Separating Variables in Bivariate Polynomial Ideals: the Local Case

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April 8, 2024

1 Abstract

We present a semi-algorithm which for any irreducible $p \in \mathbb{K}[x,y]$ finds all elements of $\mathbb{K}(x) + \mathbb{K}(y)$ that are of the form qp for some $q \in \mathbb{K}(x,y)$ whose denominator is not divisible by p.

2 Introduction

The following is the continuation of the work on an elimination problem that was started in [5]. It discussed how $I \cap (\mathbb{K}[x] + \mathbb{K}[y])$ can be determined when I is an ideal of $\mathbb{K}[x,y]$. This article extends the ideas presented therein to ideals of the local ring of $\mathbb{K}[x,y]$ at an irreducible $p \in \mathbb{K}[x,y]$. The result is a semi-algorithm that takes p as input and outputs a description of all its non-trivial rational multiples that are elements of $\mathbb{K}(x) + \mathbb{K}(y)$. In contrast to the algorithm presented in [5], the semi-algorithm discussed here may not terminate. Termination depends on whether a dynamical system on the curve defined by p is periodic or not. However, if p has a non-trivial rational multiple in $\mathbb{K}(x) + \mathbb{K}(y)$, the semi-algorithm is guaranteed to terminate.

This work has several applications. One of them is found in enumerative combinatorics and the study of lattice walks restricted to cones. The question of how many there are can be approached by considering their generating functions and studying the functional equations they satisfy. A systematic study of these equations, sometimes referred to as discrete differential equations, was initiated in [4, 10] and has received a lot of attention since then. We refer to [6, 7] and the references therein for an overview of the relevant literature. In [1], it was explained how certain partial discrete differential equations can be reduced to ordinary discrete differential equations. The reduction relies on the existence of certain rational functions, so-called invariants and decoupling functions. Whether they exist, and in case they do, how to construct them, are therefore important questions. An answer to these questions is meanwhile given in [3]. The present paper addresses these questions too, for invariants, though in more generality and with different methods.

The problem discussed here also arises in the computation of $\mathbb{K}(h_1) \cap \mathbb{K}(h_2)$ for given $h_1, h_2 \in \mathbb{K}(t)$ as studied in [2]. In order to compute it, one can consider the numerators of $x - h_1$ and $y - h_2$ and eliminate the variable t to receive a polynomial $p \in \mathbb{K}[x,y]$. If there is a rational function $q \in \mathbb{K}(x,y)$

whose denominator is not divisible by p such that $qp = f - g$ for some $f \in \mathbb{K}(x)$ and $g \in \mathbb{K}(y)$, then $f(h_1) = g(h_2)$. The non-trivial rational multiples of p in $\mathbb{K}(x) + \mathbb{K}(y)$ therefore give rise to the elements of $\mathbb{K}(h_1) \cap \mathbb{K}(h_2)$.

The paper is organized as follows. In Section 3 we make precise what this article is about and give two different but equivalent formulations of the problem. In Section 4 we explain how it can be solved for the particular case when p is homogenous. And in Section 5 we show how the general case reduces to the homogenous one. The paper closes with Section 6, where we present several open questions and conjectures.

This paper comes with an implementation of the semi-algorithm in Mathematica. It can be found on https://github.com/buchacm/nearSeparation.git.

3 Problem

We assume throughout that $\mathbb{K}$ is an algebraically closed field of characteristic 0. We denote by $\mathbb{K}[x, y]$ the ring of polynomials in x and y over $\mathbb{K}$, and we write $\mathbb{K}(x, y)$ for its quotient field. Given a rational function $r \in \mathbb{K}(x, y)$ in reduced form, we write r_n and r_d for its numerator and denominator, respectively. Conversely, given two coprime polynomials $r_n, r_d \in \mathbb{K}[x, y]$, we denote by r their quotient r_n/r_d. Given $p \in \mathbb{K}[x]$, we denote by $\deg p$ its degree, and we write $\operatorname{val} p$ for its valuation, i.e. for the degree of the lowest order term of p.

Definition 1. *Let p be an irreducible polynomial of $\mathbb{K}[x, y] \setminus (\mathbb{K}[x] \cup \mathbb{K}[y])$. We write $\mathbb{K}[x, y]_p$ for the set of rational functions of $\mathbb{K}(x, y)$ whose denominator is not divisible by p. It is closed under the addition and multiplication of rational functions, and hence forms a ring. It is the **local ring** of $\mathbb{K}[x, y]$ at p.*

The polynomial p is an element of $\mathbb{K}[x, y]_p$. We denote the ideal it generates therein by $\langle p \rangle$. It consists of all rational functions of $\mathbb{K}(x, y)$ whose numerator is a multiple of p. An element of $\mathbb{K}[x, y]_p$ has a multiplicative inverse if and only if it does not belong to $\langle p \rangle$. It is therefore the unique maximal ideal in $\mathbb{K}[x, y]_p$. Furthermore, $\mathbb{K}[x, y]_p$ is a principal ideal domain and every ideal is generated by some power of p.

Problem 1. *Given an irreducible polynomial $p \in \mathbb{K}[x, y] \setminus (\mathbb{K}[x] \cup \mathbb{K}[y])$ and an ideal $I \subseteq \mathbb{K}[x, y]_p$, find a description of*

$$I \cap (\mathbb{K}(x) + \mathbb{K}(y)) .$$

Although we have not made any restriction on the ideal in Problem 1, it turns out that the problem is only interesting for $I = \langle p \rangle$.

Lemma 1. *Let $p \in \mathbb{K}[x, y] \setminus (\mathbb{K}[x] \cup \mathbb{K}[y])$ be irreducible, and let $m > 1$ be an integer. Then*

$$\langle p^m \rangle \cap (\mathbb{K}(x) + \mathbb{K}(y)) = \{0\}.$$

Proof. Assume that there is a $q \in \mathbb{K}[x, y]_p \setminus \{0\}$ such that

$$qp^m = f - g$$

for some $f \in \mathbb{K}(x)$ and $g \in \mathbb{K}(y)$. Then

$$p^m \mid f_n g_d - g_n f_d,$$

since q_d and p are relatively prime. Therefore, there is an $x_0 \in \overline{\mathbb{K}(y)} \setminus \mathbb{K}$ that is a common root of $f_n g_d - g_n f_d$ and $\frac{\partial}{\partial x}(f_n g_d - g_n f_d)$ when considered as polynomials of $\mathbb{K}(y)[x]$, and hence a root of $\frac{\partial}{\partial x} f$ too. Because f is not a constant, $\frac{\partial}{\partial x} f$ cannot be identically zero. Therefore $x_0 \in \mathbb{K}$. A contradiction. □

There is an uncertainty in the formulation of Problem 1. We asked for a "description" of the intersection of an ideal of $\mathbb{K}[x,y]_p$ with $\mathbb{K}(x) + \mathbb{K}(y)$ but we did not make clear what kind of description. In general, the intersection is not an ideal, so there is no point in asking for an ideal basis. And although it is a vector space, a vector space basis is not very helpful as it will be infinite in general. The following two propositions provide an alternative description that will turn out to be convenient.

Proposition 1. *Let $p \in \mathbb{K}[x,y] \setminus (\mathbb{K}[x] \cup \mathbb{K}[y])$ be irreducible. Then*

$$\mathrm{F}(p) := \{(f,g) \in \mathbb{K}(x) \times \mathbb{K}(y) : f - g \in \langle p \rangle\}$$

is a field with respect to componentwise addition and multiplication. It is referred to as the ***field of separated multiples*** *of p.*

Proof. Since $\mathrm{F}(p)$ is a subset of $\mathbb{K}(x) \times \mathbb{K}(y)$ and the latter is a ring with respect to componentwise addition and multiplication it is enough to note that it contains $(0,0)$ and $(1,1)$ and to observe that $\mathrm{F}(p)$ is closed under componentwise addition and multiplication to prove that it is a ring with unity. It is clearly closed under componentwise addition, and it is closed under componentwise multiplication, because for $(f,g),(f',g') \in \mathrm{F}(p)$ we have $f - g, f' - g' \in \langle p \rangle$, and so $ff' - gg' = (f-g)f' + g(f'-g') \in \langle p \rangle$. Hence $\mathrm{F}(p)$ is indeed a ring. It is also a field since if $(f,g) \in \mathrm{F}(p)$ and $f \neq 0$, then also $g \neq 0$, and $f^{-1} - g^{-1} = -f^{-1}g^{-1}(f-g) \in \langle p \rangle$ as f and g are units in $\mathbb{K}[x,y]_p$. □

Proposition 2. *Let $p \in \mathbb{K}[x,y] \setminus (\mathbb{K}[x] \cup \mathbb{K}[y])$ be irreducible. Then*

$$\mathrm{F}(p) = \mathbb{K}((f,g))$$

for some $(f,g) \in \mathbb{K}(x) \times \mathbb{K}(y)$.

Proof. The projection $\pi : \mathbb{K}(x) \times \mathbb{K}(y) \to \mathbb{K}(x)$ on the first component induces a field isomorphism between $\mathrm{F}(p)$ and its image $\pi(\mathrm{F}(p))$. By Lüroth's theorem [9] every subfield of $\mathbb{K}(x)$ that contains $\mathbb{K}$ is simple, i.e. of the form $\mathbb{K}(f)$ for some $f \in \mathbb{K}(x)$. Therefore $\mathrm{F}(p)$ is simple too. □

We can now formulate Problem 1 more precisely.

Problem 2. *Given an irreducible polynomial $p \in \mathbb{K}[x,y] \setminus (\mathbb{K}[x] \cup \mathbb{K}[y])$, find a generator of $\mathrm{F}(p)$.*

There is a formulation of Problem 2 which does not involve rational functions but only polynomials. It relies on the notion of near-separateness and near-separability.

