Problem Set 6

Problem 1: Starting from equation (4.41) in the text, derive equation (4.45) in the approximation where you neglect the $\exp\{\pm i(\omega+\omega_0)t\}$ terms in (4.41). Note that the initial conditions are assumed to be $a(0)=1$ and $b(0)=0$. **Solution:** Eqn (4.41) with the $(\omega+\omega_0)$ terms set to zero gives the two equations

$$i\dot{c} = \frac{\omega_1}{4} e^{-i(\omega-\omega_0)t} d, \qquad i\dot{d} = \frac{\omega_1}{4} e^{i(\omega-\omega_0)t} c. \tag{1}$$

Multiplying the first through by $e^{i(\omega-\omega_0)t}$, differentiating with respect to time, then using the second equation to eliminate $\dot{d}$ in favor of c gives the single second-order equation

$$0 = \left[\frac{d^2}{dt^2} + i(\omega-\omega_0)\frac{d}{dt} + \left(\frac{\omega_1}{4}\right)^2\right] c$$

which has general solution (found, eg, by factorizing the second order differential operator above)

$$c = A_+ \exp\left\{\frac{it}{2}\left[\omega_0-\omega+\sqrt{(\omega_0-\omega)^2+\omega_1^2/4}\right]\right\} + A_- \exp\left\{\frac{it}{2}\left[\omega_0-\omega-\sqrt{(\omega_0-\omega)^2+\omega_1^2/4}\right]\right\}.$$

Then from the first equation in (1) we get that

$$\begin{aligned} d = &-\frac{2A_+}{\omega_1}\left[\omega_0-\omega+\sqrt{(\omega_0-\omega)^2+\omega_1^2/4}\right]\exp\left\{\frac{it}{2}\left[\omega-\omega_0+\sqrt{(\omega_0-\omega)^2+\omega_1^2/4}\right]\right\} \\ &-\frac{2A_-}{\omega_1}\left[\omega_0-\omega-\sqrt{(\omega_0-\omega)^2+\omega_1^2/4}\right]\exp\left\{\frac{it}{2}\left[\omega-\omega_0-\sqrt{(\omega_0-\omega)^2+\omega_1^2/4}\right]\right\}. \end{aligned} \tag{2}$$

Since $a(t) = c(t)e^{-i\omega_0 t/2}$, $b(t) = d(t)e^{i\omega_0 t/2}$, and $a(0)=1$ and $b(0)=0$, it follows that $c(0)=1$ and $d(0)=0$. Plugging these into the last two equations gives

$$A_\pm = \frac{\pm(\omega-\omega_0)+\sqrt{(\omega_0-\omega)^2+\omega_1^2/4}}{2\sqrt{(\omega_0-\omega)^2+\omega_1^2/4}}. \tag{3}$$

Plugging (3) into (2) gives

$$d = \frac{-i\omega_1 e^{it(\omega-\omega_0)/2}}{2\sqrt{(\omega_0-\omega)^2+\omega_1^2/4}} \sin\left\{\frac{t}{2}\sqrt{(\omega_0-\omega)^2+\tfrac{1}{4}\omega_1^2}\right\}.$$

Now, we want to compute $|\langle-\hat{z}|\psi(t)\rangle|^2 = |b(t)|^2 = |d(t)|^2$. So we get

$$|\langle-\hat{z}|\psi(t)\rangle|^2 = \frac{\frac{1}{4}\omega_1^2}{(\omega_0-\omega)^2+\frac{1}{4}\omega_1^2}\sin^2\left\{\frac{t}{2}\sqrt{(\omega_0-\omega)^2+\tfrac{1}{4}\omega_1^2}\right\},$$

which is eqn (4.45) of the text.

