# Quadrant Walks Starting Outside the Quadrant

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### Definition

A lattice walk is a sequence $P_0, P_1, \ldots, P_n$ of points of $\mathbb{Z}^d$. We call $P_0$ and $P_n$ its starting and end point, respectively, the consecutive differences $P_{i+1} - P_i$ are its steps, and $n$ is its length.

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How does the number of walks grow as $n$ goes to infinity?

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**Problem**

Solve

$$\begin{aligned} K(x,y;t)F(x,y;t) &= x^{k+1}y^{l+1} + K(x,0;t)F(x,0;t) \\ &+K(0,y;t)F(0,y;t) - K(0,0;t)F(0,0;t) \end{aligned}$$

for $F(x,y;t)$ in $\mathbb{Q}[x,y][[t]]$, where

$$K(x,y;t) := xy(1 - tS(x,y)) \quad \text{and} \quad S(x,y) := \sum_{(i,j)\in S} x^i y^j.$$

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Is its solution rational, algebraic, D-finite or D-algebraic?

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## WALKS WITH SMALL STEPS IN THE QUARTER PLANE

MIREILLE BOUSQUET-MÉLOU AND MARNI MISHNA

Abstract. Let $\mathcal{S} \subset \{-1,0,1\}^2 \setminus \{(0,0)\}$. We address the enumeration of plane lattice walks with steps in $\mathcal{S}$, that start from $(0,0)$ and always remain in the first quadrant $\{(i,j) : i \geq 0, j \geq 0\}$. *A priori*, there are $2^8$ problems of this type, but some are trivial. Some others are equivalent to a model of walks confined to a half-plane: such models can be solved systematically using the kernel method, which leads to algebraic generating functions. We focus on the remaining cases, and show that there are 79 inherently different problems to study.

To each of them, we associate a group $G$ of birational transformations. We show that this group is finite (of order at most 8) in 23 cases, and infinite in the 56 other cases.

We present a unified way of solving 22 of the 23 models associated with a finite group. For all of them, the generating function is found to be D-finite. The 23rd model, known as Gessel's walks, has recently been proved by Bostan *et al.* to have an algebraic (and hence D-finite) solution. We conjecture that the remaining 56 models, associated with an infinite group, have a non-D-finite generating function.

Our approach allows us to recover and refine some known results, and also to obtain new results. For instance, we prove that walks with N, E, W, SS, SW and NE steps have an algebraic generating function.

### 1. Introduction

The enumeration of lattice walks is a classical topic in combinatorics. Many combinatorial objects (trees, maps, permutations, lattice polygons, Young tableaux, queues...) can be encoded by lattice walks, so that lattice path enumeration has many applications. Given a lattice, for instance the hypercubic lattice $\mathbb{Z}^d$, and a finite set of steps $\mathcal{S} \subset \mathbb{Z}^d$, a typical problem is to determine how many $n$-step walks with steps taken from $\mathcal{S}$, starting from the origin, are confined to a certain region $\mathcal{A}$ of the space. If $\mathcal{A}$ is the whole space, then the length generating function of these walks is a simple rational series. If $\mathcal{A}$ is a half-space, bounded by a rational hyperplane, the associated generating function is an algebraic series. Instances of the latter problem have been studied in many articles since at least the end of the 19th century [1, 6]. It is now understood that the *kernel method* provides a systematic solution to all such problems, which are, in essence, one-dimensional [14, 2]. Other generic approaches to half-space problems are provided in [20, 25].

A natural next class of problems is the enumeration of walks constrained to lie in the intersec-

Let

$$
\begin{aligned}
S(x, y) &= a_{-1}(x)\overline{y} + a_0(x) + a_1(x)y \\
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\end{aligned}
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Define

$$\phi : (x, y) \mapsto \left(\frac{b_{-1}(y)}{b_1(y)}\overline{x}, y\right) \quad \text{and} \quad \psi : (x, y) \mapsto \left(x, \frac{a_{-1}(x)}{a_1(x)}\overline{y}\right),$$

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and let

$$G = \langle \phi, \psi \rangle.$$

For $g \in G$, let $\mathrm{sgn}(g)$ be 1 or $-1$ depending on whether $g$ has even or odd length, respectively, when written as a word in $\phi$ and $\psi$.

