Separating variables in bivariate polynomial ideals: the local case

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What is the problem?

Given $p \in \mathbb{C}[x, y]$, determine the solutions of

$$qp = f - g$$

for $f \in \mathbb{C}(x)$, $g \in \mathbb{C}(y)$ and $q \in \mathbb{C}(x, y)$.

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Why is the problem interesting?

It appears in the enumeration of lattice walks, and the computation of intersections of fields.

Assuming p is irreducible and not univariate,

$$F(p) := \{(f, g) \in \mathbb{C}(x) \times \mathbb{C}(y) : f - g \in \langle p \rangle\}$$

is a **field** with respect to component-wise addition and multiplication, and by Lüroth's theorem it is simple.

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What is the problem?

Find $(f, g) \in \mathbb{C}(x) \times \mathbb{C}(y)$ such that

$$F(p) = \mathbb{C}((f, g)).$$

Some examples

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The associated field of separated multiples is

$$\mathbb{C}\left(\left(\frac{1-x-x^3}{x^2},\frac{1-y-y^3}{y^2}\right)\right).$$

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Yes, because its product with $\frac{x-y}{x^2y^2}$ is

$$\frac{1 - x - x^3}{x^2} - \frac{1 - y - y^3}{y^2}.$$

Is $1 + x^2 + x^4 + x^3y + y^2$ separable?

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No, as we will see in a moment.

Some definitions

A function ω on the set of terms in x and y is a **weight function** if

$$\omega(x^i y^j) = \omega_x i + \omega_y j$$

for some $\omega_x, \omega_y \in \mathbb{Z}$ and all $i, j \in \mathbb{Z}_{\geq 0}$.

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The **sign vector** of ω is

$$\mathsf{sgn}(\omega) := (\mathsf{sgn}(\omega_x), \mathsf{sgn}(\omega_y)).$$

A test for non-separability

If p is separable, then there are $q \in \mathbb{C}[x, y]$, $f_n, f_d \in \mathbb{C}[x]$ and $g_n, g_d \in \mathbb{C}[y]$ such that

$$qp = f_n g_d - g_n f_d.$$

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The Newton polygon of $f_n g_d - g_n f_d$ is of a particular shape, and hence imposes restrictions on the Newton polygon of p.

Proposition

Let $\omega_1, \omega_2 \in \mathbb{Z}^2$ be two non-zero weight functions for which the leading parts of p are different and involve at least two terms. If

$$\mathrm{sgn}(w_1) = \mathrm{sgn}(w_2),$$

then p is not separable.

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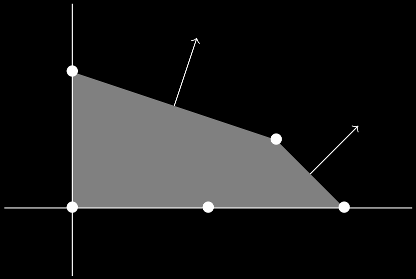
Corollary

The polynomial $1 + x^2 + x^4 + x^3y + y^2$ is not separable.

Proof



Constructing separated multiples

If there is a non-trivial $q \in \mathbb{C}(x, y)$ such that

$$qp = f - g,$$

for non-constant $f \in \mathbb{C}(x)$, $g \in \mathbb{C}(y)$, then it is enough to know the poles of f and g and their multiplicities to find it.

