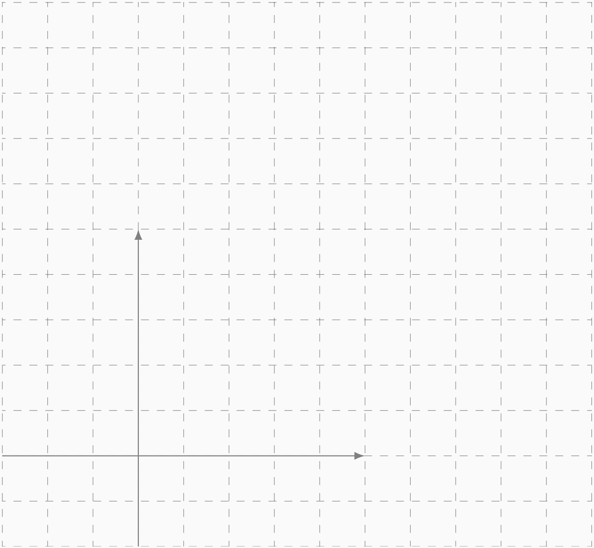
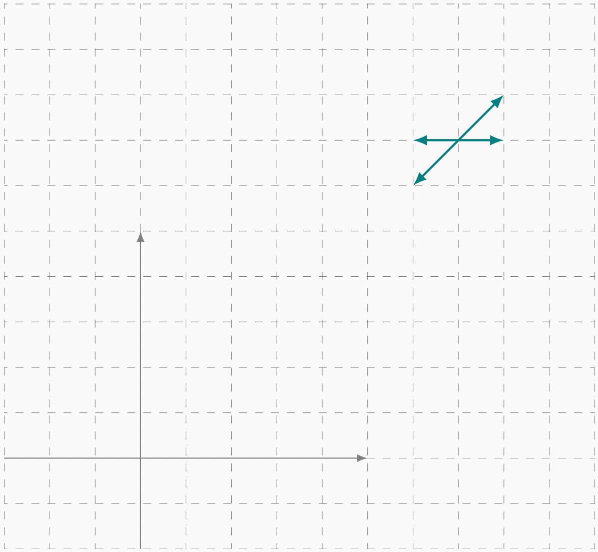
Inhomogeneous Lattice Walks

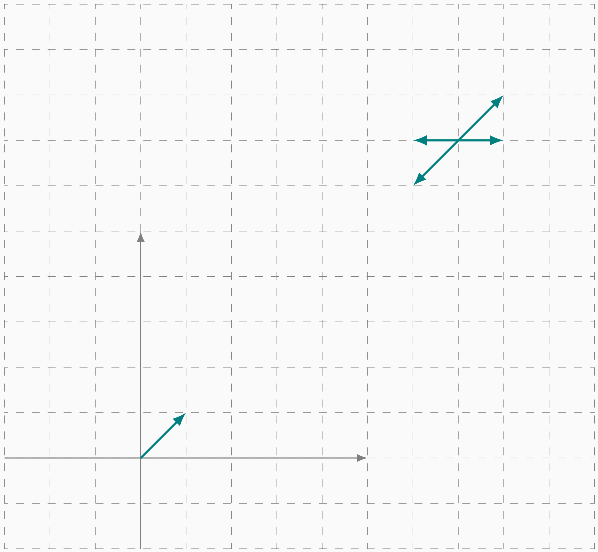
Manfred Buchacher and Manuel Kauers
Johannes Kepler Universität Linz