Definition 2. *A polynomial $p \in \mathbb{K}[x,y]$ is said to be* ***near-separated****, if there exist $f \in \mathbb{K}(x)$ and $g \in \mathbb{K}(y)$ such that $p = f_n g_d - g_n f_d$. It is called* ***near-separable****, if there is a $q \in \mathbb{K}[x,y] \setminus \{0\}$ such that qp is near-separated.*

The next lemma relates divisibility of near-separated polynomials with composition of rational functions. It will establish the existence of a distinguished near-separated multiple of a polynomial. For a proof we refer to [11, Theorem 1].

Lemma 2. *Let $f, F \in \mathbb{K}(x)$ and $g, G \in \mathbb{K}(y)$ be non-constant rational functions. Then*

$$f_n g_d - g_n f_d \mid F_n G_d - G_n F_d$$

if and only if

$$\exists\, h \in \mathbb{K}(t) : h((f, g)) = (F, G).$$

Corollary 1. *Any irreducible polynomial $p \in \mathbb{K}[x, y]$ has a near-separated multiple that divides any other near-separated multiple of p. It is unique up to multiplicative constants, and referred to as the **minimal near-separated multiple** of p. If $p \in \mathbb{K}[x, y] \setminus (\mathbb{K}[x] \cup \mathbb{K}[y])$, then $\mathrm{F}(p) = \mathbb{K}((f, g))$ if and only if $f_n g_d - g_n f_d$ is its minimal near-separated multiple.*

Proof. The first part of the statement is clearly true when p is already near-separated. So let us assume that p is not near-separated, and let (F, G) be an element of $\mathbb{K}(x) \times \mathbb{K}(y)$ such that $F_n G_d - G_n F_d$ is a near-separated multiple of it. By Proposition 2 there is an $(f, g) \in \mathbb{K}(x) \times \mathbb{K}(y)$ such that $\mathrm{F}(p) = \mathbb{K}((f, g))$. Hence there is an $h \in \mathbb{K}(t)$ such that $(F, G) = h((f, g))$. By Lemma 1, $f_n g_d - g_n f_d$ is a divisor of $F_n G_d - G_n F_d$. Since the latter was an arbitrary near-separated multiple of p, the former is a minimal near-separated multiple of it. Lemma 1 also shows that if $f_n g_d - g_n f_d$ is a minimal near-separated multiple of p, then $\mathrm{F}(p) = \mathbb{K}((f, g))$. If there were another minimal near-separated multiple, they would divide each other, and therefore differ only by a multiplicative constant. □

We can now give the aforementioned reformulation of Problem 2.

Problem 3. *Given an irreducible polynomial $p \in \mathbb{K}[x, y]$, determine its minimal near-separated multiple.*

We close this section with an example.

Example 1. *The polynomial $p = xy - x - y - x^2y^2$ is near-separable. Its minimal near-separated multiple is*

$$(x - y)p = (1 - x - x^3)y^2 - x^2(1 - y - y^3).$$

Its field of separated multiples is therefore

$$\mathrm{F}(p) = \mathbb{K}\left(\left(\frac{1 - x - x^3}{x^2}, \frac{1 - y - y^3}{y^2}\right)\right).$$

4 Homogenous case

In this section we explain how to solve Problem 2 and Problem 3 when p is homogenous. In order to do so, we first give some definitions. We introduce the notion of a weight function, tell what we mean by the leading part of a polynomial with respect to it, and recall the definition of a homogenous polynomial.

Definition 3. *A real-valued function ω on the set of terms in x and y is a* ***weight function****, if*

$$\omega(ax^iy^j) = \omega_x i + \omega_y j$$

for some $\omega_x, \omega_y \in \mathbb{Z}$ and all $i, j \in \mathbb{Z}$ and $a \in \mathbb{K} \setminus \{0\}$. In this case we write $\omega = (\omega_x, \omega_y)$. Two weight functions ω_1, ω_2 are said to be ***equivalent****, if there is a positive number $c \in \mathbb{R}$ such that $\omega_2 = c\omega_1$. Given a polynomial $p \in \mathbb{K}[x, y]$ and a weight function ω, we denote the sum of terms of p of maximal weight by $\mathrm{lp}_\omega(p)$. It is referred to as the* ***leading part*** *of p with respect to ω. It only depends on the equivalence class of the weight function, not on its representative. We say that p is* ***homogenous****, if there is a non-zero weight function ω such that $\mathrm{lp}_\omega(p) = p$. In that case p is homogenous with respect to ω. The* ***sign vector*** *of $\omega = (\omega_x, \omega_y)$ is $\mathrm{sgn}(\omega) := (\mathrm{sgn}(\omega_x), \mathrm{sgn}(\omega_y))$, where $\mathrm{sgn}(\omega_x)$, for instance, is either 1, -1 or 0, depending on whether ω_x is positive, negative or equal to 0. We will occasionally also consider the leading part $\mathrm{lp}_\omega(r)$ of (certain) rational functions $r \in \mathbb{K}(x, y)$. In this case $\mathrm{lp}_\omega(r)$ is understood to be $\mathrm{lp}_\omega(r_n)/\mathrm{lp}_\omega(r_d)$.*

Let $\omega = (\omega_x, \omega_y)$ be a non-zero weight function, and let p be a polynomial that is homogenous with respect to it. We can assume that ω_x and ω_y are different from zero, otherwise p is the product of a monomial in one and a polynomial in the other variable, and hence already near-separated. If there is a non-zero polynomial q such that qp is near-separated, we may assume that q, and therefore also qp, is homogenous. If it were not, we could replace q by $\mathrm{lp}_\omega(q)$, since $\mathrm{lp}_\omega(q)p = \mathrm{lp}_\omega(q)\mathrm{lp}_\omega(p) = \mathrm{lp}_\omega(qp)$ and the leading part of a near-separated polynomial is near-separated. Under these assumptions there are $a, b \in \mathbb{K}$ and $k, l, m, n \in \mathbb{N}$ such that

$$qp = ax^ky^l - bx^my^n.$$

We can also assume that qp is not a single term, as otherwise p as well. If ω_x, $\omega_y > 0$ and $k > m$, for instance, then $n > l$, and $ax^ky^l - bx^my^n$ is the product of x^my^l and $ax^{k-m} - by^{n-l}$ in $\mathbb{K}[x, y]$. Defining $\tilde{q}$ and $\tilde{p}$ by $q = x^{\mathrm{val}_x q}y^{\mathrm{val}_y q}\tilde{q}$ and $p = x^{\mathrm{val}_x p}y^{\mathrm{val}_y p}\tilde{p}$, we have

$$\tilde{q}\tilde{p} = ax^{k-m} - by^{n-l},$$

and we see that $\tilde{p}$ has a non-trivial multiple in $\mathbb{K}[x] + \mathbb{K}[y]$. This is not only a necessary condition for the near-separability of p, but clearly also a sufficient one. We summarize these observations in the following proposition.

Proposition 3. *Let $p \in \mathbb{K}[x, y]$ be homogenous with respect to $\omega \in \mathbb{Z}^2_{>0}$. Then p is near-separable if and only if $x^{-\mathrm{val}_x p}y^{-\mathrm{val}_y p}p$ has a non-zero multiple in $\mathbb{K}[x] + \mathbb{K}[y]$.*

A similar statement holds in case not both of ω_x, ω_y are positive. If ω_x, $\omega_y < 0$, then one can replace them by their negative, and argue as before, and if only one of them is negative, say ω_y, one needs to multiply p by a suitable Laurent monomial and substitute y^{-1} for y before one can do so.

The question of how to decide whether a polynomial of $\mathbb{K}[x, y]$ has a non-zero multiple in $\mathbb{K}[x] + \mathbb{K}[y]$, and if it does, how to find it, was discussed and solved in [5, Section 3]. If $qp = ax^m - by^n$ for some $a, b \in \mathbb{K} \setminus \{0\}$, then $p(x, 1)$ is a divisor of $ax^m - b$. Hence the roots of $p(x, 1)$ are pairwise distinct and the ratio of every two of them is a root of unity. It turns out that this is also a sufficient

condition for p to have a non-zero multiple in $\mathbb{K}[x]+\mathbb{K}[y]$, and that a bound on the degrees of such a multiple can be derived from p. The precise statement, for whose proof we refer to [5], is the following.

Proposition 4. *Let ω be a weight function, and let $p \in \mathbb{K}[x,y] \setminus (\mathbb{K}[x] \cup \mathbb{K}[y])$ satisfy $\mathrm{lp}_\omega(p) = p$. Then p has a non-zero multiple in $\mathbb{K}[x]+\mathbb{K}[y]$ if and only if*

(a) p involves a monomial only in x, and

(b) all the roots of $p(x,1)$ in $\mathbb{K}$ are distinct and the ratio of every two of them is a root of unity.

Moreover, if p has a non-zero multiple in $\mathbb{K}[x]+\mathbb{K}[y]$ and N is the minimal number such that the ratio of every pair of roots of $p(x,1)$ is an N-th root of unity, then the weight of the minimal (near-)separated multiple of p is $N\omega_x$.

