Separating Variables in Bivariate Polynomial Ideals

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joint work with
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Intersection of $\mathbb{K}$-algebras.

Let $u, v \in \mathbb{K}[t_1, \ldots, t_n]$. The intersection $\mathbb{K}[u] \cap \mathbb{K}[v]$ can be computed by determining pairs $(f, g) \in \mathbb{K}[x] \times \mathbb{K}[y]$ such that $f(u) = g(v)$, i.e. such that $f(x) - g(y) \in \langle x - u, y - v \rangle \cap \mathbb{K}[x, y]$.

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An elimination procedure for Laurent series as in Mireille Bousquet-Mélou's proof of the algebraicity of the generating function of Gessel's walks.

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Let $I \subseteq \mathbb{K}[x, y]$ be an ideal. Then

$$A(I) := \{(f, g) \in \mathbb{K}[x] \times \mathbb{K}[y] \mid f - g \in I\}$$

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Problem

Given generators of an ideal $I \subseteq \mathbb{K}[x, y]$, determine a set of generators for the algebra $A(I)$ of separated polynomials.

Examples

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$$A(\langle x(x^2 + xy + y^2)\rangle) = \mathbb{K}[(1, 1)].$$

What is $A(I)$ for the ideal I generated by

$$(x^2 - xy + y^2)(x^3 - 2xy^2 - 1) \quad \text{and} \quad (x^2 - xy + y^2)(y^3 - 2x^2y - 1)?$$

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A list of generators for $A(I)$ is x

$$(x^{12}-2x^6, y^{12}-2y^6),$$

$$(9x^{15}-26x^9+17x^3, 9y^{15}-26y^9+17y^3),$$

$$(81x^{18}-323x^6, 81y^{18}-323y^6),$$

$$(81x^{21}-539x^9+458x^3, 81y^{21}-539y^9+458y^3).$$

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2. Compute generators of $A(I_0)$ via Gröbner basis computations and linear algebra.
3. Compute generators of $A(I_1)$ by making a suitable ansatz and linear algebra.
4. Compute the intersection $A(I) = A(I_0) \cap A(I_1)$.

Zero-Dimensional Ideals

When I is zero-dimensional, there are

$$p, q \in \mathbb{K}[x, y] \setminus \{0\} \quad \text{such that}$$

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It is therefore sufficient to find all pairs $(f, g) \in A(I)$ with

$$\deg_x f < \deg_x p \quad \text{and} \quad \deg_y g < \deg_y q.$$

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3. Make an ansatz $h = \sum_{i=0}^{\deg_x p - 1} a_i x^i - \sum_{j=0}^{\deg_y q - 1} b_j y^j$ with undetermined coefficients a_i, b_j.

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6 Return $(f_1, g_1), \ldots, (f_d, g_d), (p, 0), \ldots, (x^{\deg_x p - 1} p, 0)$, $(0, q), \ldots, (0, y^{\deg_y q - 1} q)$.

Principal Ideals

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Let f, g, F, G be nonconstant polynomials. Then $f(x) - g(y)$ divides $F(x) - G(y)$ if and only if there is a polynomial r such that

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Theorem

If I is principal, then $A(I)$ is simple.

Definition

A function ω from the set of monomials in x and y to $\mathbb{R}$ is called a **weight function** if there are $\omega_x, \omega_y \in \mathbb{Z}_{>0}$ such that $\omega(x^i y^j) = \omega_x i + \omega_y j$ for all $i, j \in \mathbb{Z}_{\geq 0}$.

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If p is separable and P is its minimal separated multiple, then there is a unique weight function ω such that

(a) $\mathrm{lp}_\omega(p)$ involves at least two monomials, and

(b) the minimal separated multiple of $\mathrm{lp}_\omega(p)$ is $\mathrm{lp}_\omega(P)$.

