$$x(1 - t(x^{-1} + x))F(x;t) = x - tF(0;t)$$

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Research interests:

Algorithmic enumerative combinatorics, computer and constructive algebra, discrete differential equations and the elementary power series algebra to solve them.

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Goal: Solve the equation for $F(x;t)$ in $\mathbb{Q}[x][[t]]$.

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where Δ acts on $\mathbb{Q}[x][[t]]$ by

$$\Delta F(x;t) := (F(x;t) - F(0;t))/x.$$

Combinatorial meaning

Definition

A lattice walk is a sequence $P_0, P_1, \dots, P_n$ of points of $\mathbb{Z}^d$. We call P_0 and P_n its starting and end point, respectively, the consecutive differences $P_{i+1} - P_i$ are its steps, and n is its length.

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Example

The sequence $(0, 1, 0, 1, 2, 1, 2, 1, 2, 3)$ is a lattice walk on $\mathbb{N}$ that starts at 0, ends at 3, has length 9, and whose steps are $1, -1, 1, 1, -1, 1, -1, 1, 1$.

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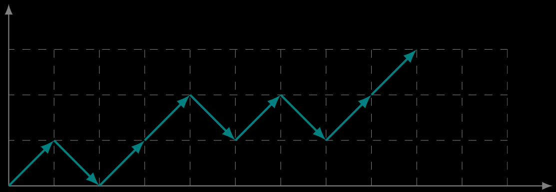

Let $f(i;n)$ be the number of walks on $\mathbb{N}$ that start at 0, end at i and consist of (presicely) n steps taken from $\{-1, 1\}$, and let

$$F(x;t) := \sum_{n \geq 0} \left(\sum_{i \geq 0} f(i;n) x^i \right) t^n.$$

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$$F(x;t) := \sum_{n\geq 0}\left(\sum_{i\geq 0} f(i;n)x^i\right)t^n.$$

Then $F(x;t)$ solves the functional equation, since

$$\begin{aligned} &f(i;n+1) = f(i-1;n) + f(i+1,n) \\ &f(i;0) = \delta_{i,0} \quad \text{and} \quad f(i;n) = 0,\ i<0. \end{aligned}$$

Existence and uniqueness

By extracting the coefficient of t^n of

$$F(x;t) = 1 + t(x^{-1} + x)F(x;t) - tx^{-1}F(0;t),$$

we find that

$$[t^n]F(x;t) = \begin{cases} 1, \text{ if } n = 0, \\ (x^{-1} + x)[t^{n-1}]F(x;t) - x^{-1}[t^{n-1}]F(0;t) \text{ else.} \end{cases}$$

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This recurrence relation has a unique solution, and therefore this is also true for the functional equation.

The map

$$F(x;t) \mapsto 1 + t(x^{-1} + x)F(x;t) - tx^{-1}F(0;t)$$

represents a contraction on $\mathbb{Q}[x][[t]]$ with respect to

$$d(F,G) := 2^{-\mathrm{val}(F-G)},$$

where

$$\mathrm{val}(F) := \min\{n : [t^n]F \neq 0\}.$$

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Banach's fixed point theorem implies the existence and uniqueness of a solution of the functional equation.

What does it mean to solve a discrete differential equation?

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It means to decide whether its solution is rational, algebraic, D-finite or D-algebraic, and in case it is, to find an algebraic or differential equation for it.

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- the formal or algebraic one: involving elementary power series algebra, constructive and computer algebra.

We focus on formal and algebraic methods, and show how discrete differential equations can be solved using only operations such as

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The latter operator acts on series by discarding all terms involving non-positive powers in x, e.g.

$$[x^>]\left(x^{-2}+x^{-1}+1+x+x^2\right)=x+x^2.$$

The classical kernel method

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where $\bar{x} := x^{-1}$, replace x by the power series root

$$x_0(t) := \frac{1 - \sqrt{1 - 4t^2}}{2t} \quad \text{of} \quad K(x;t) := x(1 - t(\bar{x} + x)),$$

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and solve the resulting equation

$$0 = x_0(t) - tF(0;t)$$

to find that

$$F(0;t) = x_0(t)/t, \qquad \text{and} \qquad F(x;t) = \frac{x - x_0(t)}{x(1 - t(\bar{x} + x))}.$$

Theorem [Bousquet-Mélou, Jehanne]

Solutions of ordinary discrete differential equations are algebraic (over a field of rational functions).

