

# Separating Variables in Bivariate Polynomial Ideals: the Local Case

Manfred Buchacher, Johannes Kepler Universität Linz

## Problem

For any irreducible  $p \in \mathbb{C}[x, y]$  that is not univariate, the set  $F(p)$  of pairs  $(f, g)$  that non-trivially solve

$$q(x, y)p = f(x) - g(y)$$

in  $\mathbb{C}(x, y)$  forms a simple field w.r.t. component-wise addition and multiplication.

We present a semi-algorithm which for any such  $p$  computes a generator of  $F(p)$

## Strategy

We reduce the non-linear problem of solving

$$qp = f - g$$

to a linear problem. The reduction will be based on the computation of the poles of  $f$  and  $g$  and their multiplicities.

## Poles

The poles are described by a dynamical system on  $\{p = 0\}$ .

If  $F(p)$  is non-trivial, then it can be assumed that

$$f(\infty) = \infty.$$

Furthermore, if  $(s_1, s_2)$  is a root of  $p$ , then

$$f(s_1) = \infty \text{ if and only if } g(s_2) = \infty.$$

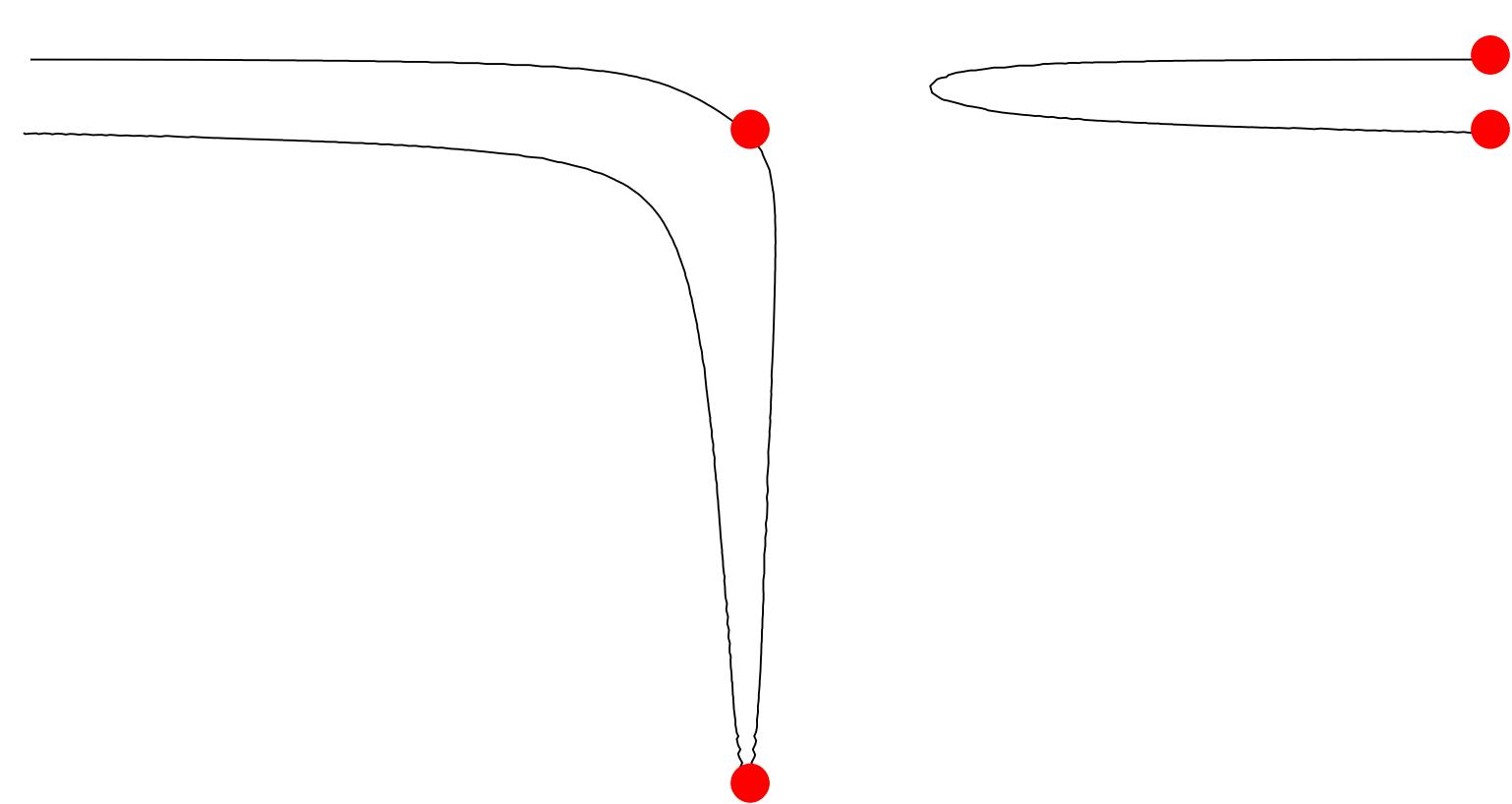
## Example

Let

$$p = xy - 1 - y - x^2y - x^2y^2.$$

The poles of a generator of  $F(p)$  are described by

$$(\infty, 0), (\infty, -1), (0, -1) \text{ and } (0, \infty).$$



## Multiplicities

The multiplicities are derived by studying  $p$  locally at pairs of poles, thereby reducing the problem to one for homogeneous polynomials.

