20598 – Finance with Big Data

Week 2 Lecture: Portfolio Theory

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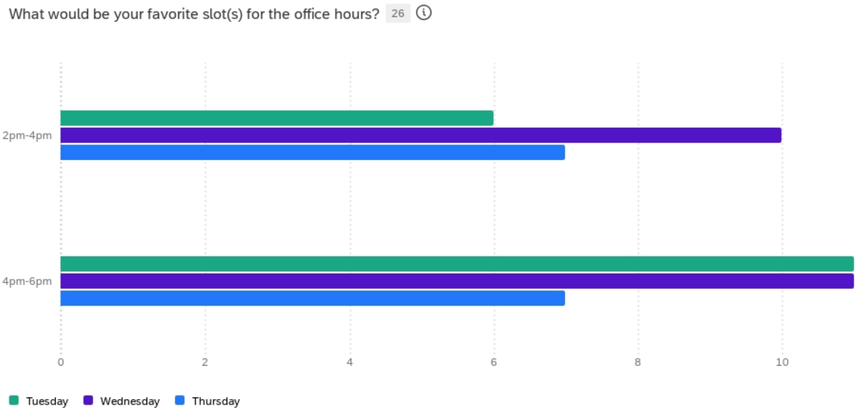
50%

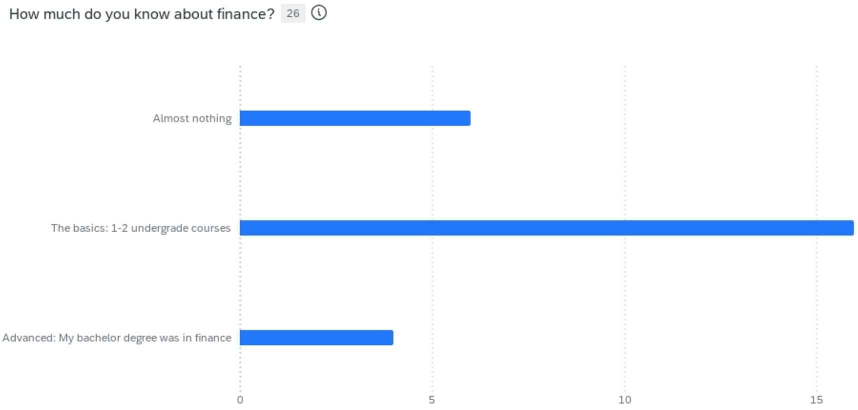
Results from the survey
#1 : Office hours



Office hours: **Wednesday 4pm-6pm**
Better if you send me an email beforehand

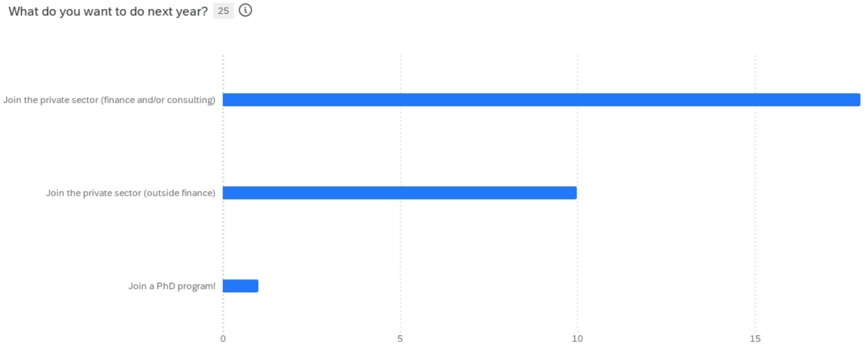
Results from the survey

#2 : Background



Results from the survey

#3 : Next year



Results from the survey

#5 : Pace was almost perfect... **on average** (avg. = 51)

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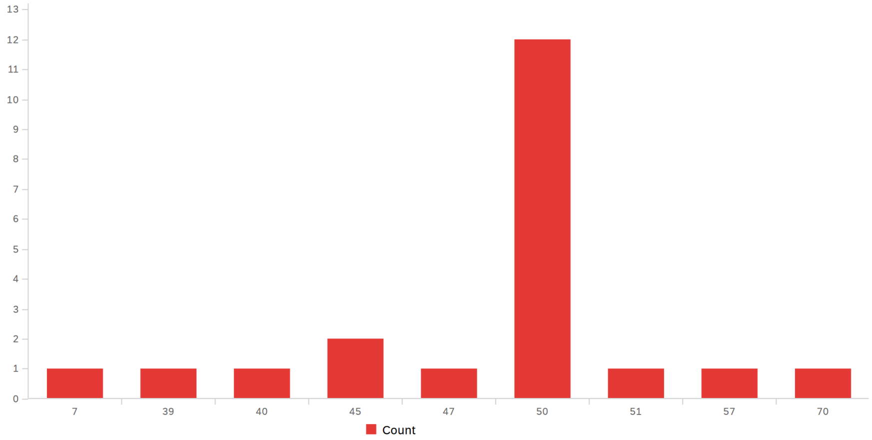

Waiting for answers about attending status!

Mark LEVIN
Beatrice LAINI
Ossama TCHINA

Outline

Asset Management : What and Why ?

Portfolio Theory

Returns : Stats. & Historical Data

Portfolio Properties

The Efficient Frontier

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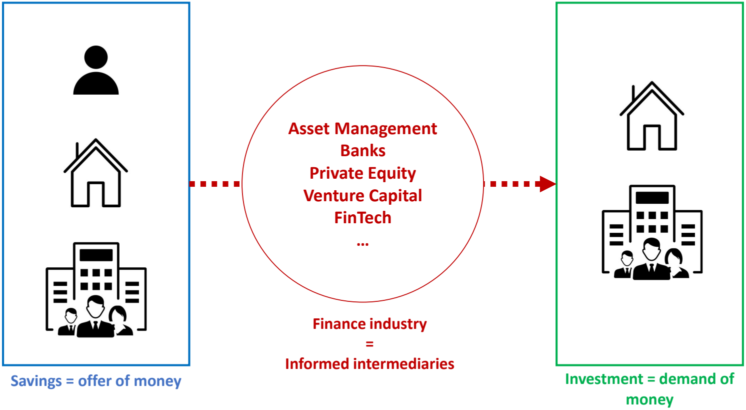
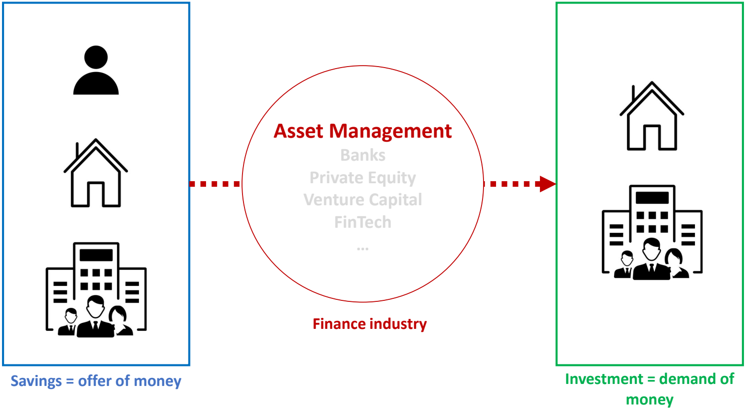
Asset Management
What & Why ?