An Example

Is the polynomial

$$p(x, y) = x^3 + x^2y + xy^2 + y^3 + x^2 + xy + y^2$$

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Its leading part $\mathrm{lp}(p)$ is $x^3 + x^2y + xy^2 + y^3$, and the minimal separated multiple of $\mathrm{lp}(p)$ is $x^4 - y^4$.

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Make an ansatz

$$P(x, y) = x^4 - y^4 + \sum_{i+j<4} P_{ij}x^iy^j$$

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The resulting linear system does not have a solution, and therefore, p is not separable.

The Homogeneous Case

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(b) all the roots of $p(x, 1)$ in $\overline{\mathbb{K}}$ are distinct and the ratio of every two of them is a root of unity.

Moreover, if p is separable and N is the minimal number such that the ratio of every pair of roots of $p(x, 1)$ is an N-th root of unity, then the weight of the minimal separated multiple is $N\omega_x$.

Reduction to the Homogeneous Case

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J. W. S. Cassels, *Factorization of polynomials in several variables*, Proceedings of the 15th Scandinavian Congress Oslo 1968, 1969

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The Galois group G of $\overline{\mathbb{K}(t)}/\mathbb{K}(t)$ acts on $\mathbb{Z}_m \times \mathbb{Z}_n$ by

$$\pi(i, j) = (i', j') \quad :\Longleftrightarrow \quad (\pi(\alpha_i), \pi(\beta_j)) = (\alpha_{i'}, \beta_{j'}).$$

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It restricts to a bijection between separated factors and (separated) invariant subsets $T \subseteq \mathbb{Z}_m \times \mathbb{Z}_n$ such that

$$\chi_T(i, \_) = \chi_T(i', \_) \quad \text{or} \quad \chi_T(i, \_) \cdot \chi_T(i', \_) = 0 \quad \text{for all } i, i' \in \mathbb{Z}_m.$$

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In particular, $\mathbb{Z}_m \times \mathbb{Z}_n$ is invariant and separated, and corresponds to the separated factor $f(x) - g(y)$.

An Example

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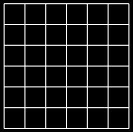
The factors of $x^6 - y^6$ in $\mathbb{Q}[x, y]$.



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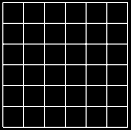

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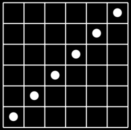
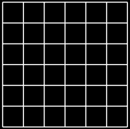


$x - y$



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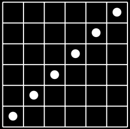
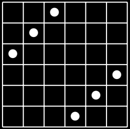
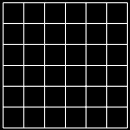


$x - y$



$x + y$



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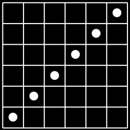

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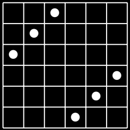

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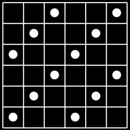
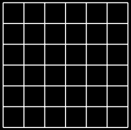


$x^2 - y^2$



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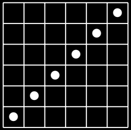

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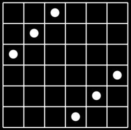

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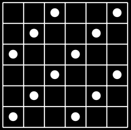

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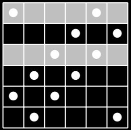
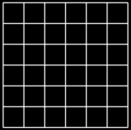


$x^2 - xy + y^2$



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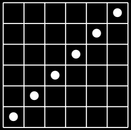

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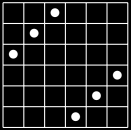

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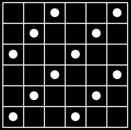
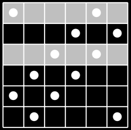
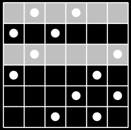


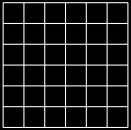
$x^2 - y^2$



$x^2 - xy + y^2$



$x^2 + xy + y^2$



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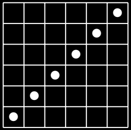

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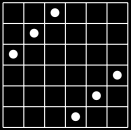
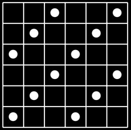
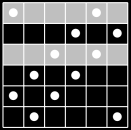