It remains to clarify how to decide whether the quotient of every pair of roots of $p(x,1)$ is a root of unity. If $p(x,1)$ is monic and a divisor of $ax^n - b$, then its constant term equals $(b/a)^{\deg p(x,1)/n}$. We can therefore consider $p(x/c,1)$ for $c = (b/a)^{1/n}$ and check whether it is square-free and its roots are roots of unity. For the former it is sufficient to see whether $p(x/c,1)$ and its derivative are co-prime. The latter can be done by computing the minimal polynomials of the roots of $p(x/c,1)$ over $\mathbb{K}$, and seeing whether they are cyclotomic. If they are given by $\phi_{n_1}, \ldots, \phi_{n_k}$, where ϕ_{n_i} is the n_i-th cyclotomic polynomial, that is, a divisor of $x^{n_i} - 1$ but not of $x^d - 1$ for $d < n_i$, then each root of $p(x/c,1)$ is an N-th root of unity for $N = \mathrm{lcm}(n_1, \ldots, n_k)$ by Proposition 4.

Example 2. *Consider the polynomial $p = x^2y^2 + xy + 1$ which is homogenous with respect to $\omega = (1,-1)$. It is the product of y^2 and*

$$\tilde{p} = x^2 + xy^{-1} + y^{-2} \in \mathbb{K}[x, y^{-1}].$$

The latter has a non-zero multiple in $\mathbb{K}[x]+\mathbb{K}[y^{-1}]$, since $\tilde{p}(x,1)$ is the third cyclotomic polynomial. Its minimal (near-)separated multiple is

$$(x - y^{-1})\tilde{p} = x^3 - y^{-3}.$$

Consequently, p is near-separable and its minimal near-separated multiple is

$$y(x - y^{-1})y^2\tilde{p} = (xy - 1)p = x^3y^3 - 1.$$

5 Reduction to the homogenous case

In this section we give a semi-algorithm that solves Problem 2 and Problem 3 and prove its correctness. We begin with presenting two necessary conditions for the near-separability of a polynomial.

Proposition 5. *If $p \in \mathbb{K}[x,y]$ is near-separable, then so is its leading part $\mathrm{lp}_\omega(p)$ with respect to any weight function $\omega \in \mathbb{Z}^2$.*

Proof. Assume that $q \in \mathbb{K}[x,y] \setminus \{0\}$ is such that qp is near-separated. Then $\mathrm{lp}_\omega(qp)$ is near-separated too, and $\mathrm{lp}_\omega(p)$ is near-separable, since $\mathrm{lp}_\omega(qp) = \mathrm{lp}_\omega(q)\mathrm{lp}_\omega(p)$. □

To compute the leading parts of a polynomial it is convenient to inspect its Newton polygon as there is a bijection between its leading parts and the faces of the polygon. The leading part which corresponds to a face is the sum of terms supported on it, that is, whose exponent vectors lie on it. Those we will be interested in consist of at least two terms. They correspond to the edges of the Newton polygon. The weight functions that give rise to them are the outward pointing normals of these edges. It happens frequently that it can be read off from the shape of its Newton polygon that a polynomial is not near-separable.

Proposition 6. *Let $p \in \mathbb{K}[x,y]$ be near-separated, and let $\omega_1, \omega_2 \in \mathbb{Z}^2$ be the outward-pointing normals of two distinct edges of its Newton polygon. Then* $\operatorname{sign}(w_1)$ *is different from* $\operatorname{sign}(w_2)$.

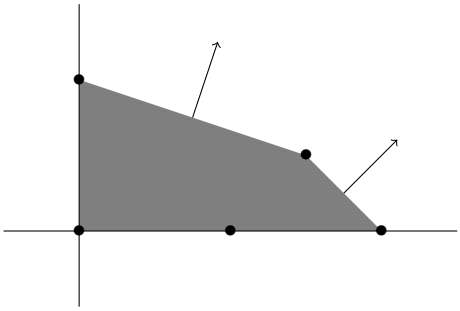
Proof. If $p \in \mathbb{K}[x] \cup \mathbb{K}[y]$, then the statement is clearly true. So let us assume that $p \in \mathbb{K}[x,y] \setminus (\mathbb{K}[x] \cup \mathbb{K}[y])$, and let $f \in \mathbb{K}(x)$ and $g \in \mathbb{K}(y)$ be such that $p = f_n g_d - g_n f_d$. We will show that the Newton polygon of p has at most one edge whose outward pointing normals have the sign vector $(1,1)$. For $(1,-1)$, $(-1,1)$ and $(-1,-1)$ the statement can be proven analogously after replacing x by x^{-1} and/or y by y^{-1} in p and multiplying by suitable powers of x and/or y. And for $(1,0)$, $(0,1)$, $(-1,0)$ and $(0,-1)$ it is clearly true, since the Newton polygon of p is convex.

Note that we can assume that $\deg f_n \neq \deg f_d$. If this were not the case, we could perform a division with remainder on f_n by f_d and move the quotient, which is just a constant, to g. This does neither alter $f-g$, nor its numerator p. Wlog we assume that $\deg f_n < \deg f_d$. We can now distinguish three cases. If $\deg g_d > \deg g_n$, then the upper-right part of the Newton polygon of p consists of (at most) three edges: possibly a horizontal edge, possibly a vertical one, and an edge whose outward pointing normal has only positive coordinates. They are spanned by, that is, the convex hulls of, the supports of $\operatorname{lt}(g_d)f_n$ and $\operatorname{lt}(f_d)g_n$ and $\operatorname{lt}(f_n)\operatorname{lt}(g_d) - \operatorname{lt}(g_n)\operatorname{lt}(f_d)$. If $\deg g_d < \deg g_n$, then the upper-right part of the Newton polygon of p consists of (at most) two edges, possibly a horizontal edge, and possibly a vertical edge, spanned by the supports of $\operatorname{lt}(g_n)f_d$ and $\operatorname{lt}(f_d)g_n$, respectively. If $\deg g_d = \deg g_n$, then the right part of the Newton polygon of p is spanned by the support of $\operatorname{lt}(f_d)g_n$. By performing a division with remainder on g_n by g_d and moving the quotient to f we can write $p = \tilde{f}_n g_d - \tilde{g}_n f_d$ where $\deg \tilde{f}_n = \deg f_d$ and $\deg g_d > \deg \tilde{g}_n$. The upper part of the Newton polygon of p is therefore spanned by the support of $\operatorname{lt}(g_d)\tilde{f}_n$. □

Lemma 3. *If $\omega \in \mathbb{R}^2$ is the outward pointing normal of an edge of the Newton polygon of $p \in \mathbb{K}[x,y]$, then it is the outward pointing normal of an edge of the Newton polygon of any non-zero multiple.*

Proof. Note that $\omega \in \mathbb{Z}^2$ is an outward pointing normal for an edge of the Newton polytope of a polynomial if and only if its leading part with respect to ω involves at least two terms. If q is a non-zero polynomial such that ω is not an outward pointing normal for an edge of the Newton polytope of qp, then $\operatorname{lp}_\omega(qp)$ were a single term, and so would be $\operatorname{lp}_\omega(p)$ since it is a divisor of it. □

An immediate consequence of this lemma and the previous proposition is the following.



The Newton polygon of $1 + x^2 + x^4 + x^3y + y^2$.

Corollary 2. *Let $\omega_1, \omega_2 \in \mathbb{Z}^2$ be the outward-pointing normals of two distinct edges of the Newton polygon of $p \in \mathbb{K}[x, y]$. If p is near-separable, then* $\operatorname{sign}(w_1)$ *and* $\operatorname{sign}(w_2)$ *cannot be equal.*

Example 3. *The Newton polygon of $p = 1 + x^4 + x^3y + y^2$ has two distinct edges whose outward pointing normals have the same sign vector. Hence p is not near-separable.*

Let $p \in \mathbb{K}[x, y] \setminus (\mathbb{K}[x] \cup \mathbb{K}[y])$ be irreducible. We now turn to the question of how to compute a generator of $\mathrm{F}(p)$ when p has a non-trivial rational multiple in $\mathbb{K}(x) + \mathbb{K}(y)$.

Recall that a **pole** s of $f \in \mathbb{K}(x)$ is either a root of f_d or ∞. The latter is the case if and only if $\deg f_n > \deg f_d$. If s is a finite pole of f, then its **multiplicity** is its order as a root of f_d. Otherwise it is $\deg f_n - \deg f_d$. We denote it by $\mathrm{m}(s, f)$ or simply by $\mathrm{m}(s)$ if it is clear from the context which rational function it is a pole of.

If there are $q \in \mathbb{K}[x, y]_p$ and $f \in \mathbb{K}(x)$ and $g \in \mathbb{K}(y)$ such that $qp = f - g$, then it is enough to know the poles of f and g and their multiplicities to determine them. For then we know the denominators of f and g and the degrees of their numerators, and by the following lemma, also the denominator of q and the degree of its numerator.

Lemma 4. *Let $p \in \mathbb{K}[x, y] \setminus (\mathbb{K}[x] \cup \mathbb{K}[y])$ be irreducible, and let $q \in \mathbb{K}[x, y]_p \setminus \{0\}$, $f \in \mathbb{K}(x)$ and $g \in \mathbb{K}(y)$ be such that $qp = f - g$. Then $q_d = f_d g_d$.*

Proof. The denominator of qp is q_d since p and q_d are relatively prime. We claim that the denominator of $f - g$ is $f_d g_d$. If this were not the case, then $f_d g_d$ and $f_n g_d - g_n f_d$ had a common factor. If this factor were a factor of f_d, then it were also one of f_n. But f_n and f_d are relatively prime. So this cannot be. Same for g_d. Hence the claim, and so $q_d = f_d g_d$, which proves the lemma. □

Making an ansatz for the numerators of q and f and g, clearing denominators in $qp = f - g$, and comparing coefficients results in a system of linear equations for the coefficients of the ansatz. Its non-trivial solutions give rise to non-zero rational functions f, g and q such that $qp = f - g$.

