

$$x(1 - t(x^{-1} + x))F(x; t) = x - tF(0; t)$$

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Research interests:

Algorithmic enumerative combinatorics, computer and constructive algebra, discrete differential equations and the elementary power series algebra to solve them.

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Goal: Solve the equation for $F(x; t)$ in $\mathbb{Q}[x][[t]]$.

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where Δ acts on $\mathbb{Q}[x][[t]]$ by

$$\Delta F(x;t) := (F(x;t) - F(0;t))/x.$$

Combinatorial meaning

Definition

A lattice walk is a sequence P_0, P_1, \dots, P_n of points of \mathbb{Z}^d . We call P_0 and P_n its starting and end point, respectively, the consecutive differences $P_{i+1} - P_i$ are its steps, and n is its length.

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Example

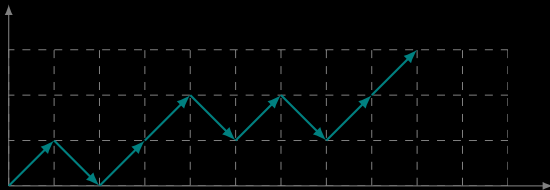
The sequence $(0, 1, 0, 1, 2, 1, 2, 1, 2, 3)$ is a lattice walk on \mathbb{N} that starts at 0, ends at 3, has length 9, and whose steps are $1, -1, 1, 1, -1, 1, -1, 1, 1$.

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Let $f(i; n)$ be the number of walks on \mathbb{N} that start at 0, end at i and consist of (precisely) n steps taken from $\{-1, 1\}$, and let

$$F(x; t) := \sum_{n \geq 0} \left(\sum_{i \geq 0} f(i; n) x^i \right) t^n.$$

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$$F(x; t) := \sum_{n \geq 0} \left(\sum_{i \geq 0} f(i; n) x^i \right) t^n.$$

Then $F(x; t)$ solves the functional equation, since

$$\begin{aligned} f(i; n+1) &= f(i-1; n) + f(i+1, n) \\ f(i; 0) &= \delta_{i,0} \quad \text{and} \quad f(i; n) = 0, \quad i < 0. \end{aligned}$$

Existence and uniqueness

By extracting the coefficient of t^n of

$$F(x; t) = 1 + t(x^{-1} + x)F(x; t) - tx^{-1}F(0; t),$$

we find that

$$[t^n]F(x; t) = \begin{cases} 1, & \text{if } n = 0, \\ (x^{-1} + x)[t^{n-1}]F(x; t) - x^{-1}[t^{n-1}]F(0; t) & \text{otherwise} \end{cases}$$

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This recurrence relation has a unique solution, and therefore this is also true for the functional equation.

The map

$$F(x; t) \mapsto 1 + t(x^{-1} + x)F(x; t) - tx^{-1}F(0; t)$$

represents a contraction on $\mathbb{Q}[x][[t]]$ with respect to

$$d(F, G) := 2^{-\text{val}(F-G)},$$

where

$$\text{val}(F) := \min\{n : [t^n]F \neq 0\}.$$

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Banach's fixed point theorem implies the existence and uniqueness of a solution of the functional equation.

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It means to decide whether its solution is **rational**, **algebraic**, **D-finite** or **D-algebraic**, and in case it is, to find an algebraic or differential equation for it.

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- the formal or **algebraic** one: involving elementary power series algebra, constructive and computer algebra.

We focus on formal and algebraic methods, and show how discrete differential equations can be solved using only operations such as

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The latter operator acts on series by discarding all terms involving non-positive powers in x , e.g.

$$[x^>] \left(x^{-2} + x^{-1} + 1 + x + x^2 \right) = x + x^2.$$

The classical kernel method

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where $\bar{x} := x^{-1}$, replace x by the power series root

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to find that

$$F(0; t) = x_0(t)/t, \quad \text{and} \quad F(x; t) = \frac{x - x_0(t)}{x(1 - t(\bar{x} + x))}.$$

Theorem [Bousquet-Mélou, Jehanne]

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Theorem [Buchacher, Kauers]

(The components of) solutions of linear systems of ordinary linear discrete differential equations are **algebraic** (over a field of rational functions).

The orbit-sum method

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and **discarding** non-positive powers in x ,

$$xF(x; t) = [x^>] \frac{x - \bar{x}}{1 - t(\bar{x} + x)}.$$

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- Given $p \in \mathbb{Q}[x_1, \dots, x_n][y]$, how can we solve

$$p(x_1, \dots, x_n, y) = 0$$

for y in terms of series in x_1, \dots, x_n ?

