta/2} + e^{+i\theta/2})|+z\rangle + \frac{1}{2}(e^{-i\theta/2} - e^{+i\theta/2})|-z\rangle = \cos\frac{\theta}{2}|+z\rangle - i\sin\frac{\theta}{2}|-z\rangle$.

Problem 6: Check that your answers to the previous 2 problems give $\widehat{R}(2\pi\widehat{z})|+z\rangle = \widehat{R}(2\pi\widehat{x})|+z\rangle = -|+z\rangle$. This shows that 2π rotations in space, which in classical mechanics are the same as identity transformations, give, instead, in quantum mechanics for spin-$\frac{1}{2}$ particles *minus* the identity — ie, all states get multiplied by -1. Does this give any observable effect on measurements (eg, in the outcome of Stern-Gerlach experiments)? **Solution:** Plugging $\theta = 2\pi$ into the solutions to the last two problems gives $\widehat{R}(2\pi\widehat{z})|+z\rangle = (\cos\pi - i\sin\pi)|+z\rangle = -|+z\rangle$ and $\widehat{R}(2\pi\widehat{x})|+z\rangle = \cos\pi|+z\rangle - i\sin\pi|-z\rangle = -|+z\rangle$. No, this ''extra'' minus sign upon rotating a spin-$\frac{1}{2}$ system by 2π does not affect the outcome of any measurement, since, as discussed in chapter 1, the overall phase factor of a state vector is unobservable.

However, if, instead, one considers a system made up of many spin-$\frac{1}{2}$ particles and one rotates, say, only one of the particles by 2π while doing nothing to the other particles, then the state vector of the system instead of getting an overall phase of -1 will acquire a *relative* phase of minus one between the rotated particle and the rest. This phase *is* observable, and, indeed, is responsible for important effects, notably the Pauli exclusion principle and Fermi-Dirac statistics of identical half-odd spin particles. We will discuss these effects next semester.