# EFFECTIVE ARITHMETIC FOR MULTIVARIATE ALGEBRAIC SERIES

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Abstract. We explain how to encode an algebraic series by a finite amount of data, namely its minimal polynomial, a total order and its first terms with respect to this order, and how to do effective arithmetic on the level of these encodings. The reasoning is based on the Newton-Puiseux algorithm and an effective equality test for algebraic series. We also explain how the latter allows to derive information about the support of an algebraic series, e.g. how to compute the (finitely many) vertices and bounded faces of the convex hull of its support.

## 1. Overview

Given a polynomial $p(x, y)$ in two variables $x$ and $y$ over an algebraically closed field $\mathbb{K}$ of characteristic zero, the classical Newton-Puiseux algorithm allows to determine the first terms of a series $\phi$ over $\mathbb{K}$ for which $p(x, \phi) = 0$. Finding a series solution of a polynomial equation is one and the most apparent aspect of the algorithm. However, it also allows to encode a series by a finite amount of data and to effectively perform operations such as plus and times on the level of these encodings. The Newton-Puiseux algorithm was introduced by Newton and analyzed by Puiseux in [9], see also [4] for a modern presentation of it. It was generalized to multivariate, not necessarily bivariate polynomials over a field of characteristic zero in [7], and studied in [10] for polynomials over a field of positive characteristic. While it is well-known how to compute effectively with univariate series that are algebraic, this is not the case for algebraic multivariate series. We explain the latter here and complement the discussion of the Newton-Puiseux algorithm presented in [7]. We also show that the convex hull of the support of an algebraic series is a polyhedral set and explain how the equality test allows to compute its vertices and bounded faces. Supports of series were also studied in [2, 1], though in the more general context of series algebraic over a certain ring of series. The article comes with a Mathematica implementation of the Newton-Puiseux algorithm and a Mathematica notebook, which can be found on https://github.com/buchacm/newtonPuiseuxAlgorithm.

## 2. Preliminaries

We begin with introducing the objects this article is about: multivariate algebraic series, and the Newton-Puiseux algorithm to constructively work with them.

Let $\mathbb{K}$ be an algebraically closed field of characteristic zero, denote by $\mathbf{x} = (x_1, \dots, x_n)$ a vector of variables, and write $\mathbf{x}^I := x_1^{i_1} \cdot \dots \cdot x_n^{i_n}$ for $I = (i_1, \dots, i_n) \in \mathbb{Q}^n$. A series $\phi$ in $\mathbf{x}$ over $\mathbb{K}$ is a formal sum

$$\phi = \sum_{I \in \mathbb{Q}^n} a_I \mathbf{x}^I$$

of terms in $\mathbf{x}$ whose coefficients $a_I$ are elements of $\mathbb{K}$. Its support is defined by

$$\operatorname{supp}(\phi) = \{I \in \mathbb{Q}^n : a_I \neq 0\},$$

and we will assume throughout that there is an integer $k \in \mathbb{Z}$, a line-free cone $C \subseteq \mathbb{R}^n$ and a vector $v \in \mathbb{R}^n$ such that

$$\operatorname{supp}(\phi) \subseteq (v + C) \cap \frac{1}{k}\mathbb{Z}^n.$$

Without any restriction on their supports, the sum and product of two series is not well-defined. However, for any line-free convex cone $C \subseteq \mathbb{R}^n$ the set $\mathbb{K}_C[[x]]$ of series whose support is contained in $C$ is a ring with respect to addition and multiplication [8, Theorem 10]. Yet, it is not a field [8, Theorem 12].

Any $w \in \mathbb{R}^n$ whose components are linearly independent over $\mathbb{Q}$ defines a total order $\preceq$ on $\mathbb{Q}^n$ by

$$\alpha \preceq \beta \quad :\Longleftrightarrow \quad \langle \alpha, w\rangle \leq \langle \beta, w\rangle,$$

where $\langle \alpha, w\rangle := \sum_{i=1}^n \alpha_i w_i$ for $\alpha, w \in \mathbb{R}^n$. As usual, we write $\alpha \prec \beta$ when $\alpha \preceq \beta$ and $\alpha \neq \beta$. It naturally extends to the set of terms in $\mathbf{x}$ by saying that $a\mathbf{x}^\alpha \preceq b\mathbf{x}^\beta$ when $\alpha \preceq \beta$. We define $\operatorname{lexp}_\preceq(\phi) := \max_\preceq \operatorname{supp}(\phi)$, and we write $\operatorname{lexp}_w(\phi)$ for it when $\preceq$ is induced by $w \in \mathbb{R}^n$. A cone $C \subseteq \mathbb{R}^n$ is compatible with a total order $\preceq$ on $\mathbb{Q}^n$ when $C\cap\mathbb{Q}^n$ has a maximal element with respect to it. Given a total order $\preceq$, let $\mathcal{C}$ be the set of cones that are compatible with it. We write

$$\mathbb{K}_{\preceq}((\mathbf{x})) := \bigcup_{e\in\mathbb{Q}^n}\bigcup_{C\in\mathcal{C}} x^e\mathbb{K}_C[[\mathbf{x}]]$$

for the set of series whose support is contained in a shift of a cone compatible with $\preceq$. It is not only a ring but even a field [8, Theorem 15].