Determining poles

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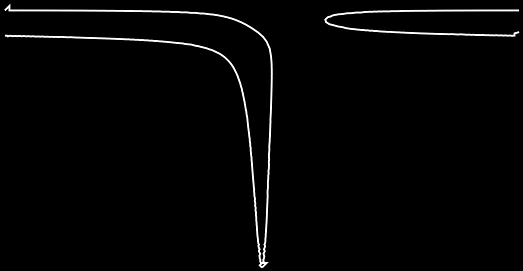
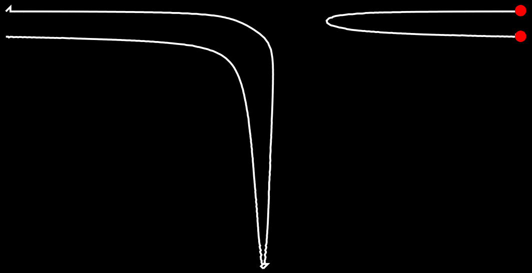
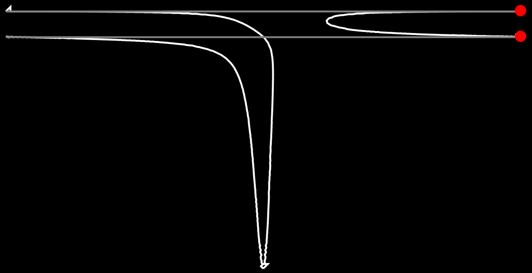
this is also true for ∞. If s is a finite pole of g, then

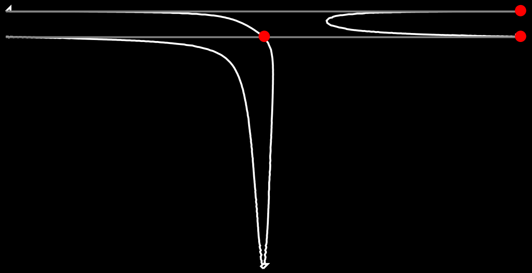
$$q_n(x, s)p(x, s) = -g_n(s)f_d(x)$$

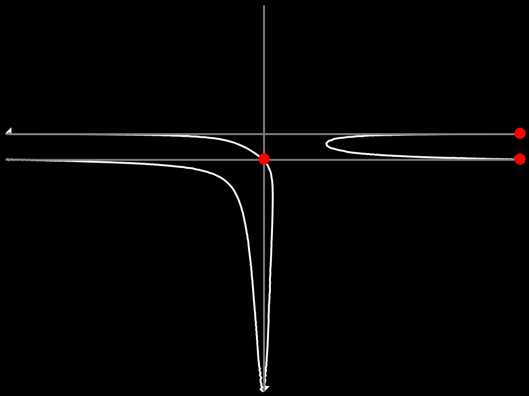
and each root of $p(x, s)$ is a pole of f.

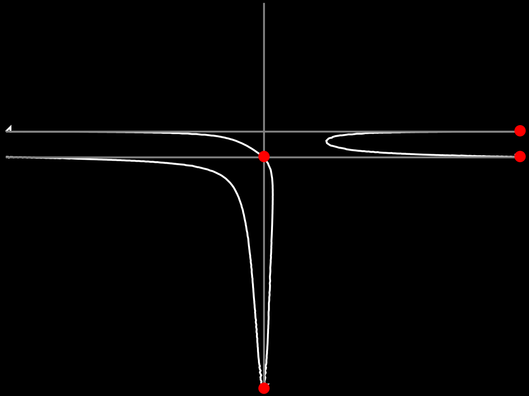












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However, if p is not separable, it might **not** terminate.

Determining multiplicities

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If p is near-separable, then so is any of its leading parts $\mathsf{lp}_\omega(p)$.

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The leading parts of p and their separated multiples provide information on the multiplicities of the singularities of f and g.

Let $f_\omega \in \mathbb{C}[x]$ and $g_\omega \in \mathbb{C}[y]$ be such that

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If $qp = f - g$, then

$$\mathsf{lp}_\omega(\mathsf{q})\mathsf{lp}_\omega(p) = \mathsf{lp}_\omega(f) - \mathsf{lp}_\omega(g),$$

and there is a $k \in \mathbb{N}$ such that

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Proposition

There is a $k \in \mathbb{N}$ such that

$$(\deg f, \deg g) = k \cdot (\deg f_\omega, \deg g_\omega).$$

Example

The points on the curve defined by

$$p := xy - \left(1 + y + x^2y + x^2y^2\right)$$

which describe the poles of a separated multiple of $f - g$ are

$$(\infty, 0),\ (\infty, -1),\ (0, -1) \quad \text{and} \quad (0, \infty).$$

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The corresponding leading parts of p (and its variants) are

$$-1 - x^2y,\ -x(1 + xy),\ -x - y \quad \text{and} \quad -y(1 + x^2y),$$

and their minimal separated multiples are

$$x^2 + y^{-1},\ x + y^{-1},\ x^{-1} + y^{-1}, \quad \text{and} \quad x^{-2} + y.$$

