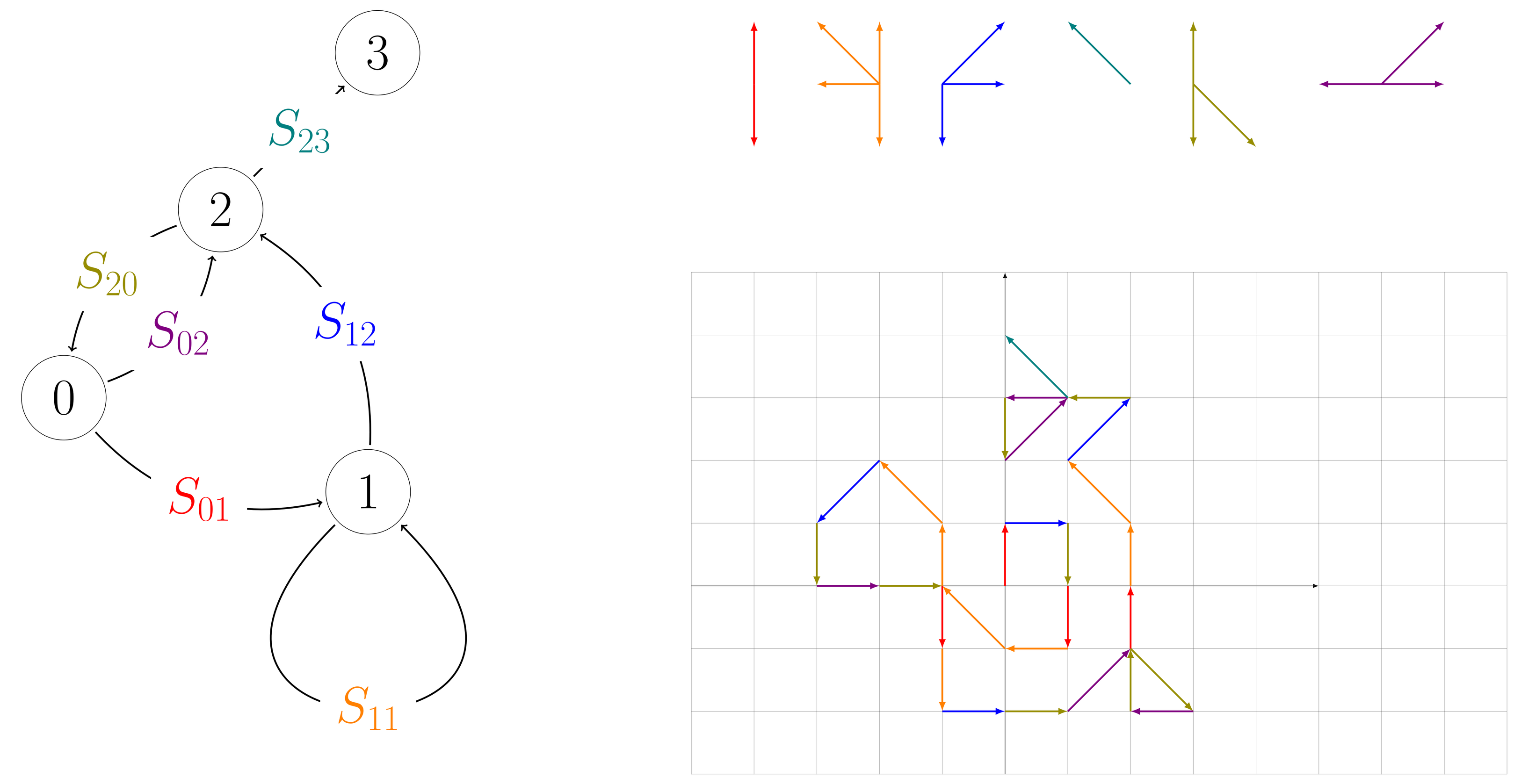
Inhomogeneous Restricted Lattice Walks

Manfred Buchacher* and Manuel Kauers, Institute for Algebra, JKU, Linz, Austria

JOHANNES KEPLER UNIVERSITÄT LINZ

Inhomogeneities via finite Automatons

A lattice walk is homogeneous with respect to a finite set $\mathbf{S} \subseteq \mathbb{Z}^d$, if each of its steps is taken from $\mathbf{S}$. It is called inhomogeneous, if the set of admissible steps is governed by a deterministic finite automaton. A finite automaton is a directed multigraph $(\mathcal{Q}, \mathcal{E})$ whose edges are labelled by letters of some alphabet. The vertices $q \in \mathcal{Q}$ are called states. One particular state $q_0 \in \mathcal{Q}$ is called the initial state, and there is a subset $\bar{\mathcal{Q}} \subseteq \mathcal{Q}$ of final states. The edges are labelled by elements of $\mathbb{Z}^d$. To be deterministic means that for every pair (q, s) with $q \in \mathcal{Q}$ and $s \in \mathbb{Z}^d$ there is at most one edge starting from q and labelled by s. A lattice walk $w = w_0, \dots, w_n$ is inhomogeneous in this respect if there is a path in the automaton starting at the initial state, ending at one of the final states, and such that the i-th edge of the path is labelled with $w_i - w_{i-1}$. We write $\mathbf{S}_{pq}$ for the set of all $s \in \mathbb{Z}^d$ which label an edge from p to q.



