Intro to QM due: October 31, 2018

Exam 2

For **problem 1** consider a system whose 2-dimensional Hilbert space has an orthonormal basis $\{|0\rangle, |1\rangle\}$. In this basis the hamiltonian for the system is given by the matrix

$$\widehat{H} = \hbar \begin{pmatrix} \omega_0 & \omega_1 \\ \omega_1 & \omega_0 \end{pmatrix}. \tag{1}$$

The energy eigenvalues, $E_\pm$, and associated normalized eigenstates, $|\pm\rangle$, are

$$E_\pm = \hbar(\omega_0 \pm \omega_1), \qquad |\pm\rangle = \frac{1}{\sqrt{2}}\Big(|0\rangle \pm |1\rangle\Big). \tag{2}$$

The state of the system is

$$|\psi(t)\rangle = c_0(t)|0\rangle + c_1(t)|1\rangle. \tag{3}$$

Problem 1: (20 points) Write down the solution of Schrödinger's equation for $c_0(t)$ and $c_1(t)$ in terms of their initial values, $c_0(0)$ and $c_1(0)$, at time $t = 0$.
Hint: Since this is a problem with a time-independent Hamiltonian, the general solution of Schrödinger's equation was derived in class in terms of the energy eigenvalues and eigenvectors.

Solution: The general solution of Schrödinger's equation for a time-independent hamiltonian is $|\psi(t)\rangle = \sum_n e^{-iE_nt/\hbar}|E_n\rangle\langle E_n|\psi(0)\rangle$. Substituting in this expression from (2) and (3) gives

$$\begin{aligned}
c_0(t)|0\rangle + c_1(t)|1\rangle &= |\psi(t)\rangle = e^{-iE_+t/\hbar}|+\rangle\langle+|\psi(0)\rangle + e^{-iE_-t/\hbar}|-\rangle\langle-|\psi(0)\rangle \\
&= \tfrac{1}{\sqrt{2}}e^{-i(\omega_0+\omega_1)t}|+\rangle\Big(\langle 0| + \langle 1|\Big)\Big(c_0(0)|0\rangle + c_1(0)|1\rangle\Big) \\
&\quad + \tfrac{1}{\sqrt{2}}e^{-i(\omega_0-\omega_1)t}|-\rangle\Big(\langle 0| - \langle 1|\Big)\Big(c_0(0)|0\rangle + c_1(0)|1\rangle\Big) \\
&= \tfrac{1}{2}e^{-i(\omega_0+\omega_1)t}\Big(|0\rangle + |1\rangle\Big)\Big(c_0(0) + c_1(0)\Big) + \tfrac{1}{2}e^{-i(\omega_0-\omega_1)t}\Big(|0\rangle - |1\rangle\Big)\Big(c_0(0) - c_1(0)\Big) \\
&= e^{-i\omega_0 t}\Big(\Big[c_0(0)\cos(\omega_1 t) - ic_1(0)\sin(\omega_1 t)\Big]|0\rangle + \Big[c_1(0)\cos(\omega_1 t) - ic_0(0)\sin(\omega_1 t)\Big]|1\rangle\Big).
\end{aligned}$$

Comparing the left and right sides then gives

$$\begin{aligned}
c_0(t) &= e^{-i\omega_0 t}\Big[c_0(0)\cos(\omega_1 t) - ic_1(0)\sin(\omega_1 t)\Big] \\
c_1(t) &= e^{-i\omega_0 t}\Big[c_1(0)\cos(\omega_1 t) - ic_0(0)\sin(\omega_1 t)\Big].
\end{aligned}$$

For **problem 2**, consider the same 2-state system with general state given by eqn. (3), but with a *time-dependent* Hamiltonian given by

$$\widehat{H}(t) = \hbar \begin{pmatrix} \omega_0 & \omega_1 \cos(\omega t) \\ \omega_1 \cos(\omega t) & \omega_0 \end{pmatrix}. \tag{4}$$

in the $\{|0\rangle, |1\rangle\}$ basis.

Problem 2: (10 points) Write down Schrödinger's equation as a coupled system of differential equations for $c_0(t)$ and $c_1(t)$. (I am *not* asking you to solve the equation!)
Solution: Schrödinger's equation is $(d/dt)|\psi(t)\rangle = -(i/\hbar)\widehat{H}|\psi(t)\rangle$. Since in the $\{|0\rangle, |1\rangle\}$ basis we have from (3) that $|\psi(t)\rangle = \begin{pmatrix} c_0(t) \\ c_1(t) \end{pmatrix}$, we get from (4)

$$\frac{d}{dt}\begin{pmatrix} c_0(t) \\ c_1(t) \end{pmatrix} = \frac{-i}{\hbar}\widehat{H}\begin{pmatrix} c_0(t) \\ c_1(t) \end{pmatrix} = -i\begin{pmatrix} \omega_0 & \omega_1 \cos(\omega t) \\ \omega_1 \cos(\omega t) & \omega_0 \end{pmatrix}\begin{pmatrix} c_0(t) \\ c_1(t) \end{pmatrix} = -i\begin{pmatrix} \omega_0 c_0(t) + \omega_1 \cos(\omega t) c_1(t) \\ \omega_1 \cos(\omega t) c_0(t) + \omega_0 c_1(t) \end{pmatrix},$$

implying

$$\begin{aligned} \frac{d}{dt}c_0(t) &= -i\omega_0 c_0(t) - i\omega_1 \cos(\omega t) c_1(t) \\ \frac{d}{dt}c_1(t) &= -i\omega_1 \cos(\omega t) c_0(t) - i\omega_0 c_1(t). \end{aligned}$$

