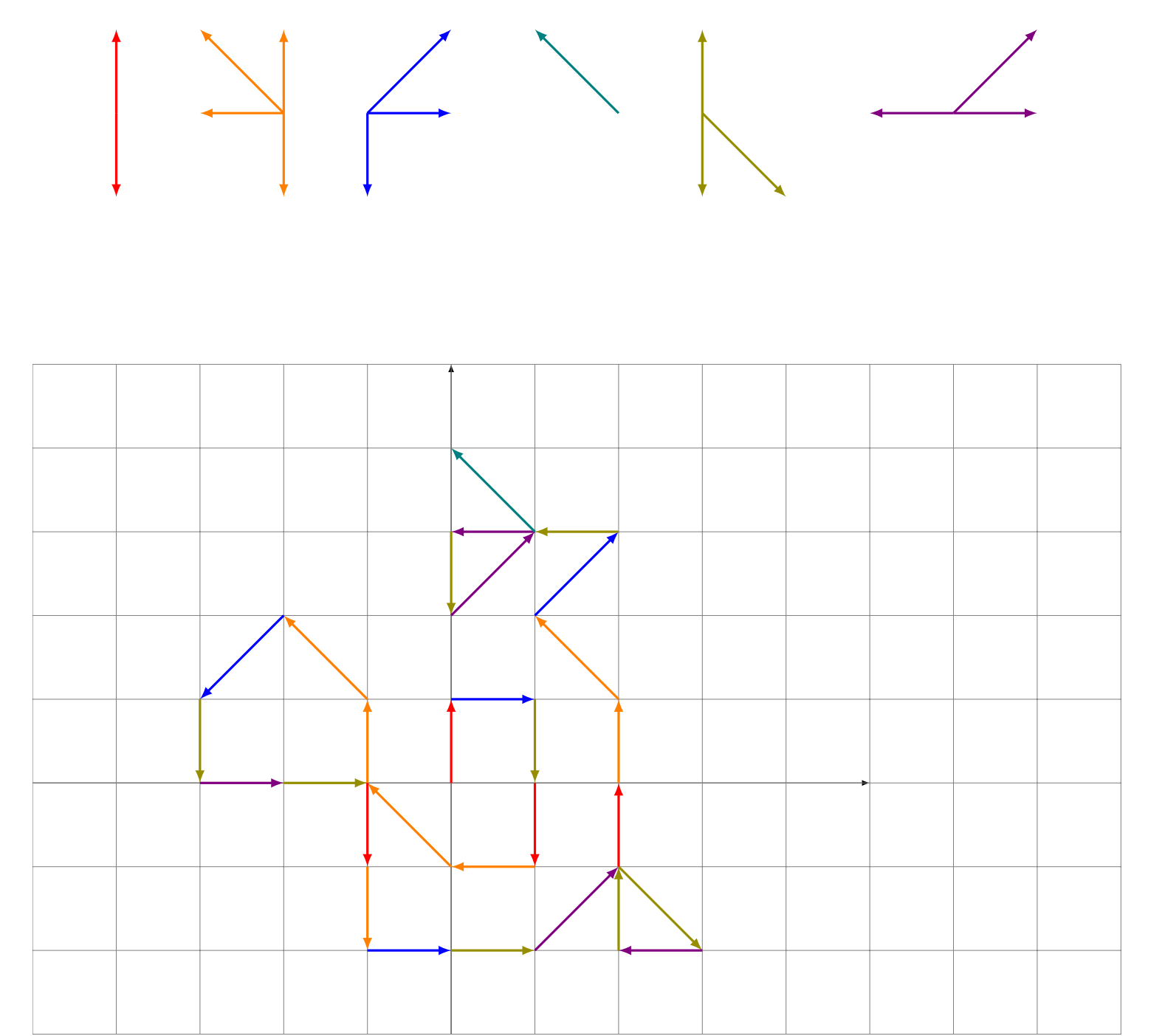