What is Asset Pricing / Management ?



What is Asset Pricing / Management ?



What is Asset Pricing / Management ?

- An asset management company is a firm that collect and invest the wealth of households and firms

- Typical informed intermediaries

What is Asset Pricing / Management ?

- Why do we need them?

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- Individual investors usually **lack the expertise and resources** to consistently produce strong investment returns over time (= information asymmetry)
- Therefore, many investors rely on asset management companies to invest capital on their behalf

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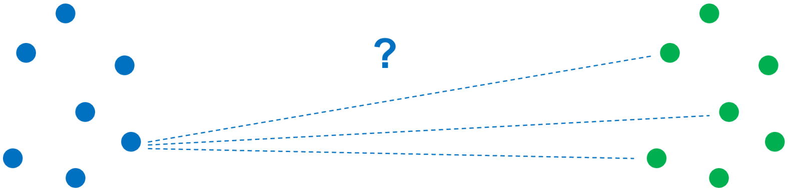
- Why do we need them?
- Individual investors usually **lack the expertise and resources** to consistently produce strong investment returns over time (= information asymmetry)
- Therefore, many investors rely on asset management companies to invest capital on their behalf
- But also:
  - economies of **scale**: average fixed cost (per $ of investment) decreases as the size increases
  - economies of **scope**: average variable cost decreases as number of products offered increases
  - Search and matching frictions

What is Asset Pricing / Management ?

- Matching buyers and sellers might be tricky
- In particular in non-centralized markets

What is Asset Pricing / Management ?



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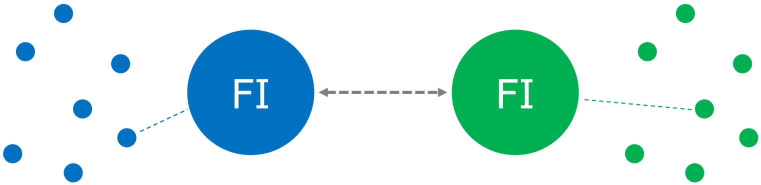
What is Asset Pricing / Management ?



- Matching buyers and sellers might be tricky
- In particular in non-centralized markets

Asset Management Companies

- Asset Management Companies come in many **different forms and structures :**
  - *Standard* Asset Management firms (MMF, Index and Pension funds)
  - Hedge funds

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- AMC differ in their **legal status, methods, investment horizons, fees, etc.**
  - Mutual funds $\rightarrow$ highly regulated (Investment Company Act of 1940)
  - Hedge funds $\rightarrow$ poorly regulated, high fees (2% of AUM + 20% of profits)

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  - Mutual funds $\rightarrow$ highly regulated (Investment Company Act of 1940)
  - Hedge funds $\rightarrow$ poorly regulated, high fees (2% of AUM + 20% of profits)
- But hedge funds come with restrictive investment rules:
  - High minimum investment (average \$1M)
  - Wealthy investors: annual income >\$200,000 and net worth (excluding house) >\$1M

Asset Management Companies

Standard Asset Management firms (2023)

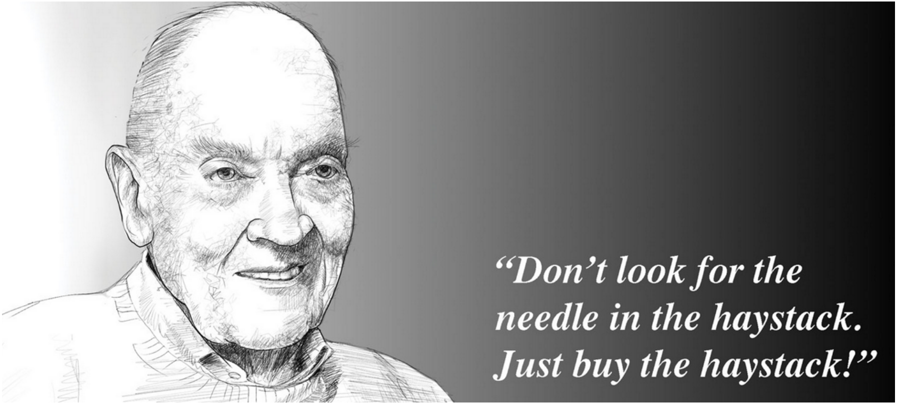
Asset Management Companies

Standard Asset Management firms (2023)

Rank	Company	Country	Total AUM, US$b	Balance sheet
1	BlackRock	US	9,570	03/31/2022
2	Vanguard Group	US	8,100	03/31/2022
3	UBS Group *	Switzerland	4,380	03/31/2022
4	Fidelity Investments	US	4,283	03/31/2022
5	State Street Global Advisors	US	4,020	03/31/2022
6	Morgan Stanley	US	3,320	03/31/2022
7	JPMorgan Chase	US	2,960	03/31/2022
8	Credit Agricole	France	2,875	12/31/2021
9	Allianz Group	Germany	2,760	03/31/2022
10	Capital Group	US	2,700	12/31/2021
11	Goldman Sachs	US	2,394	03/31/2022

Asset Management Companies

Standard Asset Management firms



- John Bogle, founder of Vanguard: advocate of passive investment

Asset Management Companies

Zoom on **BlackRock**

- Wealth Management: Aladdin (Asset, Liability and Debt and Derivative Investment Network) by BlackRock
  - As of 2020, Aladdin managed $21.6 trillion in assets
  - about 7% of the world's financial assets
  - keep track of about 30,000 investment portfolios.

Asset Management Companies

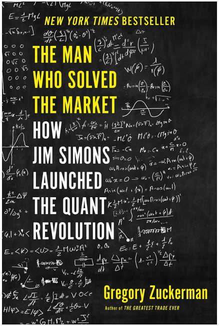
Hedge Funds = You don't know them

- Renaissance Technologies LLC (Medallion fund)
- Bridgewater Associates.
- Man Group Ltd.
- Citadel LLC.
- TCI Fund Management Ltd.

Asset Management Companies

Hedge Funds = You don't know them

- Renaissance Technologies LLC (Medallion fund)
  - $\approx$ \$106 bn AUM
  - Average gains of $\approx$ **70% per year** from 1988 to 2018
  - Must-read : The Man Who Solved the Market

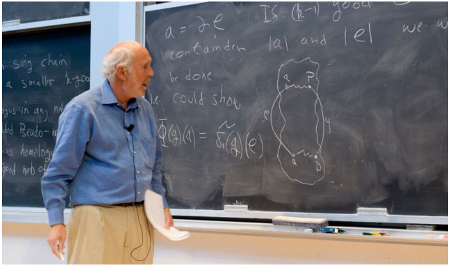


- Bridgewater Associates.
- Man Group Ltd.
- Citadel LLC.
- TCI Fund Management Ltd.

