The Newton-Puiseux algorithm and effective algebraic series

Manfred Buchacher

What is the **Newton-Puiseux algorithm?**

Problem

Given a polynomial $p \in \mathbb{C}[x_1, \ldots, x_n][Y]$, solve the equation

$$p(Y) = 0$$

for Y in terms of series in $x_1, \ldots, x_n$.

What is a **series?**

A series is a **formal sum** of the form

$$\phi = \sum_{I \in \mathbb{Q}^n} a_I \mathbf{x}^I, \quad a_I \in \mathbb{C},$$

its **support** is

$$\operatorname{supp}(\phi) := \{I \in \mathbb{Q}^n : a_I \neq 0\},$$

and we assume that

$$\operatorname{supp}(\phi) \subseteq (v + C) \cap \frac{1}{k}\mathbb{Z}^n,$$

for some $v \in \mathbb{Q}^n$, a strictly convex cone $C \subseteq \mathbb{R}^n$, and some $k \in \mathbb{N}$.

Examples / non-examples

The formal sums

$$\phi_1 := 1 + x + x^2 + x^3 + \dots$$

and

$$\phi_2 := 1 + x^{-1} + x^{-2} + x^{-3} + \dots$$

are series in this sense, but their sum $\phi_1 + \phi_2$ is not, and their product $\phi_1\phi_2$ is not even meaningful.

Fields of Puiseux series

Let $\preceq$ be an **additive total order** on $\mathbb{Q}^n$ defined by

$$\alpha \preceq \beta \quad :\Longleftrightarrow \quad \langle \alpha, w \rangle \leq \langle \beta, w \rangle,$$

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Then $\mathbb{C}_{\preceq}((\mathbf{x}))$ is an algebraically closed **field**.

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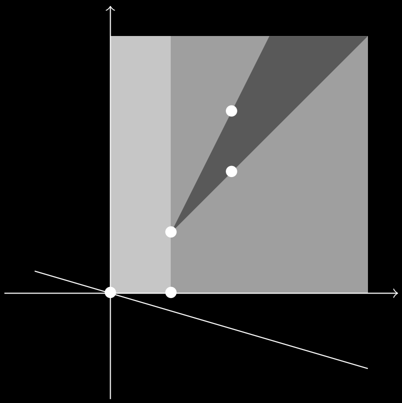
where $N \geq k$ is minimal such that the series solutions constructed from e can be distinguished by their first N terms.



Encoding of series

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This encoding is not unique. It is therefore natural to ask for an **effective equality test**.

Example

The two tuples

$$(p(Z), (-2 + 1/\sqrt{2}, -1), y) \quad \text{and} \quad (p(Z), (-1 + 1/\sqrt{2}, -2), x)$$

represent two series solutions ϕ_1 and ϕ_2 of

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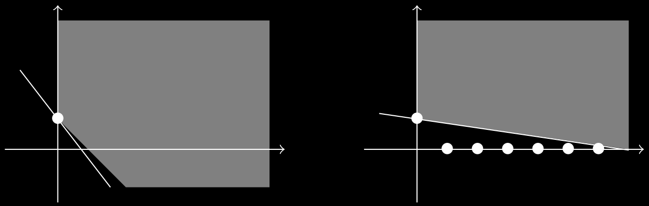
Ordering these terms with respect to $(-2 + 1/\sqrt{2}, -1)$ results in

$$y, x, -x^2, x^3, -x^4, x^5, -x^6.$$



The estimate of the supports of ϕ_1, ϕ_2 given by the NPA show that there is a field $\mathbb{C}_{\preceq}((\mathbf{x}))$ both are elements of. Hence

$$\phi_1 = \phi_2.$$

Definition

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Conjecture

Let $c_1\mathbf{x}^{\alpha_1} + \cdots + c_N\mathbf{x}^{\alpha_N}$ be the first few terms of a series root output by the NPA, and let C be the cone that goes with it. If $p(Y)$ is content-free, and if N is sufficiently large, then C is the minimal cone such that

$$\mathrm{supp}(\phi) \subseteq \{\alpha_1, \ldots, \alpha_{N-1}\} \cup (\alpha_N + C).$$

The **support** of an algebraic series

If e is an edge of $\mathrm{NP}(\mathrm{p}(\mathrm{Y}))$ that gives rise to ϕ, then

$$-\mathrm{S}(e) = \mathrm{lexp}_w(\phi)$$

for some $w \in \mathbb{R}^n$, and so $-\mathrm{S}(e)$ is a **vertex** of $\mathrm{conv}(\mathrm{supp}(\phi))$.

