Problem Set 17

All problems in this problem set will be worth 10 points instead of the usual 1 point.

Problem 1: Recall from lecture that upon traversing a closed path γ in the presence of a magnetic field, the wave function of a particle with electric charge q picks up a phase

$$\psi(\vec{x}) \xrightarrow{\gamma} \exp\left\{i\frac{q}{\hbar c}\alpha\right\} \psi(\vec{x}) \qquad \text{with} \qquad \alpha = \oint_\gamma \vec{A}(x)\cdot d\vec{x} \tag{1}$$

upon returning to its starting point, $\vec{x}$. Here $\vec{A}(x)$ is the vector potential, related to the magnetic field by $\vec{B} = \vec{\nabla}\times\vec{A}$. Assume the magnetic field is due to a *magnetic monopole* of magnetic charge g at the origin, i.e.,

$$\vec{B}(\vec{x}) = g\frac{\widehat{r}}{r^2}, \tag{2}$$

where $\widehat{r}$ is a unit vector pointing in the radial direction and $r = |\vec{x}|$ is the radial distance from the origin.

By considering moving the charged particle along any closed path γ that avoids the origin, and by computing the resulting phase change of its wave function in two different ways, show that electric and magnetic charges must satisfy

$$qg = \frac{\hbar c}{2}n \quad \text{with } n \text{ an integer.} \tag{3}$$

Hint: There are many ways of doing this problem, but the easiest is to use Stokes theorem, which states

$$\oint_\gamma \vec{A}(x)\cdot d\vec{x} = \int_S \vec{\nabla}\times\vec{A}(x)\cdot d\vec{S} \tag{4}$$

where S is any surface bounded by γ and with orientation determined by that of γ. Use this to evaluate the phase ψ picks up by choosing two different S's bounded by γ. It may also help to pick a convenient γ, and to remember Gauss's law from electrostatics and apply it to this magnetostatic situation.

Problem 2: The hamiltonian of a charged particle in a magnetic field is

$$H[\vec{A}] = \frac{1}{2m}\left(\vec{p} - \frac{q}{c}\vec{A}(\vec{x})\right)^2. \tag{5}$$

I will no longer write the "hats" on operators; here H, $\vec{p}$, and $\vec{x}$ are the usual operators. I have written the hamiltonian as a functional of the vector potential for the magnetic field because the vector potential is ambiguous. If

$$\vec{A}' = \vec{A} + \vec{\nabla}\chi \tag{6}$$

for any function $\chi(\vec{x})$, then both $\vec{A}'$ and $\vec{A}$ describe the same magnetic field. $\vec{A}$ and $\vec{A}'$ are said to be related by a *gauge transformation*.

Show that if $\psi(\vec{x})$ is the wave function of an eigenstate of $H[\vec{A}]$ of energy E, then

$$\psi'(\vec{x}) = \exp\{iq\chi(\vec{x})/(\hbar c)\}\psi(\vec{x}) \tag{7}$$

is an eigenstate with the same energy of $H[\vec{A}']$. This shows that the spectrum of H does not change under gauge transformations, or is *gauge invariant*.

Show that

$$\langle\psi'|\vec{x}|\psi'\rangle = \langle\psi|\vec{x}|\psi\rangle. \tag{8}$$

Thus $\vec{x}$ is gauge invariant.

Show that

$$\langle\psi'|\vec{p}|\psi'\rangle = \langle\psi|\vec{p}|\psi\rangle + \frac{q}{c}\langle\psi|\vec{\nabla}\chi(\vec{x})|\psi\rangle \qquad \text{and} \qquad \langle\psi'|\vec{\pi}'|\psi'\rangle = \langle\psi|\vec{\pi}|\psi\rangle, \tag{9}$$

where $\vec{\pi} := \vec{p} - \frac{q}{c}\vec{A}(\vec{x})$. Thus $\vec{p}$ is not gauge invariant, and so is not a physically meaningful operator. $\vec{\pi}$ is the correct gauge-invariant operator describing the particle momentum.

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For the rest of the problems in this problem set, consider a thin rectangular sheet of metal in the x-y plane, with some number of conduction electrons per unit area, in the presence of a uniform magnetic field in the z-direction, $\vec{B} = B\hat{z}$. Recall that electrons have electric charge $-e$, mass m, and are spin-$\frac{1}{2}$ identical fermions. We will make some (drastic) simplifying assumptions:

- the electrons form a free gas, i.e., they do not interact with each other;
- the metal sheet is so thin that its extent in the z-direction can be neglected, so we will consider the electrons to move only in 2 dimensions, namely, the x-y plane;
- the metal sheet is so large that the effects of its boundaries can be ignored.