They indicate the following table for the multiplicities:

f		g	
s	m	s	m
∞	$2k$	∞	k
0	$2k$	0	k
		-1	$2k$

Making the ansatz

$$f = \frac{f_0 + f_1 x + \cdots + f_4 x^4}{x^2} \quad \text{and} \quad g = \frac{g_0 + g_1 y + \cdots + g_4 y^4}{y(1+y)^2}$$

and

$$q = \frac{q_{0,0} + q_{1,0} x + \cdots + q_{2,1} x^2 y}{x^2 y (1+y)^2},$$

we find that

$$F(p) = \mathbb{C}\left(\left(\frac{(1-x)^2(1+x+x^2)}{x^2}, -\frac{(1+y+y^2)^2}{y(1+y)^2}\right)\right).$$

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The minimal separated multiples of the leading parts of p give rise to a one-parameter family of multiplicities parametrized by $k \in \mathbb{N}$.

Does the choice of $k = 1$ give the right multiplicities?

Definition

A polynomial of the form

$$f_n g_d - g_n f_d, \quad f_n, f_d \in \mathbb{C}[x],\ g_n, g_d \in \mathbb{C}[y],$$

is said to be **near-separated**.

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The Galois group G of $\overline{\mathbb{C}(t)}/\mathbb{C}(t)$ acts on $\mathbb{Z}_m \times \mathbb{Z}_n$ by

$$\pi(i,j) = (i',j') \quad :\Longleftrightarrow \quad (\pi(\alpha_i), \pi(\beta_j)) = (\alpha_{i'}, \beta_{j'}).$$

Definition

A subset $T \subseteq \mathbb{Z}_m \times \mathbb{Z}_n$ is said to be **invariant** if

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If T is invariant and for all $i, i' \in \mathbb{Z}_m$

$$\chi_T(i, \_) = \chi_T(i', \_) \quad \text{or} \quad \chi_T(i, \_) \cdot \chi_T(i', \_) = 0,$$

then it is called **separated**.

Consider the map

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Proposition

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It restricts to a bijection between **near-separated factors** and **separated invariant subsets.**

It is **monoton**.

Definition

Let T be an invariant subset of $\mathbb{Z}_m \times \mathbb{Z}_n$. The **separable closure** of T is

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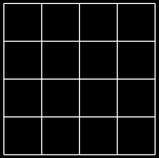
If T is the invariant subset corresponding to p, then the invariant subset corresponding to its minimal near-separated multiple is T^{sep}.

If $f_n g_d - g_n f_d$ is the minimal near-separated multiple of p, then

$$T^{\mathrm{sep}} = \mathbb{Z}_m \times \mathbb{Z}_n.$$

There are four invariant sets that can be associated with the numerator of

$$\frac{(1-x)^2(1+x+x^2)}{x^2}+\frac{(1+y+y^2)^2}{y(1+y)^2}.$$



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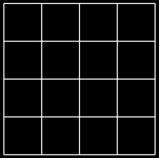

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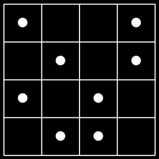
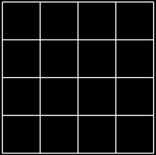


$xy - 1 - y - x^2y - x^2y^2$



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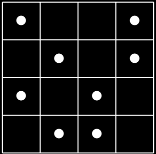

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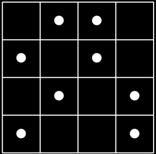