A series $\phi$ is said to be algebraic if there is a non-zero polynomial $p\in\mathbb{K}[\mathbf{x},y]$ such that

$$p(\mathbf{x},\phi)=0.$$

It is said to be D-finite if for every $i\in\{1,\dots,n\}$ there are $q_0,\dots,q_r\in\mathbb{K}[\mathbf{x}]$ such that

$$q_0\phi+q_1\frac{\partial}{\partial x_i}\phi+\cdots+q_r\frac{\partial^r}{\partial x_i^r}\phi=0.$$

Every algebraic series is D-finite [6, Theorem 6.1], and as do algebraic series, D-finite series satisfy many closure properties. For instance, the sum $\phi_1+\phi_2$ of two D-finite series $\phi_1$ and $\phi_2$ is D-finite [6, Theorem 7.2], and so is the restriction

$$[\phi]_C(\mathbf{x}) := \sum_{I\in C\cap\mathbb{Z}^n}\left([\mathbf{x}^I]\phi\right)\mathbf{x}^I.$$

of a D-finite series $\phi$ to a finitely generated rational convex cone $C$ [3]. These closure properties are effective in the sense that systems of differential equations for $\phi_1+\phi_2$ and $[\phi]_C$ can be computed from the differential equations satisfied by $\phi_1$, $\phi_2$ and $\phi$.

Having a univariate D-finite series $\phi(t) := \sum_{k\ge k_0}\phi_k t^k\in\mathbb{K}((t))$ and a differential equation satisfied by it, it is easy to check whether $\phi$ is identically zero. The differential equation for $\phi$ translates into a recurrence relation for its coefficients,

$$p_0(k)\phi_k+p_1(k)\phi_{k+1}+\cdots+p_r(k)\phi_{k+r}=0,\quad p_0,\dots,p_r\in\mathbb{K}[k],$$

so that $\phi=0$ if and only if $\phi_k=0$ for finitely many $k$, the actual number depending on the largest integer root of $p_r$.

Given $p\in\mathbb{K}[\mathbf{x},y]$ and a total order $\preceq$ on $\mathbb{Q}^n$ the Newton-Puiseux algorithm allows to determine the series solutions of $p(\mathbf{x},y)=0$ in $\mathbb{K}_{\preceq}((\mathbf{x}))$. We collect a few definitions before we present it. The Newton polytope of $p$ is the convex hull of the support of $p$,

$$\mathrm{NP}(p) := \mathrm{conv}(\mathrm{supp}(p)).$$

If $e$ is an edge of $\mathrm{NP}(p)$ that connects two vertices $v_1$ and $v_2$, we simply write $e=\{v_1,v_2\}$. It is called admissible, if $v_{1,n+1}\neq v_{2,n+1}$. If $v_{1,n+1}<v_{2,n+1}$, we call $v_1$ and $v_2$ the minor and major vertex of $e$, respectively, and denote them by $\mathrm{m}(e)$ and $\mathrm{M}(e)$. Let $P_e$ be the projection on $\mathbb{R}^{n+1}$ that projects on $\mathbb{R}^n\times\{0\}$ along lines parallel to $e$. The barrier cone of $e$, denoted by $C(e)$, is the smallest line-free convex cone that contains $P_e(\mathrm{supp}(p))-P_e(e)$. It is identified with its projection on the first $n$ coordinates in $\mathbb{R}^n$. Its dual cone is

$$C(e)^* := \{w\in\mathbb{R}^n : \langle w,v\rangle\le 0 \text{ for all } v\in C(e)\}.$$

A vector $w\in\mathbb{R}^n$ defining a total order on $\mathbb{Q}^n$ is said to be compatible with $e$ when $w\in C(e)^*$. Given an admissible edge $e=\{v_1,v_2\}$, we denote its slope with respect to its last coordinate by

$$\mathrm{S}(e) := \frac{1}{v_{2,n+1}-v_{1,n+1}}(v_{2,1}-v_{1,1},\dots,v_{2,n}-v_{1,n}).$$

We now present the Newton-Puiseux algorithm. For details, in particular for a proof of its correctness, we refer to [7, Theorem 3.5].

**Algorithm 1** (Newton-Puiseux Algorithm)**.** *Input: A square-free and non-constant polynomial* $p\in\mathbb{K}[\mathbf{x},y]$*, an admissible edge* $e$ *of its Newton polytope, an element* $w$ *of the dual of its barrier cone* $C(e)$ *defining a total order on* $\mathbb{Q}^n$*, and an integer* $k$*.*
*Output: A list of* $\mathrm{M}(e)_{n+1}-\mathrm{m}(e)_{n+1}$ *many pairs* $(c_1\mathbf{x}^{\alpha_1}+\cdots+c_N\mathbf{x}^{\alpha_N},C)$ *with* $c_1\mathbf{x}^{\alpha_1},\dots,c_N\mathbf{x}^{\alpha_N}$ *being the first* $N$ *terms of a series solution* $\phi$ *of* $p(\mathbf{x},\phi)=0$*, ordered with respect to* $w$*, and* $C$ *being a line-free cone such that* $\mathrm{supp}(\phi)\subseteq\{\alpha_1,\dots,\alpha_{N-1}\}\cup(\alpha_N+C)$*, where* $N\ge k$ *is minimal such that the series solutions can be distinguished by their first* $N$ *terms.*

1 *Compute the roots* $c$ *of* $p_e(t)=\sum_I a_I t^{I_{n+1}-\mathrm{m}(e)_{n+1}}$*, where* $a_I=[(\mathbf{x},y)^I]p$ *and the sum runs over all* $I$ *in* $e\cap\mathrm{supp}(p)$*, set* $L$ *equal to the list of pairs* $(\phi,e)$ *with* $\phi=cx^{-\mathrm{S}(e)}$ *and* $N$ *equal to* $1$*.*