To summarize, we are considering a free 2d electron gas in a perpendicular magnetic field of constant strength B. This problem is clearly translationally and rotationally invariant in the plane of the gas.

Problem 3: The hamiltonian of a single electron in the 2d gas is

$$H = \frac{1}{2m}\left(\vec{p} + \frac{e}{c}\vec{A}(\vec{x})\right)^2 + \frac{e}{mc}\vec{S}\cdot\vec{B}. \tag{10}$$

$\vec{S}$ is the vector of spin angular momentum operators of the electron. $\vec{p}$ and $\vec{x}$ are 2d vectors (in the x-y plane) while $\vec{A}$, $\vec{B}$, and $\vec{S}$ are 3d vectors.

Check that the vector potential,

$$\vec{A}(\vec{x}) = xB\widehat{y}, \tag{11}$$

where $\widehat{y}$ is the unit vector in the y-direction, gives the correct magnetic field.

Solve for the energy eigenvalues and eigenvectors with this choice of $\vec{A}$. Write them in terms of the following convenient quantities,

$$\omega_B := \frac{eB}{mc} \qquad \text{and} \qquad \ell_B := \sqrt{\frac{\hbar c}{eB}}, \tag{12}$$

known as the cyclotron frequency and magnetic length, respectively. Give the energy eigenvectors in an x-y position basis (i.e., as wave functions) tensored with the electron spin state.

You should find that the energy eigenvalues are

$$E_n = \hbar\omega_B n \qquad \text{for} \qquad n \in \{0, 1, 2, 3, \ldots\}, \tag{13}$$

and that each energy level is infinitely degenerate.

Hint: The energy eigenstates you find should be a product of harmonic oscillator eigenstates with plane wave states (in different variables).

Problem 4: The same as **problem 3**, but now with, in addition, a uniform electric field in the x-direction. For simplicity, from now on we will <u>ignore the electron spin</u>. Thus now our single-electron hamiltonian is

$$H = \frac{1}{2m}\left(\vec{p} + \frac{e}{c}\vec{A}(\vec{x})\right)^2 + eEx, \tag{14}$$

where E is a constant (the magnitude of the electric field), and x in the last term is the x-coordinate (not the $\vec{x}$ vector). Find the eigenvalues and eigenstates of H.

Hint: The eigenstate wave functions are related to those of **problem 3** simply by an E-dependent shift in x.

Problem 5: By making the gauge choice (11), we treated the x- and y-directions asymmetrically, and so hid the rotational invariance of the problem. So now let's re-do **problem 3** with the more symmetrical gauge choice

$$\vec{A} = \frac{1}{2}\left(xB\widehat{y} - yB\widehat{x}\right), \tag{15}$$

which you can check reproduces the correct magnetic field. Again, ignore the electron spin, so the 1-electron hamiltonian is just

$$H = \frac{1}{2m}\left(\vec{p} + \frac{e}{c}\vec{A}(\vec{x})\right)^2. \tag{16}$$

Diagonalize this hamiltonian by the following trick. Define the two 2-component vector operators

$$\vec{\pi}^{\pm} := \vec{p} \pm \frac{e}{c}\vec{A}(\vec{x}), \tag{17}$$

and from them define the operators

$$\begin{aligned} a &:= \frac{\ell_B}{\hbar}\frac{1}{\sqrt{2}}\Big(\pi_x^+ - i\pi_y^+\Big), & a^\dagger &:= \frac{\ell_B}{\hbar}\frac{1}{\sqrt{2}}\Big(\pi_x^+ + i\pi_y^+\Big), \\ b &:= \frac{\ell_B}{\hbar}\frac{1}{\sqrt{2}}\Big(\pi_x^- + i\pi_y^-\big), & b^\dagger &:= \frac{\ell_B}{\hbar}\frac{1}{\sqrt{2}}\Big(\pi_x^- - i\pi_y^-\Big). \end{aligned} \tag{18}$$

Show that these operators form 2 commuting sets of creation and annihilation operators. In other words, show that

$$[a,a^\dagger] = [b,b^\dagger] = 1 \qquad \text{and} \qquad [a,b] = [a^\dagger,b] = 0. \tag{19}$$