Problems 2 through 9 consider the spin states of an electron in a uniform but increasing magnetic field in the $\hat{x}$-direction, $\vec{B} = (t/\tau_0)B_0\hat{x}$, where B_0 and τ_0 are constants. As usual, define the frequency $\omega_0 = egB_0/2mc$ from the electron charge, magnetic moment g-factor, and mass, and express all your answers in terms of ω_0 and τ_0. At time $t = 0$, the the electron is in its "spin down" state, $|\psi(0)\rangle = |\frac{1}{2}, -\frac{1}{2}\rangle$, in the usual $\widehat{J}_z$ basis.

Problem 2: What is the Hamiltonian for this system, and what are its energy levels (ie, the energy eigenvalues) as a function of time? **Solution:** `From the text,` $\widehat{H} = (eg/2mc)\vec{B}\cdot\vec{J} = (\omega_0 t/\tau_0)\widehat{J}_x$. `The eigenvalues of` $\widehat{J}_x$ `are` $\pm\hbar/2$`, so the energy levels are` $\pm\hbar\omega_0 t/2\tau_0$.

Problem 3: Write down the Schrödinger equation for $|\psi(t)\rangle$. If $|\psi(t)\rangle$ in the $\widehat{J}_z$ basis is

$$|\psi(t)\rangle = a(t)|\tfrac{1}{2}, \tfrac{1}{2}\rangle + b(t)|\tfrac{1}{2}, -\tfrac{1}{2}\rangle, \tag{4}$$

write the Schrödinger equation as a coupled system of differential equations for $a(t)$ and $b(t)$. What are the initial conditions at $t = 0$ for these equations? **Solution:** `Schrödinger's equation is` $\frac{d}{dt}|\psi(t)\rangle = -(i/\hbar)\widehat{H}|\psi(t)\rangle = -(i\omega_0 t/\tau_0\hbar)\widehat{J}_x|\psi(t)\rangle$. `In the` $\widehat{J}_z$ `basis` $\widehat{J}_x = (\hbar/2)\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$`, so SE becomes`

$$\begin{pmatrix}\dot{a}\\ \dot{b}\end{pmatrix} = -\frac{i\omega_0 t}{2\tau_0}\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix} = -\frac{i\omega_0 t}{2\tau_0}\begin{pmatrix}b\\ a\end{pmatrix}. \tag{5}$$

$|\psi(0)\rangle = |\frac{1}{2}, -\frac{1}{2}\rangle$ `implies` $a(0) = 0$ `and` $b(0) = 1$.

Problem 4: Check that

$$\begin{aligned} a &= iA\cos\left(\frac{\omega_0}{4\tau_0}t^2\right) - iB\sin\left(\frac{\omega_0}{4\tau_0}t^2\right), \\ b &= A\sin\left(\frac{\omega_0}{4\tau_0}t^2\right) + B\cos\left(\frac{\omega_0}{4\tau_0}t^2\right), \end{aligned} \tag{6}$$

solve equation (6) for any values of the constants A and B. The initial condition at $t = 0$ specifies what values for A and B? **Solution:** `This is just a matter of plugging into (5).` $a(0) = 0$ `implies` $A = 0$`, and` $b(0) = 1$ `implies` $B = 1$.

Problem 5: What is the probability of measuring $J_z = -\hbar/2$ at time $t = 2t_0$? Approximate your answer for small t_0 to include the leading correction away from probability 1. **Solution:**

$$\mathcal{P}(J_z{=}-\hbar/2, t{=}2t_0) = \left|\langle\tfrac{1}{2}, \tfrac{1}{2}|\psi(2t_0)\rangle\right|^2 = |b(2t_0)|^2 = \cos^2\left(\frac{\omega_0}{\tau_0}t_0^2\right),$$

`where we used (4), (6), and the initial conditions` $A = 0$, $B = 1$. `Thus`

$$\mathcal{P}(-, 2t_0) \approx 1 - \frac{\omega_0^2}{\tau_0^2}t_0^4.$$

Problem 6: If $J_z = -\hbar/2$ is measured at time $t = t_0$, what is the state of the system immediately after the measurement? **Solution:** `The state will be projected onto the eigenstate of the observed eigenvalue, so` $\psi(t_0) \to |\frac{1}{2}, -\frac{1}{2}\rangle$.