The Man Who Solved the Market

Renaissance Technologies LLC

- James Simmons, Renaissance Founder and CEO.
- MSc from MIT, PhD in Mathematics from UC Berkeley. See his TED Talk here
- Worst year was 21%. 98% return in 2008
- James Simmons himself made \$2.5bn of income in 2008



Asset Management Companies

Hedge Funds

Hedge Fund Strategies		
Strategy	**Objectives**	**Investment Types**
Long/Short	– Purchase undervalued stocks – Short sell overvalued stocks	– Equities
Global Macro	– Opportunistically sell and purchase investments related to national and international political and economic events – Maintain global diversification	– Equities – Bonds – ETFs – Currencies and commodities – Options – Futures
Market-Neutral	– Invest in an equal amount of short stocks and long stocks in an effort to avoid the risk	– Equities

Asset Management Companies

Who are the clients ?

- They invest on behalf of **various types of clients**, such as :
  - Retail investors
  - Institutional investors (Banks, pension funds, insurance companies)
  - Public sector (government organizations)
  - Private sector
  - High-net-worth clients ("sophisticated" investors with income of at least $200,000 or $1 million net worth)

What is Asset Pricing ?

- Asset Pricing $\approx$ **Investment Theory**
- A sub-field of finance that **explores the factors determining the prices** of (and returns on) financial and real assets, including :
  - equity (stocks), bonds, currencies, real estate, etc.
  - Very complex valuation

What is Asset Pricing ?

- Asset Pricing $\approx$ **Investment Theory**
- A sub-field of finance that **explores the factors determining the prices** of (and returns on) financial and real assets, including :
  - equity (stocks), bonds, currencies, real estate, etc.
  - Very complex valuation
- Guides the decision-making process of choosing investments

$\rightarrow$ key knowledge for AMC !

Key questions

1. How to allocate/invest the money ?
2. How to predict asset returns ?

A long-term perspective on Asset Pricing

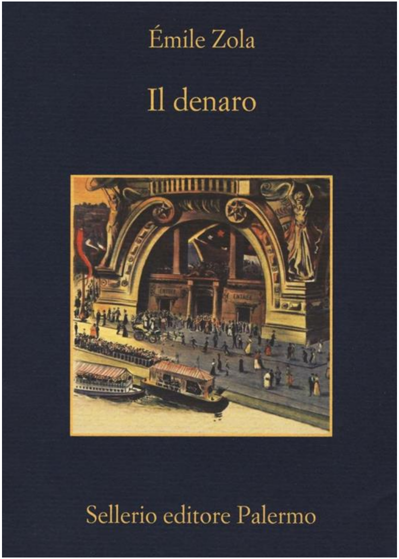
You!

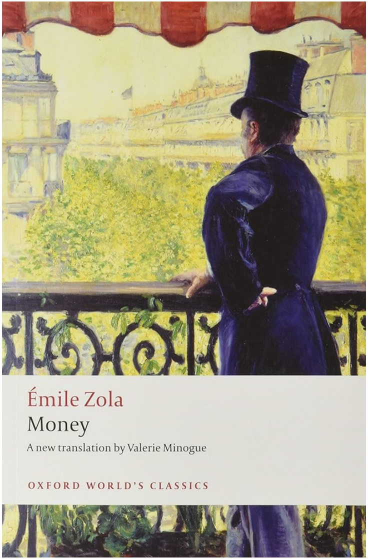
Pre-1950

2022

The « Chaos »

A long-term perspective on Asset Pricing





Link to the novel

A long-term perspective on Asset Pricing

You!

Pre-1950

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A long-term perspective on Asset Pricing

Portfolio Theory

APT

You!

Pre-1950

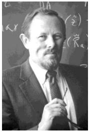
1964

1952

1976

2022

The « Chaos »



CAPM

Normative Theory

A long-term perspective on Asset Pricing

Portfolio Theory

APT

Bloomberg = data!
1981

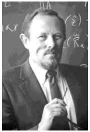
IBM-Microsoft
1980

Empirics strike back!

You!

Pre-1950 | 1952 | 1964 | 1976 | 1993 | 2022

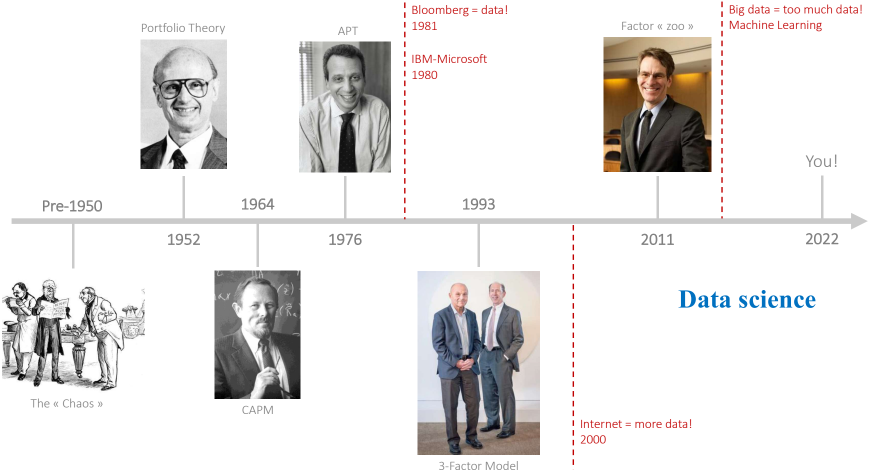
The « Chaos »



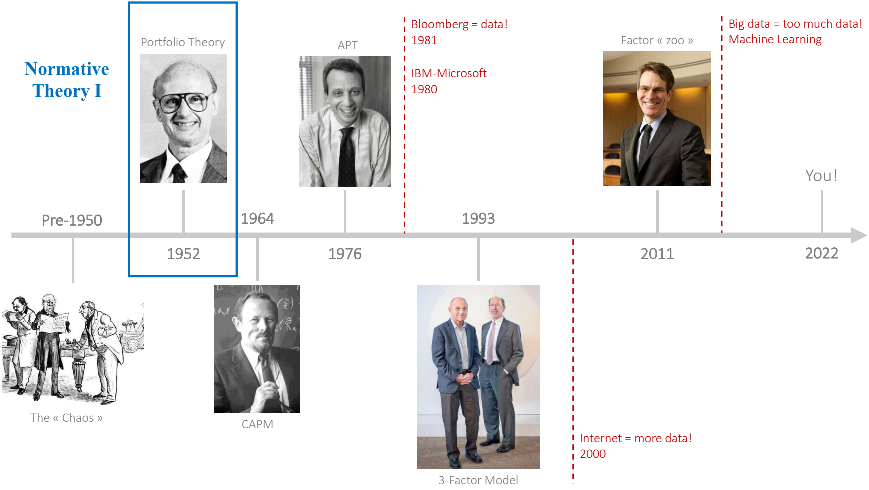
CAPM

3-Factor Model

A long-term perspective on Asset Pricing



A long-term perspective on Asset Pricing



Key questions

1. **How to allocate/invest the money ? $\rightarrow$ Portfolio Theory** (today)
2. How to predict asset returns ? $\rightarrow$ CAPM and APT (next week(s))