Show also that the hamiltonian and the z-component of angular momentum, $L_z = xp_y - yp_x$, can be written as

$$H = \hbar\omega_B\left(a^\dagger a + \tfrac{1}{2}\right) \qquad \text{and} \qquad L_z = \hbar\left(a^\dagger a - b^\dagger b\right). \tag{20}$$

Using the results (19) and (20) it is now easy to find the eigenvalues and eigenvectors of H. Define a state $|0,0\rangle$ as the one annihilated by both annihilation operators,

$$a|0,0\rangle = b|0,0\rangle = 0, \tag{21}$$

and define a basis of states by

$$|n,m\rangle = \frac{a^{\dagger n} b^{\dagger m}}{\sqrt{n!m!}}|0,0\rangle, \qquad n,m \in \{0,1,2,\dots\}. \tag{22}$$

Show that the $|n,m\rangle$ are orthonormal, that they are eigenstates of both H and L_z, and compute their eigenvalues.

Problem 6: Show that the wave functions of the lowest energy ($n=0$) eigenstates, $|0,m\rangle$, are

$$\psi_{0,m}(z,\bar{z}) \sim \bar{z}^m e^{-|z|^2/4\ell_B^2}, \tag{23}$$

where I have defined the complex coordinates in the x-y-plane by

$$z = x + iy, \qquad \bar{z} = x - iy. \tag{24}$$

In (23) I have dropped the overall normalization factors; you can too.

Hint: This goes easiest if you first rewrite a, $a^\dagger$, b, and $b^\dagger$, in terms of ℓ_B, z, $\bar{z}$, and the complex derivatives

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \qquad \text{and} \qquad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right), \tag{25}$$

and then use (21) and (22) from the last problem.

Problem 7: Now consider not just a single electron in the gas, but N such electrons. They are identical fermions, and recall that we are ignoring their spin degrees of freedom (for simplicity). Show that

$$\psi(z_1,\ldots,z_N) = P\cdot\exp\left\{-\frac{1}{4\ell_B^2}\sum_{i=1}^{N}|z_i|^2\right\}, \qquad \text{where} \qquad P := \prod_{1\le i<j\le N}(\bar{z}_i - \bar{z}_j), \tag{26}$$

is the wave function (up to normalization) of the state of N electrons in the 1-particle states $|0,m\rangle$ for $m = 0,\ldots,N-1$. (They have to be in different 1-particle states by fermi statistics.) Here we are using the notation introduced in **problems 5 and 6**.

Compute the energy of this state and show that it is one of the ground states.

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There are, of course, many other ground states, reflecting the infinite degeneracy of the lowest energy level. It is not too hard to see that any state of the form (26) but with P an arbitrary completely antisymmetric holomorphic polynomial is also a ground state. The particular choice of P in (26) is special because it gives the ground state centered on the origin with the smallest spatial extent. In the case of a finite-size 2d electron gas whose geometric center is taken to be the origin, edge effects are likely to lift the ground state degeneracy and make this state (or one "close" to it) *the* ground state.

A next step in analyzing the physics of this metal sheet in a magnetic field would be to take into account the interactions between the electrons in the 2d electron gas. This is a very hard problem which no one has solved. However someone made the *guess* that in a very strong magnetic field, and taking account of edge effects, the ground state of the interacting electron gas is given by a state of the form (26) but with the polynomial P given instead by the polynomial

$$P_s := \prod_{1\le i<j\le N}(\bar{z}_i - \bar{z}_j)^s, \qquad \text{where } s \text{ is a positive odd integer} \tag{27}$$

when $2\pi\ell_B^2 N/A = 1/s$ where A is the area of the 2d electron gas.

The power s needs to be an odd integer for the wave function to be completely antisymmetric, as required by Fermi statistics. $1/s$ is called the "filling fraction" in the literature. When s takes its smallest value, $s = 1$, we get back the non-interacting ground state discussed in **problem 7**.

The guess (27) is known to be *quantitatively* an extremely bad guess, but is, in a sense that can be made more precise, *qualitatively* an extremely good guess: it somehow captures the basic physics of what is going on. It illuminated a beautiful chapter in theoretical and experimental physics that has been the source of a great many advances of the last quarter century, and won the guessor a Nobel prize.