Unrestricted Lattice Walks

Given an inhomogeneity as above, let F be the generating function of walks in $\mathbb{Z}^d$ that start at the origin counted by their length and endpoint, and F_q be the one of those associated with paths in the finite automaton that end at final state $q \in \bar{\mathcal{Q}}$. Then $F = \sum_{q \in \bar{\mathcal{Q}}} F_q$ and the F_q's uniquely solve the following linear system of functional equations

$$F_q = [q = q_0] + t \sum_{p \in \mathcal{Q}} S_{pq} F_p, \quad q \in \mathcal{Q},$$

where $S_{pq}(x) = \sum_{i \in \mathbf{S}_{pq}} x^i$ is the step polynomial of $\mathbf{S}_{pq}$. In particular, F is a rational function.

Walks restricted to a Half-Space

Generating functions of walks restricted to $\mathbb{Z}^{d-1} \times \mathbb{Z}_{\geq 0}$ need not be rational in general, but they turn out to be always algebraic. This is a consequence of the following

Theorem

Let $\mathbb{K}$ be a field of characteristic zero, $\Delta : \mathbb{K}[x][[t]]^n \to \mathbb{K}[x][[t]]^n$ be defined by $\Delta f(x,t) = (f(x,t) - f(0,t))/x$, and let $a \in \mathbb{K}[x,t]^n$ and $B_i \in \mathbb{K}[x,t]^{n \times n}$. Then

$$f = a + t \sum_{i=0}^{k} B_i \Delta^i f,$$

has a unique solution f in $\mathbb{K}[x][[t]]^n$, and its components are algebraic over $\mathbb{K}[x,t]$.