Portfolio Theory

- How do fund managers allocate their money ? $\rightarrow$ they create **portfolios** !
- A portfolio is simply a **specific combination of assets**, usually defined by portfolio weights that sum to 1 :

$$\omega = \{w_1, w_2, ..., w_n\}$$

$$\omega_i = \frac{N_i P_i}{N_1 P_1 + N_2 P_2 + ... + N_n P_n}$$

$$1 = w_1 + w_2 + ... + w_n$$

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- Portfolio weights can sum to 0 (neutral portfolios) and weights can be positive (long positions) or negative (short positions)
- Assumption: portfolio weights summarize all relevant information

Portfolio Theory

Example 1

- You invest \$100,000 in three stocks : 200 shares of stock A, 1,000 shares of stock B, and 750 shares of stock C.
- Your portfolio is summarized by the following weights :

Asset	Shares	Price/Share	Dollar Investment	Portfolio Weight
A	200	\$50	\$10,000	10%
B	1,000	\$60	\$60,000	60%
C	750	\$40	\$30,000	30%
Total			\$100,000	100%

- You have \$50,000 and you borrow \$50,000 from the market (riskless bond)
- You invest \$100,000 in three stocks : 200 shares of stock A, 1,000 shares of stock B, and 750 shares of stock C.
- What are the new portfolio weights ?

Portfolio Theory

Example 1 with leverage

- You have \$50,000 and you borrow \$50,000 from the market (riskless bond)
- You invest \$100,000 in three stocks : 200 shares of stock A, 1,000 shares of stock B, and 750 shares of stock C.
- What are the new portfolio weights ?

Asset	Shares	Price/Share	Dollar Investment	Portfolio Weight
A	200	\$50	\$10,000	20%
B	1,000	\$60	\$60,000	120%
C	750	\$40	\$30,000	60%
Riskless Bond	–\$50,000	\$1	–\$50,000	–100%
Total			\$50,000	100%

Portfolio Theory

Example 2

- You decide to purchase a nice flat in Milan that costs \$500,000 by paying 20% of the purchase price and getting a mortgage for the remaining 80%
- What are your portfolio weights for this investment ?

Portfolio Theory

Example 2

- You decide to purchase a nice flat in Milan that costs \$500,000 by paying 20% of the purchase price and getting a mortgage for the remaining 80%
- What are your portfolio weights for this investment ?

Asset	Shares	Price/Share	Dollar Investment	Portfolio Weight
Home	1	\$500,000	\$500,000	500%
Mortgage	1	–\$400,000	–\$400,000	–400%
Total			\$100,000	100%

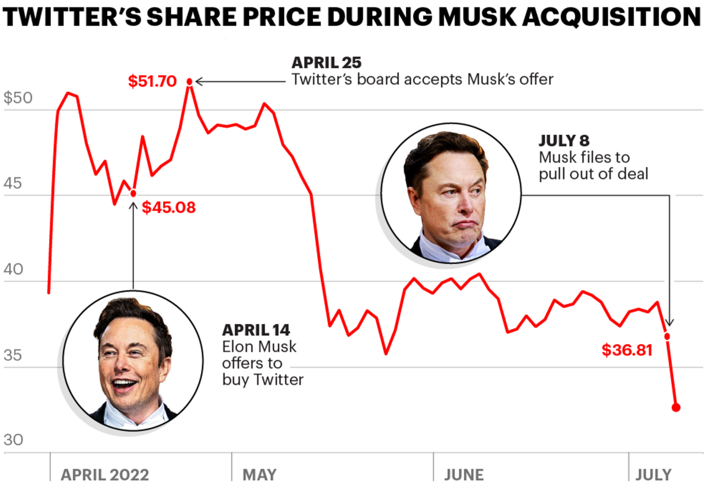
- What happens to your total assets if your home price declines by 15% ?

Portfolio Theory

Why Not Pick The Best Stock Instead of Forming a Portfolio?

Portfolio Theory

Why Not Pick The Best Stock Instead of Forming a Portfolio ?



Portfolio Theory

Why Not Pick The Best Stock Instead of Forming a Portfolio ?

- We don't know which stock is best !

Portfolio Theory

Why Not Pick The Best Stock Instead of Forming a Portfolio ?

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- Portfolios provide **diversification**, reducing unnecessary risks

Portfolio Theory

Why Not Pick The Best Stock Instead of Forming a Portfolio ?

- We don't know which stock is best !
- Portfolios provide **diversification**, reducing unnecessary risks
- Portfolios can customize and manage risk/reward trade-offs.

Portfolio Theory

How Do We Construct a "Good" Portfolio ?

- What does "good" mean ?

Portfolio Theory

How Do We Construct a "Good" Portfolio ?

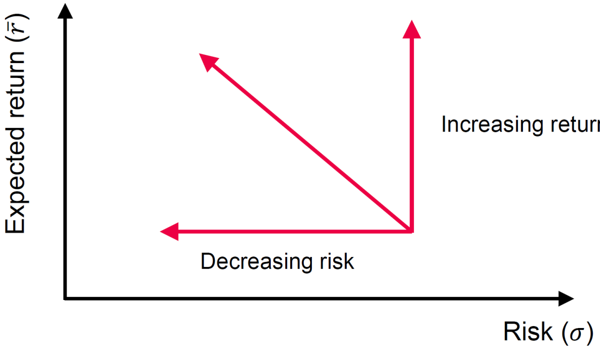
- What does "good" mean ?
- What characteristics do we care about for a given portfolio ?
  - **Risk** and **reward**
- Remember from Lecture 0:
  - Investors like higher reward (more money is better)
  - Investors dislike risk (less risk is better)

$\Rightarrow$ **Mean-variance preferences:** investors care only about the expected return and volatility of their overall portfolio

Portfolio Theory

Measuring risk and reward

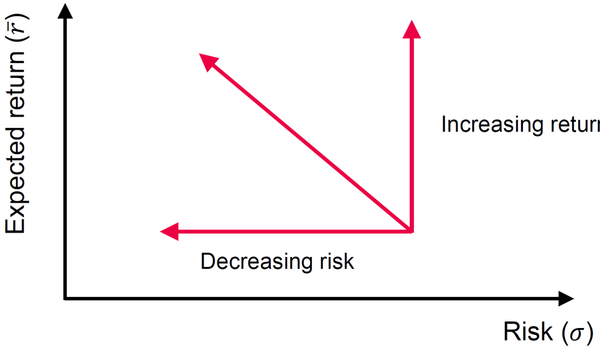
⇒ **Mean-variance preferences:** Investors care only about the expected return and volatility of their overall portfolio



Portfolio Theory

Measuring risk and reward

⇒ **Mean-variance preferences:** Investors care only about the expected return and volatility of their overall portfolio



- But what if reward and risk are unknown ?