In the next subsection we explain how to compute the poles of a generator of $\mathrm{F}(p)$, and in the subsection thereafter we show how to determine their multiplicities. The last subsection provides the arguments for the correctness of the resulting semi-algorithm.

5.1 Poles

Let us assume that $\mathrm{F}(p) = \mathbb{K}((f,g))$ for non-constant $f \in \mathbb{K}(x)$ and $g \in \mathbb{K}(y)$. We may assume that ∞ is a pole of f. If $\deg f_n$ were smaller than $\deg f_d$, then we could replace (f,g) by its reciprocal (f^{-1}, g^{-1}), and if $\deg f_n$ and $\deg f_d$ were equal, then we could perform a division with remainder on f_n by f_d, and move the quotient, which is just a constant, to g, and then consider the corresponding reciprocal. To find the other poles of f and g, assume that $q \in \mathbb{K}[x,y]_p$ is such that $qp = f - g$. Then $\mathrm{lc}_x(p)$ is a divisor of g_d, and hence each root of $\mathrm{lc}_x(p)$ is a pole of g. Furthermore, if $\deg \mathrm{lc}_x(p) < \deg_y p$, then ∞ is a pole of g. This holds, since if $\deg \mathrm{lc}_x(p) < \deg_y p$, then the Newton polygon of p has an edge whose outward pointing normals have only positive coordinates. By Lemma 3, this is also true for the Newton polygon of $f_n g_d - g_n f_d$. Hence $\deg g_n > \deg g_d$. If s is a finite pole of f, then $p(s,y)$ is a divisor of g_d and each root of $p(s,y)$ is a pole of g. If s is a finite pole of f and $\deg p(s,y) < \deg_y p(x+s,y)$, then ∞ is a pole of g. Similar with the roles of f and g switched.

These observations give rise to a procedure for computing the poles of f and g that defines a dynamical system on the (projective) curve C associated with (the bi-homogenization of) p. It starts with those points of C whose first coordinate is ∞, takes the horizontal lines through them and intersects them with the curve, then takes the vertical lines through these intersection points and determines their intersections, and continues in this way ad infinitum. Under the assumption that p has a non-trivial separated multiple, and that (f,g) is a generator of $\mathrm{F}(p)$ as before, there are only finitely many points that can be constructed in this way, because each point encountered in this process is a pair of poles for f and g, and f and g have only finitely many poles. If such a multiple does not exist, then there may be infinitely many such points, and then the semi-algorithm does not terminate.

Let $C \subseteq (\mathbb{K} \cup \{\infty\})^2$ be the projective curve defined by the bi-homogenization of p, and let $\pi_i : (\mathbb{K} \cup \{\infty\})^2 \to \mathbb{K} \cup \{\infty\}$ denote the projection on the i-th coordinate. For $p \in \mathbb{K} \cup \{\infty\}$, we define

$$C_v(p) = C \cap (\{p\} \times (\mathbb{K} \cup \{\infty\})) \quad \text{and} \quad C_h(p) = C \cap ((\mathbb{K} \cup \{\infty\}) \times \{p\}).$$

Using this notation we can now formulate the following algorithm.

Algorithm 1. *Input: an irreducible polynomial $p \in \mathbb{K}[x,y] \setminus (\mathbb{K}[x] \cup \mathbb{K}[y])$ which has a non-trivial rational multiple in $\mathbb{K}(x) + \mathbb{K}(y)$.*
Output: a subset S of $(\mathbb{K} \cup \{\infty\})^2$ whose elements are points of the curve defined by p such that $\pi_1(S)$ is the set of poles of some $f \in \mathbb{K}(x)$ and $\pi_2(S)$ is the set of poles of some $g \in \mathbb{K}(y)$ such that $\mathrm{F}(p) = \mathbb{K}((f,g))$.

Compute $S = C_h(\pi_2(C_v(\infty)))$, and
determine $\tilde{S} = C_h(\pi_2(C_v(\pi_1(S))))$.
While $\tilde{S} \neq S$, do:
  set $S = \tilde{S}$, and
  compute $\tilde{S} = C_h(\pi_2(C_v(\pi_1(S))))$.
Return S.

We will see later that the algorithm is correct, i.e. determines all poles of a generator of $\mathrm{F}(p)$.

5.2 Multiplicities

We now explain how the leading parts of p provide information on the multiplicities of the poles of a separated multiple.

Let $qp = f - g$ be a non-trivial separated multiple of p, and let $(s_1, s_2) \in \{0, \infty\}^2$ be a pair of poles as constructed in Subsection 5.1. Then there is a weight function $\omega \equiv \omega_{s_1,s_2} \in \mathbb{Z}^2$ whose i-th coordinate is positive or negative depending on whether s_i is ∞ or 0 such that $\mathrm{lp}_\omega(p)$ involves at least two terms. If $s_1 = s_2 = \infty$, this was already observed in the previous subsection. If $s_1 = \infty$ and $s_2 = 0$ and s_2 is a root of $\mathrm{lc}_x(p)$, then this is clearly also true. If both, s_1 and s_2, are 0 and s_2 is a root of $p(s_1, y)$, then $\mathrm{val}\, p(s_1, y) > 0$, and then, by symmetry, $\mathrm{val}\, p(x, s_2) > 0$ too. In this case, the Newton polygon of p has a (unique) edge whose outward pointing normals have only negative coordinates. Similar arguments in the other cases.

Therefore,

$$\mathrm{lp}_\omega(q)\mathrm{lp}_\omega(p) = \mathrm{lp}_\omega(f) - \mathrm{lp}_\omega(g),$$

and there are polynomials $f_\omega \in \mathbb{K}[x^{\mathrm{sgn}(\omega_x)}]$ and $g_\omega \in \mathbb{K}[y^{\mathrm{sgn}(\omega_y)}]$ and a positive integer $k \equiv k_{s_1,s_2}$ such that

$$\mathrm{F}(\mathrm{lp}_\omega(p)) = \mathbb{K}((f_\omega, g_\omega))$$

and

$$\mathrm{lp}_\omega(f) - \mathrm{lp}_\omega(g) = f_\omega^k - g_\omega^k.$$

If $\mathrm{sgn}(\omega_x) = 1$, then $\mathrm{lp}_\omega(f)$ is a constant multiple of $x^{\deg f_n - \deg f_d}$, and if $\mathrm{sgn}(\omega_x) = -1$, it is a constant multiple of $x^{\mathrm{val} f_n - \mathrm{val} f_d}$. The exponent of the former is the multiplicity of ∞, and the exponent of the latter is the negative of the multiplicity of 0. Consequently,

$$\mathrm{m}(s_1, s_2) := (\mathrm{m}(s_1, f), \mathrm{m}(s_2, g)) = k \cdot (|\deg f_\omega|, |\deg g_\omega|)\,.$$

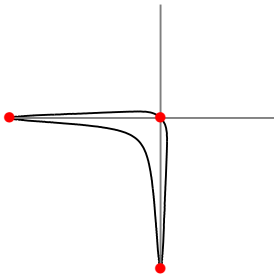
Wlog we can assume that $s_1, s_2 \in \{0, \infty\}$. If this were not the case, we can move the finite singularities to 0 and consider

$$p_{s_1,s_2}(x, y) = p\left(x + s_1 \cdot [s_1 \in \mathbb{K}], y + s_2 \cdot [s_2 \in \mathbb{K}]\right),$$

where $[s_i \in \mathbb{K}]$ is 1 or 0 depending on whether s_i is an element of $\mathbb{K}$ or not, and argue as before. For each pair of poles determined by Algorithm 1 we can therefore compute their multiplicities up to a multiplicative constant k. It turns out that we can derive a system of linear equations for these constants from which we can eliminate all but one of them. To see this, observe that if (s_1, s_2) and (t_1, t_2) are two pairs of poles of (f, g) such that $s_1 = t_1$, then

$$k_{s_1,s_2} \cdot |\deg f_{\omega_{s_1,s_2}}| = k_{t_1,t_2} \cdot |\deg f_{\omega_{t_1,t_2}}|. \tag{1}$$

An analogous equation holds when $s_2 = t_2$. And so there is a linear relation between the unknowns k_{s_1,s_2} and k_{t_1,t_2} whenever (s_1, s_2) and (t_1, t_2) have a common component. The way the set of pairs of poles was computed by Algorithm 1 implies that the solution space of these equations is at most 1-dimensional. It is not 0-dimensional since we assumed that f and g are not constants. Therefore, the statement.



The curve defined by $xy - x - y - x^2y^2$ and the points on it defined by the dynamical system

One could hope that for each choice of parameter in this 1-parameter family, there is some separated multiple of p for which the multiplicities of its poles are determined by this parameter. This is indeed the case as we will see later. For now we just formulate the algorithm these observations give rise to and illustrate it with an example.

Algorithm 2. *Input: an irreducible polynomial $p \in \mathbb{K}[x,y] \setminus (\mathbb{K}[x] \cup \mathbb{K}[y])$ which has a non-trivial rational multiple in $\mathbb{K}(x) + \mathbb{K}(y)$.*
Output: a pair $(f,g) \in \mathbb{K}(x) \times \mathbb{K}(y)$ such that $\mathrm{F}(p) = \mathbb{K}((f,g))$.