If  $\omega \in \mathbb{N}^2$  is such that

$$\text{lp}_\omega(q)\text{lp}_\omega(p) = \text{lp}_\omega(f) - \text{lp}_\omega(g),$$

and

$$F(\text{lp}_\omega(p)) = \mathbb{C}((f_\omega, g_\omega)),$$

then

$$\text{lp}_\omega(f) - \text{lp}_\omega(g) = f_\omega^k - g_\omega^k$$

for some  $k \in \mathbb{N}$ . In particular,

$$(\deg f, \deg g) = k \cdot (\deg f_\omega, \deg g_\omega).$$

These families merge to a single 1-parameter family. The smallest numbers are the multiplicities of the poles of a generator of  $F(p)$ .

## Example (continued)

The analysis of  $p$  at  $(\infty, 0)$ ,  $(\infty, -1)$ ,  $(0, -1)$  and  $(0, \infty)$  results in the following 1-parameter family for their multiplicities:

$f$	$g$
$\infty   2k$	$\infty   k$
$0   2k$	$0   k$

$-1 | -1$

Setting  $k = 1$  we find that

$$F(p) = \mathbb{C} \left( \left( \frac{(1-x)^2(1+x+x^2)}{x^2}, -\frac{(1+y+y^2)^2}{y(1+y)^2} \right) \right).$$

## Proof Ideas

Let  $f \in \mathbb{C}(x)$  and  $g \in \mathbb{C}(y)$  be non-constant. We study

$$f - g$$

by introducing a variable  $t$  and investigating

$$f = t \quad \text{and} \quad g = t.$$

We solve these equations w.r.t.  $x$  and  $y$  in  $\mathbb{C}\{\{t^{-1}\}\}$ , and we let

$$\alpha_0, \dots, \alpha_{m-1} \quad \text{and} \quad \beta_0, \dots, \beta_{n-1}$$

be their solutions. The map

$$p \mapsto T := \{(i, j) \in \mathbb{Z}_m \times \mathbb{Z}_n : p(\alpha_i, \beta_j) = 0\}$$

defines a Galois correspondence between the factors of  $f_n g_d - g_n f_d$  and the subsets of  $\mathbb{Z}_m \times \mathbb{Z}_n$  that are invariant under the Galois group of  $\mathbb{C}(t)/\mathbb{C}(t)$ .

## Example

There are four invariant subsets of  $\mathbb{Z}_4 \times \mathbb{Z}_4$  that can be associated with

$$f_n g_d - g_n f_d = (1-x)^2(1+x+x^2)y(1+y)^2 + (1+y+y^2)^2x^2.$$

The invariant set  $T$  corresponding to  $p$  is depicted below. The other invariant sets are  $\emptyset$ ,  $T^c$  and  $\mathbb{Z}_m \times \mathbb{Z}_n$  and correspond to 1, the complementary factor of  $p$ , and  $f_n g_d - g_n f_d$ .

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	•		•
•		•	
	•	•	

Each of  $\alpha_i$  and  $\beta_j$  can be associated with a pole of  $f$  and  $g$ , respectively, and any invariant set can be partitioned by grouping pairs of series according to the pairs of poles associated with them.

## Example (continued)

-1	-1	0	$\infty$
0	•		•
0		•	•
$\infty$	•		•
$\infty$		•	•

The colors in the table above highlight the different pairs of poles the pairs of series of  $T$  give rise to.

The different parts of an invariant set are the invariant sets of the leading parts of the corresponding factor and its separated multiple.

## Example (continued)

$\begin{matrix} 0 \\ \infty \\ \infty \end{matrix}$	$\begin{matrix} -1 & -1 \\ \infty & \infty \end{matrix}$	$\begin{matrix} -1 & -1 \\ 0 & 0 \end{matrix}$	$\begin{matrix} \infty \\ 0 \\ 0 \end{matrix}$
$x^2 + y^{-1}   x^2 + y^{-1}$	$x - y^{-1}   x^2 - y^{-2}$	$x^{-1} + y^{-1}   x^{-2} - y^{-2}$	$x^{-2} + y   x^{-2} + y$

The invariant sets associated with some leading parts of  $p$  and  $f_n g_d - g_n f_d$ .

## Open Problem

How can the semi-algorithm be turned into an algorithm?