Measuring risk and reward
Notation

- Let P_0 be the initial price, $\tilde{P}_1$ the price at the end of the period (random variable)
- Return on an asset over a single period is random :

$$\tilde{r}_1 = \frac{\tilde{P}_1 - P_0}{P_0} = \frac{\tilde{P}_1}{P_0} - 1$$

- Expected return :

$$\bar{r}_1 = \mathbb{E}[\tilde{r}_1] = \frac{\mathbb{E}[\tilde{P}_1]}{P_0} - 1$$

- Expected **Excess** return :

Measuring risk and reward

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- Expected **Excess** return :

$$\mathbb{E}[\tilde{r}_1^e] = \mathbb{E}[\tilde{r}_1] - r_f$$

Measuring risk and reward

Basic statistics and historical data

- In practice, $\bar{r}$ and σ^2 can be measured using **historical data**

	Return moments	Common sample estimators
Mean	$\bar{r} = \mathrm{E}[\tilde{r}]$	$\hat{r} = \frac{1}{T}\sum_{t=1}^{T} r_t$
Variance	$\sigma^2 = \mathrm{E}[(\tilde{r} - \bar{r})^2]$	$\hat{\sigma}^2 = \frac{1}{T-1}\sum_{t=1}^{T}(r_t - \hat{r})^2$
Standard deviation	$\sigma = \sqrt{\sigma^2}$	$\hat{\sigma} = \sqrt{\hat{\sigma}^2}$

Measuring risk and reward

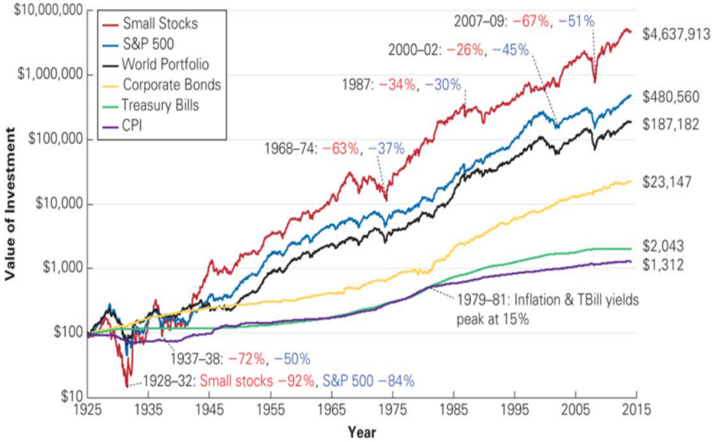
Basic statistics and historical data

- Average annual total returns from 1926 to 2018 (nominal)

Asset	Mean (%)	SD (%)
T-bills	3.4	3.1
Long term T-bonds	5.9	9.8
Long term corp. bonds	6.3	8.4
Large stocks	11.9	19.8
Small stocks	16.2	31.6
Inflation	3.0	4.0

Measuring risk and reward

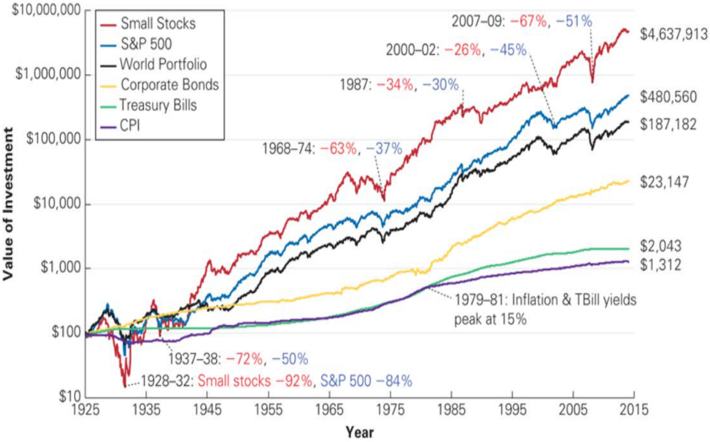
- Value at the end of 2015 of $100 invested at the end of 1925 in various assets



- Why would any investor buy bonds ?

Measuring risk and reward

- Value at the end of 2015 of $100 invested at the end of 1925 in various assets



- Why would any investor buy bonds ? It is all about the **risk-return trade-off**


5′ break

Portfolio properties

- Properties of a portfolio are determined by the **returns of its assets** and **their weight** in the portfolio
- In the mean-variance framework, we focus on the **expected value** and the **variance** of portfolio return
- Need information about **average returns on the assets** in the portfolio, and the **covariance matrix** of their returns

Portfolio properties

Expected return on the portfolio

- Expected returns on portfolio assets

Asset	**1**	**2**	$\cdots$	n
Mean Return	$\bar{r}_1$	$\bar{r}_2$	$\ldots$	$\bar{r}_n$

- Expected return on the portfolio ?

Portfolio properties

Expected return on the portfolio

- Expected returns on portfolio assets

Asset	1	2	$\cdots$	n
Mean Return	$\bar{r}_1$	$\bar{r}_2$	$\ldots$	$\bar{r}_n$

- Expected return on the portfolio ? The weighted average of expected returns on individual asset :

$$\bar{r}_p = \mathbb{E}[r_p] = w_1\bar{r}_1 + w_2\bar{r}_2 + \ldots + w_n\bar{r}_n = \sum_{i=1}^{n} w_i\bar{r}_i$$

Portfolio properties

Variance of portfolio return

- Variance is (a bit) more *complicated* :

$$\begin{aligned}\sigma_p^2 &= \mathsf{Var}[\tilde{r}_p] = \mathbb{E}\big[(\tilde{r}_p - \bar{r}_p)^2\big] \\ &= \mathbb{E}\Big[\big(w_1(\tilde{r}_1 - \bar{r}_1) + w_2(\tilde{r}_2 - \bar{r}_2) + \ldots + w_n(\tilde{r}_n - \bar{r}_n)\big)^2\Big]\end{aligned}$$

Portfolio properties

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- with any of those term equal to :

$$\begin{aligned} \mathbb{E}\Big[w_i w_j (\tilde{r}_i - \bar{r}_i)(\tilde{r}_j - \bar{r}_j)\Big] &= w_i w_j \ \mathsf{Cov}[r_i, r_j] \\ &= w_i w_j \sigma_{ij} \end{aligned}$$

Portfolio properties

Variance of portfolio return

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$$
\begin{aligned}
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&= w_i w_j \sigma_{ij}
\end{aligned}
$$

- Thus :

$$
\sigma_p^2 = \text{Var}[\tilde{r}_p] = \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \sigma_{ij}
$$

- Portfolio variance is the **weighted sum of all the variances and covariances**

Portfolio properties

Variance of portfolio return

- Portfolio variance is the **weighted sum of all the variances and covariances**
- Need to compute all elements of the covariance matrix :

	1	**2**	$\cdots$	$\boldsymbol{n}$
1	σ_1^2	σ_{12}	$\cdots$	σ_{1n}
2	σ_{21}	σ_2^2	$\cdots$	σ_{2n}
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$\boldsymbol{n}$	σ_{n1}	σ_{n2}	$\cdots$	σ_n^2

- Diagonal elements are individual return variances : $\sigma_{nn} = \sigma_n^2$
- Off-diagonal elements capture pair-wise co-movement of asset returns

Portfolio properties

Variance of portfolio return

1. How many variances? How many covariances?

Portfolio properties

Variance of portfolio return

1. How many variances ? How many covariances ? There are n variances, and $n^2 - n$ covariances
2. Covariances dominate portfolio variance
3. Positive covariances **increase portfolio variance** ;

Portfolio properties

Variance of portfolio return

1. How many variances? How many covariances? There are n variances, and $n^2 - n$ covariances
2. Covariances dominate portfolio variance
3. Positive covariances **increase portfolio variance**; negative covariances **decrease portfolio variance = diversification!**

Portfolio properties

Special case with 2 assets

- Consider the special case of 2 assets a and b

$$\begin{aligned}
\tilde{r}_p &= w_a \tilde{r}_a + w_b \tilde{r}_b \\
\bar{r}_p &= \mathbb{E}[\tilde{r}_p] = w_a \bar{r}_a + w_b \bar{r}_b \\
\sigma_p^2 &= w_a^2 \sigma_a^2 + w_b^2 \sigma_b^2 + 2 w_b w_a \sigma_{ab}
\end{aligned}$$

- with $\sigma_{ab} = \rho_{ab} \sigma_a \sigma_b$

Portfolio properties

Special case with 2 assets

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- with $\sigma_{ab} = \rho_{ab} \sigma_a \sigma_b$ (ρ_{ab} stands for the correlation)
- As correlation increases, overall portfolio risk increases !