Problem 3: (10 points) Suppose you have two particles of spins j_1 and j_2. What is the dimension of the Hilbert space describing the combined spin states of the particles?
Solution: The Hilbert space, V_j, of spin states of a particle of spin j has basis $\{|j,m\rangle\}$ where $m \in \{-j, -j+1, \dots, j\}$. Since there are $2j+1$ values for m in this set, the dimension of V_j is $\dim(V_j) = 2j+1$. The Hilbert space, V, of two particles is the tensor product $V = V_{j_1} \otimes V_{j_2}$. The dimension of the tensor product of two spaces is the product of the dimensions of the spaces, so

$$\dim(V) = \dim(V_{j_1}) \cdot \dim(V_{j_2}) = (2j_1 + 1)(2j_2 + 1).$$

Problem 4: (20 points) Suppose you have a system of two spin $j = \frac{1}{2}$ particles with hamiltonian

$$\widehat{H} = \frac{2\omega}{\hbar}\vec{\widehat{J}}_1 \cdot \vec{\widehat{J}}_2 + 2\omega\left(\widehat{J}_{1z} + \widehat{J}_{2z}\right). \tag{5}$$

What are the energy eigenvalues of this system?
Hint: Recall that $J^2 = J_1^2 + 2\vec{J}_1 \cdot \vec{J}_2 + J_2^2$.
Solution: The hint implies $2\vec{J}_1 \cdot \vec{J}_2 = J^2 - J_1^2 - J_2^2$. Also $J_z = J_{1z} + J_{2z}$. Use these to rewrite (5) as

$$\widehat{H} = \frac{\omega}{\hbar}\left(\widehat{J}^2 - \widehat{J}_1^2 - \widehat{J}_2^2\right) + 2\omega \widehat{J}_z.$$

Since particles 1 and 2 are both spin-$\frac{1}{2}$, all their spin states are in eigenstates of $\widehat{J}_1^2$ and $\widehat{J}_2^2$ with $J_i^2 = \hbar^2 j_1(j_i+1) = \hbar^2 \frac{1}{2}(\frac{1}{2}+1) = \frac{3}{4}\hbar^2$. Thus

$$\widehat{H} = \frac{\omega}{\hbar}\left(\widehat{J}^2 - \hbar^2\frac{3}{2}\right) + 2\omega \widehat{J}_z.$$

This hamiltonian is written in terms of $\widehat{J}^2$ and $\widehat{J}_z$ which are commuting observables with simultaneous eigenstates $|j,m\rangle$, as usual. Therefore, the $|j,m\rangle$ are an eigenbasis for $\widehat{H}$, and we can find the eigenvalues of $\widehat{H}$ simply by acting on them:

$$\begin{aligned}
\widehat{H}|j,m\rangle &= \frac{\omega}{\hbar}\widehat{J}^2|j,m\rangle - \hbar\omega\frac{3}{2}|j,m\rangle + 2\omega\widehat{J}_z|j,m\rangle \\
&= \frac{\omega}{\hbar}\hbar^2 j(j+1)|j,m\rangle - \hbar\omega\frac{3}{2}|j,m\rangle + 2\omega\hbar m|j,m\rangle \\
&= \hbar\omega\left[j(j+1) + 2m - \frac{3}{2}\right]|j,m\rangle,
\end{aligned}$$

so the eigenvalues are $E_{jm} = \hbar\omega\left[j(j+1) + 2m - \frac{3}{2}\right]$. It remains only to determine what values of j and m are realized in this system. But we know from the addition of angular momentum of two spin-$\frac{1}{2}$'s that only the $j=1$ triplet of states with $m \in \{-1,0,1\}$ and the $j=0$ single state with $m=0$ occur. So, putting in these values we find the energy eigenvalues

$$\begin{aligned}
E_{1,1} &= \hbar\omega\left[1(1+1) + 2\cdot 1 - \frac{3}{2}\right] = \frac{5}{2}\hbar\omega, \\
E_{1,0} &= \hbar\omega\left[1(1+1) + 2\cdot 0 - \frac{3}{2}\right] = \frac{1}{2}\hbar\omega, \\
E_{1,-1} &= \hbar\omega\left[1(1+1) + 2\cdot(-1) - \frac{3}{2}\right] = -\frac{3}{2}\hbar\omega, \\
E_{0,0} &= \hbar\omega\left[0(0+1) + 2\cdot 0 - \frac{3}{2}\right] = -\frac{3}{2}\hbar\omega.
\end{aligned}$$