*While* $|L| \neq \mathrm{M}(e)_{n+1} - \mathrm{m}(e)_{n+1}$ *or* $N < k$*, do:*
  *Set* $\tilde{L} = \{\}$ *and* $N = N + 1$.
  *For each* $(\phi, e) \in L$ *with* $\phi$ *not having* $k$ *terms or* $p_e(t)$ *not having only simple roots, do:*
    *If* $\phi$ *satisfies* $p(\mathbf{x}, \phi) = 0$*, append* $(\phi, e)$ *to* $\tilde{L}$*, otherwise compute the Newton polytope of* $p(\mathbf{x}, \phi + y)$ *and determine its unique edge path* $e_1, \ldots, e_k$ *such that* $\mathrm{m}(e_1)_{n+1}$ *equals zero, and* $\mathrm{M}(e_k)$*, but not* $\mathrm{m}(e_k)$*, lies on the line through* $e$*, and* $w \in \bigcap C^*(e_i)$.
    *For each edge* $e$ *of the edge path, do:*
      *Compute the roots* $c$ *of* $p_e(t) = \sum_I a_I t^{I_{n+1} - \mathrm{m}(e)_{n+1}}$*, where* $a_I = [(\mathbf{x}, y)^I] p(\mathbf{x}, \phi + y)$ *and the sum runs over all elements* $I$ *in* $e \cap \mathrm{supp}(p(\mathbf{x}, \phi + y))$*, and append to* $\tilde{L}$ *all pairs* $(\phi + c\mathbf{x}^{-\mathrm{S}(e)}, e)$.
  *Set* $L = \tilde{L}$.
*Replace each pair* $(\phi + c\mathbf{x}^{-\mathrm{S}(e)}, e)$ *of* $L$ *by* $(\phi + c\mathbf{x}^{-S(e)}, C)$*, where* $C$ *is the barrier cone of* $e$ *with respect to* $p(\mathbf{x}, \phi + y)$*, and return* $L$.

## 3. Finite encodings of algebraic series

The Newton-Puiseux algorithm allows to determine the series solutions of a polynomial equation term by term. The next proposition implies that it can also be used to represent a series by a finite amount of of data: its minimal polynomial, a total order, and the first few terms of it with respect to this order.

**Proposition 1.** *Let* $p \in \mathbb{K}[\mathbf{x}, y]$ *be square-free and non-constant,* $e$ *an admissible edge of its Newton polytope,* $w \in C^*(e)$ *defining a total order on* $\mathbb{Q}^n$ *and* $a_1\mathbf{x}^{\alpha_1}, \ldots, a_N\mathbf{x}^{\alpha_N}$ *the first few terms of a series solution* $\phi$ *as output by Algorithm 1 when applied to* $p, e, w$ *and* $k = 0$*. Then* $\phi$ *is the only series solution whose first terms with respect to the total order defined by* $w$ *are* $a_1\mathbf{x}^{\alpha_1}, \ldots, a_N\mathbf{x}^{\alpha_N}$.

*Proof.* By design of the algorithm, the first $N$ terms of any other series solution constructed from $e$ differ from $a_1\mathbf{x}^{\alpha_1}, \ldots, a_N\mathbf{x}^{\alpha_N}$. If $\tilde{e}$ is another edge such that $w \in C^*(\tilde{e})$, then the leading exponent of any series solution resulting from it is $-\mathrm{S}(\tilde{e})$ and different from the leading exponent $-\mathrm{S}(e)$ of $\phi$. Since all series solutions which have a leading exponent with respect to $w$ are constructed from such edges, this finishes the proof. □

We illustrate the Newton-Puiseux algorithm and Proposition 1 with a first example.

**Example 1.** We determine the first terms of a series solution of the equation

$$p(x, y, z) := 4x^2y + (x^2y + xy^2 + xy + y)^2 - z^2 = 0$$

The Newton polytope of $p$ has four admissible edges, one of which is the edge $e = \{(0, 2, 0), (0, 0, 2)\}$. Its barrier cone is $C(e) = \langle (1, 1), (2, -1) \rangle$, and $w := (-\sqrt{2}, -1)$ is an element of its dual $C^*(e)$. Its components are linearly independent over $\mathbb{Q}$, therefore it defines a total order $\preceq$ on $\mathbb{Q}^2$. By [7, Theorem 3.5], and because the projection of $e$ on its last coordinate has length 2, there are two series solutions $\phi_1$ and $\phi_2$ of $p(x, y, z) = 0$ in $\mathbb{C}_{\preceq}((x, y))$. We determine their first terms using Algorithm 1. The slope of $e$ is $\mathrm{S}(e) = (0, -1)$, so the solutions have a term of the form $cy$, for some $c \in \mathbb{C}$. The coefficients $c$ are the solutions to $-1 + t^2 = 0$. Hence, $y$ is the first term of one series solution, say $\phi_1$, and $-y$ the first term of $\phi_2$. Furthermore, their support is contained in $(0, 1) + \langle (1, 1), (2, -1) \rangle$. To compute the next term of $\phi_1$, for instance, we consider the polynomial $p(x, y, y + z)$. The edge path on its Newton polytope mentioned in Algorithm 1 consists of the single edge $e = \{(1, 2, 0), (0, 1, 1)\}$. Its slope with respect to the last coordinate is $(-1, -1)$, so its next term is of the form $cxy$, where $c$ is the root of $-2 + 2t$. By Proposition 1 the series $\phi_1$ and $\phi_2$ can be encoded by $(p, (-\sqrt{2}, -1), y)$ and $(p, (-\sqrt{2}, -1), -y)$, respectively.