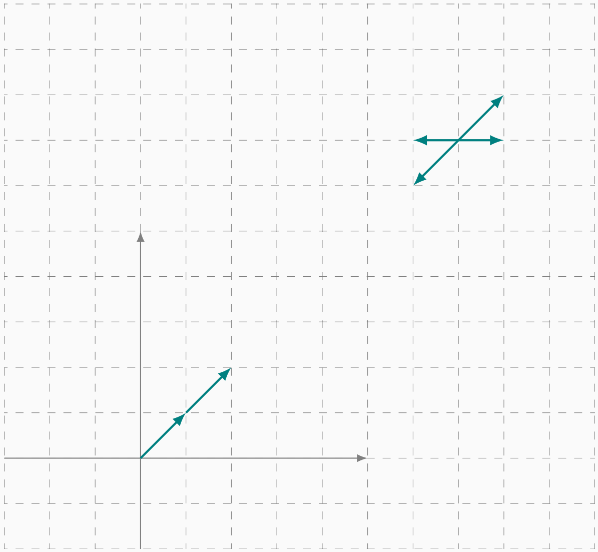
SIAM Conference on Applied Algebraic Geometry
July 11, 2019 Bern, Switzerland

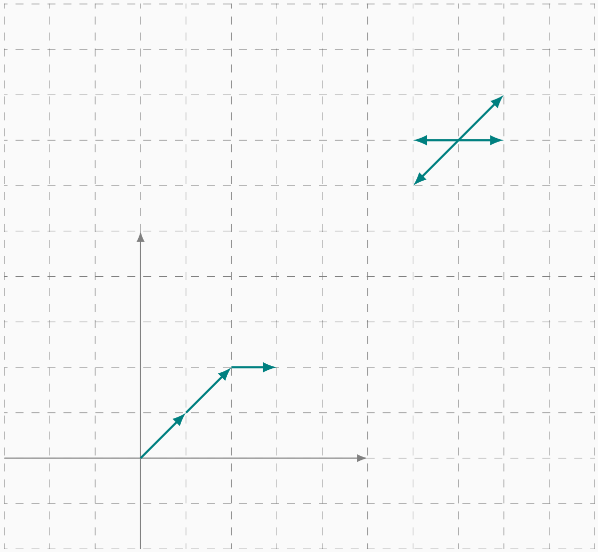
A **lattice walk** is a sequence of points of $\mathbb{Z}^d$. The consecutive differences of these points are the **steps** of the walk. If the steps are taken from a fixed set $\mathbf{S}$ we consider the lattice walk to be **homogeneous** with respect to $\mathbf{S}$.