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Under the assumption that this holds for all admissible edges, the NPA can be used to compute **all** vertices of $\mathrm{conv}(\mathrm{supp}(\phi))$.

Example

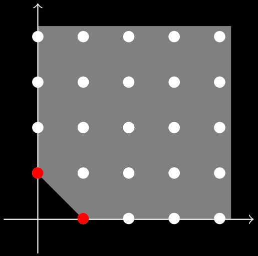
We have observed that

$$(p(Z), (-2 + 1/\sqrt{2}, -1), y) \quad \text{and} \quad (p(Z), (-1 + 1/\sqrt{2}, -2), x)$$

represent the same series solution ϕ of

$$p(Z) := x + y - (1 + x + y)Z = 0,$$

and so $(0, 1)$ and $(1, 0)$ are two vertices of $\mathrm{conv}(\mathrm{supp}(\phi))$.



Computing the **bounded faces** of $\mathrm{conv}(\mathrm{supp}(\phi))$

The bounded faces of $\mathrm{conv}(\mathrm{supp}(\phi))$ seem to be reconstructable from its vertices $v_1, \ldots, v_k$ and strictly convex cones $C_1, \ldots, C_k$ such that

$$\mathrm{conv}(\mathrm{supp}(\phi)) \subseteq v_i + C_i.$$

Example

Let $P \subseteq \mathbb{R}^2$ be a polyhedral set, let $v_1, v_2 \in \mathbb{R}^2$ be its vertices, and let $C_1, C_2 \subseteq \mathbb{R}^2$ be strictly convex cones such that

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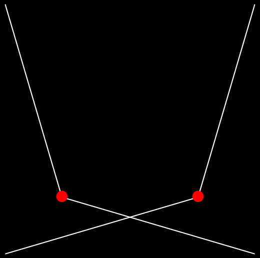
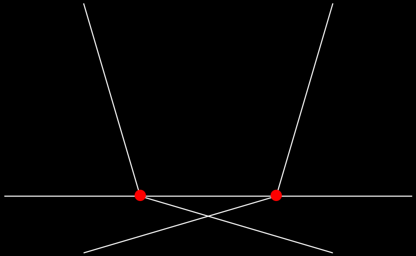
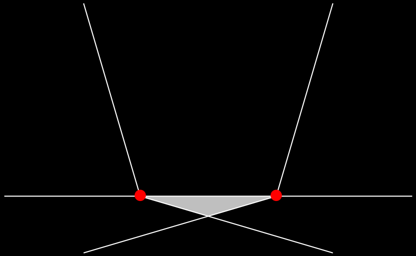
Is the line segment joining v_1 and v_2 a face of P?







We have a **sufficient** condition for the convex hull of a set of vertices to be a (bounded) face.

Is it also a **necessary** condition?

Being able to compute the bounded faces of $\mathrm{conv}(\mathrm{supp}(\phi))$, it can be decided whether an algebraic series ϕ is a (Puiseux) polynomial.

Problem

Determine the **unbounded faces** of the convex hull of the support of an algebraic series.

The **number** of series solutions

If the edge polynomials of $p(Y)$ are square-free, then $p(Y) = 0$ has only finitely many series solutions.

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Problem

Find a formula for the number of series solutions of a polynomial equation in terms of statistics of its Newton polytope.

Example

The solution of

$$p(Y) = 1 - (1 - x)Y = 0$$

is clearly the multiplicative inverse of $1 - x$.

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In $\mathbb{C}((x))$ we have

$$\phi = 1 + x + x^2 + \dots$$

but in $\mathbb{C}((x^{-1}))$

$$\phi = -x^{-1} - x^{-2} - x^{-3} + \dots$$

References

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