## 4. An effective equality test for algebraic series

The encoding of an algebraic series by its minimal polynomial, a total order, and its first terms is not unique, and so it is natural to ask if it is possible to decide whether two encodings represent the same series. We clarify this now. Assume that $\phi_1$ and $\phi_2$ are two series solutions of $p(\mathbf{x}, y) = 0$ encoded by $(p, w_1, p_1)$ and $(p, w_2, p_2)$ where $w_1$ and $w_2$ are elements of $\mathbb{R}^n$ inducing total orders $\preceq_1$ and $\preceq_2$ on $\mathbb{Q}^n$ and $p_1$ and $p_2$ are Puiseux polynomials in $\mathbf{x}$ representing the sum of the first terms of $\phi_1$ and $\phi_2$ with respect to $\preceq_1$ and $\preceq_2$, respectively. Using the Newton-Puiseux algorithm we can assume that the trailing term of $p_2$ with respect to $\preceq_2$ is smaller than the trailing term of $p_1$ with respect to $\preceq_2$. If the sequence of terms of $p_1$ does not agree with the initial sequence of terms of $p_2$ when ordered with respect to $\preceq_1$, then this proves that $\phi_1$ and $\phi_2$ are not the same. The next example demonstrates that we can also prove equality of series by comparing finitely many of their initial terms and determining an estimate of their supports.

**Example 2.** The Newton polytope of
$$p(x,y,z) := x + y - (1 + x + y)z$$
has four admissible edges from each of which we can compute the first terms of a series solution of $p(x,y,z) = 0$. These series can be encoded by
$$(p, (-1 + 1/\sqrt{2}, 1), 1) \quad \text{and} \quad (p, (-1 + 1/\sqrt{2}, -1), 1), \quad \text{and}$$
$$(p, (-1 + 1/\sqrt{2}, -2), x) \quad \text{and} \quad (p, (-2 + 1/\sqrt{2}, -1), y).$$
Since there are only three series which qualify as multiplicative inverses of $1 + x + y$, two of the above encodings have to represent the same series. We claim that the series $\phi_1$ represented by $(p, (-1 + 1/\sqrt{2}, -2), x)$ and $\phi_2$ represented by $(p, (-2 + 1/\sqrt{2}, -1), y)$ are equal. The order of $y$ with respect to $(-1 + 1/\sqrt{2}, -2)$ is $-2$, and the terms of $\phi_1$, whose order with respect to $(-1 + 1/\sqrt{2}, -2)$ is at least $-2$, are
$$x, -x^2, x^3 - x^4, x^5 - x^6, y.$$
Ordering these terms with respect to $(-2 + 1/\sqrt{2}, -1)$ results in the sequence
$$y, x, -x^2, x^3, -x^4, x^5, -x^6,$$
whose first term equals the first term of $\phi_2$. Algorithm 1 shows that the support of $\phi_1$ is contained in a shift of $\langle(0, 1), (7, -1)\rangle$ and that the support of $\phi_2$ is contained in a shift of $\langle(0, 1), (1, -1)\rangle$. These cones, $\langle(0, 1), (7, -1)\rangle$ and $\langle(0, 1), (1, -1)\rangle$, are compatible in the sense that their sum is a line-free cone. Consequently, there is a total order $\preceq$ on $\mathbb{Q}^2$ that is compatible with both of them, hence $\mathbb{C}_{\preceq}((x, y))$ contains $\phi_1$ as well as $\phi_2$. Since, by construction, $\phi_1$ and $\phi_2$ are roots of $p$, and since $p$ has degree 1 with respect to $z$ and $\mathbb{C}_{\preceq}((x, y))$ is a field and therefore can only contain at most one solution of $p(x, y, z) = 0$, the series $\phi_1$ and $\phi_2$ have to be the same.

To see that the argument just given applies in more generality, we recall that $\mathbb{K}_{\preceq_1}((\mathbf{x}))$ contains a complete set of series solutions of $p(\mathbf{x}, y) = 0$. Assume that $\phi_1$ is constructed from an edge $e$ of the Newton polytope of $p$ for which $w_1 \in C^*(e)$. We claim that $e$ can be extended to a unique maximal edge path $\{e_1, \ldots, e_k\}$ for which $w \in \bigcap C^*(e_i)$ so that the series resulting from these edges form a complete set of series solutions of $p(\mathbf{x}, y) = 0$ in $\mathbb{K}_{\preceq_1}((\mathbf{x}))$. Starting with the edge $e$, let $e'$ and $e''$ be those edges of the Newton polytope of $p(\mathbf{x}, y)$ with $\mathrm{M}(e') = \mathrm{m}(e)$ and $\mathrm{m}(e'') = \mathrm{M}(e)$ for which each of $\langle w_1, \mathrm{m}(e')\rangle$ and $\langle w_1, \mathrm{M}(e'')\rangle$ is maximal. Extending $e'$ and $e''$ analogously as well as the subsequent edges until none of them can be extended results in a path with the claimed property. The uniqueness of this path is immediate: if it were not unique then $\mathbb{K}_{\preceq_1}((\mathbf{x}))$ would contain more than $\deg_y(p)$ many series solutions, contradicting $\mathbb{K}_{\preceq_1}((\mathbf{x}))$ being a field. Assume that the sequences of initial terms of $\phi_1$ and $\phi_2$ are the same when ordered with respect to $\preceq_1$, and let $C_2$ be the cone output by Algorithm 1 such that $\mathrm{supp}(\phi_2) \subseteq \{\beta_1, \ldots, \beta_{M-1}\} \cup (\beta_M + C_2)$ where $\beta_M$ is the trailing exponent of $p_1$ with respect to $w_2$. If $w_1 \in C_2^*$ then $\phi_2$ is an element of $\mathbb{K}_{\preceq_1}((\mathbf{x}))$ and it follows that $\phi_2 = \phi_1$ as in Example 2. If we assume that $C_2$ is the minimal cone for which $\mathrm{supp}(\phi_2) \subseteq \{\beta_1, \ldots, \beta_{M-1}\} \cup (\beta_M + C_2)$ then we can also conclude that $\phi_2 \neq \phi_1$ if $w_1 \notin C_2^*$. The latter holds because $C_2$ being minimal and $w_1$ not being an element of $C_2^*$ implies that there is an element of $\mathrm{supp}(\phi_2)$ in $\beta_M + C_2$ that is larger than $\beta_M$ with respect to $w_1$. But if $\phi_1 = \phi_2$, the set of elements of $\mathrm{supp}(\phi_2)$ not smaller than $\beta_M$ is $\mathrm{supp}(p_1)$, and by construction $\mathrm{supp}(p_1)$ is a subset of $\{\beta_1, \ldots, \beta_M\}$.