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- How can we do effective arithmetic for multivariate algebraic series?
- How can we compute the convex hull of the support of a multivariate algebraic series?

Wiener-Hopf factorization

The coefficient polynomial of $F(x; t)$ in

$$x(1 - t(\bar{x} + x))F(x; t) = x - tF(0; t)$$

factors into

$$-t(x - x_0)(x - x_1) = -t(1 - \bar{x}x_0)x(x - x_1),$$

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where

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So

$$-t(x - x_1)F(x; t) = \frac{x - tF(0; t)}{(1 - \bar{x}x_0)x}$$

with the lhs in $\mathbb{Q}[x][[t]]$ and the rhs in $\mathbb{Q}[\bar{x}][[t]]$.

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and so

$$F(x; t) = -\frac{1}{t(x - x_1)}.$$

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- If there is such a factorization, how can it be determined?

Compositional inverses and Lagrange inversion

Replacing t in

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by the power series root

$$t_0(x) := \frac{x}{1 + x^2} \quad \text{of} \quad x(1 - t(\bar{x} + x)) \in \mathbb{Q}(x)[t],$$

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i.e. $\frac{x}{1+x^2}$ and $xF(0; x)$ are compositional inverse to each other.

Theorem [Lagrange Inversion]

Let $H(t) \in \mathbb{Q}[[t]]$ be of valuation 1, and let $G(t)$ be its compositional inverse, i.e. such that $G(H(t)) = t$. Then

$$[t^n]G(t) = \frac{1}{n}[t^{-1}]H^{-n}(t), \quad n \neq 0.$$

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Applying this to

$$H(t) = (t^{-1} + t)^{-1}$$

we find that

$$[t^{n+1}]tF(0;t) = \frac{1}{n+1}[t^{-1}](t^{-1} + t)^{n+1},$$

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$$[t^{n+1}]tF(0;t) = \frac{1}{n+1}[t^{-1}](t^{-1} + t)^{n+1},$$

which means that

$$f(0;n) = \begin{cases} 0, & \text{if } n \not\equiv 0 \pmod{2}, \\ \frac{1}{n/2+1} \binom{n}{n/2} & \text{otherwise.} \end{cases}$$

The series $F(0; t)$ is algebraic (and consequently also $F(x; t)$) because

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Therefore

$$P(t, F(0; t)) = 0 \quad \text{for} \quad P(t, s) = t^2 s^2 - s + 1.$$

Guess and prove

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2. computing a truncation

$$F_0(x; t) := \sum_{n=0}^{n_0} \left(\sum_{i \geq 0} f(i; n) x^i \right) t^n$$

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3. setting coefficients of $x^i t^j$, $j \leq n_0$, in $P(x, t, F_0(x; t))$ equal to zero, and solving the resulting linear system.

For $d = 2$ and $n_0 = 8$ we find

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$$F_{\text{cand}}(x; t) \in \mathbb{Q}[x][[t]], \text{ and}$$

2. verifying that $F_{\text{cand}}(x, t)$ satisfies the functional equation.

The former can be done using the **Newton-Puiseux algorithm**.

For the latter we perform closure properties of algebraic functions to find that

$$A(y) := y(1 - 4t^2 - t^2y^2)$$

is an annihilating polynomial for

$$x(1 - t(\bar{x} + x))F_{\text{cand}}(x; t) - x - tF_{\text{cand}}(0; t).$$

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By plugging some truncation of this series into the second factor of A we find that y is its minimal polynomial.

Looking for an additional equation

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The additional equation allows to determine $F(x; t)$ by solving a system of polynomial equations.

Note: Although the second equation is equivalent to the first, the latter suggests to write $F(0;t)$ as a continued fraction

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This indicates that there is a connection between the enumeration of restricted lattice walks and continued fractions and orthogonal polynomials.

The reflection principle

The orbit-sum method showed that

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The combinatorial interpretation of this equation is that

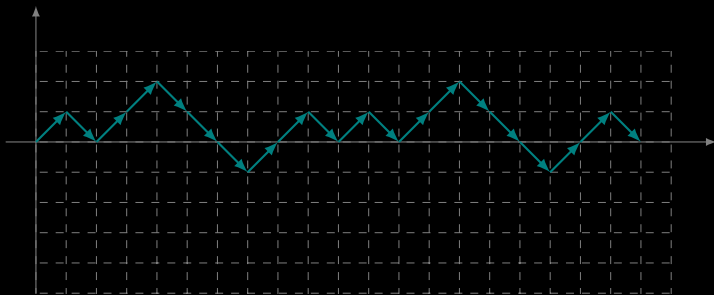
walks in \mathbb{Z} from 0 to i not lying in \mathbb{N}