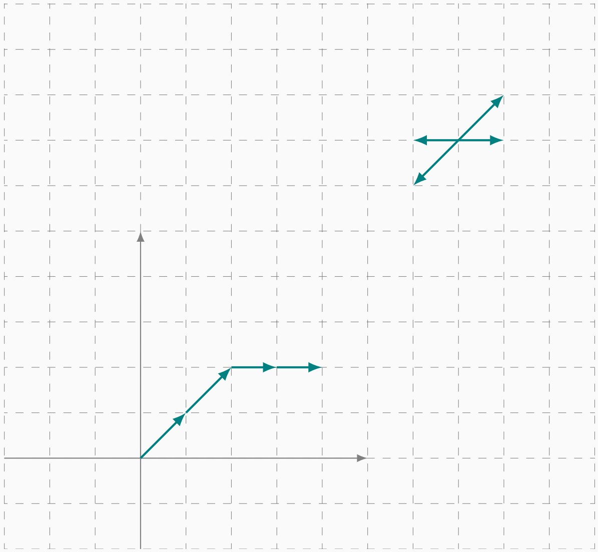


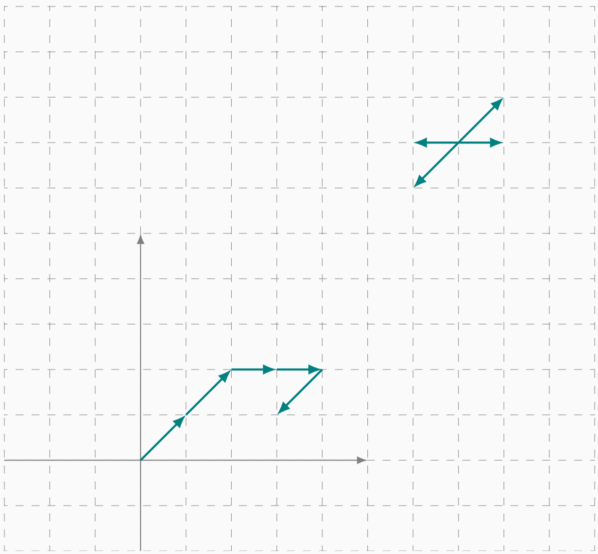


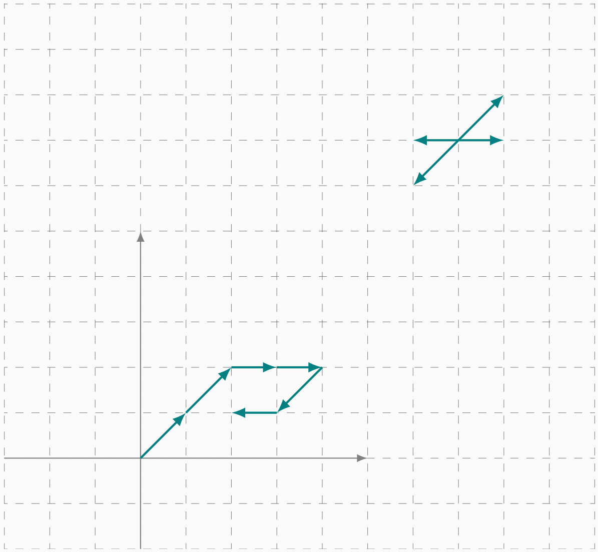


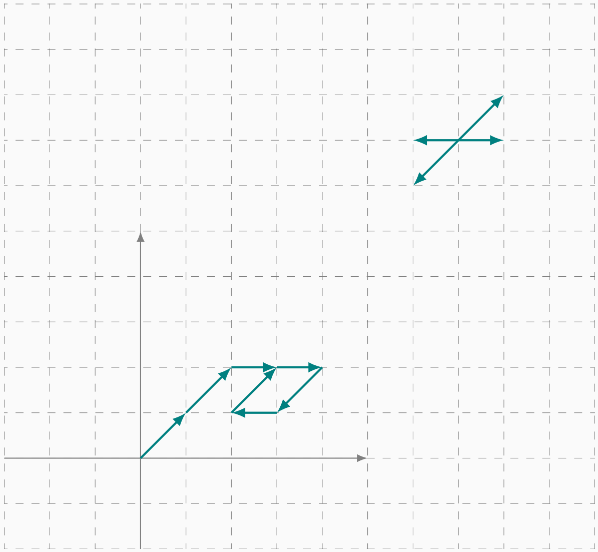


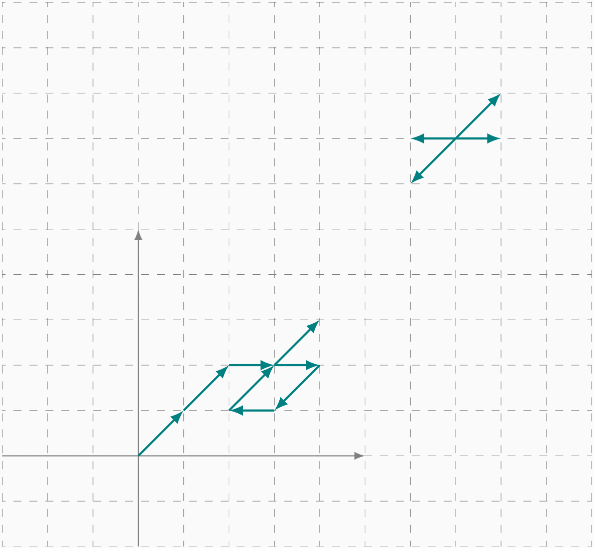


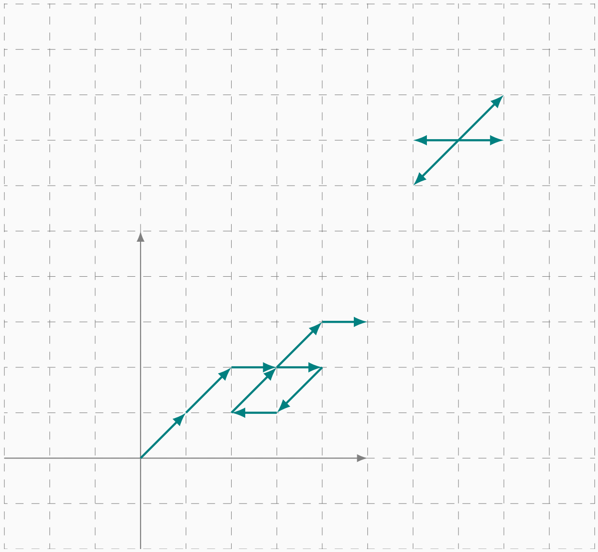




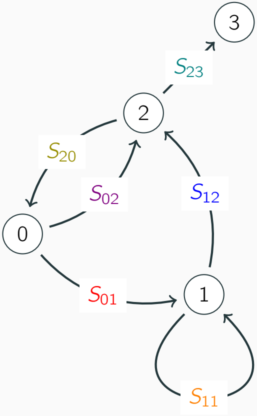


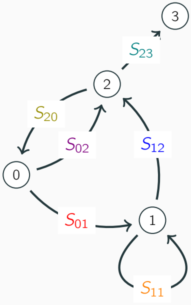




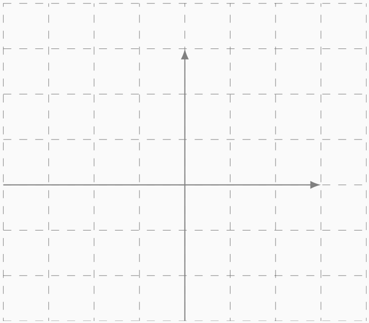


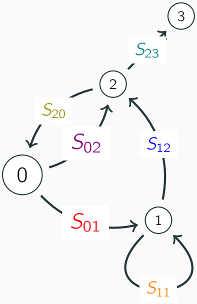
A lattice walk is called **inhomogeneous**, if the set of admissible steps is governed by a **deterministic finite automaton**.