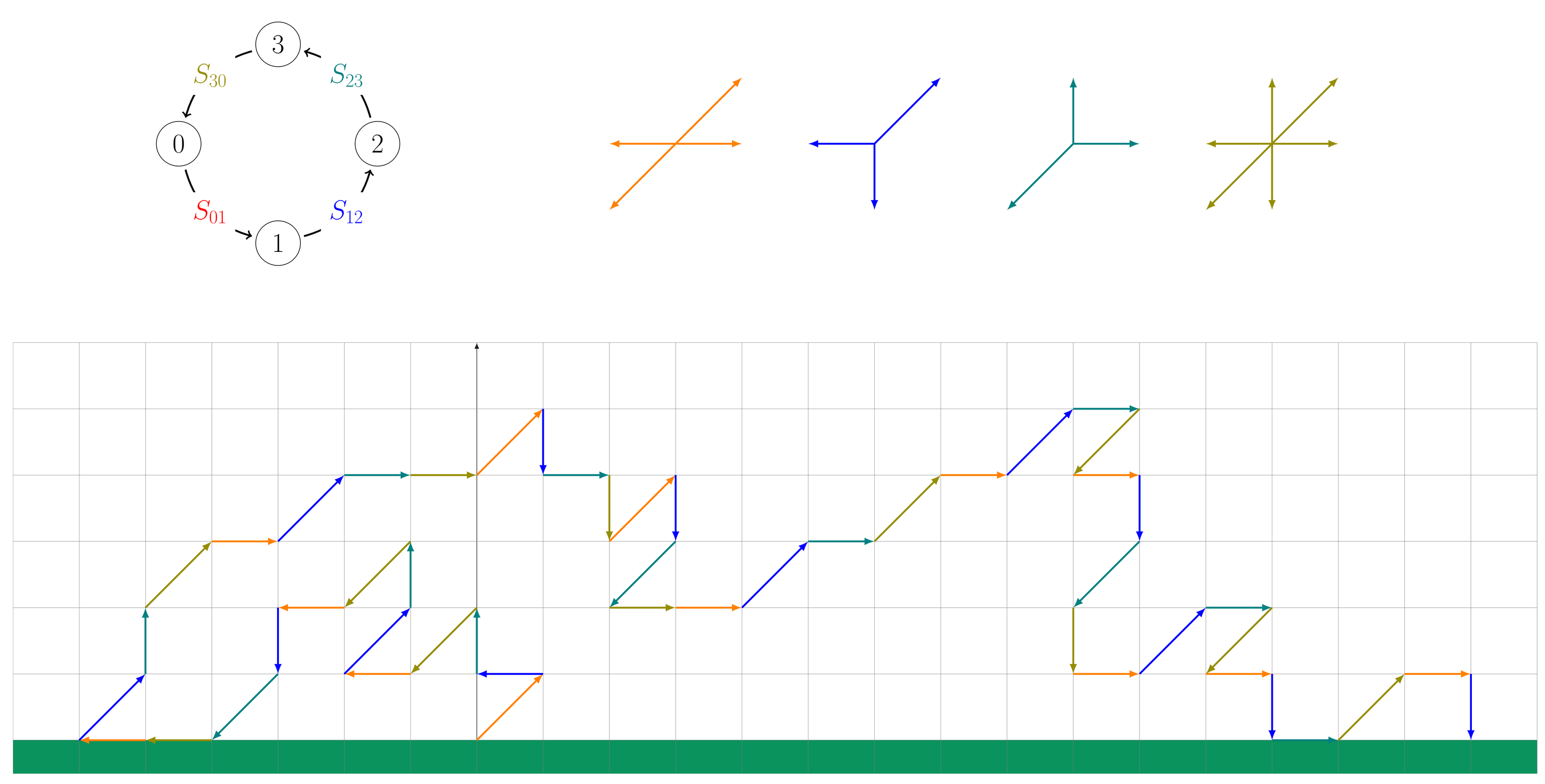
Sketch of Proof

1. Rewrite the equation in terms of evaluations of derivatives of f:

$$\left(x^k I_n - t \sum_{i=0}^{k} x^{k-i} B_i\right) f(x,t) = x^k a - t \sum_{j=0}^{k-1} \left(\sum_{i=j+1}^{k} \frac{x^{k+j-i}}{j!} B_i \right) f^{(j)}(0,t).$$

2. Eliminate $f(x,t)$ by
   a) replacing x by a root $x(t)$ of $\det\left(x^k I_n - t \sum_{i=0}^{k} x^{k-i} B_i\right)$, and
   b) multiplying the equation by elements of the co-kernel of the matrix.
3. Solve the resulting linear systems for the $f^{(j)}(0,t)$'s and $f(x,t)$.

Uniqueness of the solution of the linear system for the $f^{(j)}(0,t)$'s is not necessarily guaranteed, but can be assured by a perturbation argument.



