Separated Variables on Plane Algebraic Curves

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What's the problem?

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Given $r \in \mathbb{C}(x, y)$ and an irreducible $p \in \mathbb{C}[x, y]$, determine all solutions of

$$r + q(x, y)p = f(x) - g(y)$$

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It has applications in *computer vision*, *parameter identification* in ODE models and *algebraic (in)dependence* of solutions of ODEs.

Some (combinatorial) context

The enumeration of *lattice walks restricted to cones* leads to the study of *DDEs* and the problem of locating their solutions in the hierarchy of *rational*, *algebraic*, *D-finite* and *D-algebraic functions*.

Problem

Given $P \in \mathbb{Q}[x, y]$ and $Q \in \mathbb{Q}[v_0, v_1, v_2, \ldots, x, y, t]$, solve

$$F = P(x, y) + tQ\left(F, \Delta_x F, \Delta_y F, \ldots, x, y, t\right)$$

for $F(x, y; t)$ in $\mathbb{Q}[x, y][[t]]$,

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for $F(x, y; t)$ in $\mathbb{Q}[x, y][[t]]$, where

$$\Delta_x F(x, y; t) := \frac{F(x, y; t) - F(0, y; t)}{x}$$

and Δ_y is defined analogously.

Example

Let $f(i; n)$ be the number of walks on $\mathbb{N}$ that start at 0, end at i and consist of (presicely) n steps taken from $\{-1, 1\}$, and define

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Then

$$F(x) = 1 + t(x^{-1} + x)F(x) - tx^{-1}F(0).$$

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Theorem (Bousquet-Mélou, Jehanne)

The solution of an *ordinary* DDE is *algebraic*.

It is much more complicated for (linear) partial DDEs.

Let $S(x, y)$ be a Laurent polynomial such that

$$\mathrm{supp}(S) \subseteq \{-1, 0, 1\}^2$$

and define

$$K(x, y, t) = xy(1 - tS(x, y)).$$

The nature of the solution of a DDE of the form

$$\begin{aligned} K(x,y)F(x,y) &= xy + K(x,0)F(x,0) \\ &\quad + K(0,y)F(0,y) - K(0,0)F(0,0) \end{aligned}$$

can be very diverse.

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Theorem

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has a non-trivial solution. It is *algebraic* iff it is D-finite and

$$xy + q(x, y)K = f(x) - g(y)$$

has a non-trivial solution. Furthermore, if it is not D-finite, then it is *D-algebraic* iff the latter equation has a non-trivial solution.

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in $\mathbb{C}(x,y)$.

Let $\mathbb{C}[x, y]_p$ be the *local ring* at p, and let $\langle p \rangle$ be the ideal generated therein.

Define

$$F(r, p) = \{(f, g) \in \mathbb{C}(x) \times \mathbb{C}(y) : f - g \in r + \langle p \rangle\},$$

and let $(f, g) \in F(r, p)$.

Then

$$F(r, p) = (f, g) + F(0, p).$$

We refer to

$$F(p) \equiv F(0, p)$$

as the set of *separated multiples* of p. It is a *field* with respect to component-wise addition and multiplication. Furthermore,

$$F(p) = \mathbb{C}((f, g))$$

for some $(f, g) \in \mathbb{C}(x) \times \mathbb{C}(y)$. If $F(p)$ is not isomorphic to $\mathbb{C}$, we say that p is *separable*.

The problem splits into two subproblems:

1) find a generator of $\mathrm{F}(\mathrm{p})$, and
2) determine an element of $\mathrm{F}(\mathrm{r}, \mathrm{p})$.

The non-linear problem of solving

$$r + qp = f - g$$

will be reduced to a *linear* problem. The reduction will be based on the computation of the *poles* of f and g and their *multiplicities*.

Some definitions

A function ω on the set of terms in x and y is a *weight function* if

$$\omega(ax^i y^j) = \omega_x i + \omega_y j$$

for some $\omega_x, \omega_y \in \mathbb{Z}$ and all $i, j \in \mathbb{Z}_{\geq 0}$.

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The *sign vector* of ω is

$$\mathsf{sgn}(\omega) := (\mathsf{sgn}(\omega_x), \mathsf{sgn}(\omega_y)).$$

A test for non-separability

If p is separable, then there are $q_n \in \mathbb{C}[x, y]$, $f_n, f_d \in \mathbb{C}[x]$ and $g_n, g_d \in \mathbb{C}[y]$ such that

$$q_n p = f_n g_d - g_n f_d.$$

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The Newton polygon of $f_n g_d - g_n f_d$ is of a particular shape, and hence imposes restrictions on the Newton polygon of p.

Proposition

Let $\omega_1, \omega_2 \in \mathbb{Z}^2$ be two non-zero weight functions for which the leading parts of p are different and involve at least two terms. If

$$\operatorname{sgn}(w_1) = \operatorname{sgn}(w_2),$$

then p is not separable.

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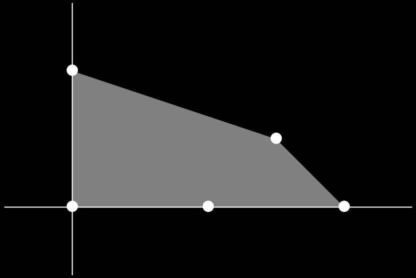
Corollary

The polynomial $1 + x^2 + x^4 + x^3 y + y^2$ is not separable.



The Newton polygon of $1 + x^2 + x^4 + x^3y + y^2$.