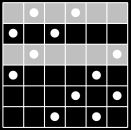
$x + y$



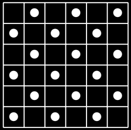
$x^2 - y^2$



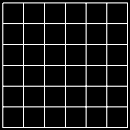
$x^2 - xy + y^2$



$x^2 + xy + y^2$



$x^3 - y^3$



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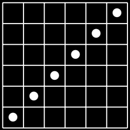

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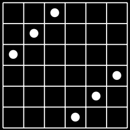

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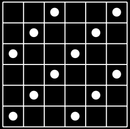

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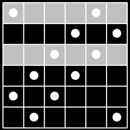

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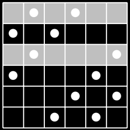
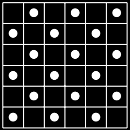
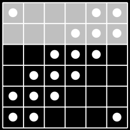


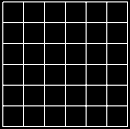
$x^2 + xy + y^2$



$x^3 - y^3$



$x^3 - 2x^2y + 2xy^2 - y^3$



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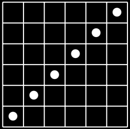
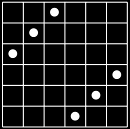
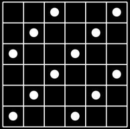


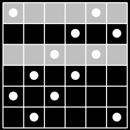
$x - y$



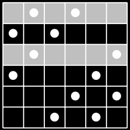
$x + y$



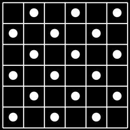
$x^2 - y^2$



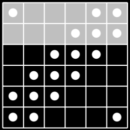
$x^2 - xy + y^2$



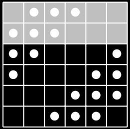
$x^2 + xy + y^2$



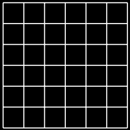
$x^3 - y^3$



$x^3 - 2x^2y + 2xy^2 - y^3$



$x^3 + 2x^2y + 2xy^2 + y^3$



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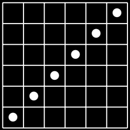
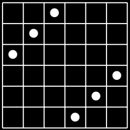
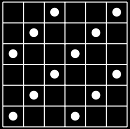


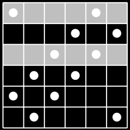
$x - y$



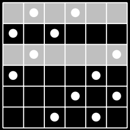
$x + y$



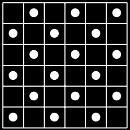
$x^2 - y^2$



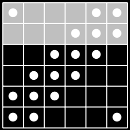
$x^2 - xy + y^2$



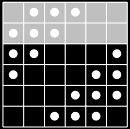
$x^2 + xy + y^2$



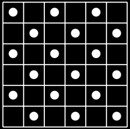
$x^3 - y^3$



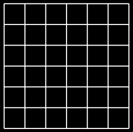
$x^3 - 2x^2y + 2xy^2 - y^3$



$x^3 + 2x^2y + 2xy^2 + y^3$



$x^3 + y^3$



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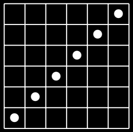
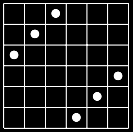
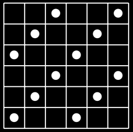


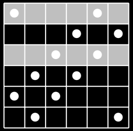
$x - y$



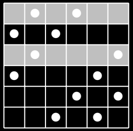
$x + y$



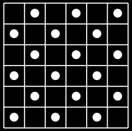
$x^2 - y^2$



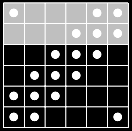
$x^2 - xy + y^2$



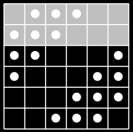
$x^2 + xy + y^2$



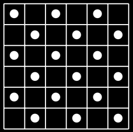
$x^3 - y^3$



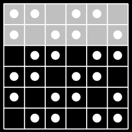
$x^3 - 2x^2y + 2xy^2 - y^3$



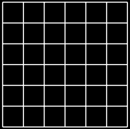
$x^3 + 2x^2y + 2xy^2 + y^3$



$x^3 + y^3$



$x^4 + x^2y^2 + y^4$



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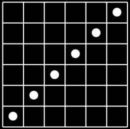
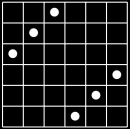
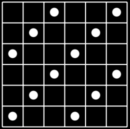