1 *Call Algorithm 1 on p, and let S be its output.*
2 *Set $M = \emptyset$, $E = \emptyset$.*
3 *For each $(s_1, s_2) \in S$ do:*
4 *compute a generator $(f_{\omega_{s_1,s_2}}, g_{\omega_{s_1,s_2}})$ of $\mathrm{F}(\mathrm{lp}_{\omega_{s_1,s_2}}(p))$ and enlarge M by the pair consisting of (s_1, s_2) and $k_{s_1,s_2} \cdot \left(|\deg f_{\omega_{s_1,s_2}}|, |\deg g_{\omega_{s_1,s_2}}|\right)$.*
4 *For any two elements (s_1, s_2) and (t_1, t_2) of S do:*
5 *if $s_1 = t_1$, append equation (1) to E, and if $s_2 = t_2$ replace $f_{\omega_{s_1,s_2}}$ and $f_{\omega_{t_1,t_2}}$ therein by $g_{\omega_{s_1,s_2}}$ and $g_{\omega_{t_1,t_2}}$, respectively, and append it to E.*
6 *Solve E over $\mathbb{N}$, and substitute the generator of its solution space into M.*
7 *Determine f_d, g_d and q_d, and make an ansatz for f_n, g_n and q_n according to the poles and multiplicities specified in M.*
8 *Equate the coefficients in $q_n p - f_n g_d + g_n f_d$ to zero and solve the resulting linear system for them.*
9 *Determine a non-trivial (f,g) corresponding to a solution and return it.*

Example 4. *Let us again consider the polynomial $p = xy - x - y - x^2y^2$ from Example 1. We already know that it is near-separable, and we know how a generator of $\mathrm{F}(p)$ looks like. Let us compute this generator (f,g) again, now using the ideas presented in this section. To do so, we first determine the poles of f and g. They appear in pairs, and are points on the curve defined by p. Among them is $(\infty, 0)$, since $\mathrm{lc}_x(p) = -y^2$, and $(0,0)$ and $(0,\infty)$ as $p(x,0) = -x$ and $\deg p(0,y) < \deg_y p$, respectively. Since $p(0,y) = -y$, there are no further such pairs. See the figure above for a drawing of the curve and the pairs of poles on it. Next, we derive information on their multiplicities. For each pair (s_1, s_2) just found, there is a weight function ω whose i-th component is positive or negative, depending on whether s_i is ∞ or not, such that $\mathrm{lp}_\omega(p)$ consists of at least two*

terms. They are $(2,-1)$, $(-1,-1)$ *and* $(-1,2)$, *and the corresponding leading parts are*

$$\mathrm{lp}_{(2,-1)}(p) = -x - x^2y^2, \quad \mathrm{lp}_{(-1,-1)}(p) = -x - y, \quad \mathrm{lp}_{(-1,2)}(p) = -y - x^2y^2.$$

For each of them, we solve Problem 2. Since each leading part $\mathrm{lp}_\omega(p)$ *is already near-separated, that is, of the form* $f_n g_d - g_n f_d$, *we find that* (f,g) *is a generator of* $\mathrm{F}(\mathrm{lp}_\omega(p))$. *The fields of separated multiples for the leading parts are therefore*

$$\mathbb{K}((x,-y^{-2})), \quad \mathbb{K}((x,-y)) \quad \textit{and} \quad \mathbb{K}((x^{-2},-y)).$$

Their generators show that there are $k_1, k_2, k_3 \in \mathbb{N}$ *such that*

$$\mathrm{m}(\infty,0) = k_1 \cdot (1,2), \quad \mathrm{m}(0,0) = k_2 \cdot (1,1) \quad \textit{and} \quad \mathrm{m}(0,\infty) = k_3 \cdot (2,1).$$

The numbers k_1, k_2 *and* k_3 *are not independent from each other, there are linear relations between them, since the second components of* $\mathrm{m}(\infty,0)$ *and* $\mathrm{m}(0,0)$ *and the first components of* $\mathrm{m}(0,0)$ *and* $\mathrm{m}(0,\infty)$ *are the same. We have* $2k_1 = k_2$ *and* $2k_3 = k_2$. *The solutions* (k_1,k_2,k_3) *of these equations over* $\mathbb{N}$ *are positive multiples of* $(1,2,1)$, *and so there is a* $k \in \mathbb{N}$ *such that the multiplicities of* ∞ *and* 0 *as poles of both* f *and* g *are* k *and* $2k$, *respectively. Setting* $k = 1$ *and making the ansatz*

$$f = \frac{f_0 + f_1x + f_2x^2 + f_3x^3}{x^2} \quad \textit{and} \quad g = \frac{g_0 + g_1y + g_2y^2 + g_3y^3}{y^2}$$

and

$$q = \frac{q_{00} + q_{10}x + q_{01}y}{x^2y^2},$$

clearing denominators in $qp = f - g$ *and comparing coefficients, results in a system of linear equations for the undetermined coefficients whose solutions correspond to the rational functions*

$$f = \frac{u - ux - ux^3}{x^2} \quad \textit{and} \quad g = \frac{u - uy - uy^3}{y^2}$$

and

$$q = \frac{ux - uy}{x^2y^2},$$

for $u \in \mathbb{K}$. *In particular, we find that*

$$\frac{x-y}{x^2y^2}p = \frac{1 - x - x^3}{x^2} - \frac{1 - y - y^3}{y^2}.$$

5.3 Correctness

In this subsection we prove the correctness of Algorithm 2. In particular, we show that it succeeds in determining the poles of a generator of $\mathrm{F}(p)$ as well as their multiplicities when p is near-separable. Our arguments generalize the arguments in [5, Section 3].

Let $f \in \mathbb{K}(x)$ and $g \in \mathbb{K}(y)$ be non-constant rational functions. We study the near-separated polynomial

$$f_n g_d - g_n f_d$$

by introducing a new variable t and investigating the auxiliary equations

$$f = t \quad \text{and} \quad g = t.$$

We solve these equations with respect to x and y in $\overline{\mathbb{K}(t)}$, the algebraic closure of $\mathbb{K}(t)$. Let their solutions be $\alpha_0, \ldots, \alpha_{m-1}$ and $\beta_0, \ldots, \beta_{n-1}$, respectively, where $m = \max\{\deg f_n, \deg f_d\}$ and $n = \max\{\deg g_n, \deg g_d\}$. We will throughout view $\overline{\mathbb{K}(t)}$ as a subfield of $\mathbb{K}\{\{t^{-1}\}\}$, the field of Puiseux series in descending powers of t. The α_i's and β_j's are then of the form

$$c_1 t^{d_1} + c_2 t^{d_2} + \ldots,$$

where $c_i \in \mathbb{K}$ and $d_1 > d_2 > \ldots$ are rational numbers which have a common denominator. The Newton-Puiseux algorithm [12] shows that their leading terms encode the poles of f and g as well as their multiplicities in the following sense.

Proposition 7. *Let $f \in \mathbb{K}(x)$, and let $s \in \mathbb{K} \cup \{\infty\}$ be a pole of f of multiplicity m. If $s = \infty$ or $s = 0$, then for each root c of $\mathrm{lc}(f_d) + \mathrm{lc}(f_n)t^m$ and $\mathrm{lc}(f_n(x^{-1})) + \mathrm{lc}(f_d(x^{-1}))t^m$, respectively, there is a root of $f - t$ in $\mathbb{K}\{\{t^{-1}\}\}$ whose leading term is $ct^{1/m}$ and $ct^{-1/m}$. If $s \neq \infty, 0$, then there are m series roots whose leading term is s. If α such a series, we say that it is associated with s.*

Every element π of $\mathrm{Gal}(\overline{\mathbb{K}(t)}/\mathbb{K}(t))$, the Galois group of $\overline{\mathbb{K}(t)}$ over $\mathbb{K}(t)$, acts on $\mathbb{Z}_m \times \mathbb{Z}_n$ by

$$\pi(i,j) := (i',j') \quad :\Longleftrightarrow \quad (\pi(\alpha_i), \pi(\beta_j)) = (\alpha_{i'}, \beta_{j'}).$$

Let $G \subseteq \mathrm{S}_m \times \mathrm{S}_n$ be the group of permutations induced on $\mathbb{Z}_m \times \mathbb{Z}_n$ by this action. In the following we study subsets $T \subseteq \mathbb{Z}_m \times \mathbb{Z}_n$ that are invariant under the action of G and investigate how they relate to factors of $f_n g_d - g_n f_d$.

For a subset $T \subseteq \mathbb{Z}_m \times \mathbb{Z}_n$, and $(i,j) \in \mathbb{Z}_m \times \mathbb{Z}_n$, we introduce

$$T_{i,*} = \{k \mid (i,k) \in T\} \quad \text{and} \quad T_{*,j} = \{k \mid (k,j) \in T\}.$$

As in [5], we have the following two lemmas.

Lemma 5. *Let $T \subseteq \mathbb{Z}_m \times \mathbb{Z}_n$ be invariant under the action of G. Then*

$$|T_{0,*}| = |T_{1,*}| = \cdots = |T_{m-1,*}| \quad \textit{and} \quad |T_{*,0}| = |T_{*,1}| = \cdots = |T_{*,n-1}|.$$

Proof. We show that $|T_{0,*}| = |T_{1,*}|$, the rest is analogous. We observe that $f_n - tf_d$ is irreducible over $\mathbb{K}(t)$. If it were not, it would be reducible over $\mathbb{K}[t]$ due to Gauss's lemma. This impossible because $f_n - tf_d$ is linear in t, and does not have factors in $\mathbb{K}[x]$ since f_n and f_d are relative prime. The irreducibility of $f_n - tf_d$ implies that its Galois group acts transitively on its roots. In particular, there exists $\pi \in \mathrm{Gal}(\overline{\mathbb{K}(t)}/\mathbb{K}(t))$ such that $\pi(\alpha_0) = \alpha_1$. Hence π maps $T_{0,*}$ to $T_{1,*}$, and we have $|T_{0,*}| \leqslant |T_{1,*}|$. The reverse inequality is analogous. □

Lemma 6. *The map*

$$p \quad \mapsto \quad T := \{(i,j) \in \mathbb{Z}_m \times \mathbb{Z}_n : p(\alpha_i, \beta_j) = 0\}$$

defines a bijection between the set of factors of $f_n g_d - g_n f_d$ and the set of subsets of $\mathbb{Z}_m \times \mathbb{Z}_n$ that are invariant under the action of G.

