Intro to QM due: October 8, 2018

Problem Set 5

For problems 1-7 we are considering the spin states of an electron. This is a spin $j = 1/2$ particle with electric charge e, magnetic moment g-factor g, and mass m. It is in a uniform magnetic field $\vec{B} = B_0\widehat{n}$ with $\widehat{n} = \cos\theta\widehat{x} + \sin\theta\widehat{y}$ and with B_0 and θ constant in time. Define the frequency $\omega_0 = egB_0/2mc$. At time $t = 0$, the spin state of the electron is $|\psi(0)\rangle = |\frac{1}{2}, \frac{1}{2}\rangle$ in the usual angular momentum $|j, m\rangle$ eigenbasis (ie, the simultaneous eigenbasis of $\widehat{J}^2$ and $\widehat{J}_z$). Express all your answers in terms of ω_0 and θ.

Problem 1: What is $\langle J_z\rangle$ at $t = 0$? **Solution:** Since the electron is the in the $J_z = +\hbar/2$ eigenstate at $t = 0$, $\langle J_z\rangle = +\hbar/2$ at $t = 0$.

Problem 2: What is the hamiltonian operator, $\widehat{H}$, in terms of the angular momentum operators $\widehat{J}_x$, $\widehat{J}_y$, and $\widehat{J}_z$? **Solution:** From the text, $\widehat{H} = (eg/2mc)\vec{B}\cdot\vec{J} = \omega_0\widehat{n}\cdot\vec{J} = \omega_0(\cos\theta\widehat{J}_x + \sin\theta\widehat{J}_y)$.

Problem 3: What are the eigenvalues, E_n, and an orthonormal basis of eigenvectors, $\{|E_n\rangle\}$, of $\widehat{H}$? **Solution:** In the $\widehat{J}_z$ basis,

$$\widehat{J}_x = \frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \qquad \widehat{J}_y = \frac{\hbar}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix};$$

see eqns (3.88)-(3.89) of the text. Then the hamiltonian is

$$\widehat{H} = \frac{\hbar\omega_0}{2}\begin{pmatrix}0 & e^{-i\theta}\\ e^{i\theta} & 0\end{pmatrix}, \tag{1}$$

where I used that $\cos\theta + i\sin\theta = e^{i\theta}$. The eigenvalues of this matrix are $E_1 = \hbar\omega_0/2$ and $E_2 = -\hbar\omega_0/2$, and the associated normalized eignvectors are

$$|E_1\rangle = \tfrac{1}{\sqrt{2}}\begin{pmatrix}1\\ +e^{i\theta}\end{pmatrix} = \tfrac{1}{\sqrt{2}}\left(|\tfrac{1}{2},\tfrac{1}{2}\rangle + e^{i\theta}|\tfrac{1}{2},-\tfrac{1}{2}\rangle\right), \quad |E_2\rangle = \tfrac{1}{\sqrt{2}}\begin{pmatrix}1\\ -e^{i\theta}\end{pmatrix} = \tfrac{1}{\sqrt{2}}\left(|\tfrac{1}{2},\tfrac{1}{2}\rangle - e^{i\theta}|\tfrac{1}{2},-\tfrac{1}{2}\rangle\right).$$

Problem 4: Find the components, $\langle E_n|\psi(0)\rangle$, of $|\psi(0)\rangle$ in the energy basis. **Solution:**

$$\langle E_1|\psi(0)\rangle = \tfrac{1}{\sqrt{2}}\left(\langle\tfrac{1}{2},\tfrac{1}{2}| + e^{-i\theta}\langle\tfrac{1}{2},-\tfrac{1}{2}|\right)|\tfrac{1}{2},\tfrac{1}{2}\rangle = \frac{1}{\sqrt{2}},$$
$$\langle E_2|\psi(0)\rangle = \tfrac{1}{\sqrt{2}}\left(\langle\tfrac{1}{2},\tfrac{1}{2}| - e^{-i\theta}\langle\tfrac{1}{2},-\tfrac{1}{2}|\right)|\tfrac{1}{2},\tfrac{1}{2}\rangle = \frac{1}{\sqrt{2}}.$$

Problem 5: Compute $|\psi(t)\rangle$. **Solution:**

$$
\begin{aligned}
|\psi(t)\rangle &= \sum_n e^{-iE_n t/\hbar}|E_n\rangle\langle E_n|\psi(0)\rangle = e^{-i\omega_0 t/2}|E_1\rangle\frac{1}{\sqrt{2}} + e^{i\omega_0 t/2}|E_2\rangle\frac{1}{\sqrt{2}} \\
&= \frac{1}{2}e^{-i\omega_0 t/2}\left(|\tfrac{1}{2},\tfrac{1}{2}\rangle + e^{i\theta}|\tfrac{1}{2},-\tfrac{1}{2}\rangle\right) + \frac{1}{2}e^{i\omega_0 t/2}\left(|\tfrac{1}{2},\tfrac{1}{2}\rangle - e^{i\theta}|\tfrac{1}{2},-\tfrac{1}{2}\rangle\right) \\
&= \cos(\omega_0 t/2)|\tfrac{1}{2},\tfrac{1}{2}\rangle - ie^{i\theta}\sin(\omega_0 t/2)|\tfrac{1}{2},-\tfrac{1}{2}\rangle.
\end{aligned}
$$