Constructing separated multiples

Poles

There is a generator (f, g) of $F(p)$ such that

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If (s_1, s_2) is a root of p, then

$$f(s_1) = \infty \quad \text{if and only if} \quad g(s_2) = \infty.$$

Let $\sim$ be the smallest equivalence relation on the curve such that

$$(x_0, y_0) \sim (x_1, y_1) \quad \text{whenever} \quad x_0 = x_1 \text{ or } y_0 = y_1.$$

The equivalence class of (x_0, y_0) is called the *orbit* of (x_0, y_0).

Proposition

The coordinates of the orbit of ∞ are poles of f and g, respectively. The orbit is finite, and it is exhaustive. If

$$F(p) \cong \mathbb{C},$$

it might however be infinite.

Multiplicities

Proposition

If $\mathfrak{p}$ is separable, then so is any of its leading parts.

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If p is separable, then so is any of its leading parts.

The leading parts of p and their separated multiples provide information on the multiplicities of the poles of f and g.

Assume that $\omega \in \mathbb{N}^2$ is such that

$$\mathrm{lp}_\omega(\mathsf{q})\mathrm{lp}_\omega(\mathsf{p}) = \mathrm{lp}_\omega(\mathsf{f}) - \mathrm{lp}_\omega(\mathsf{g}),$$

and let

$$\mathrm{F}(\mathrm{lp}_\omega(\mathsf{p})) = \mathbb{C}\left((\mathsf{f}_\omega, \mathsf{g}_\omega)\right).$$

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$$\mathrm{lp}_\omega(q)\mathrm{lp}_\omega(p) = \mathrm{lp}_\omega(f) - \mathrm{lp}_\omega(g),$$

and let

$$F(\mathrm{lp}_\omega(p)) = \mathbb{C}\left((f_\omega, g_\omega)\right).$$

Then there some $k \in \mathbb{Z}$ such that

$$\mathsf{lp}_\omega(f) - \mathsf{lp}_\omega(g) = f_\omega^k - g_\omega^k.$$

In particular,

$$(\mathrm{m}(\infty, f), \mathrm{m}(\infty, g)) = k \cdot (\deg f_\omega, \deg g_\omega).$$

Example

Let

$$p = xy - \left(1 + y + x^2y + x^2y^2\right).$$

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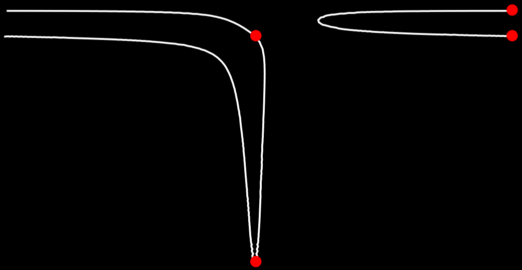
The elements of the orbit of ∞ are

$$(\infty, 0),\ (\infty, -1),\ (0, -1) \quad \text{and} \quad (0, \infty).$$



The leading part of p associated with $(\infty, 0)$ is

$$\mathrm{lp}_\omega(p) = -1 - x^2 y.$$

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Since

$$F(\mathrm{lp}_\omega(p)) = \mathbb{C}((x^2, -y^{-1})),$$

there is a $k \in \mathbb{N}$ such that

$$(\mathrm{m}(\infty, f), \mathrm{m}(0, g)) = k \cdot (2, 1).$$

The analysis of the other poles is done analogously. It results in the following 1-parameter family for their multiplicities.

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f		g	
∞	$2k$	∞	k
0	$2k$	0	k
		-1	$2k$

Making the ansatz

$$f = \frac{f_0 + f_1 x + \cdots + f_4 x^4}{x^2} \quad \text{and} \quad g = \frac{g_0 + g_1 y + \cdots + g_4 y^4}{y(1+y)^2}$$

and

$$q = \frac{q_{0,0} + q_{1,0} x + \cdots + q_{2,1} x^2 y}{x^2 y (1+y)^2}$$

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$$q = \frac{q_{0,0} + q_{1,0} x + \cdots + q_{2,1} x^2 y}{x^2 y (1+y)^2}$$

we find that

$$F(p) = \mathbb{C}\left(\left(\frac{(1-x)^2(1+x+x^2)}{x^2}, -\frac{(1+y+y^2)^2}{y(1+y)^2}\right)\right).$$

Proposition

If p is separable, then the choice of $k = 1$ results in the multiplicities of the poles of a generator of $F(p)$.

Determining an element of $F(r, p)$

We restrict

$$r + qp = f - g$$

to the curve defined by p and relate the poles of r to those of f and g and connect their multiplicities to the asymptotics of r.

Let (s_1, s_2) be a root of p. Then

$$f(s_1) = \infty \quad \text{or} \quad g(s_2) = \infty \quad \text{if} \quad r(s_1, s_2) = \infty.$$

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If

$$r(s_1, s_2) < \infty, \quad \text{then} \quad f(s_1) = \infty \quad \text{iff} \quad g(s_2) = \infty.$$

Proposition

If $\mathfrak{p}$ is separable, then every orbit is finite.

It is not enough to consider the orbits of the poles of r. It is also necessary to consider the orbits of its roots if p is separable. And the (finitely many) finite orbits if p is not separable.

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Proposition [Bell, Moosa, Topaz, Bellaïche]

If p is not separable, then the number of finite orbits is finite.

The multiplicities are determined by analyzing r at these points, and computing $\omega(\mathrm{r})$ and $\mathrm{F}(\mathrm{lp}_\omega(\mathrm{p}))$ for certain weights ω.

References

- *Separating variables in bivariate polynomial ideals*
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- *Separated variables on plane algebraic curves*
- *Galoisian structure of large steps walks confined in the first quadrant*