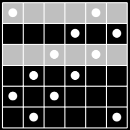
$x - y$



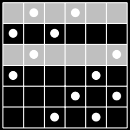
$x + y$



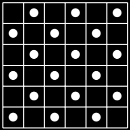
$x^2 - y^2$



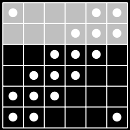
$x^2 - xy + y^2$



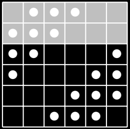
$x^2 + xy + y^2$



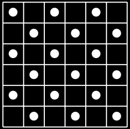
$x^3 - y^3$



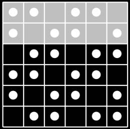
$x^3 - 2x^2y + 2xy^2 - y^3$



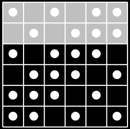
$x^3 + 2x^2y + 2xy^2 + y^3$



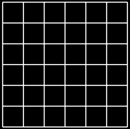
$x^3 + y^3$



$x^4 + x^2y^2 + y^4$



$x^4 - x^3y + xy^3 - y^4$



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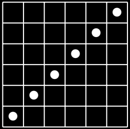
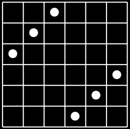
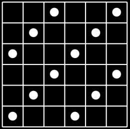


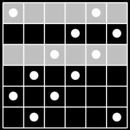
$x - y$



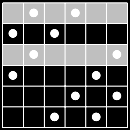
$x + y$



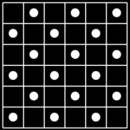
$x^2 - y^2$



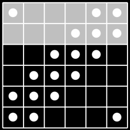
$x^2 - xy + y^2$



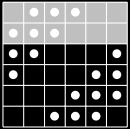
$x^2 + xy + y^2$



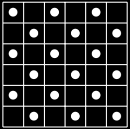
$x^3 - y^3$



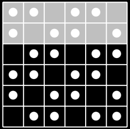
$x^3 - 2x^2y + 2xy^2 - y^3$



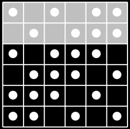
$x^3 + 2x^2y + 2xy^2 + y^3$



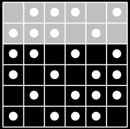
$x^3 + y^3$



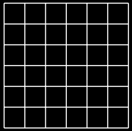
$x^4 + x^2y^2 + y^4$



$x^4 - x^3y + xy^3 - y^4$



$x^4 + x^3y - xy^3 - y^4$



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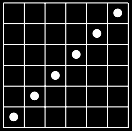
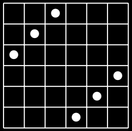
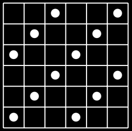