Proof. Let p be a divisor of $f_n g_d - g_n f_d$, and let T be the corresponding subset of $\mathbb{Z}_m \times \mathbb{Z}_n$. If $(i,j) \in T$, then $p(\alpha_i, \beta_j) = 0$, and so $p(\pi(\alpha_i), \pi(\beta_j)) = 0$ for any $\pi \in \mathrm{Gal}(\overline{\mathbb{K}(t)}/\mathbb{K}(t))$. Therefore, $\pi(i,j) \in T$, and T is G-invariant.

Let now T be a G-invariant subset of $\mathbb{Z}_m \times \mathbb{Z}_n$, and let $T_{0,*} = \{j_1, \ldots, j_s\}$. Since $f(\alpha_0) = t$, we have $\mathbb{K}(\alpha_0) \supseteq \mathbb{K}(t)$, so T is invariant with respect to the action of the Galois group $\mathrm{Gal}(\overline{\mathbb{K}(t)}/\mathbb{K}(\alpha_0))$. If α_0 is fixed, then $\beta_{j_1}, \ldots, \beta_{j_s}$ are permuted. Therefore, $(y - \beta_{j_1})(y - \beta_{j_2}) \ldots (y - \beta_{j_s})$ is invariant under the action of $\mathrm{Gal}(\overline{\mathbb{K}(t)}/\mathbb{K}(\alpha_0))$. Hence by the fundamental theorem of Galois theory, it is a polynomial in $\mathbb{K}(\alpha_0)[y]$. By construction, its numerator divides $f_n(\alpha_0) g_d(y) - g_n(y) f_d(\alpha_0)$ in $\mathbb{K}[\alpha_0, y]$. Replacing α_0 by x results in a polynomial $p \in \mathbb{K}[x,y]$ that divides $f_n g_d - g_n f_d$ in $\mathbb{K}[x,y]$.

It remains to show that the two constructions are inverse to each other. In order to do so, we first prove that the invariant set associated with the polynomial p just constructed equals T. Let $(i,j) \in \mathbb{Z}_m \times \mathbb{Z}_n$. Since $\mathrm{Gal}(\overline{\mathbb{K}(t)}/\mathbb{K}(t))$ acts transitively on the roots of $f - t$, there is an automorphism π with $\pi(\alpha_i) = \alpha_0$. Let $\beta_{j'} = \pi(\beta_j)$. We then have

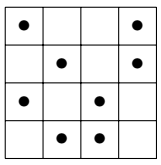
$$p(\alpha_i, \beta_j) = 0 \iff p(\alpha_0, \beta_{j'}) = 0 \iff j' \in T_{0,*} \iff (i,j) \in T.$$

We now show that p is the unique factor of $f_n g_d - g_n f_d$ whose associated invariant set is T. Assume that $\tilde{p}$ is another divisor such that $\tilde{p}(\alpha_i, \beta_j) = 0$ if and only if $(i,j) \in T$. The same argument which proved that the polynomial constructed from T is a divisor of $f_n g_d - g_n f_d$ applies to show that p is a divisor of $\tilde{p}$ in $\mathbb{K}[x,y]$, and vice versa. Hence they only differ by a multiplicative constant. □

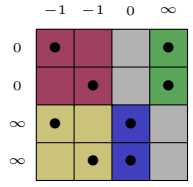
Example 5. *There are four invariant subsets of $\mathbb{Z}_4 \times \mathbb{Z}_4$ that can be associated with*

$$f_n g_d - g_n f_d = (1-x)^2(1+x+x^2)y(1+y)^2 + (1+y+y^2)^2 x^2.$$

The diagram in the left figure below illustrates the invariant set T that corresponds to $p = xy - 1 - y - x^2 y - x^2 y^2$. Its rows are numbered by the roots of $f - t$, and the roots of $g - t$ number its columns. A dot in the i-th row and j-th column indicates that p annihilates (α_i, β_j). The other invariant sets are $\emptyset$, T^c and $\mathbb{Z}_m \times \mathbb{Z}_n$. The first and last correspond to the trivial factors 1 and $f_n g_d - g_n f_d$, the second one is associated with the complementary factor of p.



$xy - 1 - y - x^2 y - x^2 y^2$



$xy - 1 - y - x^2 y - x^2 y^2$

Definition 4. *Let p be a factor of $f_n g_d - g_n f_d$, and let T be the corresponding invariant set. Let s_1 and s_2 be poles of f and g, respectively, and let $T^{s_1,s_2} \subseteq T$ be such that $(i,j) \in T^{s_1,s_2}$ if and only if (α_i, β_j) is associated with (s_1, s_2) in the sense of Proposition 7. We refer to T^{s_1,s_2} as the* ***component*** *of T associated with (s_1, s_2).*

Example 6. *Continuing with Example 5, the invariant set associated with* $xy - 1 - y - x^2y - x^2y^2$ *and* $f_n g_d - g_n f_d$ *has four non-empty components. The above figure on the right depicts its diagram again. Its rows and columns are now not only numbered by the series roots of* $f - t$ *and* $g - t$*, respectively, but also labeled by the poles they encode. Its non-empty components are highlighted in color.*

The non-empty components of an invariant set associated with a factor p of $f_n g_d - g_n f_d$ have an interpretation on the level of their leading parts.

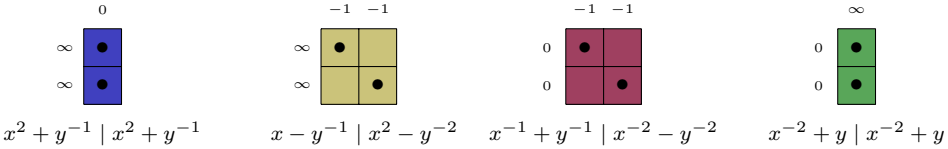
Definition 5. *Let* (s_1, s_2) *be a pair of poles of* $(f, g) \in \mathbb{K}(x) \times \mathbb{K}(y)$*, and let* (α, β) *be a pair of roots of* $f - t$ *and* $g - t$ *in* $\mathbb{K}\{\{t^{-1}\}\}$ *that is associated with it. We say that* $\omega \in \mathbb{Z}^2$ *is associated with* (s_1, s_2) *and* (f, g)*, if it is a positive multiple of* $(\deg \alpha, \deg \beta)$*.*

Lemma 7. *Let* T *be the invariant set of* p *and* $f_n g_d - g_n f_d$*, and assume that* $(s_1, s_2) \in \{0, \infty\}^2$ *is a pair of poles of* (f, g)*. Let* $\omega \in \mathbb{Z}^2$ *be associated with* (s_1, s_2)*. Then* $T^{s_1,s_2} \neq \emptyset$ *if and only if* $\mathrm{lp}_\omega(p)$ *is not a single term. In that case the invariant set of* $\mathrm{lp}_\omega(p)$ *and* $\mathrm{lp}_\omega(f) - \mathrm{lp}_\omega(g)$ *can be identified with* T^{s_1,s_2}*.*

Proof. Assume that $T^{s_1,s_2} \neq \emptyset$, and let $(i, j) \in T^{s_1,s_2}$. Then $p(\alpha_i, \beta_j) = 0$, and therefore $\mathrm{lp}_\omega(p)(\overline{\alpha}_i, \overline{\beta}_j) = 0$, where $\overline{\alpha}_i$ and $\overline{\beta}_j$, respectively, denotes the leading term of α_i and β_j. So $\mathrm{lp}_\omega(p)$ involves at least two terms, and since $\mathrm{lp}_\omega(p)$ is a divisor of $\mathrm{lp}_\omega(f_n g_d - g_n f_d)$, so does the latter. Therefore, and because $\deg \alpha_i, \deg \beta_j \neq 0$, we have $\mathrm{lp}_\omega(f_n g_d - g_n f_d) = \mathrm{lp}_\omega(f_n)\mathrm{lp}_\omega(g_d) - \mathrm{lp}_\omega(g_n)\mathrm{lp}_\omega(f_d)$. Hence taking leading terms of series induces a bijection between T^{s_1,s_2} and the invariant set associated with $\mathrm{lp}_\omega(p)$ and $\mathrm{lp}_\omega(f) - \mathrm{lp}_\omega(g)$.