=

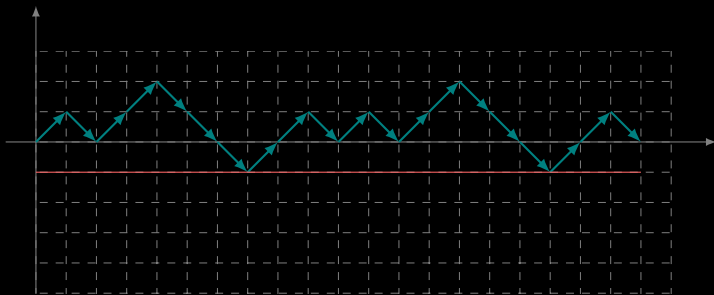
walks in \mathbb{Z} from -2 to i

A bijection is given by the reflection principle:

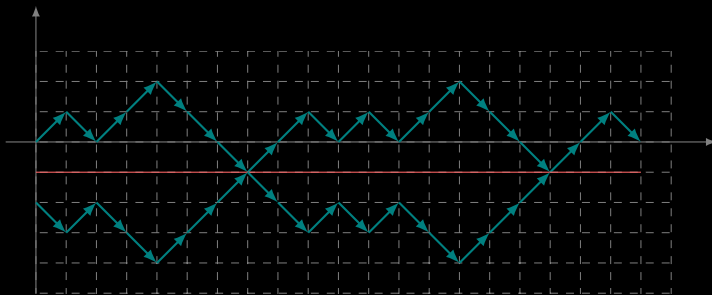
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This implies that

$$\begin{aligned} f(i; n) &= \# \text{ walks in } \mathbb{Z} \text{ from } 0 \text{ to } i \text{ of length } n - \\ &\quad \# \text{ walks in } \mathbb{Z} \text{ from } -2 \text{ to } i \\ &= \binom{n}{\frac{n+i}{2}} - \binom{n}{\frac{n+i+2}{2}} = \frac{1}{\frac{n+i}{2} + 1} \binom{n}{\frac{n+i}{2}}. \end{aligned}$$

The cycle lemma / Spitzer's lemma

Langrange inversion implies that

$$f(0; n) = \frac{1}{n+1} [t^{-1}] (t^{-1} + t)^{n+1}$$

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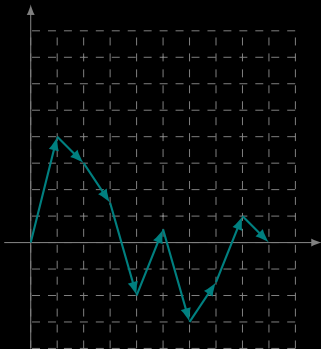
Is there a combinatorial explanation for this identity?

Lemma [Spitzer's lemma]

Let $a_1, a_2, \dots, a_N \in \mathbb{R}$ be s.t. $a_1 + a_2 + \dots + a_N = 0$ and no other partial sum of consecutive a_i 's read cyclically vanishes. Then there is a unique cyclic permutation $a_i, a_{i+1}, \dots, a_N, a_1, \dots, a_{i-1}$ s.t. the sum of the first j numbers is non-negative for $j = 1, 2, \dots, N$.

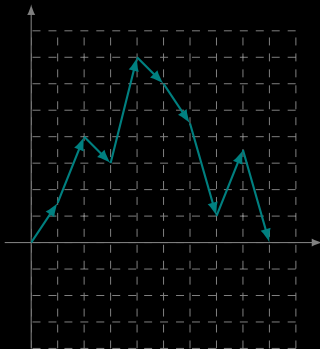
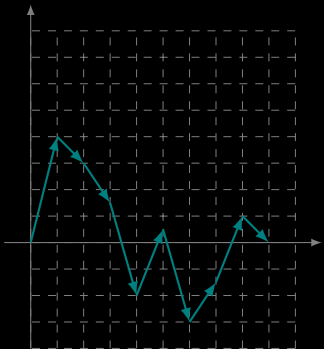
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Theorem

Let $r, s \in \mathbb{N}_+$ be relative prime. Then the number of walks in \mathbb{Z}^2 from $(0,0)$ to (r,s) with steps taken from $\{(1,0), (0,1)\}$ staying weakly below $ry = sx$ equals $\frac{1}{r+s} \binom{r+s}{r}$.

Thank you!