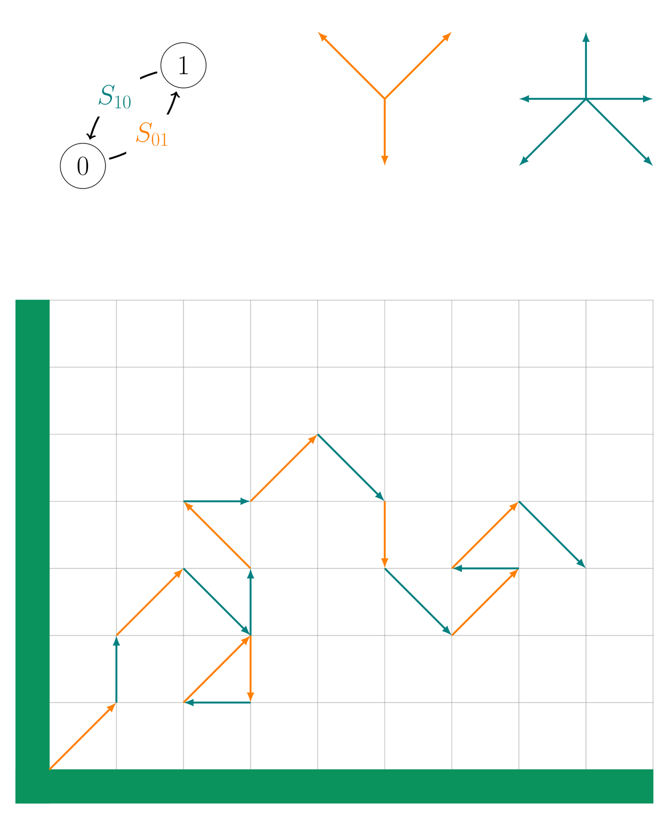
Walks restricted to the Orthant

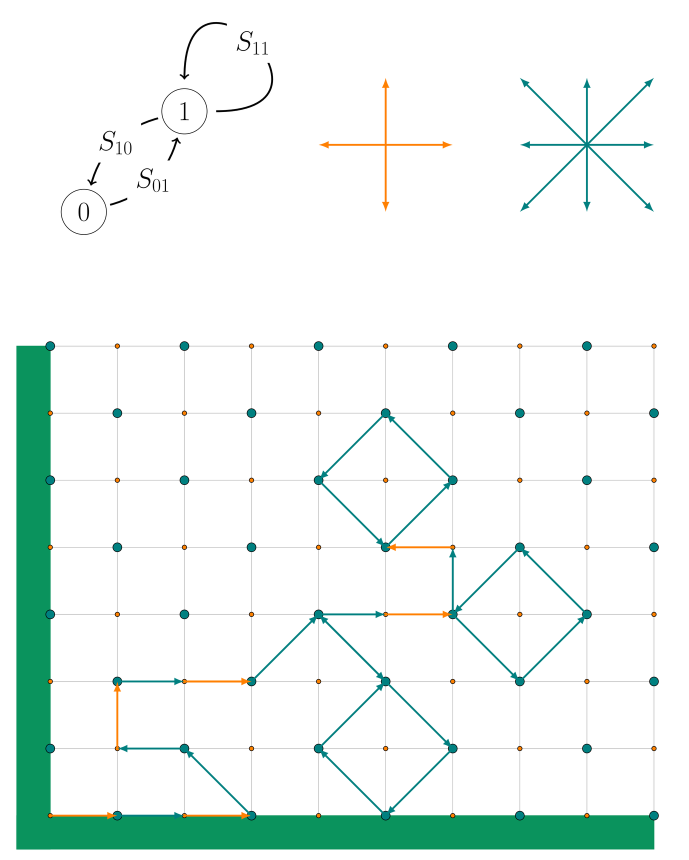
The nature of generating functions of walks restricted to $\mathbb{Z}^d_{\geq 0}$ is more diverse when $d \geq 2$: it can be rational, algebraic but non-rational, D-finite but non-algebraic, or non-D-finite.

Methods for deciding their nature and finding expressions for them carry over from the homogeneous setting to the inhomogeneous one such as, for instance, the notion of dimension, the decomposition into and projection onto lower dimensional models, proofs of D-finiteness via the kernel method, proofs of non-D-finiteness via the computation of the asymptotics of their coefficients...

But for many models existing methods do not apply.

We investigated time-inhomogeneous and space-inhomogeneous models whose step sets are contained in $\{-1,0,1\}^2 \setminus \{(0,0)\}$, experimentally.





We computed at least the first 10000 terms of the corrsponding length generating functions and tried to guess a differential equation. If one was found, we also searched for an algebraic equation.

The classification is available at the accompanying website.

References

[1] Manfred Buchacher and Manuel Kauers. *"Inhomogeneous restricted lattice walks"*. In: *Proceedings of FPSAC'19*. 2019.

[2] Mireille Bousquet-Mélou and Arnaud Jehanne. *"Polynomial equations with one catalytic variable, algebraic series and map enumeration"*. In: *Journal of Combinatorial Theory, Series B*. 96.5 (2006). pp. 661-704.

[3] Qing-Hu Hou and Toufik Mansour. *"The kernel method and systems of functional equations with several conditions"*. In: *Journal of Computational and Applied Mathematics* 235(2011), pp. 1205-1212.

manfred.buchacher@jku.at