The above reasoning relied on the cones output by Algorithm 1 not being too big. Although they are not always minimal, we believe that they are always in certain situations. We will have more to say about this in Section 5, but for the moment we just state the following conjecture.

**Conjecture 1.** *If the polynomial $p \in \mathbb{K}[\mathbf{x}][y]$ is primitive and the integer $k \in \mathbb{N}$ input to Algorithm 1 is large enough, then the cones $C$ output by it are minimal.*

**Theorem 1.** *The equality of two multivariate algebraic series can be decided effectively.*

If Conjecture 1 were correct, then Theorem 1 were just a corollary of it. In the following, however, we give a proof of Theorem 1 that is independent of Conjecture 1, based on properties of D-finite functions.

*Proof.* We complete the equality test for the case that $w_1 \notin C_2^*$ and $C_2$ is not minimal. Algorithm 1 provides a cone $C_1$ such that $w_1 \in C_1^*$ and $\mathrm{supp}(\phi_1) \subseteq \{\alpha_1, \ldots, \alpha_{N-1}\} \cup (\alpha_N + C_1)$, where $\alpha_1, \ldots, \alpha_N$ are the exponents of the first few terms of $\phi_1$ with respect to $w_1$. W.l.o.g. we assume that $\alpha_N = \beta_M$. If $\mathrm{supp}(\phi_2) \setminus \{\beta_1, \ldots, \beta_{M-1}\} \subseteq \beta_M + C_1$, then $\phi_2 \in \mathbb{C}_{\preceq_1}((x))$, and therefore $\phi_2 = \phi_1$. If $\mathrm{supp}(\phi_2) \setminus \{\beta_1, \ldots, \beta_{M-1}\} \nsubseteq \beta_M + C_1$, then $\phi_2 \neq \phi_1$. It therefore remains to explain how to decide whether the support of an algebraic series $\phi$ is contained in a cone $C$. We do so by using basic properties of D-finite series. To simplify the argument we assume that $\operatorname{supp}(\phi) \subseteq \mathbb{Z}^n$. Let $\omega = (\omega_1, \dots, \omega_n) \in \mathbb{Z}^n$ be such that for each $i \in \mathbb{Z}$ there are only finitely many $\alpha \in \operatorname{supp}(\phi)$ for which $\langle \alpha, \omega \rangle = i$ and none if $i < 0$, and consider the series

$$\tilde{\phi}(\mathbf{x}, t) := \phi(x_1 t^{\omega_1}, \dots, x_n t^{\omega_n})$$

Since $\phi$ is algebraic, so is $\tilde{\phi}$, and because every algebraic series is D-finite, so is $\tilde{\phi}$. Let

$$[\phi]_C(\mathbf{x}) := \sum_{I \in C \cap \mathbb{Z}^n} \left( [\mathbf{x}^I]\phi \right) \mathbf{x}^I$$

be the restriction of $\phi$ to $C$ and let $[\tilde{\phi}]_C(\mathbf{x}, t) := [\phi]_C(x_1 t^{\omega_1}, \dots, x_n t^{\omega_n})$ be the restriction of $\tilde{\phi}$ to $C \times \mathbb{R}_{\geq 0}$, by abuse of notation. By closure properties of D-finite functions $[\tilde{\phi}]_C$ is D-finite, and so is the difference $\tilde{\phi} - [\tilde{\phi}]_C$. In particular, when viewed as a series in $t$, the coefficients of $\tilde{\phi} - [\tilde{\phi}]_C$ satisfiy a linear recurrence relation of the form

$$q_0(k)c_k + q_1(k)c_{k+1} + \cdots + q_r(k)c_{k+r} = 0$$

with $q_0, \dots, q_r \in \mathbb{K}[\mathbf{x}][k]$. Verifying whether $\operatorname{supp}(\phi) \subseteq C$ amounts to checking if $\tilde{\phi} - [\tilde{\phi}]_C = 0$ which can be done by comparing finitely many initial terms of $\tilde{\phi} - [\tilde{\phi}]_C$ to zero. □

**Remark 1.** *The equality test for algebraic series is effective because the closure properties of D-finite functions it is based on can be performed effectively. However, as was explained in* [3]*, it can be computationally quite expensive to do so.*

## 5. The support of an algebraic series

In Example 2 we saw that the number of admissible edges of the Newton polytope of a polynomial equation is not necessarily bounded by the number of series solutions the equation has. It can happen that different edges give rise to the same series solution. For the purpose of encoding a series any edge is as good as any other edge as long as it gives rise to the same series. However, when interested in information about the support of a series solution, it is advisable to inspect all the edges which give rise to this series solution. We explain how the effective equality test for algebraic series allows to derive information about the convex hull of the support of an algebraic series.