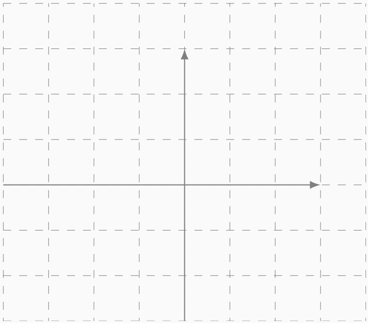

3
S_{23}
2
S_{20}
S_{02}
S_{12}
0
S_{01}
1
S_{11}

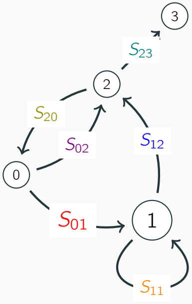


3
S_{23}
2
S_{20}
S_{02}
S_{12}
0
S_{01}
1
S_{11}

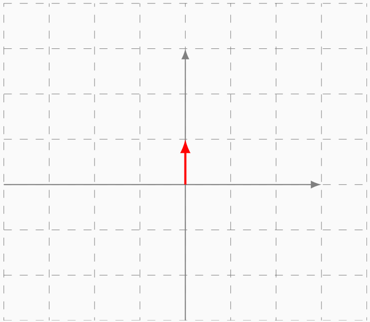


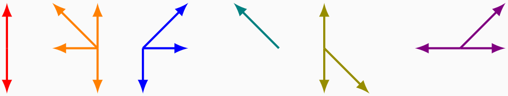

3
S23
2
S20
S02
S12
0
S01
1
S11

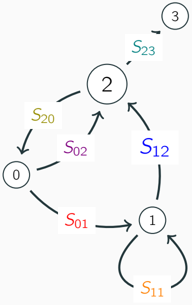




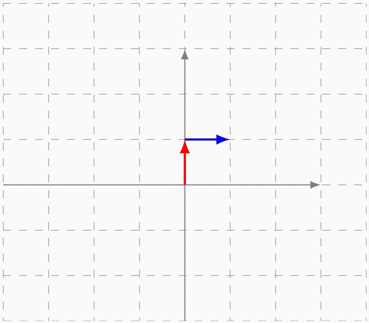
3
S_{23}
2
S_{20}
S_{02}
S_{12}
0
S_{01}
1
S_{11}