Let us now assume that $\mathrm{lp}_\omega(p)$ involves at least two terms. We have to show that there is some pair (α, β) of series solutions that is associated with (s_1, s_2) and that is annihilated by p. Since $f_n g_d - g_n f_d$ is near-separated, so is $\mathrm{lp}_\omega(f_n g_d - g_n f_d)$. Hence it is the product of a monomial and, by Lemma 1, a polynomial that is square-free. If we assume that this polynomial involves a term in x and y, respectively, only, then it factors into $\prod_i (a_i x - y^{\omega_x/\omega_y})$, where the a_i's are non-zero and pairwise different, and i ranges from 1 to $\mathrm{m}(s_1)$. Consider now the pairs $(\overline{\alpha}, \overline{\beta})$ of leading terms, where β is some root of $g - t$ associated with s_2 and α ranges over the roots of $f - t$ associated with s_1. By Proposition 7, there are $\mathrm{m}(s_1)$ such pairs, and they are pairwise distinct. Since each of them is annihilated by $\mathrm{lp}_\omega(f_n g_d - g_n f_d)$, there is some pair (α, β) associated with (s_1, s_2) such that $\mathrm{lp}_\omega(p)$ annihilates $(\overline{\alpha}, \overline{\beta})$ but not its complementary factor in $\mathrm{lp}_\omega(f_n g_d - g_n f_d)$. Consequently, p annihilates (α, β). □

Example 7. *In Example 6 we observed that the invariant set* T *associated with* p *and* $f_n g_d - g_n f_d$ *partitions into four non-empty components. Two of them can be associated with their leading parts with respect to* $\omega_1 = (1, -2)$ *and* $\omega_2 = (-1, 2)$*. The other two are related to the leading parts of* $p(x, -1 + y)$ *and* $f_n(x) g_d(-1 + y) - g_n(-1 + y) f_d(x)$ *with respect to* $\omega_3 = (1, -1)$ *and* $\omega_4 = (-1, -1)$*. The diagrams and the pairs of polynomials they are corresponding to are depicted below.*



$x^2 + y^{-1} \mid x^2 + y^{-1}$ $\quad$ $x - y^{-1} \mid x^2 - y^{-2}$ $\quad$ $x^{-1} + y^{-1} \mid x^{-2} - y^{-2}$ $\quad$ $x^{-2} + y \mid x^{-2} + y$

The diagrams of the components above have the same heights and lengths, respectively, when their vertical and horizontal sides are labeled by the same poles. This is not a coincidence.

Lemma 8. *Let T be the invariant subset associated with p and $f_n g_d - g_n f_d$, and let s, s_1, $s_2 \in \mathbb{K} \cup \{\infty\}$ be such that T^{s,s_1}, $T^{s,s_2} \neq \emptyset$. Then*

$$\bigcup_i T^{s,s_1}_{*,i} = \bigcup_j T^{s,s_2}_{*,j}.$$

Proof. Wlog we assume that $s, s_1, s_2 \in \{0, \infty\}$. Let $(i_k, j_k) \in T^{s,s_k}$, and define $\omega_k = (\deg \alpha_{i_k}, \deg \beta_{j_k})$ for $k \in \{1, 2\}$. Then T^{s,s_k} is the invariant set associated with $\mathrm{lp}_{\omega_k}(p)$ and $\mathrm{lp}_{\omega_k}(f) - \mathrm{lp}_{\omega_k}(g)$. Since $\deg \alpha_{i_k}$ is independent of k, so is $\mathrm{lp}_{\omega_k}(f)$. The statement of the lemma now follows from the definition of the invariant set associated with a polynomial and its near-separated multiple and Lemma 5. □

Lemma 6 showed that there is a bijection between factors of $f_n g_d - g_n f_d$ and G-invariant subsets of $\mathbb{Z}_m \times \mathbb{Z}_n$. We next give a characterization of near-separated factors of $f_n g_d - g_n f_d$ in terms of properties of the invariant subsets associated with them.

Definition 6. *A subset $T \subseteq \mathbb{Z}_m \times \mathbb{Z}_n$ is called **separated** if*

$$\forall\ i, j \in \mathbb{Z}_m : (T_{i,*} \cap T_{j,*} = \emptyset)\ \text{or}\ (T_{i,*} = T_{j,*}).$$

Lemma 9. *Let p be a factor of $f_n g_d - g_n f_d$, and let $T \subseteq \mathbb{Z}_m \times \mathbb{Z}_n$ be the corresponding invariant set. Then p is near-separated if and only if T is separated.*

Proof. If $p = \tilde{f}_n \tilde{g}_d - \tilde{g}_n \tilde{f}_d$, then $(i, j) \in T$ if and only if $\tilde{f}(\alpha_i) = \tilde{g}(\beta_j)$. Hence (i, j), (i, j'), $(i', j) \in T$ implies that $(i', j') \in T$. This shows the only-if part of the statement. Let us now assume that T is separated, and let us show that the polynomial

$$p(x, y) = a_s(x) y^s + a_{s-1}(x) y^{s-1} + \cdots + a_0(x),$$

that corresponds to it is near-separated. By construction $a_i(\alpha_j)/a_s(\alpha_j)$ is, up to sign, the $(s - i)$-th elementary symmetric polynomial in $\{\beta_k : k \in T_{j,*}\}$ for each $0 \leq i < s$ and $0 \leq j < m$. Let us assume for the moment that $\operatorname{val} a_s > 0$. If i_0 is such that $\operatorname{val} a_{i_0} = 0$, then there are $c_i \in \mathbb{K}$ such that $\operatorname{val}(a_i - c_i a_{i_0}) > 0$. The number of non-zero roots of

$$\frac{a_i(x) - c_i a_{i_0}(x)}{a_s(x)} - \frac{a_i(\alpha_0) - c_i a_{i_0}(\alpha_0)}{a_s(\alpha_0)}$$

is at most

$$\max\{\deg(a_i - c_i a_{i_0}), \deg a_s\} - \min\{\operatorname{val}(a_i - c_i a_{i_0}), \operatorname{val} a_s\},$$

and therefore smaller than $\deg_x p$. If $j \in T_{0,*}$, then for each $k \in T_{*,j}$ the series α_k is a root of it. Since these roots are non-zero and pairwise distinct, and because there are $\deg_x p$ of them, the rational function is identically zero. Hence there are $d_i \in \mathbb{K}$ such that

$$a_i(x) = c_i a_{i_0}(x) + d_i a_s(x).$$

Consequently,

$$p(x,y) = \sum_{i=0}^{s} a_i(x)y^i = a_{i_0}(x)\sum_{i=0}^{s} c_i y^i + a_s(x)\sum_{i=0}^{s} d_i y^i,$$

that is, p is near-separated. If $\operatorname{val} a_s = 0$ and a_s is not just a single term, then it has a root $c \in \mathbb{K}$. The leading coefficient of $p(x+c,y)$ with respect to y has positive valuation, and we can argue as before to show that $p(x+c,y)$, and hence also p, is near-separated. If $\operatorname{val} a_s = 0$ and a_s is just a single term, then a_s is a constant, and $\deg a_s < \deg_x p$. Choosing i_0 such that $\deg a_{i_0} = \deg_x p$ and $c_i \in \mathbb{K}$ such that $\deg(a_i - c_i a_{i_0}) < \deg_x p$, we can argue as before to show that a_i is a linear combination of a_{i_0} and a_s to conclude that p is near-separated. □

We present another definition and another lemma before we come to the main theorem. The proof of Lemma 10 is taken literally from [5, Lemma 3.13.] and included here for convenience of the reader.

Definition 7. *Let T be an invariant subset of $\mathbb{Z}_m \times \mathbb{Z}_n$. The **separable closure** of T is*

$$T^{\mathrm{sep}} := \bigcap_{\substack{S \supseteq T \\ S \text{ sep}}} S.$$

Lemma 10. *Let $T \subseteq \mathbb{Z}_m \times \mathbb{Z}_n$ be invariant with respect to $G \subseteq \mathrm{S}_m \times \mathrm{S}_n$. Then T^{sep} is also G-invariant.*

Proof. Let $\pi = (\sigma, \tau) \in \mathrm{S}_m \times \mathrm{S}_n$, and let $S \subseteq \mathbb{Z}_m \times \mathbb{Z}_n$ be a separated set. Since $\pi(S)_{i,*} = \tau(S_{\sigma(i),*})$, we find that $\pi(S)$ is separated as well.

Assume that T^{sep} is not G-invariant, that is, there exists a $\pi \in G$ such that $\pi(T^{\mathrm{sep}}) \neq T^{\mathrm{sep}}$. As we have shown, $\pi(T^{\mathrm{sep}})$ is separated, hence so is $S := T^{\mathrm{sep}} \cap \pi(T^{\mathrm{sep}})$. Observe that, since $\pi(T^{\mathrm{sep}}) \neq T^{\mathrm{sep}}$, $S \subsetneq T^{\mathrm{sep}}$. Since T is G-invariant, $T \subseteq \pi(T^{\mathrm{sep}})$, so $T \subseteq S$. This contradicts the minimality of T^{sep}. □

Theorem 1. *Let $p \in \mathbb{K}[x,y] \setminus (\mathbb{K}[x] \cup \mathbb{K}[y])$ be irreducible, and assume that $\mathrm{F}(p)$ is not isomorphic to $\mathbb{K}$. Then Algorithm 2 terminates on input p, and outputs a pair $(f,g) \in \mathbb{K}(x) \times \mathbb{K}(y)$ such that $\mathrm{F}(p) = \mathbb{K}((f,g))$.*

Proof. Let $f_n g_d - g_n f_d$ be the minimal near-separated multiple of p with $\deg f_n > \deg f_d$, and let T be the invariant set associated with them. We first prove the correctness of Algorithm 1 which is used as a subroutine. If it did not succeed in finding all poles of f and g, then, after a permutation of its rows and columns, T would be the union of two subsets T_0 and T_1 of $\{0,1,\ldots,m_0\} \times \{0,1,\ldots,n_0\}$ and $\{m_0+1,\ldots,m-1\} \times \{n_0+1,\ldots,n-1\}$, respectively. To see this, let (s_1,s_2) be a pair of poles of (f,g) of which s_1 is known. It can therefore be assumed to be equal to 0 or ∞. We assume that it is zero. If $(i,j) \in T^{s_1,s_2}$, then $p(\alpha_i,\beta_j) = 0$, and therefore $\mathrm{lp}_\omega(p)(\overline{\alpha}_i,\overline{\beta}_j) = 0$, where $\omega = (\deg \alpha_i, \deg \beta_j)$. Since α_i and β_j are different from zero, the leading part $\mathrm{lp}_\omega(p)$ involves at least two terms. Therefore, ω is an outward pointing normal of an edge of the Newton polygon of p. If $s_2 = \infty$, then $\deg p(0,y) < \deg_y p$, and if $s_2 \in \mathbb{K}$ then $p(0,s_2) = 0$. Therefore, if Algorithm 1 finds a pole s_1 of f, then it also finds the

poles s_2 of g for which $T^{s_1,s_2} \neq \emptyset$. If $T = T_0 \cup T_1$ for subsets T_0, $T_1 \subseteq \mathbb{Z}_m \times \mathbb{Z}_n$ as above, then $T^{\text{sep}} = T_0^{\text{sep}} \cup T_1^{\text{sep}}$ is a proper subset of $\mathbb{Z}_m \times \mathbb{Z}_n$. This, however, contradicts the assumption that $f_n g_d - g_n f_d$ is the minimal near-separated multiple of p. Therefore, the algorithm succeeds in finding all poles of f and g.

