

# Algorithms for the Enumeration of Lattice Walks

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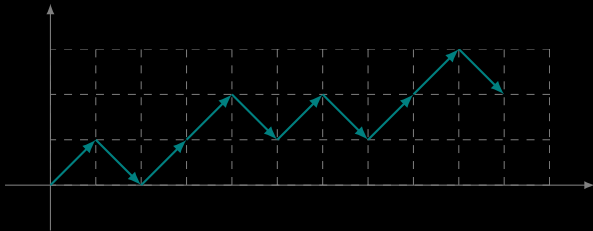
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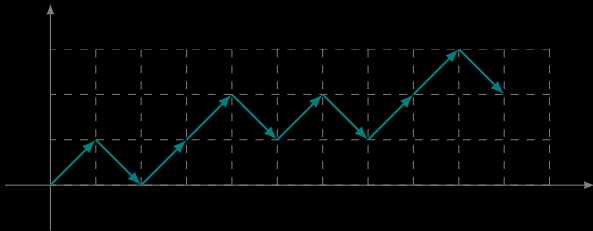
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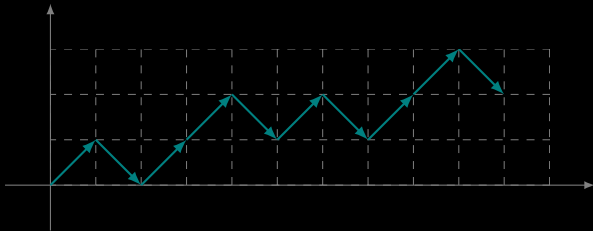


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Then

$$F(x; t) = 1 + tx F(x; t) + t \frac{F(x; t) - F(0; t)}{x}.$$

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Given  $P \in \mathbb{Q}[x, y]$  and  $Q \in \mathbb{Q}[v_0, \dots, v_{k+l}, x, y, t]$ , solve

$$F = P(x, y) + tQ\left(F, \Delta_x F, \dots, \Delta_x^{(k)} F, \Delta_y F, \dots, \Delta_y^{(l)} F, x, y, t\right)$$

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Is its solution rational, algebraic, D-finite or D-algebraic?

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We show how the **classical kernel-method**, the **orbit-sum method** and the **obstinate kernel method** generalize and / or algorithmize.

# Part 1

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## The Classical Kernel-Method

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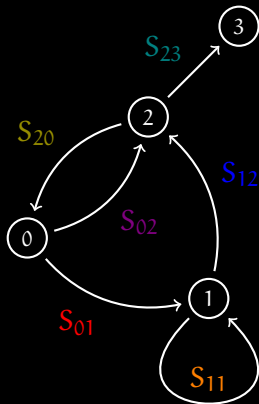
$$F(0; t) = x_0(t)/t, \quad \text{and} \quad F(x; t) = \frac{x - x_0(t)}{x(1 - t(\bar{x} + x))}.$$

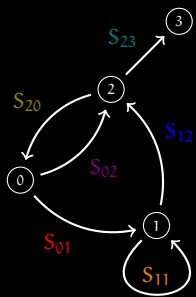
## Systems of Discrete Differential Equations

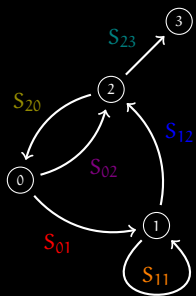
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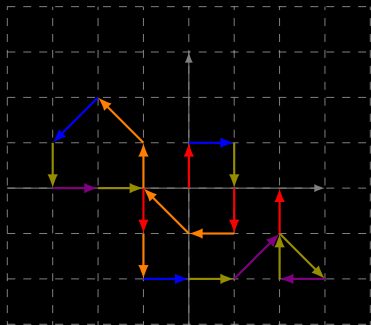
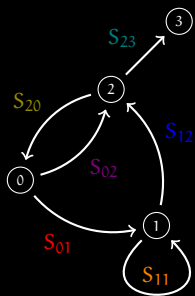
## Definition

A lattice walk is called **inhomogeneous**, if the set of admissible steps is governed by a deterministic finite automaton.