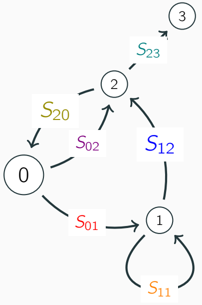




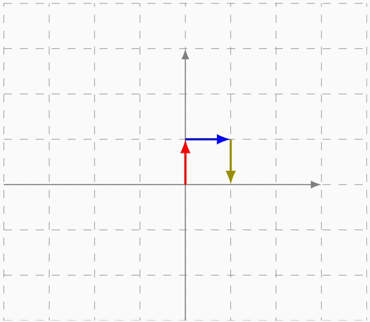


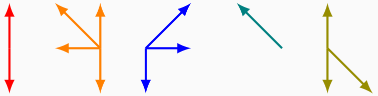
3
S_{23}
2
S_{20}
S_{02}
S_{12}
0
S_{01}
1
S_{11}

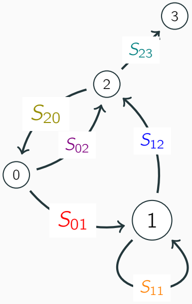


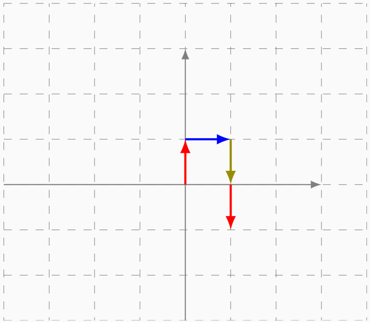


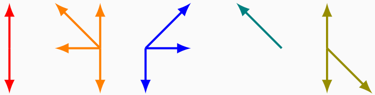
3
S_{23}
2
S_{20}
S_{02}
S_{12}
0
S_{01}
1
S_{11}

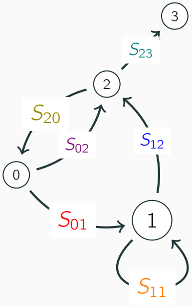




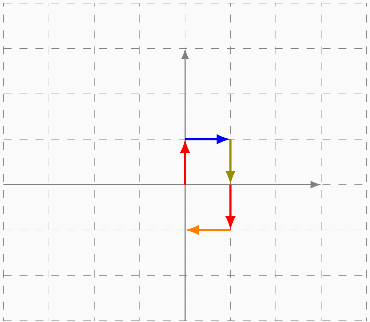

3
S_{23}
2
S_{20}
S_{02}
S_{12}
0
S_{01}
1
S_{11}

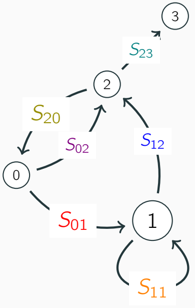




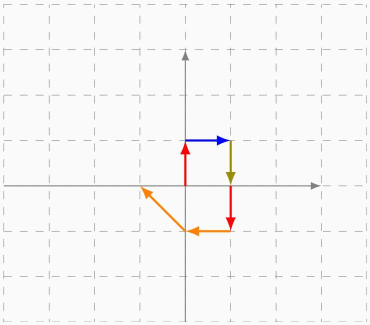


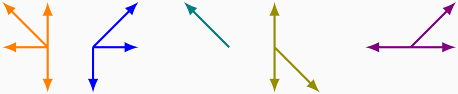
3
S_{23}
2
S_{20}
S_{02}
S_{12}
0
S_{01}
1
S_{11}