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Theorem [Buchacher, Kauers]

(The components of) solutions of systems of ordinary linear discrete differential equations are algebraic (over a field of rational functions).

The orbit-sum method

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subtracting the latter equation from the former,

$$xF(x;t) - \bar{x}F(\bar{x};t) = \frac{x - \bar{x}}{1 - t(\bar{x} + x)},$$

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$$xF(x;t) - \bar{x}F(\bar{x};t) = \frac{x - \bar{x}}{1 - t(\bar{x} + x)},$$

and discarding non-positive powers in x,

$$xF(x;t) = [x^{>}]\frac{x - \bar{x}}{1 - t(\bar{x} + x)}.$$

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- How can we compute in a finite field extension of $\mathbb{Q}(x_1, \ldots, x_n)$ in an effective and efficient way?
- Given $p \in \mathbb{Q}[x_1, \ldots, x_n][y]$, how can we solve

$$p(x_1, \ldots, x_n, y) = 0$$

for y in terms of series in $x_1, \ldots, x_n$?

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- How can we do effective arithmetic for multivariate algebraic series?
- How can we compute the convex hull of the support of a multivariate algebraic series?

Wiener-Hopf factorization

The coefficient polynomial of $F(x;t)$ in

$$x(1 - t(\bar{x} + x))F(x;t) = x - tF(0;t)$$

factors into

$$-t(x - x_0)(x - x_1) = -t(1 - \bar{x}x_0)x(x - x_1),$$

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where

$$(1 - \bar{x}x_0)^{\pm 1} \in \mathbb{Q}[\bar{x}][[t]] \quad \text{and} \quad (x - x_1)^{\pm 1} \in \mathbb{Q}[x][[t]].$$

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So

$$-t(x-x_1)F(x;t) = \frac{x-tF(0;t)}{(1-\bar{x}x_0)x}$$

with the lhs in $\mathbb{Q}[x][[t]]$ and the rhs in $\mathbb{Q}[\bar{x}][[t]]$.

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$$-t(x-x_1)F(x;t) = [x^0]\frac{x-tF(0;t)}{(1-\bar{x}x_0)x} = 1,$$

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factors into

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and so

$$F(x;t) = -\frac{1}{t(x - x_1)}.$$

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- Given $A \in \mathbb{Q}(x,t)^{n\times n}$, does it factor into

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- If there is such a factorization, how can it be determined?

Compositional inverses and Lagrange inversion

Replacing t in

$$x(1 - t(\bar{x} + x))F(x;t) = x - tF(0;t),$$

by the power series root

$$t_0(x) := \frac{x}{1 + x^2} \quad \text{of} \quad x(1 - t(\bar{x} + x)) \in \mathbb{Q}(x)[t],$$

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$$\frac{x}{1+x^2} F\left(0; \frac{x}{1+x^2}\right) = x,$$

i.e. $\frac{x}{1+x^2}$ and $xF(0;x)$ are compositional inverse to each other.

Theorem [Lagrange Inversion]

Let $H(t) \in \mathbb{Q}[[t]]$ be of valuation 1, and let $G(t)$ be its compositional inverse, i.e. such that $G(H(t)) = t$. Then

$$[t^n]G(t) = \frac{1}{n}[t^{-1}]H^{-n}(t), \quad n \neq 0.$$

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Applying this to

$$H(t) = (t^{-1} + t)^{-1}$$

we find that

$$[t^{n+1}]tF(0;t) = \frac{1}{n+1}[t^{-1}](t^{-1} + t)^{n+1},$$

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$$[t^{n+1}]tF(0;t) = \frac{1}{n+1}[t^{-1}](t^{-1} + t)^{n+1},$$

which means that

$$f(0;n) = \begin{cases} 0, \text{ if } n \not\equiv 0 \mod 2, \\ \frac{1}{n/2+1}\binom{n}{n/2} \text{ else.} \end{cases}$$

