Intro to QM due: February 18, 2019

Exam 4

All questions are about a hydrogen atom. Write all your answers in terms of the reduced mass, μ, of the hydrogen atom, the fine structure constant, α, and fundamental constants $\hbar$ and c. If it's more convenient, you can use the Bohr radius a_0, which is the combination $a_0 = \hbar/(\mu c\alpha)$.

Problem 1: (5 points) The hamiltonian, $\widehat{H}$, for the relative motion of the electron and proton in a hydrogen atom is $\widehat{H} = \frac{\widehat{p}^2}{2\mu} - \frac{\alpha\hbar c}{\widehat{r}}$. Write this hamiltonian as a differential operator as it would act on the wave function in spherical coordinates of a state with a definite value, ℓ, of the total angular momentum quantum number.

Solution:

$$\widehat{H} = -\frac{\hbar^2}{2\mu}\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} + \frac{\hbar^2\ell(\ell+1)}{2\mu r^2} - \frac{\alpha\hbar c}{r}.$$

Problem 2: (5 points) A hydrogen atom is in its ground state, whose wave function is

$$\psi(r,\theta,\phi) = \frac{1}{\sqrt{\pi a_0^3}}e^{-r/a_0},$$

where a_0 is the Bohr radius. Calculate $\langle r\rangle$ and $\langle r^{-1}\rangle$ in this state, where r is the radius (distance from the origin). It might be helpful to recall that $\int_0^\infty dz\, z^n e^{-\beta z} = n!/\beta^{n+1}$.

Solution: `Since the wave function has no angular dependence, the angular integrations just give` 4π`, so`

$$\langle r\rangle = \langle\psi|\widehat{r}|\psi\rangle = 4\pi\int_0^\infty \psi^* r\psi r^2 dr = \frac{4\pi}{\pi a_0^3}\int_0^\infty e^{-2r/a_0}r^3 dr = \frac{4}{a_0^3}\frac{3!}{(2/a_0)^4} = \frac{3}{2}a_0,$$
$$\langle r^{-1}\rangle = \langle\psi|\widehat{r}^{-1}|\psi\rangle = 4\pi\int_0^\infty \psi^*\frac{1}{r}\psi r^2 dr = \frac{4\pi}{\pi a_0^3}\int_0^\infty e^{-2r/a_0}r\,dr = \frac{4}{a_0^3}\frac{1!}{(2/a_0)^2} = \frac{1}{a_0}.$$

Problem 3: (5 points) Suppose that at time $t = 0$ the hydrogen atom is in the (normalized) state

$$|\psi(0)\rangle = \frac{1}{2}\Big(|4,2,-1\rangle + \sqrt{2}i\,|2,2,-2\rangle + i\,|2,2,-1\rangle\Big), \qquad (1)$$

where we are using the standard $|n,\ell,m\rangle$ notation for the simultaneous eigenbasis of $\widehat{H}$, $\widehat{L}^2$, $\widehat{L}_z$. What is this state at time t?

Solution: `Since` $E_n = -\mu\alpha^2c^2/(2n^2)$`,`

$$\begin{aligned}|\psi(t)\rangle &= \frac{1}{2}\left(e^{-iE_4t/\hbar}|4,2,-1\rangle + \sqrt{2}i\,e^{-iE_2t/\hbar}|2,2,-2\rangle + i\,e^{-iE_2t/\hbar}|2,2,-1\rangle\right)\\ &= \frac{1}{2}\left(e^{i\mu\alpha^2c^2t/(32\hbar)}|4,2,-1\rangle + \sqrt{2}i\,e^{i\mu\alpha^2c^2t/(8\hbar)}|2,2,-2\rangle + i\,e^{i\mu\alpha^2c^2t/(8\hbar)}|2,2,-1\rangle\right).\end{aligned}$$

Problem 4: (5 points) If $\widehat{L}^2$ were measured at time $t=0$ in the state (1), what would be the possible outcomes of the measurement, and what would their probabilities be?

Solution: $|\psi(0)\rangle$ is a superposition of $\widehat{L}^2$ eigenstates all with $\ell=2$, so the only possible value of $\widehat{L}^2$ is $\hbar^2\ell(\ell+1)=6\hbar^2$. Since there is only one possible outcome, its probability must be 1.

Problem 5: (5 points) What is $\langle H\rangle$ for the particle in the state (1) at time $t=0$?

Solution: Since energy eigenstates are orthonormal,

$$\begin{aligned}\langle H\rangle &= \left|\frac{1}{2}\right|^2 \langle 4,2,-1|\widehat{H}|4,2,-1\rangle + \left|\frac{\sqrt{2}}{2}i\right|^2 \langle 2,2,-2|\widehat{H}|2,2,-2\rangle + \left|\frac{1}{2}i\right|^2 \langle 2,2,-1|\widehat{H}|2,2,-1\rangle \\ &= \frac{1}{4}E_4 + \frac{1}{2}E_2 + \frac{1}{4}E_2 = -\frac{\mu\alpha^2c^2}{2}\left(\frac{1}{4}\cdot\frac{1}{4^2}+\frac{3}{4}\cdot\frac{1}{2^2}\right) = -\frac{13\mu\alpha^2c^2}{128}.\end{aligned}$$

Problem 6: (5 points) What is $\langle L_x\rangle$ for the particle in the state (1)? Recall that $\widehat{L}_\pm := \widehat{L}_x \pm i\widehat{L}_y$ and that $\widehat{L}_\pm|\ell,m\rangle = \hbar\sqrt{(\ell\mp m)(\ell\pm m+1)}|\ell,m\pm 1\rangle$.

Solution: $\widehat{L}_x = \frac{1}{2}(\widehat{L}_+ + \widehat{L}_-)$, so

$$\begin{aligned}\langle L_x\rangle &= \langle\psi_1(0)|\widehat{L}_x|\psi_1 0\rangle \\ &= \frac{1}{8}\Big(\langle 4,2,-1| - \sqrt{2}i\,\langle 2,2,-2| - i\,\langle 2,2,-1|\Big)\,(\widehat{L}_+ + \widehat{L}_-)\,\Big(|4,2,-1\rangle + \sqrt{2}i\,|2,2,-2\rangle + i\,|2,2,-1\rangle\Big) \\ &= \frac{1}{8}\left(-i\,\langle 2,2,-1|\right)\widehat{L}_+\left(\sqrt{2}i\,|2,2,-2\rangle\right) + \frac{1}{8}\left(-\sqrt{2}i\,\langle 2,2,-2|\right)\widehat{L}_-\left(i\,|2,2,-1\rangle\right)\end{aligned}$$

because $\widehat{L}_\pm$ can only connect states with the same ℓ and n, and m's differing by 1. Thus

$$\begin{aligned}\langle L_x\rangle &= \frac{\sqrt{2}}{8}\Big(\langle 2,2,-1|\widehat{L}_+|2,2,-2\rangle + \langle 2,2,-2|\widehat{L}_-|2,2,-1\rangle\Big) \\ &= \frac{\sqrt{2}}{8}\Big(\langle 2,2,-1|L_+|2,2,-2\rangle + \langle 2,2,-1|L_+|2,2,-2\rangle^*\Big) = \frac{\sqrt{2}}{4}\mathrm{Re}\Big[\langle 2,2,-1|L_+|2,2,-2\rangle\Big] \\ &= \frac{\sqrt{2}}{4}\mathrm{Re}\Big[\hbar\sqrt{(2-(-2))(2+(-2)+1)}\;\;\langle 2,2,-1|2,2,-1\rangle\Big] = \hbar\frac{\sqrt{2}}{2}.\end{aligned}$$