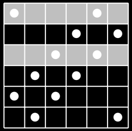
$x - y$



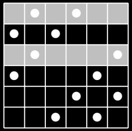
$x + y$



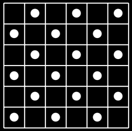
$x^2 - y^2$



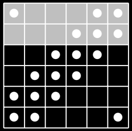
$x^2 - xy + y^2$



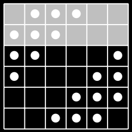
$x^2 + xy + y^2$



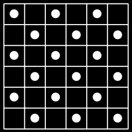
$x^3 - y^3$



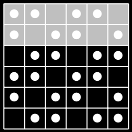
$x^3 - 2x^2y + 2xy^2 - y^3$



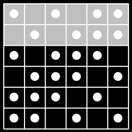
$x^3 + 2x^2y + 2xy^2 + y^3$



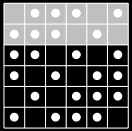
$x^3 + y^3$



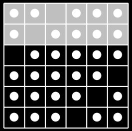
$x^4 + x^2y^2 + y^4$



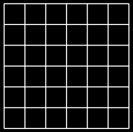
$x^4 - x^3y + xy^3 - y^4$



$x^4 + x^3y - xy^3 - y^4$



$x^5 - x^4y + \cdots - y^5$



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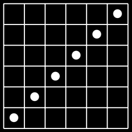
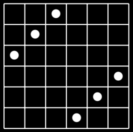
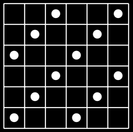


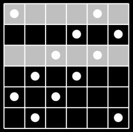
$x - y$



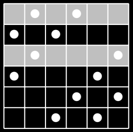
$x + y$



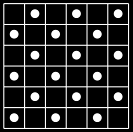
$x^2 - y^2$



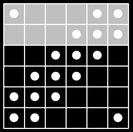
$x^2 - xy + y^2$



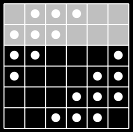
$x^2 + xy + y^2$



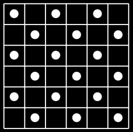
$x^3 - y^3$



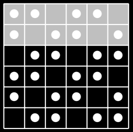
$x^3 - 2x^2y + 2xy^2 - y^3$



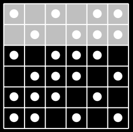
$x^3 + 2x^2y + 2xy^2 + y^3$



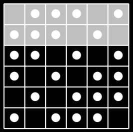
$x^3 + y^3$



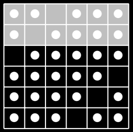
$x^4 + x^2y^2 + y^4$



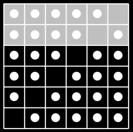
$x^4 - x^3y + xy^3 - y^4$



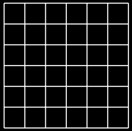
$x^4 + x^3y - xy^3 - y^4$



$x^5 - x^4y + \cdots - y^5$



$x^5 + x^4y + \cdots + y^5$



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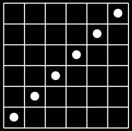
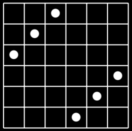
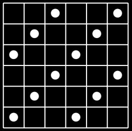


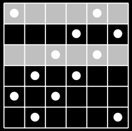
$x - y$



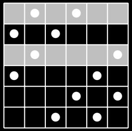
$x + y$



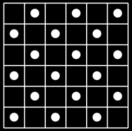
$x^2 - y^2$



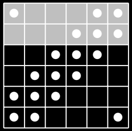
$x^2 - xy + y^2$



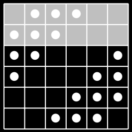
$x^2 + xy + y^2$



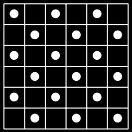
$x^3 - y^3$



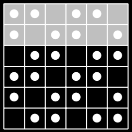
$x^3 - 2x^2y + 2xy^2 - y^3$



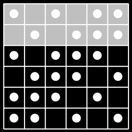
$x^3 + 2x^2y + 2xy^2 + y^3$



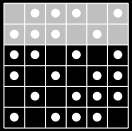
$x^3 + y^3$



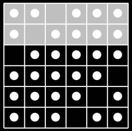
$x^4 + x^2y^2 + y^4$



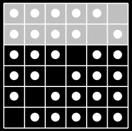
$x^4 - x^3y + xy^3 - y^4$



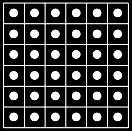
$x^4 + x^3y - xy^3 - y^4$



$x^5 - x^4y + \cdots - y^5$



$x^5 + x^4y + \cdots + y^5$



$x^6 - y^6$

Definition

Definition

Let T be an invariant subset of $\mathbb{Z}_n \times \mathbb{Z}_m$. The **separable closure** $\mathsf{T}^{\mathrm{sep}}$ of T is defined by

$$\mathsf{T}^{\mathrm{sep}} := \bigcap_{\substack{S \supseteq \mathsf{T} \\ S \text{ inv, sep}}} S.$$

Theorem

If p is separable and P is its minimal separated multiple, then there is a unique weight function ω such that

(a) $\mathrm{lp}_\omega(p)$ involves at least two monomials, and

(b) the minimal separated multiple of $\mathrm{lp}_\omega(p)$ is $\mathrm{lp}_\omega(P)$.