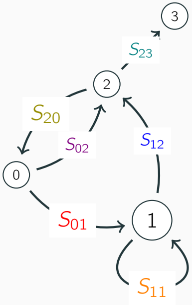




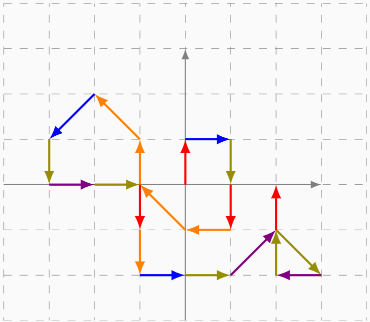
3
S_{23}
2
S_{20}
S_{02}
S_{12}
0
S_{01}
1
S_{11}



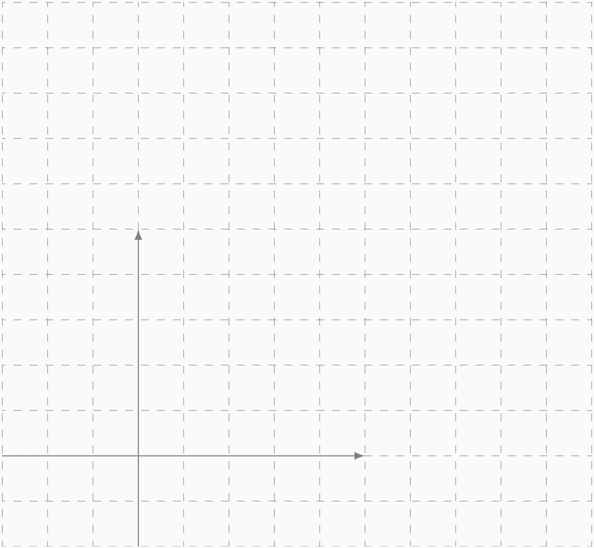


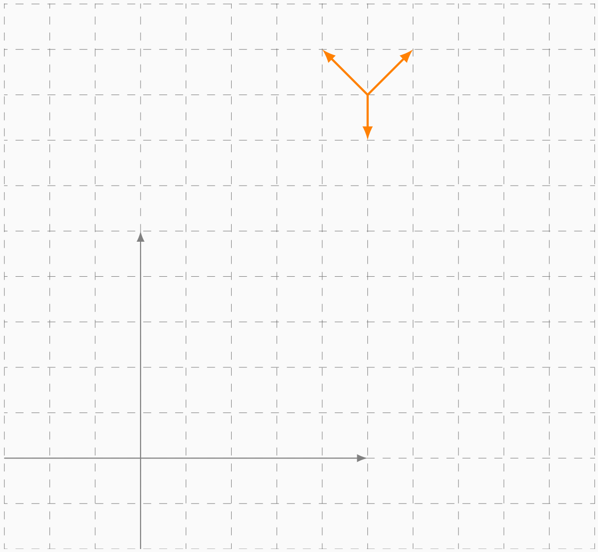


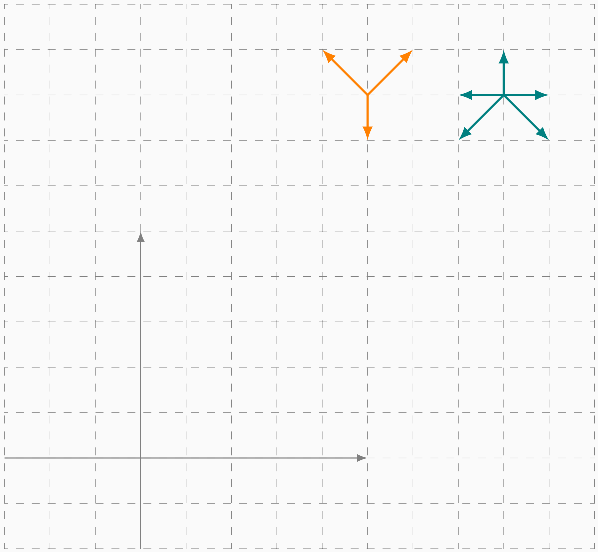
3
S_{23}
2
S_{20}
S_{02}
S_{12}
0
S_{01}
1
S_{11}