It remains to show that the multiplicities computed by Algorithm 2 are indeed the multiplicities of the poles of f and g. Let T^{s_1,s_2} be a non-empty component of T, and assume it is the invariant set associated with $\text{lp}_\omega(p)$ and $\text{lp}_\omega(f_n g_d - g_n f_d)$ for some $\omega \in \mathbb{Z}^2$. If (f_ω, g_ω) is a generator of $\text{F}(\text{lp}(p))$, then there is an integer k such that the invariant set $T_k^{s_1,s_2}$ associated with $\text{lp}_\omega(p)$ and $f_\omega^k - g_\omega^k$ can be identified with T^{s_1,s_2}. If for each pair (s_1, s_2) of poles $k = k(s_1, s_2)$ is such that $T_k^{s_1,s_2}$ can be identified with T^{s_1,s_2}, then the diagrams associated with $T_k^{s_1,s_2}$ have to be compatible in the sense of Lemma 8. This compatibility does not uniquely determine the k's but only gives rise to a 1-parameter family of sets T_k one of which can be identified with T. Since $f_n g_d - g_n f_d$ is the minimal near-separated multiple of p, the separated closure of T equals $\mathbb{Z}_m \times \mathbb{Z}_n$. We claim that the only k for which the separated closure of $T_k \subseteq \mathbb{Z}_{km'} \times \mathbb{Z}_{kn'}$ equals $\mathbb{Z}_{km'} \times \mathbb{Z}_{kn'}$ is 1. In order to prove it we compare $T_1^{s_1,s_2}$ with $T_k^{s_1,s_2}$ and T_1 with T_k. Let

$$\alpha_{k,i} = \exp\left(\frac{2\pi\imath i}{km_1}\right) a^{-\frac{1}{m_1}} t^{\frac{1}{km_1}} \quad \text{and} \quad \beta_{k,j} = \exp\left(\frac{2\pi\imath j}{km_2}\right) b^{-\frac{1}{m_2}} t^{\frac{1}{km_2}}.$$

be the solutions of the auxiliary equations $f_\omega^k - t = 0$ and $g_\omega^k - t = 0$. Since

$$t^{\frac{1}{km_1m_2}} \quad \mapsto \quad \exp\left(\frac{2\pi\imath}{km_1m_2}\right) t^{\frac{1}{km_1m_2}}$$

is an element of $\text{Gal}(\mathbb{K}(t^{\frac{1}{km_1m_2}})/\mathbb{K}(t))$, we find that $\text{lp}_\omega(p)$ annihilates the pair $(\alpha_{k,i+1 \bmod km_1}, \beta_{k,j+1 \bmod km_2})$ whenever it annihilates $(\alpha_{k,i}, \beta_{k,j})$. Since

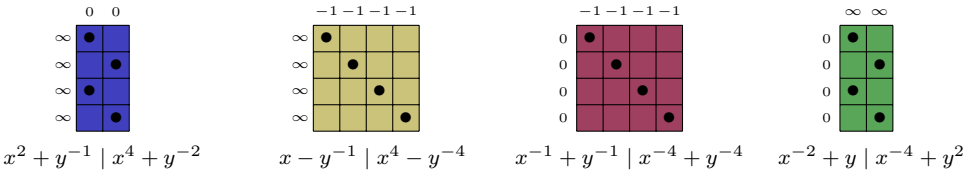
$$\alpha_{k,ki}(t) = \alpha_{1,i}(t^{1/k}) \quad \text{and} \quad \beta_{k,kj}(t) = \beta_{1,j}(t^{1/k})$$

and $\text{lp}_\omega(p)$ is homogenous with respect to $\omega = (\deg \alpha_{1,i}, \deg \beta_{1,j})$, we also find that $\text{lp}_\omega(p)$ annihilates $(\alpha_{k,ki}, \beta_{k,kj})$ if and only if it annihilates $(\alpha_{1,i}, \beta_{1,j})$. From this one can deduce that the permutation of $\mathbb{Z}_{km_1} \times \mathbb{Z}_{km_2}$ given by

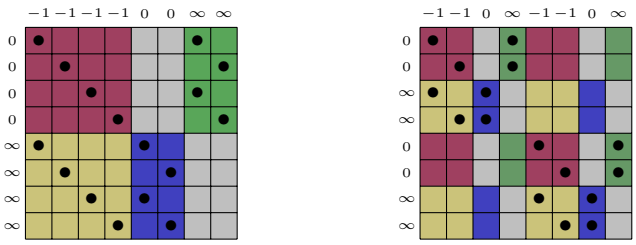
$$(u_1 k + v_1, u_2 k + v_2) \mapsto (v_1 m_1 + u_1, v_2 m_2 + u_2),$$

where $u_1 \in \{0, \ldots, m_1 - 1\}$, $u_2 \in \{0, \ldots, m_2 - 1\}$ and $v_1, v_2 \in \{0, \ldots, k - 1\}$ permutes the rows and columns of $T_k^{s_1,s_2}$ such that the associated diagram is of block diagonal form with each block equal to the diagram associated with $T_1^{s_1,s_2}$. These permutations indexed by the pairs (s_1, s_2) of poles of (f, g) make up a permutation of the rows and columns of T_k such that the diagram of the component associated with (s_1, s_2) is of block diagonal form as above. These blocks can be permuted such that the corresponding diagram is of block diagonal form with each block now equal to the diagram associated with T_1. Let us write $\text{diag}_k(T)$ for it, or more generally for a diagram in block diagonal form, consisting of k blocks of a diagram T. Then the diagram of the separated closure is $\text{diag}_k(T^{\text{sep}})$. If $k > 1$, the separated closure of T_k is a proper subset of $\mathbb{Z}_{km'} \times \mathbb{Z}_{kn'}$. □

Example 8. *In Example 14 we computed the invariant sets associated with* x^2+y^{-1}, $x^{-2}+y$, $x-y^{-1}$ *and* $x^{-1}+y^{-1}$ *as factors of* x^2+y^{-1}, $x^{-2}+y$, x^2-y^{-2} *and* $x^{-2}-y^{-2}$, *respectively, and observed that they are the components of the invariant set associated with* $p = xy-1-y-x^2y-x^2y^2$ *and* $f_n g_d - g_n f_d = (1-x)^2(1+x+x^2)y(1+y)^2 + (1+y+y^2)^2x^2$. *The invariant sets associated with* x^2+y^{-1}, $x^{-2}+y$, $x-y^{-1}$ *and* $x^{-1}+y^{-1}$ *as factors of* x^4+y^{-2}, $x^{-4}+y^2$, x^4-y^{-4} *and* $x^{-4}-y^{-4}$, *respectively, are depicted below.*



They make up the invariant set $T_2 \subseteq \mathbb{Z}_8 \times \mathbb{Z}_8$ *associated with* p *and* f^2-g^2. *A permutation of the rows and columns of* T_2 *results in a diagram of block diagonal form with its two blocks corresponding to* T.



6 Open questions

In the previous section we discussed a semi-algorithm that takes as input an irreducible polynomial $p \in \mathbb{K}[x,y] \setminus (\mathbb{K}[x] \cup \mathbb{K}[y])$ and outputs a generator of $\mathrm{F}(p)$ whenever it terminates. We observed that it does terminate, if p is near-separable, and we claimed that it may not if p is not near-separable. The following conjecture claims even more. Compare with [8, Theorem 1] and [3, Theorem 4.3]

Conjecture 1. *Let* $p \in \mathbb{K}[x,y] \setminus (\mathbb{K}[x] \cup \mathbb{K}[y])$ *be irreducible. Then* p *is near-separable if and only if the dynamical system associated with* p *in Subsection 5.1 is finite.*

We have seen two necessary conditions for a polynomial p to be near-separable: the sign vectors of the outward pointing normals of any two distinct edges of the Newton polygon of p need to be different, and the leading parts of p need to be near-separable. These conditions are not sufficient for a polynomial to be near-separable. It remains an open question how to prove that a polynomial is not near-separable when it satisfies these conditions. We do not have a solution to this problem. We only have the following conjecture. Compare with [8, Remark 5.1].

Conjecture 2. *Let* $p \in \mathbb{K}[x,y] \setminus (\mathbb{K}[x] \cup \mathbb{K}[y])$ *be irreducible. Then there is an integer* k *that can be computed from* p *such that the orbit of a point under the*

dynamical system associated with it in Subsection 5.1 is infinite if and only if it consists of at least k points.

7 Acknowledgements

Thanks go to the Johann Radon Institute for Computational and Applied Mathematics of the Austrian Academy of Sciences at which the author was employed while considerable progress on this work was made. Thanks also go to the Johannes Kepler University Linz which supported this work with the grant LIT-2022-11-YOU-214.

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