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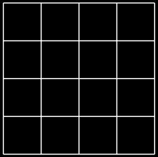

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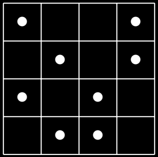

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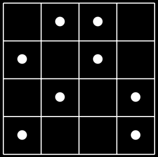

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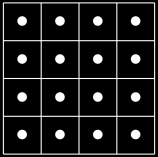
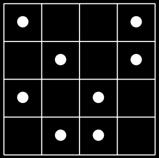


$f_n g_d - g_n f_d$

Let us have a closer look at one of these invariant sets.



$$xy - 1 - y - x^2y - x^2y^2$$

$$
\begin{array}{c|c|c|c|c|}
 & -1 & -1 & 0 & \infty \\
\hline
0 & \bullet & & & \bullet \\
\hline
0 & & \bullet & & \bullet \\
\hline
\infty & \bullet & & \bullet & \\
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\infty & & \bullet & \bullet & \\
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\end{array}
$$

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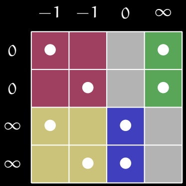

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Let $f_n g_d - g_n f_d$ be a near-separated polynomial, and let T be the invariant set associated with one of its factors p.

Let (s_1, s_2) be a pair of poles associated with $f - g$, and let $T^{s_1, s_2} \subseteq T$ be the pairs of roots of $f - t$ and $g - t$ related to it.

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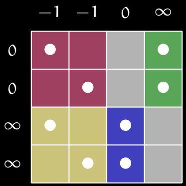
Assume that $s_1, s_2 \in \{0, \infty\}$, and let ω be the weight function associated with (s_1, s_2). Then T^{s_1,s_2} and the invariant set of

$$\mathsf{lp}_\omega(p) \mid \mathsf{lp}_\omega(f_n g_d - g_n f_d)$$

can be identified via

$$(\alpha, \beta) \mapsto (\bar{\alpha}, \bar{\beta}).$$



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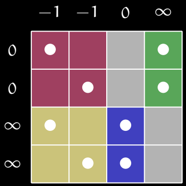

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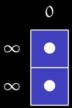
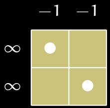
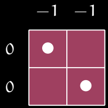


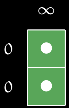
$x^2 + y^{-1} \mid x^2 + y^{-1}$



$x - y^{-1} \mid x^2 - y^{-2}$

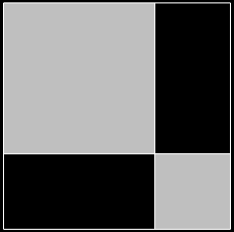


$x^{-1} + y^{-1} \mid x^{-2} - y^{-2}$



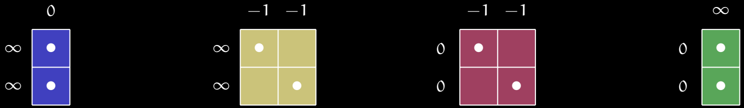
$x^{-2} + y \mid x^{-2} + y$

The set of poles of f and g found by inspecting the leading parts of p (and its variants) is exhaustive.



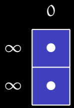
The choice of $k = 1$ results in the multiplicities of the poles of a minimal separated multiple of p.

Let us consider the diagrams associated with the leading parts of $p := xy - 1 - y - x^2y - x^2y^2$ and their near-separated multiples.

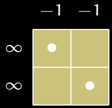


$x^2 + y^{-1} \mid x^2 + y^{-1}$ $\quad$ $x - y^{-1} \mid x^2 - y^{-2}$ $\quad$ $x^{-1} + y^{-1} \mid x^{-2} - y^{-2}$ $\quad$ $x^{-2} + y \mid x^{-2} + y$

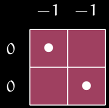
Let us consider the diagrams associated with the leading parts of $p := xy - 1 - y - x^2y - x^2y^2$ and their near-separated multiples.