**Example 3.** In Example 2 we saw that the Newton polytope of

$$p(x, y, z) = x + y - (1 + x + y)z$$

has two admissible edges,

$$e_1 = \{(0,0,1), (1,0,0)\} \quad \text{and} \quad e_2 = \{(0,0,1), (0,1,0)\},$$

that give rise to two encodings,

$$(p, (-1 + 1/\sqrt{2}, -2), x) \quad \text{and} \quad (p, (-2 + 1/\sqrt{2}, -1), y),$$

of one and the same series solution $\phi$ of $p(x, y, z) = 0$. The barrier cones of these edges are

$$C(e_1) = \langle (1,0), (-1,1) \rangle \quad \text{and} \quad C(e_2) = \langle (0,1), (1,-1) \rangle,$$

so that, by Algorithm 1, we have

$$\operatorname{supp}(\phi) \subseteq (1,0) + C(e_1) \quad \text{as well as} \quad \operatorname{supp}(\phi) \subseteq (0,1) + C(e_2).$$

The next proposition shows that the convex hull of the support of an algebraic series has only finitely many vertices and indicates how they can be found.

**Proposition 2.** *For any series root $\phi$ of a non-zero square-free polynomial $p \in \mathbb{K}[\mathbf{x}, y]$, there is a surjection from the set of edges of the Newton polytope of $p$ which give rise to the series solution $\phi$ to the set of vertices of the convex hull of its support. In particular, the convex hull of the support of an algebraic series is a polyhedral set.*

*Proof.* We claim that the function that maps an edge $e$ to its slope $-\mathrm{S}(e)$ has the required properties. If $e$ is an edge that gives rise to the series solution $\phi$, then $-\mathrm{S}(e)$ is necessarily a vertex of its support as it is the maximal element with respect to a total order induced by some irrational vector, so the function is well-defined. The function is also surjective because for any vertex $\alpha$ of the convex hull of $\operatorname{supp}(\phi)$ there is some $w \in \mathbb{R}^n$ that induces a total order $\preceq$ on $\mathbb{Q}^n$ with respect to which $\alpha$ is the maximal element of $\operatorname{supp}(\phi)$. In particular, $\phi$ is an element of $\mathbb{K}_{\preceq}((\mathbf{x}))$. Since all series solutions of $p(\mathbf{x}, y) = 0$ in $\mathbb{K}_{\preceq}((\mathbf{x}))$ can be constructed by the Newton-Puiseux algorithm, there is also an edge $e$ from which $\phi$ can be constructed. □

The vertices of the convex hull of the support of $\phi$ can consequently be determined by identifying the admissible edges $e$ of the Newton polytope of $p(\mathbf{x}, y)$ that give rise to it. To determine also the bounded faces of the convex hull of the support of $\phi$ it is helpful to be able to compute, for each of its vertices $v$, a line-free cone $C$ such that $\operatorname{supp}(\phi) \subseteq v + C$. For each such edge $e$ that gives rise to $\phi$ and for any $w \in C^*(e)$ that defines a total order, Algorithm 1 provides its first $N$ terms $a_1\mathbf{x}^{\alpha_1}, \dots, a_N\mathbf{x}^{\alpha_N}$ with respect to $w$ and a line-free cone $C$ compatible with $w$ such that $\operatorname{supp}(\phi) \subseteq \{\alpha_1, \dots, \alpha_{N-1}\} \cup (\alpha_N + C)$. The cone $\tilde{C}$ generated by $C$ and $\{\alpha_2 - \alpha_1, \dots, \alpha_N - \alpha_1\}$ has the property that $\operatorname{supp}(\phi) \subseteq \alpha_1 + \tilde{C}$. It is also line-free since it is compatible with $w$.

**Example 4.** We continue Example 3. Proposition 2 implies that the vertices of the convex hull of the support of $\phi$ are $v_1 = (1, 0)$ and $v_2 = (0, 1)$. Apart from the vertices $v_1$ and $v_2$ itself, the only possible bounded face is the convex hull of $v_1$ and $v_2$. Since

$$\operatorname{supp}(\phi) \subseteq v_1 + \langle (1, 0), (-1, 1) \rangle \quad \text{and} \quad \operatorname{supp}(\phi) \subseteq v_2 + \langle (0, 1), (1, -1) \rangle,$$

the line through $v_1$ and $v_2$ supports $\operatorname{conv}(\operatorname{supp}(\phi))$, so $\operatorname{conv}(\{v_1, v_2\})$ is a face of it.

In general, the candidates for the bounded faces of the convex hull of $\operatorname{supp}(\phi)$ are the convex hulls of subsets of its set of vertices. Whether the convex hull of a subset $V$ of vertices is indeed a face of $\operatorname{conv}(\operatorname{supp}(\phi))$ can be decided by computing for each $v \in V$ a line-free cone $C_v$ such that $\operatorname{supp}(\phi) \subseteq v + C_v$ and checking if there is a hyperplane $H$ containing $\operatorname{conv}(V)$ whose complement is the union of two half-spaces one of which has bounded intersection with $\bigcap_{v \in V}(v + C_v)$ and does not contain any vertices of $\operatorname{conv}(\operatorname{supp}(\phi))$. Clearly, there is such a hyperplane if and only if $\operatorname{conv}(V)$ is a face.

We already pointed out that for deciding whether two algebraic series are equal or not, it is convenient that the cones output by Algorithm 1 are not too big. The minimality of the cones is also important for getting a good estimate of the convex hull of the support of such a series. The following example, however, shows that they do not need to be minimal.

**Example 5.** One of the two series solutions of

$$p(x, y, z) = (1 - x)((1 - y)z - 1) = 0$$

is the geometric series

$$\phi = 1 + y + y^2 + \dots$$

Though the convex hull of its support is the cone generated by $(1, 0)$, Algorithm 1 only shows that

$$\operatorname{supp}(\phi) \subseteq \langle (1, 0), (0, 1) \rangle.$$

The difference between the two cones is caused by the polynomial $p \in \mathbb{C}[x, y][z]$ not being primitive: first getting rid of its content, and then applying Algorithm 1 results in a cone that is minimal.

The non-primitivity is not the only possible reason for a cone output by Algorithm 1 not being minimal.