The series $F(0;t)$ is algebraic (and consequently also $F(x;t)$) because

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Therefore

$$P(t, F(0;t)) = 0 \quad \text{for} \quad P(t, s) = t^2 s^2 - s + 1.$$

Guess and prove

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for some $d \in \mathbb{N}$ and undetermined p_{ijk}'s,

2. computing a truncation

$$F_0(x;t) := \sum_{n=0}^{n_0} \left(\sum_{i \geq 0} f(i;n) x^i \right) t^n$$

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   of $F(x;t)$ up to some order n_0, and

3. setting coefficients of $x^i t^j$, $j \leq n_0$, in $P(x,t,F_0(x;t))$ equal to zero, and solving the resulting linear system.

For $d = 2$ and $n_0 = 8$ we find

$$P(x, t, y) = 1 - (1 - 2xt)y - xt(1 - t(x^{-1} + x))y^2.$$

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$$F_{\mathsf{cand}}(x; t) \in \mathbb{Q}[x][[t]], \text{ and}$$

2. verifying that $F_{\mathsf{cand}}(x, t)$ satisfies the functional equation.

Theorem [Implicit Function Theorem]

Let $\mathbb{K}$ be a field of characteristic 0, and let $A(x, y) \in \mathbb{K}[[t, y]]$ s.t.

$$A(0,0) = 0 \quad \text{and} \quad \frac{\partial A}{\partial y}(0,0) \neq 0.$$

Then there is a unique $f(f) \in \mathbb{K}[[t]]$ with

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This guarantees the existence of a root $F_{\mathsf{cand}}(x; t)$ of P in $\mathbb{Q}(x)[[t]]$, which actually lies in $\mathbb{Q}[x][[t]]$.

By performing closure properties of algebraic functions we observe that

$$A(y) := y(1 - 4t^2 - t^2y^2)$$

is an annihilating polynomial for

$$x(1 - t(\bar{x} + x))F_{\mathsf{cand}}(x;t) - x - tF_{\mathsf{cand}}(0;t).$$

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$$x(1 - t(\bar{x} + x))F_{\mathsf{cand}}(x;t) - x - tF_{\mathsf{cand}}(0;t).$$

By plugging some truncation of this series into the second factor of A we find that y is its minimal polynomial.

Looking for an additional equation

Instead of solving the seemingly underdetermined equation

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The additional equation allows to determine $F(x; t)$ by solving a system of polynomial equations.

Note: Although the second equation is equivalent to the first, the latter suggests to write $F(0;t)$ as a continued fraction

$$F(0;t) = \cfrac{1}{1 - \cfrac{t^2}{1 - \cfrac{t^2}{1 - \dots}}}.$$

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$$F(0;t) = \cfrac{1}{1 - \cfrac{t^2}{1 - \cfrac{t^2}{1 - \dots}}}.$$

This indicates that there is a connection between the enumeration of restricted lattice walks and continued fractions and orthogonal polynomials.

The reflection principle

The orbit-sum method showed that

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The combinatorial interpretation of this equation is that

$$\#\text{ walks in }\mathbb{Z}\text{ from }0\text{ to }i\text{ not lying in }\mathbb{N}$$

$$=$$

$$\#\text{ walks in }\mathbb{Z}\text{ from }-2\text{ to }i$$

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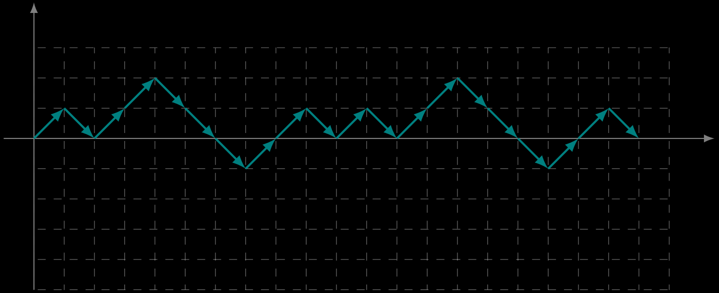