## Theorem [Buchacher, Kauers]

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Let  $\mathbb{K}$  be a field of characteristic zero, and let  $\Delta$  be given by

$$\Delta : \mathbb{K}[\mathbf{x}][[\mathbf{t}]]^n \rightarrow \mathbb{K}[\mathbf{x}][[\mathbf{t}]]^n$$

$$\Delta f(\mathbf{x}, \mathbf{t}) := (f(\mathbf{x}, \mathbf{t}) - f(0, \mathbf{t}))/\mathbf{x}.$$

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Then, for  $a \in \mathbb{K}[x, t]^n$  and  $B_i \in \mathbb{K}[x, t]^{n \times n}$ ,

$$f = a + t \sum_{i=0}^k B_i \Delta^i f$$

has a unique solution  $f$  in  $\mathbb{K}[x][[t]]^n$ , and its components are algebraic over  $\mathbb{K}[x, t]$ .

## Sketch of Proof

## Existence and Uniqueness

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The equation

$$f = a + t \sum_{i=0}^k B_i \Delta^i f$$

has a unique solution  $f \in \mathbb{K}[x][[t]]^n$  because the coefficients of  $f$  can be computed recursively via

$$[t^0]f = [t^0]a$$

$$[t^{n+1}]f = [t^{n+1}]a + \sum_{i=0}^k \sum_{j=0}^n [t^j]B_i \Delta^i [t^{n-j}]f.$$

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1 Rewrite the equation in the form

$$\left(x^k I_n - t \sum_{i=0}^k x^{k-i} B_i\right) f(x; t) = x^k a - t \sum_{j=0}^{k-1} \left( \sum_{i=j+1}^k \frac{x^{k+j-i}}{j!} B_i \right) f^{(j)}(0; t).$$

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- b) multiplying the equation by elements of the co-kernel of the matrix.

3 Solve the resulting linear systems for the  $f^{(j)}(0; t)$ 's and for  $f(x; t)$ .

To avoid difficulties arising from the linear system possibly being singular one solves a perturbation of the original equation:

$$\tilde{f} = a(x; t^2) + \epsilon t E \Delta^k \tilde{f} + t^2 \sum_{i=0}^k B_i(x; t^2) \Delta^i \tilde{f}.$$

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Algebraicity of  $\tilde{f}$  then implies algebraicity of  $f$ , since

$$f(x; t^2) = [\epsilon^0] \tilde{f}(x; t).$$

Problem

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Prove (some variant of) the theorem using matrix factorization theory and functional analysis.

## References

- Donald E. Knuth, *The Art of Computer Programming. Volume 1. Fundamental Algorithms*, 1968
- Mireille Bousquet-Mélou and Arnaud Jehanne, *Polynomial equations with one catalytic variable, algebraic series and map enumeration*, 2006
- Manfred Buchacher and Manuel Kauers, *Inhomogeneous Restricted Lattice Walks*, 2019

# Part 2

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## The Orbit-Sum Method

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$$xF(x; t) - \bar{x}F(\bar{x}; t) = \frac{x - \bar{x}}{1 - t(\bar{x} + x)}$$

and **discarding** non-positive powers in  $x$ ,

$$xF(x; t) = [x^>] \frac{x - \bar{x}}{1 - t(\bar{x} + x)}.$$

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**Question:** How can we do linear algebra with algebraic functions?

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### Corollary

Let  $L/K$  be as before. Then there is a polynomial  $m \in K[X]$  such that

$$L \cong K[X]/\langle m \rangle.$$

### Theorem [Shape Lemma]

Let  $K$  be a field with  $\text{char}(K) = 0$  and let  $I \subseteq K[x_1, \dots, x_n]$  be a 0-dimensional radical ideal in normal  $x_n$ -position. Then  $I$  has a Gröbner basis w.r.t. lex order of the form

$$\{x_1 - g_1, \dots, x_{n-1} - g_{n-1}, g_n\}$$

for  $g_1, \dots, g_n \in K[x_n]$ .