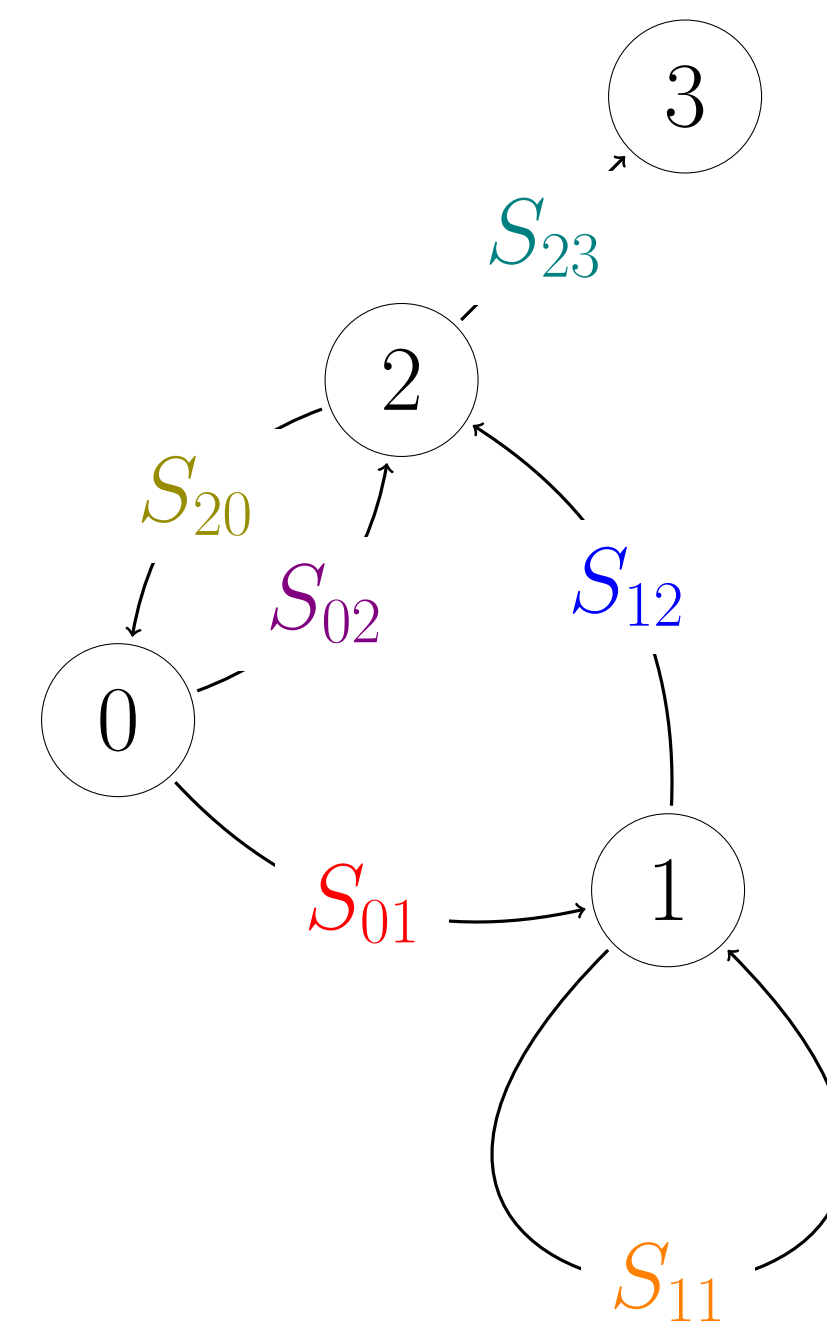