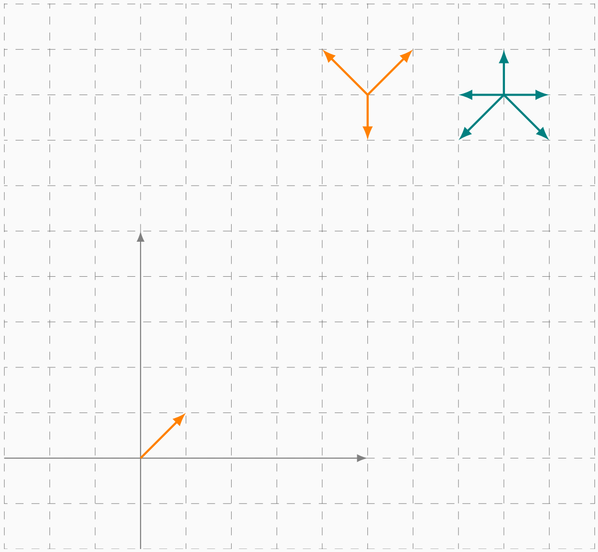


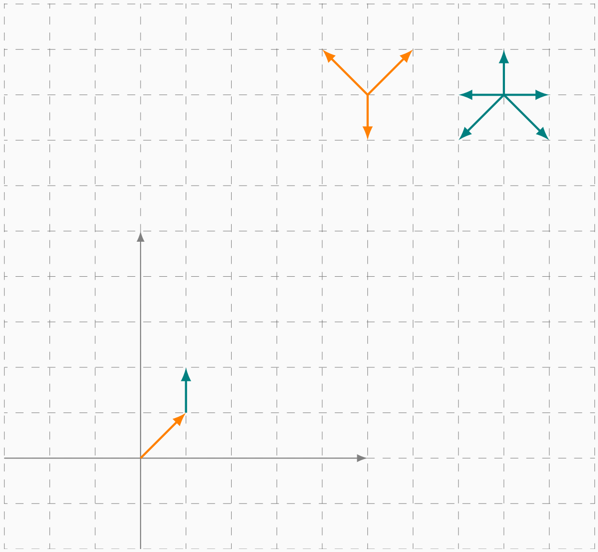
A few **examples** of what being inhomogeneous can mean...