Problem 6: What is $\langle J_z\rangle$ at time t? **Solution:**

$$
\begin{aligned}
\langle J_z\rangle &= \langle\psi(t)|\widehat{J}_z|\psi(t)\rangle = \langle\psi(t)|\widehat{J}_z\left[\cos(\omega_0 t/2)|\tfrac{1}{2},\tfrac{1}{2}\rangle - ie^{i\theta}\sin(\omega_0 t/2)|\tfrac{1}{2},-\tfrac{1}{2}\rangle\right] \\
&= \langle\psi(t)|\left[\tfrac{\hbar}{2}\cos(\omega_0 t/2)|\tfrac{1}{2},\tfrac{1}{2}\rangle - (-\tfrac{\hbar}{2})ie^{i\theta}\sin(\omega_0 t/2)|\tfrac{1}{2},-\tfrac{1}{2}\rangle\right] \\
&= \tfrac{\hbar}{2}\left[\cos(\omega_0 t/2)\langle\tfrac{1}{2},\tfrac{1}{2}| + ie^{-i\theta}\sin(\omega_0 t/2)\langle\tfrac{1}{2},-\tfrac{1}{2}|\right]\cdot\left[\cos(\omega_0 t/2)|\tfrac{1}{2},\tfrac{1}{2}\rangle + ie^{i\theta}\sin(\omega_0 t/2)|\tfrac{1}{2},-\tfrac{1}{2}\rangle\right] \\
&= \tfrac{\hbar}{2}\left[\cos^2(\omega_0 t/2) - \sin^2(\omega_0 t/2)\right] = \tfrac{\hbar}{2}\cos(\omega_0 t)\,.
\end{aligned}
$$

Problem 7: What is the probability of observing $J_z = \hbar/2$ at time t? **Solution:**

$$
\begin{aligned}
\mathcal{P}(J_z = \hbar/2)(t) &= \left|\langle\tfrac{1}{2},\tfrac{1}{2}|\psi(t)\rangle\right|^2 = \left|\langle\tfrac{1}{2},\tfrac{1}{2}|\Big(\cos(\omega_0 t/2)|\tfrac{1}{2},\tfrac{1}{2}\rangle - ie^{i\theta}\sin(\omega_0 t/2)|\tfrac{1}{2},-\tfrac{1}{2}\rangle\Big)\right|^2 \\
&= \cos^2(\omega_0 t/2) = \frac{1}{2}\left[1 - \cos(\omega_0 t)\right].
\end{aligned}
$$

For problems 8-11 we are considering a system described by a 3-dimensional Hilbert space, with hamiltonian given by the matrix elements

$$
\widehat{H} = \begin{pmatrix} h & 0 & if \\ 0 & g & 0 \\ -if & 0 & h \end{pmatrix}
$$

with respect to some orthonormal basis $\{|1\rangle, |2\rangle, |3\rangle\}$. (Since $\widehat{H}$ is hermitian, f, g, and h are all real numbers.)

Problem 8: Compute the eigenvalues, E_n, of $\widehat{H}$ in terms of f, g, and h. **Solution:**

$$
\begin{aligned}
0 = \det\begin{pmatrix} h-\lambda & 0 & if \\ 0 & g-\lambda & 0 \\ -if & 0 & h-\lambda \end{pmatrix} &= (h-\lambda)^2(g-\lambda) - (-if)(if)(g-\lambda) \\
= (g-\lambda)(\lambda - h - f)(\lambda - h + f)
\end{aligned}
$$

so the eigenvalues of $\widehat{H}$ are

$$
E_1 = g, \qquad E_2 = h + f, \qquad E_3 = h - f.
$$

Problem 9: Find an orthonormal energy eigenbasis, $\{|E_n\rangle\}$, of $\widehat{H}$. **Solution:** `Solve for the eigenvectors inthe usual way, to get`

$$|E_1\rangle \propto \begin{pmatrix}0\\1\\0\end{pmatrix} = |2\rangle, \qquad |E_2\rangle \propto \begin{pmatrix}1\\0\\-i\end{pmatrix} = |1\rangle - i|3\rangle, \qquad |E_3\rangle \propto \begin{pmatrix}1\\0\\i\end{pmatrix} = |1\rangle + i|3\rangle. \tag{2}$$