Inhomogeneities via finite Automata

A lattice walk is **homogeneous** with respect to a finite set $\mathbf{S} \subseteq \mathbb{Z}^d$, if each of its steps is taken from \mathbf{S} . It is called **inhomogeneous**, if the set of admissible steps is governed by a **deterministic finite automaton**. A finite automaton is a directed multigraph $(\mathcal{Q}, \mathcal{E})$ whose edges are labelled by letters of some alphabet. The vertices $q \in \mathcal{Q}$ are called states. One particular state $q_0 \in \mathcal{Q}$ is called the initial state, and there is a subset $\mathcal{Q} \subseteq \mathcal{Q}$ of final states. The edges are labelled by elements of \mathbb{Z}^d . To be deterministic means that for every pair (q, s) with $q \in \mathcal{Q}$ and $s \in \mathbb{Z}^d$ there is at most one edge starting from q and labelled by s . A lattice walk $w = w_0, \dots, w_n$ is inhomogeneous in this respect if there is a path in the automaton starting at the initial state, ending at one of the final states, and such that the i -th edge of the path is labelled with $w_i - w_{i-1}$. We write \mathbf{S}_{pq} for the set of all $s \in \mathbb{Z}^d$ which label an edge from p to q .



Unrestricted Lattice Walks

Given an inhomogeneity as above, let F be the generating function of walks in \mathbb{Z}^d that start at the origin counted by their length and endpoint, and F_q be the one of those associated with paths in the finite automaton that end at final state $q \in \bar{\mathcal{Q}}$. Then $F = \sum_{q \in \bar{\mathcal{Q}}} Q F_q$ and the F_q 's uniquely solve the following linear system of functional equations

$$F_q = [q = q_0] + t \sum_{p \in \mathcal{Q}} S_{pq} F_p, \quad q \in \mathcal{Q},$$

where $S_{pq}(x) = \sum_{i \in \mathbf{S}_{pq}} x^i$ is the step polynomial of \mathbf{S}_{pq} . In particular, F is a **rational function**.

Walks restricted to a Half-Space

Generating functions of walks restricted to $\mathbb{Z}^{d-1} \times \mathbb{Z}_{\geq 0}$ need not be rational in general, but they turn out to be always **algebraic**. This is a consequence of the following

Theorem

Let \mathbb{K} be a field of characteristic zero, $\Delta : \mathbb{K}[x][[t]]^n \rightarrow \mathbb{K}[x][[t]]^n$ be defined by $\Delta f(x, t) = (f(x, t) - f(0, t))/x$, and let $a \in \mathbb{K}[x, t]^n$ and $B_i \in \mathbb{K}[x, t]^{n \times n}$. Then

$$f = a + t \sum_{i=0}^k B_i \Delta^i f,$$

has a unique solution f in $\mathbb{K}[x][[t]]^n$, and its components are algebraic over $\mathbb{K}[x, t]$.

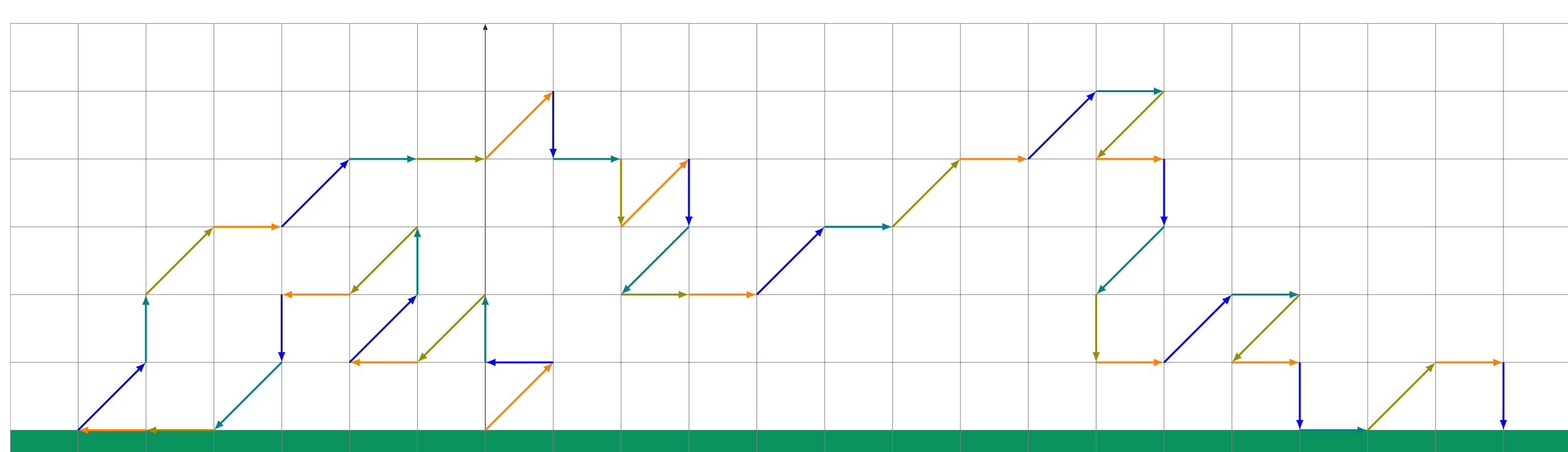
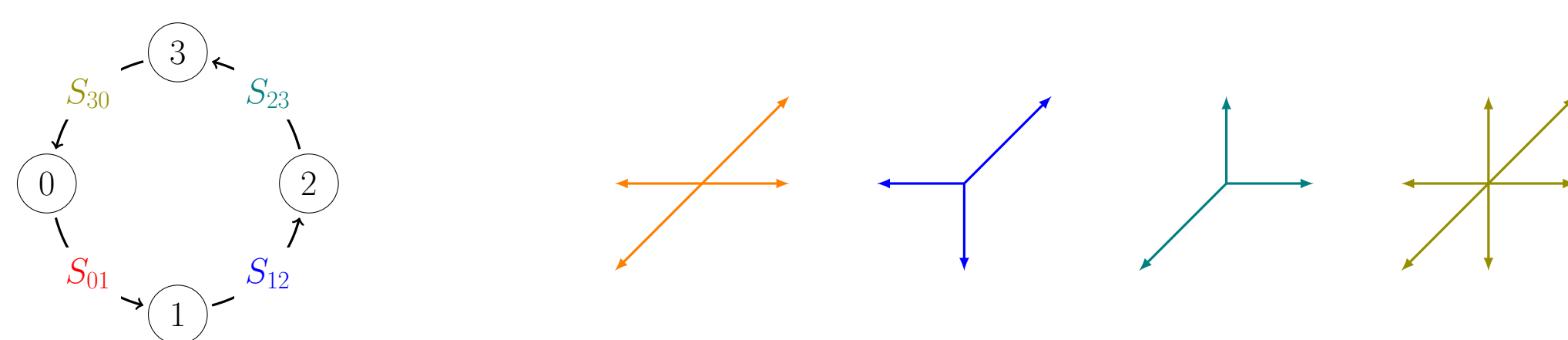
Sketch of Proof