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Theorem [Mishna, Bousquet-Mélou, Bostan, Kauers, Rechnitzer, Melczer, Raschel, Salvy, Fayolle, Kurkova]

- $|G| < \infty \iff F(x, y; t)$ is D-finite.
- $|G| < \infty$ and $\mathrm{OS}(x^{k+1}y^{l+1}) = 0 \iff F(x, y; t)$ is algebraic.

Are there any examples for which

$$|G| < \infty \quad \text{and} \quad \mathrm{OS}(x^{k+1}y^{l+1}) = 0,$$

but $F(x, y; t)$ is not algebraic?

No.

# No.

Otherwise the theorem would be wrong.

Maybe.

# Maybe.

There might be some in higher dimensions.

Yes.

# Yes.

If the starting point $(k, l)$ is allowed to lie in $\mathbb{Z}^2 \setminus \mathbb{N}^2$.

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$$\mathrm{OS}(x^{k+1}y^{l+1}) = x^{k+1}y^{l+1} - \bar{x}^{k+1}y^{l+1} + \bar{x}^{k+1}\bar{y}^{l+1} - x^{k+1}\bar{y}^{k+1}$$

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$$\begin{aligned} \mathrm{OS}(x^{k+1}y^{l+1}) &= x^{k+1}y^{l+1} - \bar{x}^{k+1}y^{l+1} + \bar{x}^{k+1}\bar{y}^{l+1} - x^{k+1}\bar{y}^{k+1} \\ &= 0, \quad \text{if } (k, l) = (-1, -1). \end{aligned}$$

$$
\begin{aligned}
&(1 - t(x + y + \overline{x} + \overline{y}))F(x, y; t) = \\
&\qquad \overline{xy} - t\overline{x}F(0, y; t) - t\overline{y}F(x, 0; t)
\end{aligned}
$$

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What do we mean by $F(0, y; t)$ and $F(x, 0; t)$?

$$F(x, 0; t) := [y^0]F(x, y; t)$$
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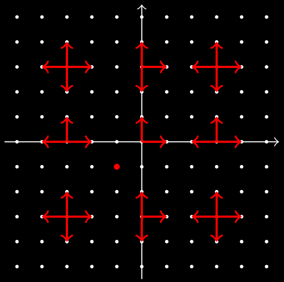

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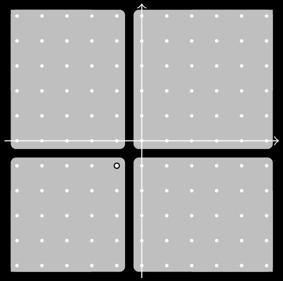

$$F(x, y; t) =$$

$$[x^< y^<] \frac{xy - \overline{x}y + \overline{xy} - x\overline{y}}{1 - tS} +$$

$$t\overline{y}[x^<]\Big(\big([y^>]\frac{y - \overline{y}}{1 - St}\big)\big([\overline{y}]\frac{xy - \overline{x}y - x\overline{y} + \overline{xy}}{1 - tS}\big)\Big) +$$

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$$\overline{xy}t^2[y^>]\Big(\big([\overline{x}]\frac{(y - \overline{y})[\overline{y}]\frac{xy - \overline{x}y + \overline{xy} - x\overline{y}}{1 - tS}}{1 - tS}\big)\big([x^>]\frac{x - \overline{x}}{1 - tS}\big)\Big) +$$

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The coefficient sequence of $F(1,1;t)$ satisfies the recurrence

$$\begin{aligned}&(2+n)(4+n)(6+n)(-1+2n+n^2)a_{n+2}\\&-4(3+n)(-18+4n+9n^2+2n^3)a_{n+1}\\&-16(1+n)(2+n)(3+n)(2+4n+n^2)a_n=0.\end{aligned}$$

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\end{aligned}$$

Its only asymptotic solutions are of the form

$$a_n \sim 4^n n^{-1} \quad \text{and} \quad a_n \sim (-4)^n n^{-3},$$

which are not compatible with algebraicity of $F(x,y;t)$.

$$F(x,0;t) := [x^{\geq}y^{0}]F(x,y;t)$$
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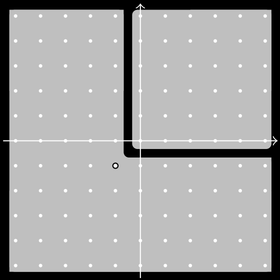

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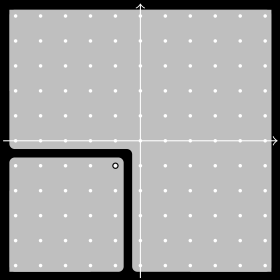

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However, we could show that

$$F(1,1;t) = A(t) + T(t),$$

where $A(t)$ is algebraic and $T(t)$ is D-finite but transcendental.