Problem 7: What is the probability of observing $J_z = -\hbar/2$ at time $t = 2t_0$ if $J_z = -\hbar/2$ is also measured at time $t = t_0$? How does this compare to the probability computed in problem 5 for small t_0? **Solution:** If $J_z = -\hbar/2$ is measured at $t = t_0$, then $\psi(t_0) \to \widetilde{\psi}(t_0) = |\frac{1}{2}, -\frac{1}{2}\rangle$. After $t = t_0$, the state then evolves, according to (6), but with the initial condition $\widetilde{b}(t_0) = 1$ and $\widetilde{a}(t_0) = 0$. (I use $\widetilde{a}$ and $\widetilde{b}$ to denote the components of the state $|\widetilde{\psi}\rangle$ that results from the measurement at $t = t_0$.) The solution of (6) with these initial conditions has

$$A = \sin\left(\frac{\omega_0}{4\tau_0} t_0^2\right), \qquad B = \cos\left(\frac{\omega_0}{4\tau_0} t_0^2\right),$$

so for $t > t_0$

$$\begin{aligned}
\widetilde{a} &= i\sin\left(\frac{\omega_0}{4\tau_0}t_0^2\right)\cos\left(\frac{\omega_0}{4\tau_0}t^2\right) - i\cos\left(\frac{\omega_0}{4\tau_0}t_0^2\right)\sin\left(\frac{\omega_0}{4\tau_0}t^2\right) = -i\sin\left(\frac{\omega_0}{4\tau_0}(t^2 - t_0^2)\right), \\
\widetilde{b} &= \sin\left(\frac{\omega_0}{4\tau_0}t_0^2\right)\sin\left(\frac{\omega_0}{4\tau_0}t^2\right) + \cos\left(\frac{\omega_0}{4\tau_0}t_0^2\right)\cos\left(\frac{\omega_0}{4\tau_0}t^2\right) = \cos\left(\frac{\omega_0}{4\tau_0}(t^2 - t_0^2)\right).
\end{aligned}$$

So in this state, the probability of measuring $J_z = -\hbar/2$ at $t = 2t_0$ is

$$\mathcal{P}(J_z{=}-\hbar/2, t{=}2t_0) = \left|\langle \tfrac{1}{2}, -\tfrac{1}{2}|\widetilde{\psi}(2t_0)\rangle\right|^2 = \left|\widetilde{b}(2t_0)\right|^2 = \cos^2\left(\frac{3\omega_0}{4\tau_0}t_0^2\right). \tag{7}$$

For small t_0 this gives

$$\mathcal{P}(-, 2t_0|-, t_0) \approx 1 - \frac{9\omega_0^2}{16\tau_0^2}t_0^4,$$

so is greater than that found in problem 5.

Problem 8: What is the probability of observing $J_z = -\hbar/2$ at time $t = 2t_0$ if instead $J_z = +\hbar/2$ is measured at time $t = t_0$? How does this compare to the probability computed in problem 5 for small t_0? **Solution:** If $J_z = +\hbar/2$ is measured at $t = t_0$, then $\psi(t_0) \to \widetilde{\psi}(t_0) = |\frac{1}{2}, \frac{1}{2}\rangle$. After $t = t_0$, the state then evolves, according to (6), but with the initial condition $\widetilde{b}(t_0) = 0$ and $\widetilde{a}(t_0) = 1$. The solution of (6) with these initial conditions has

$$A = -i\cos\left(\frac{\omega_0}{4\tau_0} t_0^2\right), \qquad B = +i\sin\left(\frac{\omega_0}{4\tau_0} t_0^2\right),$$