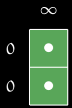
$x^2 + y^{-1} \mid x^2 + y^{-1}$



$x - y^{-1} \mid x^2 - y^{-2}$

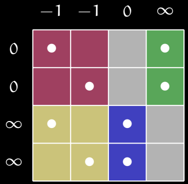


$x^{-1} + y^{-1} \mid x^{-2} - y^{-2}$

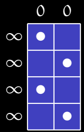


$x^{-2} + y \mid x^{-2} + y$

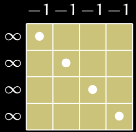
They fit together nicely (and make up the diagram associated with $p \mid f_n g_d - g_n f_d$).



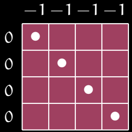
However, there are many other diagrams that can be associated with the leading parts of p (and its variants), for instance:



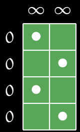
$x^2 + y^{-1} \mid x^4 + y^{-2}$



$x - y^{-1} \mid x^4 - y^{-4}$

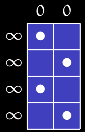


$x^{-1} + y^{-1} \mid x^{-4} + y^{-4}$

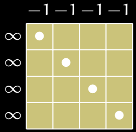


$x^{-2} + y \mid x^{-4} + y^2$

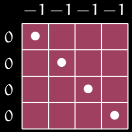
However, there are many other diagrams that can be associated with the leading parts of p (and its variants), for instance:



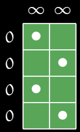
$x^2 + y^{-1} \mid x^4 + y^{-2}$



$x - y^{-1} \mid x^4 - y^{-4}$

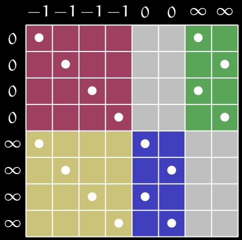


$x^{-1} + y^{-1} \mid x^{-4} + y^{-4}$



$x^{-2} + y \mid x^{-4} + y^2$

Again, they fit together to make up a (bigger) diagram.



It turns out that the first diagram is distinguished among the family of diagrams that can be constructed in this way.

	-1	-1	0	∞
0	•			•
0		•		•
∞	•		•	
∞		•	•	

$$
\begin{array}{c|c|c|c|c|}
 & -1 & -1 & 0 & \infty \\
\hline
0 & \bullet & & & \bullet \\
\hline
0 & & \bullet & & \bullet \\
\hline
\infty & \bullet & & \bullet & \\
\hline
\infty & & \bullet & \bullet & \\
\hline
\end{array}
\qquad\qquad
\begin{array}{c|c|c|c|c|}
 & -1 & -1 & 0 & \infty \\
\hline
0 & \bullet & \bullet & \bullet & \bullet \\
\hline
0 & \bullet & \bullet & \bullet & \bullet \\
\hline
\infty & \bullet & \bullet & \bullet & \bullet \\
\hline
\infty & \bullet & \bullet & \bullet & \bullet \\
\hline
\end{array}
$$

	-1	-1	-1	-1	0	0	∞	∞
0	●						●	
0		●						●
0			●				●	
0				●				●
∞	●				●			
∞		●				●		
∞			●		●			
∞				●		●		

	-1	-1	-1	-1	0	0	∞	∞
0	●						●	
0		●						●
0			●				●	
0				●				●
∞	●				●			
∞		●				●		
∞			●		●			
∞				●		●		

	-1	-1	-1	-1	0	0	∞	∞
0	●		●		●		●	
0		●		●		●		●
0	●		●		●		●	
0		●		●		●		●
∞	●		●		●		●	
∞		●		●		●		●
∞	●		●		●		●	
∞		●		●		●		●

Manfred Buchacher, *Separating variables in bivariate polynomial ideals: the local case*

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Olivier Bernardi, Mireille Bousquet-Mélou, and Kilian Raschel, *Counting quadrant walks via Tutte's invariant method*, 2020

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