Let

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and the minimal polynomials

$$m_1, m_2, m_3, m \in \mathbb{Q}(x, y)[Z]$$

of  $p_1(\alpha), p_2(\alpha), p_3(\alpha)$  and  $\alpha$ , respectively?

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- 3 What are their supports?

Example

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The positive part depends on over which field of Laurent series,

$$\mathbb{Q}((x)) \quad \text{or} \quad \mathbb{Q}((\bar{x})),$$

the equation is solved.

## Proposition

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$$\mathbb{C}_{\preceq}((\mathbf{x})) := \left\{ \phi \mid \phi \in \mathbb{C}^{\mathbb{Q}^n} \text{ and } \exists C \in \mathcal{C} \text{ s.t. } \text{supp}(\phi) \subseteq C \right\}$$

is a field with respect to addition and multiplication.

## Algorithm [Generalized Newton-Puiseux Algorithm]

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$$(c_{\alpha_1} \mathbf{x}^{\alpha_1} + \dots + c_{\alpha_N} \mathbf{x}^{\alpha_N}, \preceq, C)$$

**Algorithm** [Generalized Newton-Puiseux Algorithm]

**Input:** A square-free polynomial  $m(x_1, \dots, x_n, x_{n+1})$  over  $\mathbb{C}$  and a positive integer  $k$ .

**Output:** The list of series solutions of

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Furthermore, the integer  $N \geq k$  is minimal.

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To summarize, we have an effective **sufficient** condition for

$$F(x, y; t) = [x^{\geq} y^{\geq}] \text{rat}(x, y, t)$$

to hold.

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# Part 3

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## The Obstinate Kernel Method

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- 1 Let  $\Gamma$  be a finite matrix group of  $GL(n; \mathbb{K})$  which naturally acts on  $\mathbb{K}[\mathbf{x}] := \mathbb{K}[x_1, \dots, x_n]$  via

$$\Gamma \times \mathbb{K}[\mathbf{x}] \rightarrow \mathbb{K}[\mathbf{x}], \quad (A, p(\mathbf{x})) \mapsto p(A^{-1}\mathbf{x}).$$

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Given an ideal  $I$  of  $\mathbb{K}[\mathbf{x}]$ , determine its invariant elements.

- 2 Given an ideal  $I$  of  $\mathbb{K}[x_1, \dots, x_n, y_1, \dots, y_m]$ , determine

$$I \cap (\mathbb{K}[x_1, \dots, x_n] + \mathbb{K}[y_1, \dots, y_m]).$$

## Separating Variables in Bivariate Polynomial Ideals

## Definition

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Let  $p \in \mathbb{K}[x, y]$ . It is **separated**, if there is a  $(f, g) \in \mathbb{K}[x] \times \mathbb{K}[y]$  such that

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Let  $I \subseteq \mathbb{K}[x, y]$  be an ideal. Then

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## Problem

Given generators of an ideal  $I \subseteq \mathbb{K}[x, y]$ , determine a set of generators for the algebra  $A(I)$  of separated polynomials.

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The polynomial  $x(x^2 + xy + y^2)$  is not separable,  
since it is a multiple of  $x$  and involves  $y$ .

## Principal Ideals

Assume that

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Let  $f, g, F, G$  be nonconstant polynomials. Then  $f(x) - g(y)$  divides  $F(x) - G(y)$  if and only if there is a polynomial  $r$  such that

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**Corollary**

If  $I$  is principal, then  $A(I)$  is simple.

**Theorem** [Buchacher, Kauers, Pogudin]

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If  $p$  is separable and  $P$  is its minimal separated multiple, then there is a unique weight function  $\omega$  such that

- (a)  $lp_\omega(p)$  involves at least two monomials, and
- (b) the minimal separated multiple of  $lp_\omega(p)$  is  $lp_\omega(P)$ .