Portfolio properties

Example with 2 (risky) assets

- Two assets : Google and Amazon (Monthly returns, 01/2008–12/2018)
  - Google : average monthly return of 1.75% and a std. dev. of 9.73%
  - Amazon : average monthly return of 1.08% and a std. dev. of 6.23%
  - Their correlation is 0.37

$$\begin{aligned}\mathbb{E}[\tilde{r}_p] &= w_{Google} \times 1.75 + w_{Amazon} \times 1.08 \\ \sigma_p^2 &= w_{Google}^2 \times 9.73^2 + w_{Amazon}^2 \times 6.23^2 \\ &\quad + 2w_{Google}w_{Amazon}(6.23 \times 9.73 \times 0.37)\end{aligned}$$

w_{Google}	w_{Amazon}	$\mathbb{E}[r_p]$	Var(r_p)	SD(r_p)
0	1	1.08	38.8	6.23
0.25	0.75	1.25	36.2	6.01
0.5	0.5	1.42	44.6	6.68
0.75	0.25	1.58	64.1	8.00
1	0	1.75	94.6	9.73

Portfolio properties

Example with 2 (risky) assets

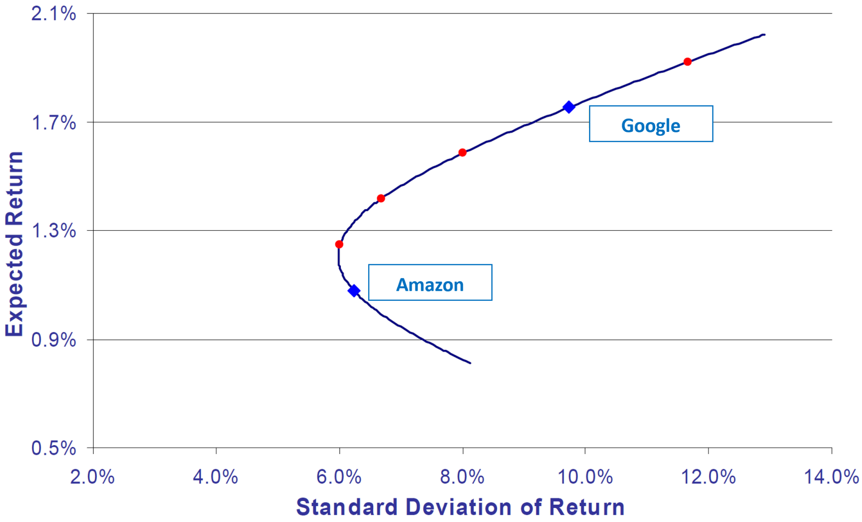
- Let's have a look to the 25/75 portfolio :

w_{Google}	w_{Amazon}	$\mathbb{E}[r_p]$	Var(r_p)	SD(r_p)
0	1	1.08	38.8	6.23
0.25	**0.75**	**1.25**	**36.2**	**6.01**
0.5	0.5	1.42	44.6	6.68
0.75	0.25	1.58	64.1	8.00
1	0	1.75	94.6	9.73

- Portfolio is less volatile than either of the two stocks individually !

Portfolio properties

Example with 2 (risky) assets



Portfolio properties

Example with 1 risky and 1 safe asset

- Two assets: stock market (S&P 500) and a safe bond (T-bill)
  - Stock market : avg. monthly return of 0.75% and a SD. of 4.25%
  - T-bill : avg. monthly return of 0.12%

Portfolio properties

Example with 1 risky and 1 safe asset

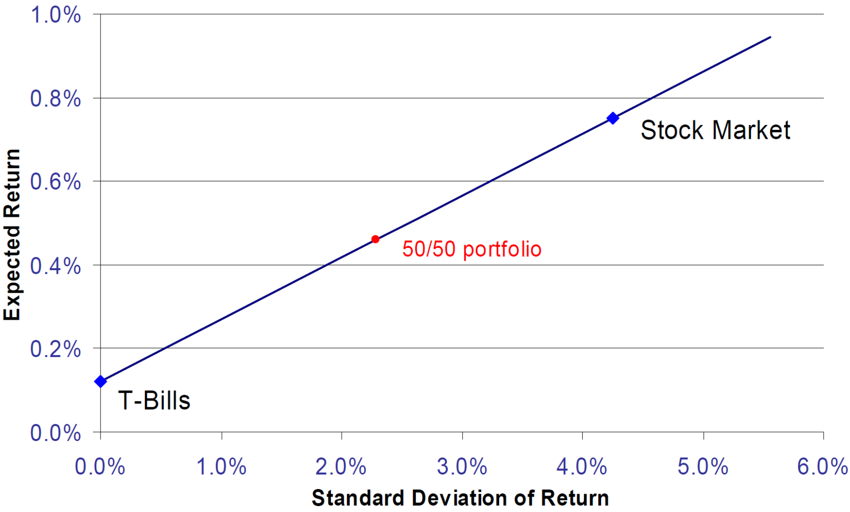
- Two assets: stock market (S&P 500) and a safe bond (T-bill)
  - Stock market : avg. monthly return of 0.75% and a SD. of 4.25%
  - T-bill : avg. monthly return of 0.12% (and a SD. of 0%)
  - Their correlation is 0

$$
\begin{aligned}
\mathbb{E}[\tilde{r}_p] &= w_{Stk} \times 0.75 + w_{Tbill} \times 0.12 \\
\sigma_p^2 &= w_{Stk}^2 \times 4.25^2 + w_{Tbill}^2 \times 0^2 \\
&\quad + 2 w_{Stk} w_{Tbill} (0 \times 0 \times 4.25)
\end{aligned}
$$

w_{Stk}	w_{Tbill}	$\mathbb{E}[r_p]$	Var(r_p)	SD(r_p)
0	1	0.12	0.00	0.00
0.33	0.67	0.33	1.97	1.40
0.67	0.33	0.54	8.11	2.85
1	0	0.75	18.06	4.25

Portfolio properties

Example with 1 risky and 1 safe asset



Portfolio properties
Summary

1. Portfolio expected return is **a weighted average of stocks' expected returns**
2. Portfolio risk is **smaller** if stocks' correlation is **lower**
3. Portfolio risk is smaller than the weighted average of stock's risk
4. Investors should care more about the risk that is common to many stocks ; **risks that are unique to each stock can be diversified away**

Portfolio properties

Certain risks cannot be diversified away

1. Diversification is effective up to a certain limit — risk cannot be fully eliminated through diversification
2. Remaining risk is known as **non-diversifiable** (also called market risk, systematic risk, common risk).