Let $r \in \mathbb{N}$, and $p_0(t), p_1(t), \ldots, p_r(t) \in \mathbb{Q}(t)$.

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How can a differential relation

$$p_0(t)f(t) + p_1(t)f'(t) + \cdots + p_r(t)f^{(r)}(t) = 0$$

imply the transcendence of a function $f(t)$?

# Step 1

Change your point of view.

Let $f(t)$ be a function, and let $L \in \mathbb{Q}(t)[D]$ be such that

$$L \cdot f = 0.$$

Let $\mathbb{Q}(t)[D]$ be the set of differential operators of the form

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It is naturally endowed with the structure of a non-commutative ring and **right-Euclidean domain**.

For all $U, V \in \mathbb{Q}(t)[D]$ with $V \neq 0$ there are unique $Q, R \in \mathbb{Q}(t)[D]$ with $\operatorname{ord}(R) < \operatorname{ord}(V)$ such that

$$U = QV + R.$$

We write

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The greatest common right divisor and least common left multiple of $U$ and $V$ are denoted by

$$\operatorname{gcrd}(U, V) \quad \text{and} \quad \operatorname{lclm}(U, V).$$

# Step 2

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Check if $L$ is completely reducible.

An operator $L \in \mathbb{Q}(t)[D]$ is called **irreducible**, if

$$L = L_1 L_2$$

for $L_1, L_2 \in \mathbb{Q}(t)[D]$ implies

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In this case, and if this factorization is minimal,

$$\mathrm{sol}(L) = \mathrm{sol}(L_1) \oplus \cdots \oplus \mathrm{sol}(L_n).$$

# Step $3$

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Check which of the $L_i$'s only have algebraic solutions.

An irreducible operator has either only algebraic solutions or the only algebraic solution is $0$.

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If $\mathrm{gcrd}(M, L_i) \neq 1$, then

$$L_i = \mathrm{gcrd}(M, L_i),$$

and $L_i$ has only algebraic solutions.

How to detect an operator which has transcendental solutions?

Inspect its generalized series solutions:

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$$
\begin{aligned}
\exp(s_1 x^{-1/v} + s_2 x^{-2/v} + \cdots + s_u x^{-u/v}) \\
\times x^{\alpha}((c_{0,0} + c_{0,1} x^{1/v} + c_{0,2} x^{2/v} + \dots) \\
+ (c_{1,0} + c_{1,1} x^{1/v} + c_{1,2} x^{2/v} + \dots) \log(x) \\
+ \dots \\
+ (c_{m,0} + c_{m,1} x^{1/v} + c_{m,2} x^{2/v} + \dots) \log(x)^m)
\end{aligned}
$$

Assume that $k \in \{1, \ldots, n\}$ is such that $L_1, \ldots, L_{k-1}$ only have algebraic solutions, while the non-zero solutions of $L_k, \ldots, L_n$ are transcendental.

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If

$$L \cdot f = 0,$$

there are functions $f_1, \ldots, f_n$ such that

$$f = f_1 + \cdots + f_n \quad \text{and} \quad L_i \cdot f_i = 0.$$

# Step 4

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Check if $f_k + \cdots + f_n$ is transcendental.

If $f$ were algebraic, then so would be

$$M \cdot f = M \cdot f_k \neq 0,$$

for

$$M = \mathrm{lclm}(L_1, \ldots, L_{k-1}, L_{k+1}, \ldots, L_n).$$

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This cannot be the case, if one can show that

$$P = \mathrm{rquo}(L, M) \quad \text{with} \quad P \cdot (M \cdot f_k) = 0$$

does not have any non-zero algebraic solutions.

## Summary

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1) Differential Operators

## Summary

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4) Transcendence and Complete Reducible Operators

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