The polynomial

$$p(x, y) = x^3 + x^2y + xy^2 + y^3 + x^2 - xy + y^2$$

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Make an ansatz

$$P(x, y) = x^4 - y^4 + \sum_{i+j < 4} P_{ij} x^i y^j$$

for the minimal separated multiple  $P$  of  $p$ , divide it by  $p$ , and set the coefficients of the remainder equal to zero.

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The resulting linear system does not have a solution, and therefore,  $p$  is not separable.

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Given  $f, g \in \mathbb{K}[t_1, \dots, t_n]$ , determine a set of generators for the algebra

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## Open Problems

- Determine  $A(I)$  when  $I$  is an ideal of

$$\mathbb{K}[x, y]_K := \left\{ \frac{N(x, y)}{D(x, y)} : K(x, y) \nmid D(x, y) \right\},$$

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- How can  $A(I)$  be determined when  $I$  is an ideal of a not necessarily bivariate polynomial ring?

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# Part 4

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## Characterization of Solutions of Discrete Differential Equations

## Problem

Solve

$$\begin{aligned} K(x, y; t)Q(x, y; t) &= x^{k+1}y^{l+1} + K(x, 0; t)Q(x, 0; t) \\ &\quad + K(0, y; t)Q(0, y; t) - K(0, 0; t)Q(0, 0; t) \end{aligned}$$

over  $\mathbb{Q}[x, y][[t]]$  given that

$$K(x, y; t) := xy(1 - tS(x, y)), \quad S(x, y) \in \mathbb{Q}[x, y, \bar{x}, \bar{y}],$$

$$\text{supp}(S) \subseteq \{-1, 0, 1\}^2, \quad \text{and} \quad (k, l) \in \mathbb{N}^2.$$

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Is its solution rational, algebraic, D-finite or D-algebraic?

Let

$$\begin{aligned} S(x, y) &= a_{-1}(x)\overline{y} + a_0(x) + a_1(x)y \\ &= b_{-1}(y)\overline{x} + b_0(y) + b_1(y)x \end{aligned}$$

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be such that

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Define

$$\phi : (x, y) \mapsto \left( \frac{b_{-1}(y)}{b_1(y)}\overline{x}, y \right) \quad \text{and} \quad \psi : (x, y) \mapsto \left( x, \frac{a_{-1}(x)}{a_1(x)}\overline{y} \right),$$

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and let

$$G = \langle \phi, \psi \rangle.$$

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- If  $|G| < \infty$ , then  $Q(x, y, t)$  is D-finite.
- If  $|G| < \infty$  and  $\text{OS}(x^{k+1}y^{l+1}) = 0$ , then  $Q(x, y, t)$  is algebraic.

Are there any examples for which

$$|G| < \infty \quad \text{and} \quad \text{OS}(x^{k+1}y^{l+1}) = 0,$$

but  $Q(x, y, t)$  is not algebraic?

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The associated group is finite,

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$$OS(x^{k+1}y^{l+1}) = x^{k+1}y^{l+1} - \bar{x}^{k+1}y^{l+1} + \bar{x}^{k+1}\bar{y}^{l+1} - x^{k+1}\bar{y}^{k+1}$$

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$$\begin{aligned} \text{OS}(x^{k+1}y^{l+1}) &= x^{k+1}y^{l+1} - \bar{x}^{k+1}y^{l+1} + \bar{x}^{k+1}\bar{y}^{l+1} - x^{k+1}\bar{y}^{l+1} \\ &= 0, \quad \text{if } (k, l) = (-1, -1). \end{aligned}$$

$$(1 - t(x + y + \bar{x} + \bar{y}))Q(x, y; t) = \\ \bar{x}\bar{y} - t\bar{x}Q(0, y; t) - t\bar{y}Q(x, 0; t)$$

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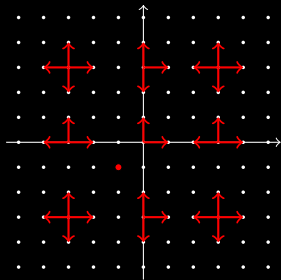
What do we mean by  $Q(0, y; t)$  and  $Q(x, 0; t)$ ?

