Problem Set 10

All problems consider a simple harmonic oscillator with hamiltonian $\widehat{H} = \widehat{p}^2/(2m) + (m\omega^2/2)\widehat{x}^2$, for which $|n\rangle$, $n = 0, 1, 2, \ldots$, is an orthonormal energy eigenbasis whose eigenvalues are $E_n := \hbar\omega(n + \frac{1}{2})$.

Problem 1: Compute $\langle\widehat{x}\rangle$, $\langle\widehat{p}\rangle$, $\langle\widehat{x}^2\rangle$, $\langle\widehat{p}^2\rangle$, and $\Delta x \cdot \Delta p$ in the state $|n\rangle$. For which n does $\Delta x \cdot \Delta p$ saturate the Heisenberg uncertainty bound? **Solution:**

$$\begin{aligned}
\langle\widehat{x}\rangle &= \langle n|\widehat{x}|n\rangle = \left(\frac{\hbar}{2m\omega}\right)^{1/2} \langle n|(a^\dagger + a)|n\rangle = \left(\frac{\hbar}{2m\omega}\right)^{1/2} \langle n| \left(\sqrt{n+1}|n+1\rangle + \sqrt{n}|n-1\rangle\right) \\
&= \left(\frac{\hbar}{2m\omega}\right)^{1/2} \left(\sqrt{n+1}\langle n|n+1\rangle + \sqrt{n}\langle n|n-1\rangle\right) = 0,
\end{aligned}$$

since $\langle n|n-1\rangle = \langle n|n+1\rangle = 0$ by orthogonality of the energy eigenstates. Similarly,

$$\begin{aligned}
\langle\widehat{p}\rangle &= \langle n|\widehat{p}|n\rangle = i\left(\frac{m\omega\hbar}{2}\right)^{1/2} \langle n|(a^\dagger - a)|n\rangle \propto \langle n| \left(\sqrt{n+1}|n+1\rangle - \sqrt{n}|n-1\rangle\right) \\
&\propto \sqrt{n+1}\langle n|n+1\rangle - \sqrt{n}\langle n|n-1\rangle = 0.
\end{aligned}$$

Therefore

$$\begin{aligned}
(\Delta x)^2 &= \langle n|\widehat{x}^2|n\rangle = \frac{\hbar}{2m\omega}\langle n|(a^\dagger + a)^2|n\rangle = \frac{\hbar}{2m\omega}\langle n| \left(a^2 + aa^\dagger + a^\dagger a + (a^\dagger)^2\right)|n\rangle \\
&= \frac{\hbar}{2m\omega}\left(\langle n|a\sqrt{n+1}|n+1\rangle + \langle n|a^\dagger\sqrt{n}|n-1\rangle\right) \\
&= \frac{\hbar}{2m\omega}\left(\sqrt{n+1}\sqrt{n+1}\langle n|n\rangle + \sqrt{n}\sqrt{n}\langle n|n\rangle\right) = \frac{(2n+1)\hbar}{2m\omega},
\end{aligned}$$

where I dropped the a^2 and $(a^\dagger)^2$ terms in the first line since $a^2|n\rangle \propto |n-2\rangle$ and $(a^\dagger)^2|n\rangle \propto |n+2\rangle$ which are orthogonal to $|n\rangle$, and in the last step I used the normalization condition that $\langle n|n\rangle = 1$. Similarly,

$$\begin{aligned}
(\Delta p)^2 &= \langle n|\widehat{p}^2|n\rangle = -\frac{m\omega\hbar}{2}\langle n|(a^\dagger - a)^2|n\rangle = -\frac{m\omega\hbar}{2}\langle n| \left(a^2 - aa^\dagger - a^\dagger a + (a^\dagger)^2\right)|n\rangle \\
&= \frac{m\omega\hbar}{2}\langle n|(aa^\dagger + a^\dagger a)|n\rangle = \frac{(2n+1)m\omega\hbar}{2},
\end{aligned}$$

from the $(\Delta x)^2$ calculation. Therefore

$$\Delta x \cdot \Delta p = \sqrt{2n+1}\left(\frac{\hbar}{2m\omega}\right)^{1/2} \cdot \sqrt{2n+1}\left(\frac{m\omega\hbar}{2}\right)^{1/2} = (2n+1)\frac{\hbar}{2}.$$

This saturates the uncertainty bound only for $n = 0$, the ground state.