Portfolio properties

Certain risks cannot be diversified away

1. Diversification is effective up to a certain limit — risk cannot be fully eliminated through diversification
2. Remaining risk is known as **non-diversifiable** (also called market risk, systematic risk, common risk).
3. Sources of non-diversifiable risks include :
   - Business cycle (Covid-19, Ukraine war)
   - Inflation
   - Liquidity

Portfolio properties

What determines limits of diversification ?

- Consider an equally-weighted portfolio of n assets.
- Portfolio variance is the sum of all the terms in the matrix below :

	1	$\cdots$	n
1	$w_1^2\sigma_1^2$	$\cdots$	$w_1 w_n \sigma_{1n}$
$\vdots$	$\vdots$	$\ddots$	$\vdots$
n	$w_n w_1 \sigma_{n1}$	$\cdots$	$w_n^2\sigma_n^2$

– A typical variance term : $(\frac{1}{n})^2\sigma_{ii}$ – total number of variance terms is n

– A typical covariance term : $(\frac{1}{n})^2\sigma_{ij}$ – total number of covariance terms is $n^2 - n$

Portfolio properties

What determines limits of diversification ?

- Let's add all the terms :

$$
\begin{aligned}
\sigma_p^2 &= \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \sigma_{ij} = \sum_{i=1}^{n}(\frac{1}{n})^2\sigma_{ii} + \sum_{i=1}^{n}\sum_{i\neq j}^{n}(\frac{1}{n})^2\sigma_{ij} \\
&= (\frac{1}{n})(\frac{1}{n}\sum_{i=1}^{n}\sigma_{ii}) + (\frac{n^2-n}{n^2})(\frac{1}{n^2-n}\sum_{i=1}^{n}\sum_{i\neq j}^{n}\sigma_{ij}) \\
&= (\frac{1}{n}) \times \textbf{average variance} + (1-\frac{1}{n}) \times \textbf{average covariance}
\end{aligned}
$$

- What happens when n becomes very large ?

Portfolio properties

What determines limits of diversification ?

- Let's add all the terms :

$$\sigma_p^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \sigma_{ij} = \sum_{i=1}^{n} (\frac{1}{n})^2 \sigma_{ii} + \sum_{i=1}^{n}\sum_{i \neq j}^{n} (\frac{1}{n})^2 \sigma_{ij}$$

$$= (\frac{1}{n})(\frac{1}{n}\sum_{i=1}^{n} \sigma_{ii}) + (\frac{n^2 - n}{n^2})(\frac{1}{n^2 - n}\sum_{i=1}^{n}\sum_{i \neq j}^{n} \sigma_{ij})$$

$$= (\frac{1}{n}) \times \textbf{average variance} + (1 - \frac{1}{n}) \times \textbf{average covariance}$$

- What happens when n becomes very large ?
  - contribution of variance $\rightarrow$ 0
  - contribution of **covariance terms goes to average covariance**

Portfolio properties

What determines limits of diversification ?

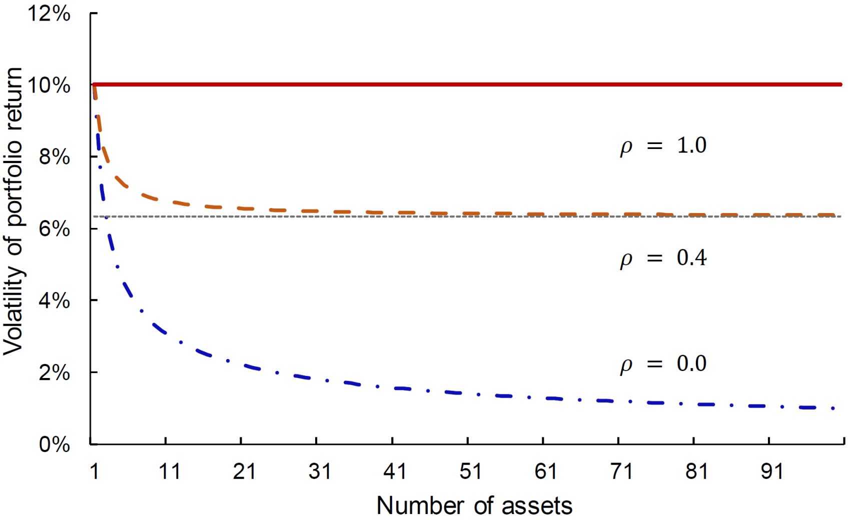
- The average US stock has a monthly standard deviation of 10% and the average correlation between stocks is 40%
- If you invest the same amount in each stock, what is variance of the portfolio ?

$$
\begin{aligned}
\sigma_{ij} &= \rho_{ij}\sigma_i\sigma_j = 0.40 \times 0.10 \times 0.10 = 0.004 \\
\sigma_p^2 &= (\frac{1}{n}) \times 0.10^2 + (1 - \frac{1}{n}) \times 0.004 \approx 0.004 \\
\sigma_p &\approx 6.3\%
\end{aligned}
$$

  - $\sigma_p \leq$ average stock risk
  - But still 6.3% that can't be diversified

Portfolio properties

What determines limits of diversification ?



Portfolio properties

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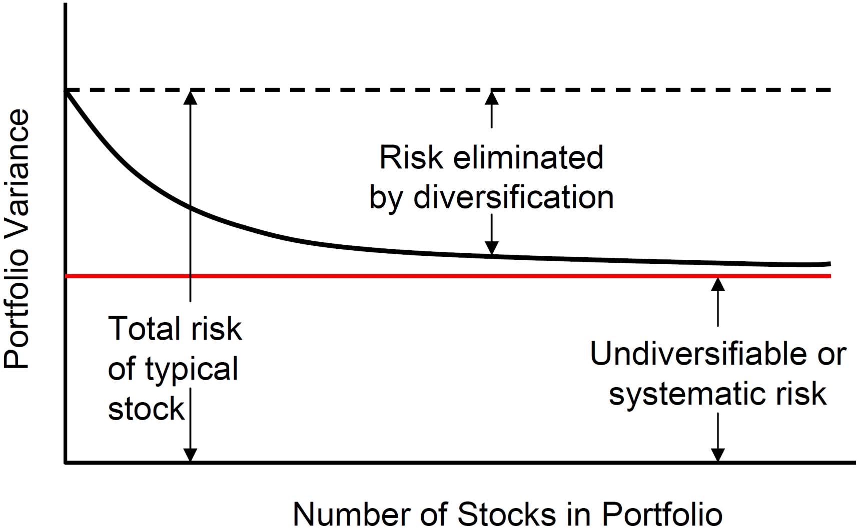

How to choose weights ?