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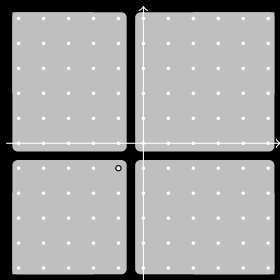
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$$Q(x, y; t) =$$

$$\begin{aligned}
& [x^{<}y^{<}] \frac{xy - \bar{x}y + \bar{x}\bar{y} - x\bar{y}}{1 - tS} + \\
& t\bar{y}[x^{<}] \left( ([y^{>}] \frac{y - \bar{y}}{1 - S t}) ([\bar{y}] \frac{xy - \bar{x}y - x\bar{y} + \bar{x}\bar{y}}{1 - tS}) \right) + \\
& t\bar{x}[y^{<}] \left( ([x^{>}] \frac{x - \bar{x}}{1 - S t}) ([\bar{x}] \frac{xy - \bar{x}y - x\bar{y} + \bar{x}\bar{y}}{1 - tS}) \right) + \\
& \bar{x}\bar{y}t^2[y^{>}] \left( ([\bar{x}] \frac{(y - \bar{y})[\bar{y}]}{1 - tS} \frac{xy - \bar{x}y + \bar{x}\bar{y} - x\bar{y}}{1 - tS}) ([x^{>}] \frac{x - \bar{x}}{1 - tS}) \right) + \\
& \bar{x}\bar{y}t^2[x^{>}] \left( ([\bar{y}] \frac{(x - \bar{x})[\bar{x}]}{1 - tS} \frac{xy - \bar{x}y + \bar{x}\bar{y} - x\bar{y}}{1 - tS}) ([y^{>}] \frac{y - \bar{y}}{1 - tS}) \right)
\end{aligned}$$

The coefficient sequence of  $Q(1, 1; t)$  satisfies the recurrence

$$\begin{aligned} & (2 + n)(4 + n)(6 + n)(-1 + 2n + n^2)a_{n+2} \\ & - 4(3 + n)(-18 + 4n + 9n^2 + 2n^3)a_{n+1} \\ & - 16(1 + n)(2 + n)(3 + n)(2 + 4n + n^2)a_n = 0. \end{aligned}$$

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Its only asymptotic solutions are of the form

$$a_n \sim 4^n n^{-1} \quad \text{and} \quad a_n \sim (-4)^n n^{-3},$$

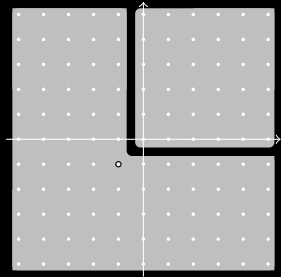
which are not compatible with algebraicity of  $Q(x, y; t)$ .

$$Q(x, 0; t) := [x^{\geq} y^0] Q(x, y; t)$$

$$Q(0, y; t) := [x^0 y^{\geq}] Q(x, y; t)$$

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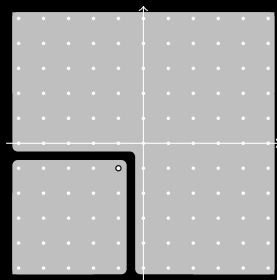


$$Q(x, 0; t) := [x^< y^0] Q(x, y; t)$$

$$Q(0, y; t) := [x^0 y^<] Q(x, y; t)$$

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The asymptotics were compatible with  $Q(1, 1; t)$  being algebraic.  
However, we could show, on the level of differential operators, that

$$Q(1, 1; t) = A(t) + T(t),$$

where  $A(t)$  is algebraic and  $T(t)$  is D-finite but transcendental.

Open Problem

## Open Problem

Do the group and the orbit-sum in this context encode any information about the nature of the generating function?

## References

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