**Example 6.** One of the solutions of

$$p(x, y, z) = 1 + x + y + (1 + xy + 2y)z + yz^2 = 0$$

is

$$\frac{-1 - 2y - xy + \sqrt{1 - 2xy + 4xy^2 + x^2y^2}}{2y}.$$

It has a series expansion $\phi$ whose first terms with respect to $w = (-1 + 1/\sqrt{2}, -1)$ are

$$-1 - x + xy + x^2y^2 + \dots.$$

The closed form of $\phi$ together with Newton's generalized binomial theorem implies that the minimal cone containing $\operatorname{supp}(\phi)$ is $\langle (1, 0), (1, 2) \rangle$ though Algorithm 1 only shows that it is contained in $\langle (1, 0), (0, 1) \rangle$. However, the algorithm also shows that

$$\operatorname{supp}(\phi) \subseteq \{(0, 0)\} \cup ((1, 0) + \langle (1, 1), (0, 1) \rangle),$$

where now $\langle (1, 1), (0, 1) \rangle$ is the minimal cone having this property, and computing another term and another cone, we find that

$$\operatorname{supp}(\phi) \subseteq \{(0, 0), (1, 0)\} \cup ((1, 1) + \langle (1, 1), (1, 2) \rangle),$$

where the cone $\langle (1, 1), (1, 2) \rangle$ is not only minimal but also has the property that $(1, 1) + \mathbb{R}_{\geq 0} \cdot (1, 1)$ and $(1, 1) + \mathbb{R}_{\geq 0} \cdot (1, 2)$ contain infinitely many elements of $\operatorname{supp}(\phi)$.

For the series in the previous examples we could easily decide whether the corresponding cones given by the Newton-Puiseux algorithm were minimal or not, because the series were algebraic of degree 1 and 2, respectively, and therefore very explicit. It remains to clarify how this can be decided in general. We have just seen that for the first terms $a_1\mathbf{x}^{\alpha_1}, \dots, a_N\mathbf{x}^{\alpha_N}$ of an algebraic series $\phi$, the line-free cone $C$ output by Algorithm 1 for which $\operatorname{supp}(\phi) \subseteq \{\alpha_1, \dots, \alpha_{N-1}\} \cup (\alpha_N + C)$ is not necessarily minimal. However, if $C$ is minimal and if for each edge of $\alpha_N + C$ the exponent $\alpha_N$ is not the only element of $\operatorname{supp}(\phi)$ it contains, then its minimality can be verified simply by determining some of these elements.

**Example 7.** We continue Example 6. The cone $\langle(1,1),(1,2)\rangle$ for which $\operatorname{supp}(\phi) \subseteq \{(0,0),(1,0)\} \cup ((1,1)+\langle(1,1),(1,2)\rangle)$ is minimal because the first terms of $\phi$ with respect to $w = (-1+1/\sqrt{2}, -1)$ are

$$-1 - x + xy + x^2y^2 - x^2y^3 + \dots$$

and $(2,2) \in (1,1) + \mathbb{R}_{>0} \cdot (1,1)$ and $(2,3) \in (1,1) + \mathbb{R}_{>0} \cdot (1,2)$.

When the cone $C$ output by Algorithm 1 is minimal but $\alpha_N$ is the only element of $\operatorname{supp}(\phi)$ that lies on an edge of $\alpha_N + C$ we are not able to show its minimality. Proving its minimality relates to the following open problem which closes this section.

**Problem 1.** *Given an algebraic series in terms of its minimal polynomial, a total order and its first terms with respect to this order, determine the unbounded faces of the convex hull of its support.*

## 6. Effective arithmetic for algebraic series

The effectivity of the equality test for algebraic series implies the effectivity of their arithmetic: given encodings of two series $\phi_1$ and $\phi_2$, we discuss how to decide whether their sum $\phi_1 + \phi_2$ and product $\phi_1\phi_2$ are well-defined / algebraic, and in case they are, how to determine finite encodings for them.

We first explain that in general the sum and product of two algebraic series do not need to be well-defined / algebraic.

**Example 8.** The polynomial

$$p(x,y) := (1-x)y - 1$$

has two series roots

$$\phi_1 = 1 + x + x^2 + \dots \quad \text{and} \quad \phi_2 = -x^{-1} - x^{-2} - x^{-3} - \dots$$

both of which are algebraic, by construction, but neither is their sum nor their product. Their product is not well-defined, since its coefficients involve infinite sums, and their sum is not algebraic, because its powers are not well-defined.

The sum and product of two algebraic series $\phi_1$ and $\phi_2$ are algebraic if and only if there is a vector $w \in \mathbb{R}^n$ that induces a total order $\preceq$ on $\mathbb{Q}^n$ with respect to which both $\operatorname{supp}(\phi_1)$ and $\operatorname{supp}(\phi_2)$ have a minimal element. To decide whether such a vector exists, and in case it does, to find it, one would need to determine $\operatorname{conv}(\operatorname{supp}(\phi_1))$ and $\operatorname{conv}(\operatorname{supp}(\phi_2))$. However, we are only able to determine an estimate of the convex hull of the support of an algebraic series, see Section 5, in particular Problem 1. Consequently, we can find such a vector only sometimes but not always when it exists, and of course, in case we cannot find one, it does not mean there is none. Yet, to prove that the sum (and product) of $\phi_1$ and $\phi_2$ is not algebraic, one could instead compute an annihilating polynomial $p(\mathbf{x}, y)$ for the potentially algebraic series $\phi_1 + \phi_2$ and observe that none of its series roots equals the sum of $\phi_1$ and $\phi_2$.