1. Rewrite the equation in terms of evaluations of derivatives of f :

$$\left(x^k I_n - t \sum_{i=0}^k x^{k-i} B_i\right) f(x, t) = x^k a - t \sum_{j=0}^{k-1} \left(\sum_{i=j+1}^k \frac{x^{k+j-i}}{j!} B_i\right) f^{(j)}(0, t).$$

2. Eliminate $f(x, t)$ by
 - a) replacing x by a root $x(t)$ of $\det(x^k I_n - t \sum_{i=0}^k x^{k-i} B_i)$, and
 - b) multiplying the equation by elements of the co-kernel of the matrix.
3. Solve the resulting linear systems for the $f^{(j)}(0, t)$'s and $f(x, t)$.

Uniqueness of the solution of the linear system for the $f^{(j)}(0, t)$'s is not necessarily guaranteed, but can be assured by a perturbation argument.



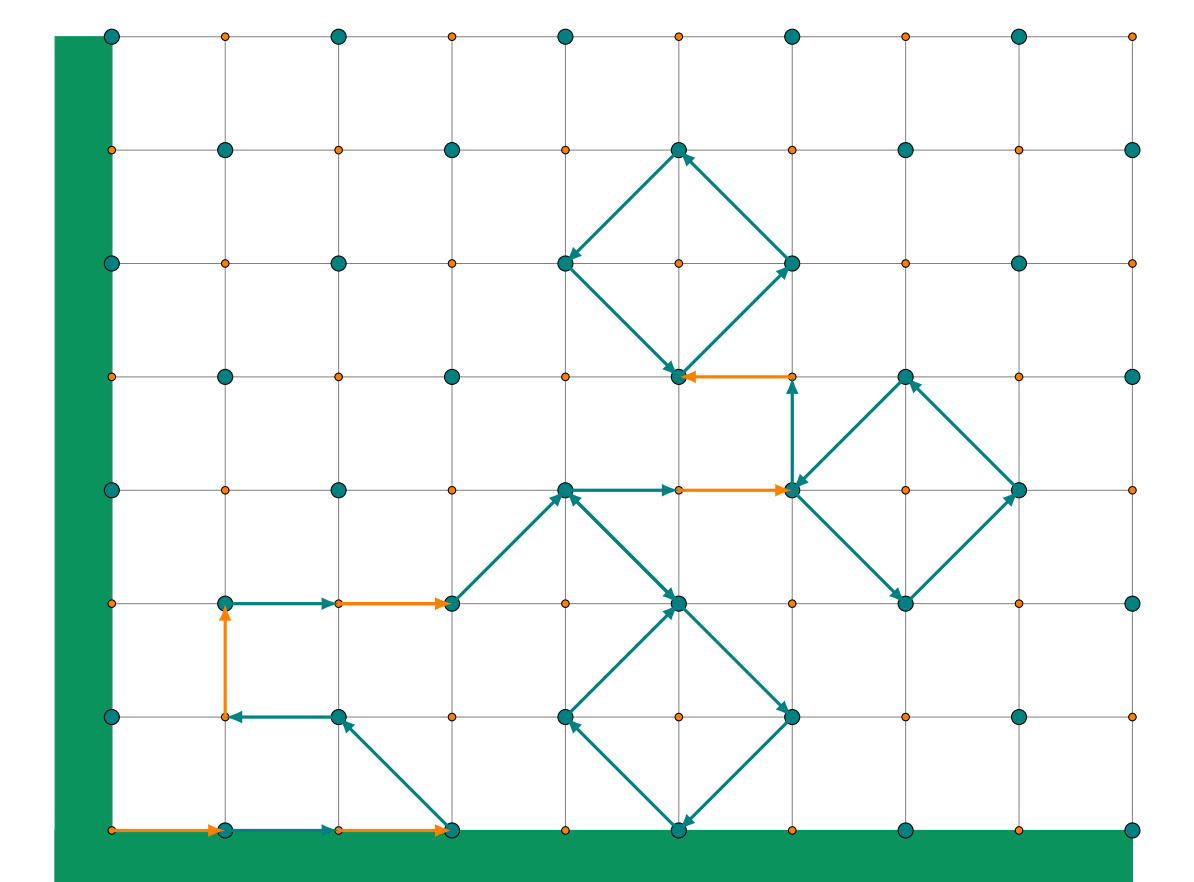
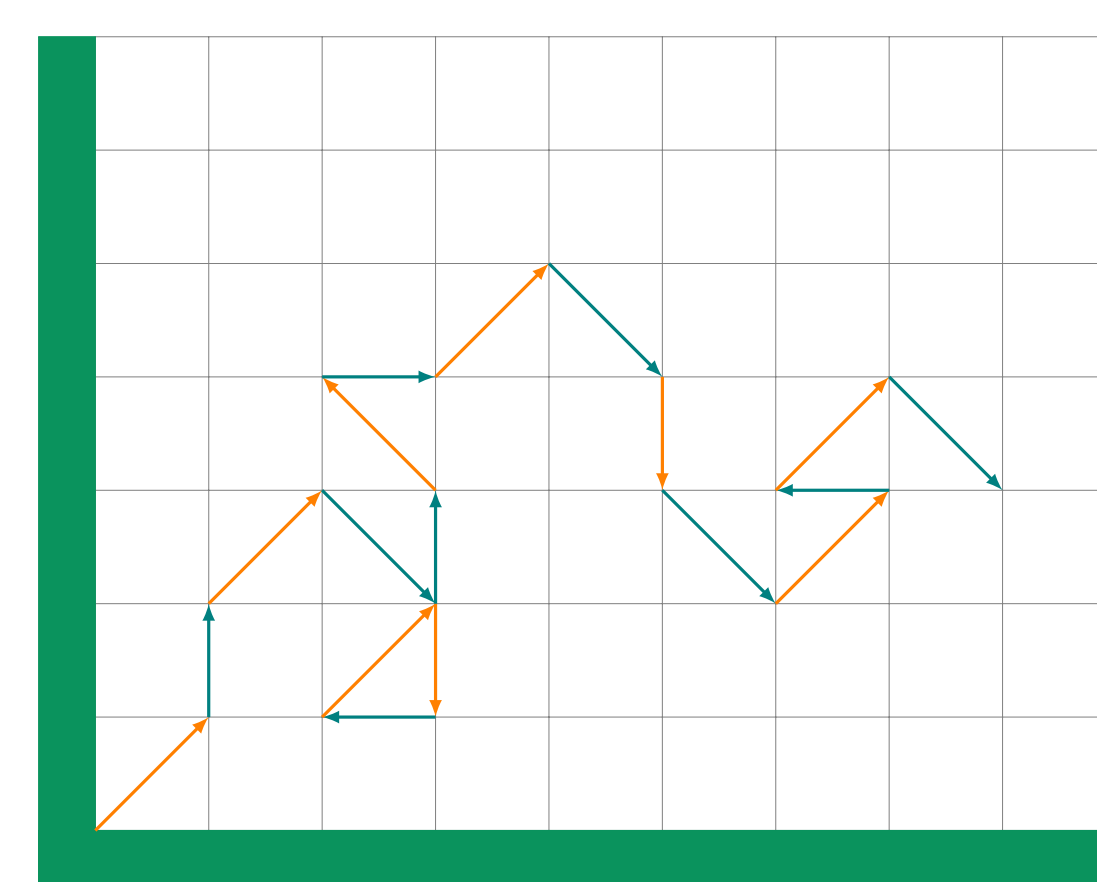