- Given Portfolio Expected Returns and Variances :

$$\bar{r}_p = \sum_{i=1}^{n} w_i \bar{r}_i$$

$$\sigma_p^2 = \sum_{i=1}^{n} w_i^2 \sigma_{ii} + \sum_{i=1}^{n} \sum_{i \neq j}^{n} w_i w_j \sigma_{ij}$$

- How should we choose the best weights ?

How to choose weights?

Optimization problem

- How should we choose the best weights ?
- Mean-variance static portfolio choice is a simple and commonly used formulation
- Formally, solve the following problem :

$$\underset{w_1,\ldots,w_n}{\text{Minimize}}\ \sigma_p^2 = \sum_{i=1}^{n} w_i^2 \sigma_{ii} + \sum_{i=1}^{n} \sum_{i \neq j}^{n} w_i w_j \sigma_{ij}$$

$$\text{Subject to :}(1) \sum_{i=1}^{n} w_i = 1$$

$$(2) \sum_{i=1}^{n} w_i \bar{r}_i = \bar{r}_p$$

- Intuitively : of all the portfolios [`constraint (1)`] with an expected return of [`constraint(2)`], find the one that has the lowest variance.

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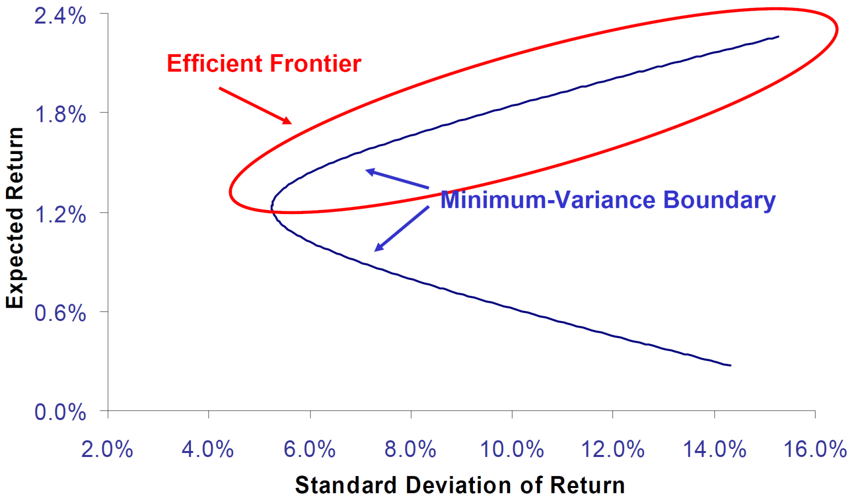
How to choose weights ?

Summary

- Given an expected return, the portfolio that minimizes risk (measured by variance) is a mean-variance frontier portfolio
- The **set of all frontier portfolios** in the mean-variance framework is called **portfolio frontier**
- The upper part of the portfolio frontier gives the efficient frontier portfolios
- To obtain the efficient portfolios, we need to solve the constrained optimization problem (P).
- Use matrix mathematics to solve analytically !
- Use a numerical solver (e.g., Python, R, Matlab) to solve numerically

The Efficient Frontier



Portfolio frontier with a safe asset (54K+ citations)

PORTFOLIO SELECTION*

HARRY MARKOWITZ
The Rand Corporation

THE PROCESS OF SELECTING a portfolio may be divided into two stages. The first stage starts with observation and experience and ends with beliefs about the future performances of available securities. The second stage starts with the relevant beliefs about future performances and ends with the choice of portfolio. This paper is concerned with the second stage. We first consider the rule that the investor does (or should) maximize discounted expected, or anticipated, returns. This rule is rejected both as a hypothesis to explain, and as a maximum to guide investment behavior. We next consider the rule that the investor does (or should) consider expected return a desirable thing *and* variance of return an undesirable thing. This rule has many sound points, both as a

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- When a safe (risk-free) asset exists, **each portfolio should consist of the risk-free asset and risky assets.**
  - A portfolio of risk-free and risky assets can be viewed as a portfolio of 2 portfolios : (i) the risk-free asset and (ii) a portfolio of only risky assets
- All **efficient portfolios are linear combinations of the riskless asset and a unique portfolio of stocks, called the tangency portfolio**

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- All **efficient portfolios are linear combinations of the riskless asset and a unique portfolio of stocks, called the tangency portfolio**
- T be the tangency portfolio, and p all portfolios combining T and the risk-free asset :

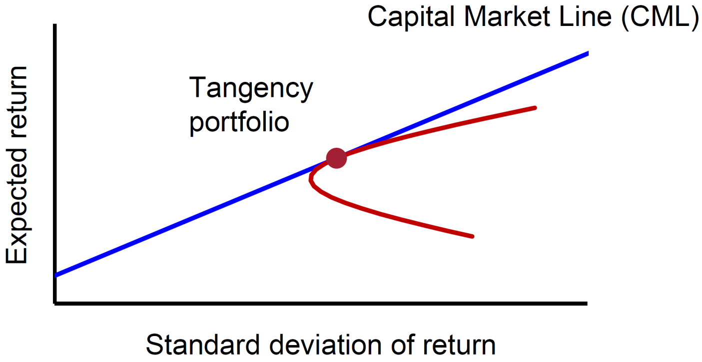
$$\bar{r}_p = (1-w)\, r_f + w\, r_T$$
$$\sigma_p^2 = w^2\, \sigma_T^2$$

- Which implies :

$$\bar{r}_p = r_f + \frac{r_T - r_f}{\sigma_T}\, \sigma_p$$

Portfolio frontier with a safe asset (54K+ citations)

- In this case, efficient frontier becomes straight line (CML)



Sharpe ratio

- A measure of a **portfolio's risk-return trade-off**, equal to the portfolio's risk premium divided by its volatility :

$$\text{Sharpe ratio} = \frac{\mathbb{E}[\tilde{r}_p] - r_f}{\sigma_p}$$
$$= \frac{\bar{r}_p - r_f}{\sigma_p} \qquad \textbf{Higher is better!}$$

- Question : What portfolio has the highest possible Sharpe ratio ?

Sharpe ratio

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$$\text{Sharpe ratio} = \frac{\mathbb{E}[\tilde{r}_p] - r_f}{\sigma_p} = \frac{\bar{r}_p - r_f}{\sigma_p} \qquad \textbf{Higher is better!}$$

- Question : What portfolio has the highest possible Sharpe ratio ?
- The **tangency portfolio has the highest possible Sharpe ratio of all portfolios**. So are all the portfolios on the CML.
- Measure of risk-adjusted performance: key indicator for funds !

Portfolio Theory

Key Take Away

1. Diversification reduces risk: The standard deviation of a portfolio is always less than the average standard deviation of the individual stocks in the portfolio

2. In diversified portfolios, covariances among stocks are more important than individual variances: only systematic risk matters

Portfolio Theory

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4. With a riskless asset, all investors should hold the tangency portfolio. This portfolio maximizes the trade-off between risk and expected return

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3. Investors should try to hold portfolios on the efficient frontier. These portfolios maximize expected return for a given level of risk
4. With a riskless asset, all investors should hold the tangency portfolio. This portfolio maximizes the trade-off between risk and expected return
5. However, the Markowitz mean-variance framework has never been fully adopted by practitioners. Why? Hint: read this blog post

Discussion