`Normalize to find`

$$|E_1\rangle = |2\rangle, \qquad |E_2\rangle = \frac{1}{\sqrt{2}}\left(|1\rangle - i|3\rangle\right), \qquad |E_3\rangle = \frac{1}{\sqrt{2}}\left(|1\rangle + i|3\rangle\right). \tag{3}$$

Problem 10: If the system is initially in the state $|\psi(0)\rangle = |2\rangle$, what is $|\psi(t)\rangle$? **Solution:** `Since` $|2\rangle$ `is an energy eigenstate (namely,` $|E_1\rangle$`),`

$$|\psi(t)\rangle = e^{-i\widehat{H}t/\hbar}|\psi(0)\rangle = e^{-i\widehat{H}t/\hbar}|2\rangle = e^{-i\widehat{H}t/\hbar}|E_1\rangle = e^{-iE_1t/\hbar}|E_1\rangle = e^{-igt/\hbar}|2\rangle.$$

Problem 11: If the system is initially in the state $|\psi(0)\rangle = |1\rangle$, what is $|\psi(t)\rangle$? **Solution:**

$$|\psi(t)\rangle = \sum_n e^{-iE_nt/\hbar}|E_n\rangle\langle E_n|\psi(0)\rangle = \sum_n e^{-iE_nt/\hbar}|E_n\rangle\langle E_n|1\rangle.$$

`From problem 9 we compute`

$$\langle E_1|1\rangle = 0, \qquad \langle E_2|1\rangle = \frac{1}{\sqrt{2}}\left(\langle 1| + i\langle 3|\right)|1\rangle = \frac{1}{\sqrt{2}}, \qquad \langle E_3|1\rangle = \frac{1}{\sqrt{2}}\left(\langle 1| - i\langle 3|\right)|1\rangle = \frac{1}{\sqrt{2}}.$$

`So`

$$\begin{aligned}
|\psi(t)\rangle &= \sum_n e^{-iE_nt/\hbar}|E_n\rangle\langle E_n|1\rangle = e^{-iE_2t/\hbar}|E_2\rangle\langle E_2|1\rangle + e^{-iE_3t/\hbar}|E_3\rangle\langle E_3|1\rangle \\
&= \frac{1}{\sqrt{2}}\left[e^{-i(h+f)t/\hbar}\tfrac{1}{\sqrt{2}}\left(|1\rangle - i|3\rangle\right) + e^{-i(h-f)t/\hbar}\tfrac{1}{\sqrt{2}}\left(|1\rangle + i|3\rangle\right)\right] \\
&= e^{-iht/\hbar}\frac{1}{2}\left[\left(e^{ift/\hbar} + e^{-ift/\hbar}\right)|1\rangle + i\left(e^{ift/\hbar} - e^{-ift/\hbar}\right)|3\rangle\right] \\
&= e^{-iht/\hbar}\Big(\cos(ft/\hbar)|1\rangle - \sin(ft/\hbar)|3\rangle\Big).
\end{aligned}$$

For problems 12 and 13 we are considering an arbitrary system with hamiltonian $\widehat{H}$ with eigenvalues E_n, $n = 0, 1, 2, \ldots$, and corresponding orthonormal eigenbasis $\{|E_n\rangle\}$. Furthermore, we will name the energy eigenvalues in increasing order, so that $E_0 \leq E_1 \leq E_2 \leq \cdots$. ($|E_0\rangle$ then called the "ground state" and E_0 is called the "ground state energy".)

Problem 12: Show that $\langle H\rangle = \sum_{n=0}^{\infty} E_n|\langle E_n|\psi\rangle|^2$ for any state $|\psi\rangle$. **Solution:**

$$\langle H\rangle = \langle\psi|\widehat{H}|\psi\rangle = \langle\psi|\widehat{H}\left(\sum_{n=0}^{\infty}|E_n\rangle\langle E_n|\right)|\psi\rangle = \sum_{n=0}^{\infty}\langle\psi|\widehat{H}|E_n\rangle\langle E_n|\psi\rangle = \sum_{n=0}^{\infty}E_n\langle\psi|E_n\rangle\langle E_n|\psi\rangle = \sum_{n=0}^{\infty}E_n\left|\langle E_n|\psi\rangle\right|^2.$$

Problem 13: Use the result of problem 12 to prove that $\langle H\rangle \geq E_0$ in any state $|\psi\rangle$.
Solution:

$$\langle H\rangle = \sum_{n=0}^{\infty}E_n\left|\langle E_n|\psi\rangle\right|^2 \geq \sum_{n=0}^{\infty}E_0\left|\langle E_n|\psi\rangle\right|^2 = E_0\sum_{n=0}^{\infty}\left|\langle E_n|\psi\rangle\right|^2 = E_0\langle\psi|\left(\sum_{n=0}^{\infty}|E_n\rangle\langle E_n|\right)|\psi\rangle = E_0\langle\psi|\psi\rangle = E_0.$$