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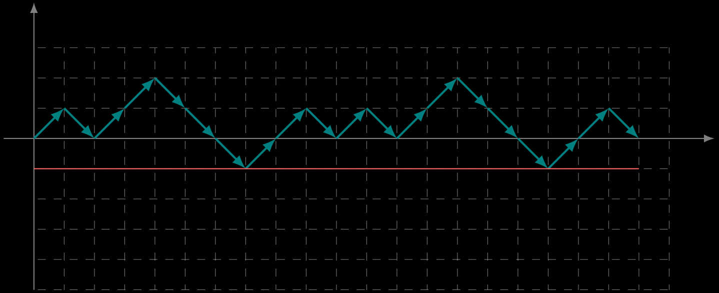

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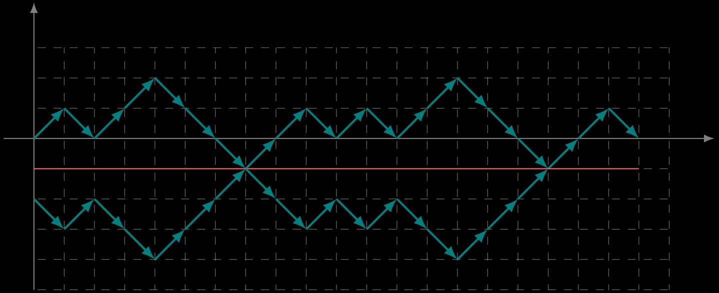

This implies that

$$\begin{aligned} f(i;n) = {} & \#\text{ walks in } \mathbb{Z} \text{ from } 0 \text{ to } i \text{ of length } n - \\ & \#\text{ walks in } \mathbb{Z} \text{ from } -2 \text{ to } i \\ = {} & \binom{n}{\frac{n+i}{2}} - \binom{n}{\frac{n+i+2}{2}} = \frac{1}{\frac{n+i}{2}+1}\binom{n}{\frac{n+i}{2}}. \end{aligned}$$

The cycle lemma / Spitzer's lemma

Langrange inversion implies that

$$f(0; n) = \frac{1}{n+1}[t^{-1}](t^{-1} + t)^{n+1}$$

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Is there a combinatorial explanation for this identity?

Lemma [Spitzer's lemma]

Let $a_1, a_2, \ldots, a_N \in \mathbb{R}$ be s.t. $a_1 + a_2 + \cdots + a_N = 0$ and no other partial sum of consecutive a_i's read cyclically vanishes. Then there is a unique cyclic permutation $a_i, a_{i+1}, \ldots, a_N, a_1, \ldots, a_{i-1}$ s.t. the sum of the first j numbers is non-negative for $j =, 1, 2, \ldots, N$.

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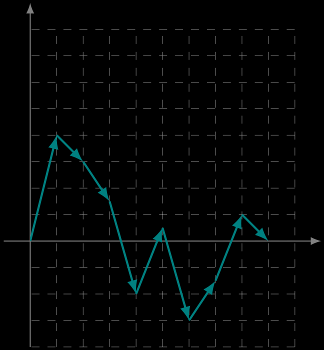

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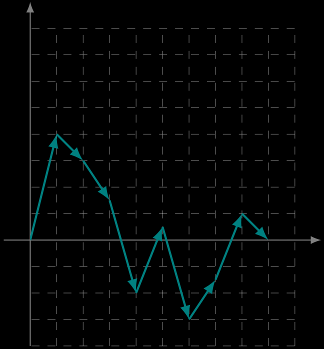

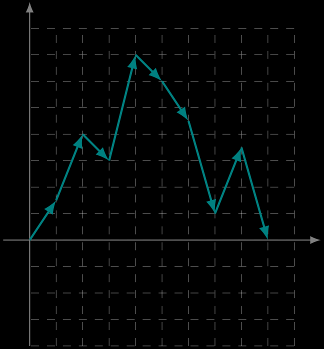

Theorem

Let $r, s \in \mathbb{N}_+$ be relative prime. Then the number of walks in $\mathbb{Z}^2$ from $(0,0)$ to (r,s) with steps taken from $\{(1,0),(0,1)\}$ staying weakly below $ry = sx$ equals $\frac{1}{r+s}\binom{r+s}{r}$.