Problem 2: Write down the energy eigenvalue equation in both the position basis and the momentum basis and compare the two equations to show that the momentum basis wavefunctions can be obtained from the ones in the position basis through the substitution $x \to p$ and $m\omega \to 1/m\omega$. Thus, for example,

$$\langle x|0\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-m\omega x^2/2\hbar} \quad \text{implies} \qquad \langle p|0\rangle = \left(\frac{1}{m\omega\pi\hbar}\right)^{1/4} e^{-p^2/2\hbar m\omega}.$$

Solution: In the momentum basis $|\psi\rangle \to \psi(p)$, $\widehat{p} \to p$, and $\widehat{x} \to i\hbar(d/dp)$, so the energy eigenvalue equation

$$\left(\frac{1}{2m}\widehat{p}^2 + \frac{m\omega^2}{2}\widehat{x}^2\right)|E\rangle = E|E\rangle$$

becomes

$$\frac{1}{2m}p^2\psi(p) - \frac{m\omega^2}{2}\hbar^2\psi''(p) = E\psi(p),$$

where $\psi(p) = \langle p|E\rangle$. Compare this to the position-basis equation

$$-\frac{1}{2m}\hbar^2\psi''(x) + \frac{m\omega^2}{2}x^2\psi(x) = E\psi(x),$$

where $\psi(x) = \langle x|E\rangle$. These are the same equations with the substitutions $x \leftrightarrow p$ and $m \leftrightarrow 1/(m\omega^2)$.

Problem 3: Suppose that the oscillator is intially in the state

$$|\psi(0)\rangle = \frac{1}{\sqrt{3}}\left(|1\rangle + i\sqrt{2}\,|2\rangle\right). \tag{1}$$

What is its state at time t? **Solution:** Since $E_n = (n + \frac{1}{2})\hbar\omega$,

$$|\psi_1(t)\rangle = \frac{1}{\sqrt{3}}\left(e^{-iE_1t/\hbar}|1\rangle + e^{-iE_2t/\hbar}i\sqrt{2}\,|2\rangle\right) = \frac{1}{\sqrt{3}}\left(e^{-3i\omega t/2}|1\rangle + e^{-5i\omega t/2}i\sqrt{2}\,|2\rangle\right).$$

Problem 4: What is $\langle\widehat{p}\rangle$ for the particle in the state $|\psi(0)\rangle$ (given in equation (1))?
Solution: $\widehat{p} = i(m\omega\hbar/2)^{1/2}(a^\dagger - a)$, so

$$\begin{aligned}\langle\widehat{p}\rangle &= \langle\psi(0)|\widehat{p}|\psi(0)\rangle = \frac{i}{3}\sqrt{\frac{m\omega\hbar}{2}}\left(\langle 1| - i\sqrt{2}\,\langle 2|\right)(a^\dagger - a)\left(|1\rangle + i\sqrt{2}\,|2\rangle\right) \\ &= \frac{i}{3}\sqrt{\frac{m\omega\hbar}{2}}\left(-i\sqrt{2}\langle 1|a|2\rangle - i\sqrt{2}\langle 2|a^+|1\rangle\right) = \frac{\sqrt{2}}{3}\sqrt{\frac{m\omega\hbar}{2}}\left(\sqrt{2}\langle 1|1\rangle + \sqrt{2}\langle 2|2\rangle\right) = \frac{2}{3}\sqrt{2m\omega\hbar}.\end{aligned}$$

Problem 5: If the energy is measured in the state given in equation (1) at $t = 0$, what are the possible outcomes of the measurement, and what are their probabilities?
Solution: $E_1 = \frac{3}{2}\hbar\omega$ with probability $|\langle 1|\psi(0)\rangle|^2 = \frac{1}{3}$, or $E_2 = \frac{5}{2}\hbar\omega$ with probability $|\langle 2|\psi(0)\rangle|^2 = \frac{2}{3}$.

Problem 6: What is $\langle E\rangle$, the expectation value of the energy, of the particle in the state $|\psi(t)\rangle$ at time t? **Solution:** Since energy eigenstates are orthonormal, $\langle 1|2\rangle = 0$, etc., so

$$
\begin{aligned}
\langle E\rangle = \langle\psi(t)|\widehat{H}|\psi(t)\rangle &= \frac{1}{3}\left(e^{3i\omega t/2}\langle 1| - e^{5i\omega t/2} i\sqrt{2}\,\langle 2|\right)\widehat{H}\left(e^{-3i\omega t/2}|1\rangle + e^{-5i\omega t/2} i\sqrt{2}\,|2\rangle\right)\\
&= \frac{1}{3}\left(e^{3i\omega t/2}\langle 1| - e^{5i\omega t/2} i\sqrt{2}\,\langle 2|\right)\left(\frac{3}{2}\hbar\omega e^{-3i\omega t/2}|1\rangle + \frac{5}{2}\hbar\omega e^{-5i\omega t/2} i\sqrt{2}\,|2\rangle\right)\\
&= \frac{1}{3}\left(\frac{3}{2}\hbar\omega - \frac{5}{2}\hbar\omega (i\sqrt{2})^2\right) = \frac{13}{6}\hbar\omega.
\end{aligned}
$$

The expectation value is time-independent, reflecting the fact that the energy is conserved.