so for $t > t_0$

$$\begin{aligned}
\widetilde{a} &= \cos\left(\frac{\omega_0}{4\tau_0}t_0^2\right)\cos\left(\frac{\omega_0}{4\tau_0}t^2\right) + \sin\left(\frac{\omega_0}{4\tau_0}t_0^2\right)\sin\left(\frac{\omega_0}{4\tau_0}t^2\right) = \cos\left(\frac{\omega_0}{4\tau_0}(t^2 - t_0^2)\right), \\
\widetilde{b} &= -i\cos\left(\frac{\omega_0}{4\tau_0}t_0^2\right)\sin\left(\frac{\omega_0}{4\tau_0}t^2\right) + i\sin\left(\frac{\omega_0}{4\tau_0}t_0^2\right)\cos\left(\frac{\omega_0}{4\tau_0}t^2\right) = -i\sin\left(\frac{\omega_0}{4\tau_0}(t^2 - t_0^2)\right).
\end{aligned}$$

So in this state, the probability of measuring $J_z = -\hbar/2$ at $t = 2t_0$ is

$$\mathcal{P}(J_z{=}-\hbar/2, t{=}2t_0) = \left|\langle \tfrac{1}{2}, -\tfrac{1}{2}|\widetilde{\psi}(2t_0)\rangle\right|^2 = \left|\widetilde{b}(2t_0)\right|^2 = \sin^2\left(\frac{3\omega_0}{4\tau_0}t_0^2\right). \tag{8}$$

For small t_0 this gives

$$\mathcal{P}(-, 2t_0|+, t_0) \approx \frac{9\omega_0^2}{16\tau_0^2} t_0^4,$$

so is much smaller than that found in problem 5.

Problem 9: (This is a bit tricky!) What is the probability of observing $J_z = -\hbar/2$ at time $t = 2t_0$ if J_z is measured at time $t = t_0$? (I'm just telling you that J_z is measured at time $t = t_0$, but not specifying which outcome is actually found.) How does this compare to the probability computed in problem 5 at small t_0? **Solution:** At time t_0 the probability of measuring $J_z = -\hbar/2$ is $\mathcal{P}(-, t_0) = \cos^2\left(\frac{\omega_0}{4\tau_0} t_0^2\right)$ by problem 5. So the probability of measuring $J_z = +\hbar/2$ at the same time is $\mathcal{P}(+, t_0) = 1 - \mathcal{P}(-, t_0)$. If $J_z = -\hbar/2$ is measured at $t = t_0$, then from problem 7, the probability of measuring $J_z = -\hbar/2$ at $t = 2t_0$ is $\mathcal{P}(-, 2t_0|-, t_0)$ given in (7). Likewise, if $J_z = +\hbar/2$ is measured at $t = t_0$, then from problem 8, the probability of measuring $J_z = -\hbar/2$ at $t = 2t_0$ is $\mathcal{P}(-, 2t_0|+, t_0)$ given in (8). Thus the total probability of measuring $J_z = -\hbar/2$ at $t = 2t_0$ given a measurement of $\widehat{J}_z$ at $t = t_0$ is

$$\begin{aligned}
\mathcal{P}(-, 2t_0 \mid J_z \texttt{ at } t_0) &= \mathcal{P}(-, t_0)\mathcal{P}(-, 2t_0|-, t_0) + \mathcal{P}(+, t_0)\mathcal{P}(-, 2t_0|+, t_0) \\
&= \cos^2\left(\frac{\omega_0}{4\tau_0} t_0^2\right) \cos^2\left(\frac{3\omega_0}{4\tau_0} t_0^2\right) + \sin^2\left(\frac{\omega_0}{4\tau_0} t_0^2\right) \sin^2\left(\frac{3\omega_0}{4\tau_0} t_0^2\right) \\
&\approx \left(1 - \frac{\omega_0^2}{16\tau_0^2} t_0^4\right)\left(1 - \frac{9\omega_0^2}{16\tau_0^2} t_0^4\right) + \frac{\omega_0^2}{16\tau_0^2} t_0^4 \frac{9\omega_0^2}{16\tau_0^2} t_0^4 \\
&\approx 1 - \frac{5}{8}\frac{\omega_0^2}{\tau_0^2} t_0^4.
\end{aligned}$$

This is greater than the result found in problem 5 when no measurement is made at an intermediate time.