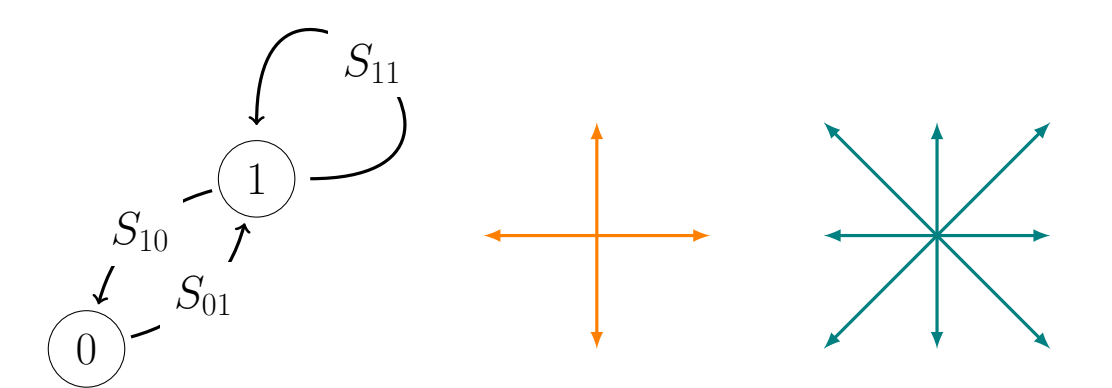
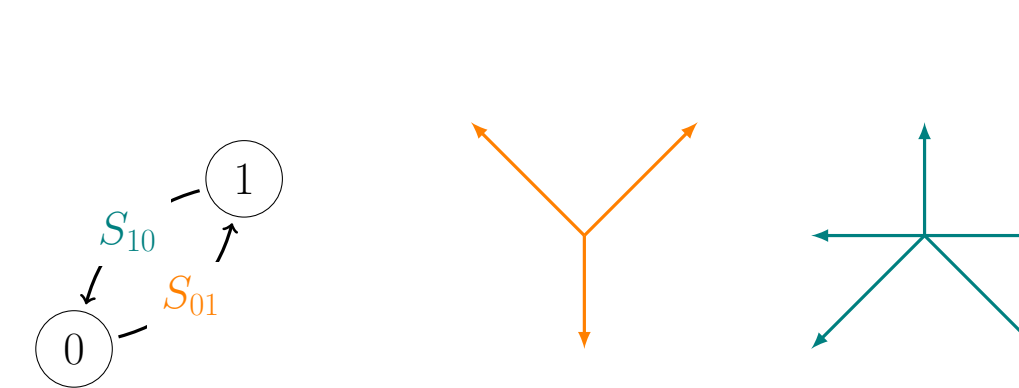
Walks restricted to the Orthant

The nature of generating functions of walks restricted to $\mathbb{Z}_{\geq 0}^d$ is more **diverse** when $d \geq 2$: it can be rational, algebraic but non-rational, D-finite but non-algebraic, or non-D-finite.

Methods for deciding their nature and finding expressions for them carry over from the homogeneous setting to the inhomogeneous one such as, for instance, the notion of **dimension**, the **decomposition** into and **projection** onto lower dimensional models, proofs of D-finiteness via the **kernel method**, proofs of non-D-finiteness via the computation of the **asymptotics** of their coefficients. . .

But for many models existing methods do not apply.

We investigated time-inhomogeneous and space-inhomogeneous models whose step sets are contained in $\{-1, 0, 1\}^2 \setminus \{(0, 0)\}$, **experimentally**.



We computed at least the first 10000 terms of the corresponding length generating functions and tried to **guess a differential equation**. If one was found, we also searched for an **algebraic equation**.

The classification is available at the [accompanying website](#).

References

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- [3] Qing-Hu Hou and Toufik Mansour. *"The kernel method and systems of functional equations with several conditions"*. In: *Journal of Computational and Applied Mathematics* 235(2011), pp. 1205-1212.