Sketch of Proof

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Assume $\alpha_i, \beta_j \in \mathbb{K}^{\text{Puiseux}}(t^{-1})$, and define

$$\overline{\alpha}_i := \mathrm{lt}(\alpha_i) \quad \text{and} \quad \overline{\beta}_j := \mathrm{lt}(\beta_j), \text{ and}$$

$$T := \{(i,j) \mid p(\alpha_i, \beta_j) = 0\} \quad \text{and} \quad \overline{T} := \{(i,j) \mid \mathrm{lp}_\omega(p)(\overline{\alpha}_i, \overline{\beta}_j) = 0\}.$$

Sketch of Proof

Assume $\alpha_i, \beta_j \in \mathbb{K}^{\text{Puiseux}}(t^{-1})$, and define

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$$T := \{(i,j) \mid p(\alpha_i, \beta_j) = 0\} \quad \text{and} \quad \overline{T} := \{(i,j) \mid \operatorname{lp}_\omega(p)(\overline{\alpha}_i, \overline{\beta}_j) = 0\}.$$

Since

$$p(\alpha_i, \beta_j) = 0 \quad \Longrightarrow \quad \operatorname{lp}_\omega(p)(\overline{\alpha}_i, \overline{\beta}_j) = 0,$$

we have

$$T \subseteq \overline{T}, \quad \text{and hence} \quad T^{\text{sep}} \subseteq \overline{T}^{\text{sep}}.$$

Sketch of Proof

Assume $\alpha_i, \beta_j \in \mathbb{K}^{\text{Puiseux}}(t^{-1})$, and define

$$\overline{\alpha}_i := \operatorname{lt}(\alpha_i) \quad \text{and} \quad \overline{\beta}_j := \operatorname{lt}(\beta_j), \text{ and}$$

$$T := \{(i,j) \mid p(\alpha_i, \beta_j) = 0\} \quad \text{and} \quad \overline{T} := \{(i,j) \mid \operatorname{lp}_\omega(p)(\overline{\alpha}_i, \overline{\beta}_j) = 0\}.$$

Since

$$p(\alpha_i, \beta_j) = 0 \quad \Longrightarrow \quad \operatorname{lp}_\omega(p)(\overline{\alpha}_i, \overline{\beta}_j) = 0,$$

we have

$$T \subseteq \overline{T}, \quad \text{and hence} \quad T^{\text{sep}} \subseteq \overline{T}^{\text{sep}}.$$

If P is the minimal separated multiple of p, then

$$T^{\text{sep}} = \mathbb{Z}_m \times \mathbb{Z}_n, \quad \text{and hence} \quad \overline{T}^{\text{sep}} = \mathbb{Z}_m \times \mathbb{Z}_n,$$

and $\operatorname{lp}_\omega(P)$ is the minimal separated multiple of $\operatorname{lp}_\omega(p)$.

Arbitrary Bivariate Ideals

Let $I = I_0 \cap I_1$ be such that I_0 is zero-dimensional and I_1 principal. Given a set of generators of $A(I_0)$ and the generator of $A(I_1)$, how can we determine a set of generators of

$$A(I) = A(I_0) \cap A(I_1)?$$

Lemma

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Let $I_0 \subseteq \mathbb{K}[x, y]$ be a zero-dimensional ideal. There is a finite-dimensional $\mathbb{K}$-subspace V of $\mathbb{K}[x] \times \mathbb{K}[y]$ such that

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Moreover, given $(f, g) \in \mathbb{K}[x] \times \mathbb{K}[y]$, we can compute $(\tilde{f}, \tilde{g}) \in V$ such that

$$(f, g) - (\tilde{f}, \tilde{g}) \in A(I_0).$$

Algorithm

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Input: $a \in \mathbb{K}[x] \times \mathbb{K}[y]$, and $A(I_0)$ and V as before, and a finite set $S = \{s_1, \dots, s_m\}$ of elements of $\mathbb{N}$.
Output: a basis of the vector space of polynomials p such that $\operatorname{supp}(p) \subseteq S$ and $p(a) \in A(I_0)$.