**Example 9.** Let

$$p_1(x,y) := \big(1 + x + x^2 - y\big)\big(x^2 - (1-x)y\big) \quad \text{and} \quad p_2(x,y) := y\left(x^2 - (x-1)y\right)$$

and consider the series

$$\phi_1 = x^2 + x^3 + x^4 + \dots \quad \text{and} \quad \phi_2 = x + 1 + x^{-1}\dots$$

encoded by

$$(p_1, -\sqrt{2}, x^2) \quad \text{and} \quad (p_2, \sqrt{2}, x).$$

We can show that $\phi_1 + \phi_2$ is not algebraic by computing the generator

$$p(x,y) = \big(1 + x + x^3 - y\big)\, y \left(-1 + x^2 + x^4 - (x-1)y\right)\left(x^3 - (1-x)y\right)$$

of the elimination ideal $\langle p_1(x,y_1), p_2(x,y_2), y_3 - (y_1+y_2)\rangle \cap \mathbb{K}(x)[y_3]$ and observing that none of its roots equals $\phi_1 + \phi_2$. For instance, the series $\phi$ represented by $(p, -\sqrt{2}, x^2)$ is different from $\phi_1 + \phi_2$ because $\operatorname{supp}(\phi_1) \subseteq 2 + \langle 1\rangle$ and $\operatorname{supp}(\phi_2) \subseteq 1 + \langle -1\rangle$ and so $1 \in \operatorname{supp}(\phi_1+\phi_2)$ but $1 \notin \operatorname{supp}(\phi)$

as $\mathrm{supp}(\phi) \subseteq 2 + \langle 1 \rangle$. The series $\phi$ encoded by $(p, \sqrt{2}, x^2 + 2x)$ does not equal $\phi_1 + \phi_2$ because the terms of $\phi_1$ and $\phi_2$ of order at least $-\sqrt{2} \cdot 2$ and $\sqrt{2} \cdot 1$ with respect to $-\sqrt{2}$ and $\sqrt{2}$ are $x^2$ and $x$, respectively, and their sum $x^2 + x$ differs from $x^2 + 2x$. Similar arguments apply for showing that the other series roots of $p$ do not equal $\phi_1 + \phi_2$, proving that the sum of $\phi_1$ and $\phi_2$ is not algebraic.

In the following we circumvent these difficulties by simply assuming that we know a $w \in \mathbb{R}^n$ that induces a total order $\preceq$ for which $\phi_1, \phi_2 \in \mathbb{K}_{\preceq}((\mathbf{x}))$ so that e.g. $\phi_1 + \phi_2$ is algebraic and we can be sure to find it among the series roots of $p(\mathbf{x}, y)$. It is natural to do so also because this is the case in applications, for instance in the context of enumerative combinatorics [5, Chapter 6, Section 12].

Assume that $\phi_1$ and $\phi_2$ are given by $(p_1(\mathbf{x}, y), w, q_1)$ and $(p_2(\mathbf{x}, y), w, q_2)$. We already noted that an annihilating polynomial $p(\mathbf{x}, y)$ for $\phi_1 + \phi_2$ can be derived from annihilating polynomials $p_1(\mathbf{x}, y)$ and $p_2(\mathbf{x}, y)$ of $\phi_1$ and $\phi_2$ by computing a generator of the elimination ideal of

$$\langle p_1(\mathbf{x}, y_1), p_2(\mathbf{x}, y_2), y_3 - (y_1 + y_2) \rangle$$

in $\mathbb{K}(\mathbf{x})[y_3]$. Whether a series root of $p(\mathbf{x}, y)$ represented by $(p(\mathbf{x}, y), w, p)$ equals $\phi_1 + \phi_2$ can be decided by computing the truncations $\tilde{q}_1$ and $\tilde{q}_2$ of $\phi_1$ and $\phi_2$ up to order $\mathrm{ord}(q, w)$, where

$$\mathrm{ord}(q, w) := \min\{\langle w, \alpha \rangle \mid \alpha \in \mathrm{supp}(q)\}.$$

If $\tilde{q}_1 + \tilde{q}_2$ does not equal $q$ when ordered with respect to $w$, the series represented by $(p(\mathbf{x}, y), w, q)$ does not equal $\phi_1 + \phi_2$. However, if it does, then $(p(\mathbf{x}, y), w, q)$ is a finite encoding of $\phi_1 + \phi_2$, and by assumption we can be sure that we find a finite encoding of it this way.

Similarly, an annihilating polynomial $p(\mathbf{x}, y)$ for $\phi_1 \phi_2$ can be determined by computing a generator of the elimination ideal of

$$\langle p_1(\mathbf{x}, y_1), p_2(\mathbf{x}, y_2), y_3 - y_1 y_2 \rangle$$

in $\mathbb{K}(\mathbf{x})[y_3]$. To find a representation $(p(\mathbf{x}, y), w, q)$ of $\phi_1 \phi_2$ compute the truncation $\tilde{q}_1$ of $\phi_1$ up to order $\mathrm{ord}(q, w) - \langle w, \mathrm{lexp}_w(\phi_2) \rangle$ and the truncation $\tilde{q}_2$ of $\phi_2$ up to order $\mathrm{ord}(q, w) - \langle w, \mathrm{lexp}_w(\phi_1) \rangle$. If $q$ does not equal the sum of the first terms of $\tilde{q}_1 \tilde{q}_2$ when ordered with respect to $w$, the series represented by $(p(\mathbf{x}, y), w, q)$ does not equal $\phi_1 \phi_2$. However, if it does, then $(p(\mathbf{x}, y), w, q)$ is a finite encoding of $\phi_1 \phi_2$, and by assumption we can be sure that we find a finite encoding of it this way.

Other closure properties for algebraic series such as taking multiplicative inverses or derivatives can be performed similarly. We just refer to [6, Theorem 6.3] for an explanation of how to the corresponding annihilating polynomials can be computed.

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