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1 For $i = 1, \ldots, m$, compute $r_i \in V$ such that $a^{s_i} - r_i \in A(I_0)$.

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1 For $i = 1, \ldots, m$, compute $r_i \in V$ such that $a^{s_i} - r_i \in A(I_0)$.
2 Compute a basis B of the space of all $(c_1, \ldots, c_m) \in \mathbb{K}^m$ with $c_1 r_1 + \cdots + c_m r_m = 0$.

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3. For every element $(c_1, \ldots, c_m) \in B$, return $c_1 t^{s_1} + \cdots + c_m t^{s_m}$.

A set of generators of $A(\mathrm{I}_0) \cap A(\mathrm{I}_1)$ can then be computed by iteratively applying the algorithm as follows:

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2 Set $G = \emptyset$, $\Delta = \emptyset$.

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```
Compute a basis of a vector space V for which
V ⊕ A(I_0) = 𝕂[x] × 𝕂[y].
Set G = ∅, Δ = ∅.
While gcd(Δ) ≠ 1, do:
   Select a set S ⊆ ℕ \ ⟨Δ⟩ with |S| > dim V and find a
   polynomial p with p(a) ∈ A(I_0) and supp(p) ⊆ S.
   G = G ∪ {p}, Δ = Δ ∪ {deg p}
Find a basis of the vector space of polynomials p with
p(a) ∈ A(I_0) and supp(p) ⊆ S = ℕ \ ⟨Δ⟩ and add the
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Set $G = \emptyset$, $\Delta = \emptyset$.
While $\gcd(\Delta) \neq 1$, do:
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Find a basis of the vector space of polynomials p with $p(a) \in A(I_0)$ and $\operatorname{supp}(p) \subseteq S = \mathbb{N} \setminus \langle \Delta \rangle$ and add the resulting polynomials to G.
Return G

An Example

To compute $A(I_0) \cap A(I_1)$ for

$$I_0 = \langle x^3 - 2xy + y^2, y^3 - 2x^2y - 1 \rangle \quad \text{and} \quad I_1 = \langle x^2 - xy + y^2 \rangle,$$

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we find $A(I_1) = \mathbb{K}(x^3, -y^3)$, and $V = \bigoplus_{i=0}^{8} \mathbb{K} \cdot (0, y^i)$ such that

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By making an ansatz for a polynomial p with $\deg(p) \leq 10$ such that $p((x^3, -y^3)) \in A(I_0)$, we find $p = t^4 - 2t^2$,

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By making an ansatz for a polynomial p with $\deg(p) \leq 10$ such that $p((x^3, -y^3)) \in A(I_0)$, we find $p = t^4 - 2t^2$, and, in the next step, a polynomial $q = 9t^5 - 26t^3 + 17$ with support in $S = \{1, 2, 3, 5, 6, 7, 9, 10, 11, 13\}$.

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Since $\gcd(4, 5) = 1$, the set $S = \mathbb{N} \setminus \langle 4, 5 \rangle$ is finite,

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Since $\gcd(4, 5) = 1$, the set $S = \mathbb{N} \setminus \langle 4, 5\rangle$ is finite, and the space of polynomials whose support is contained in S is generated by $81t^6 - 323t^3$, $81t^7 - 539t^3 + 458$, and $6561t^{11} - 191125t^3 + 184564$.

The implementation of the algorithm can be found on

http://kauers.de/software/separate.m

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Thank you for your attention.