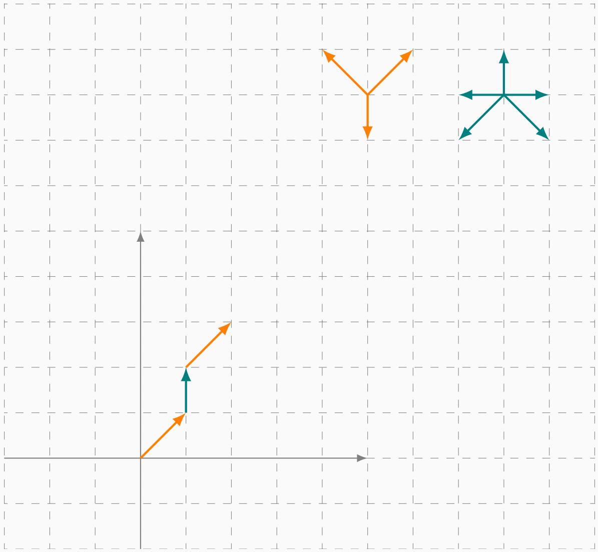


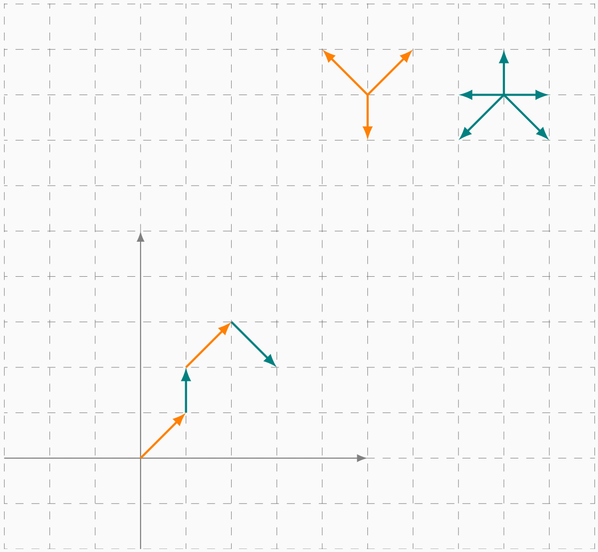


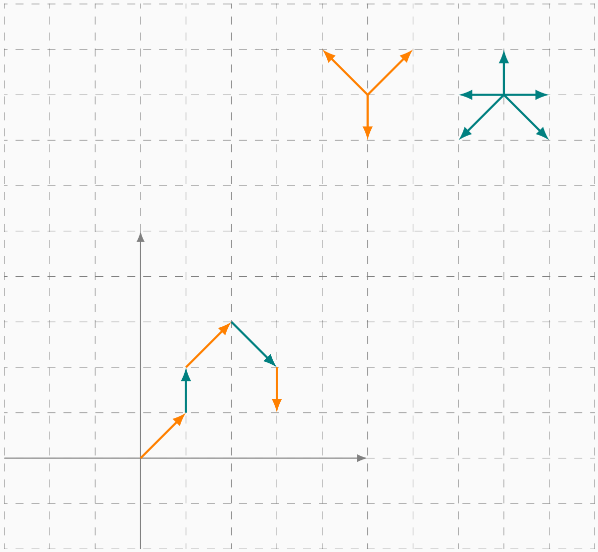


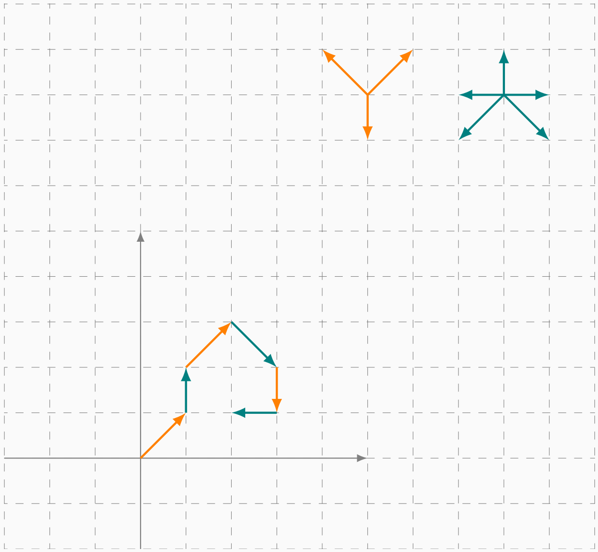


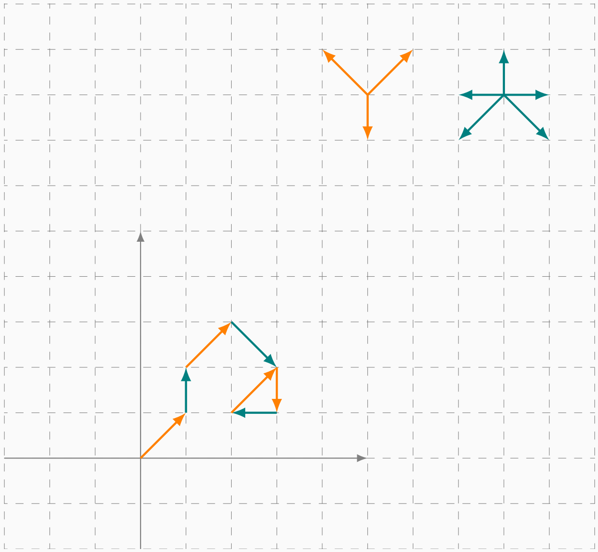


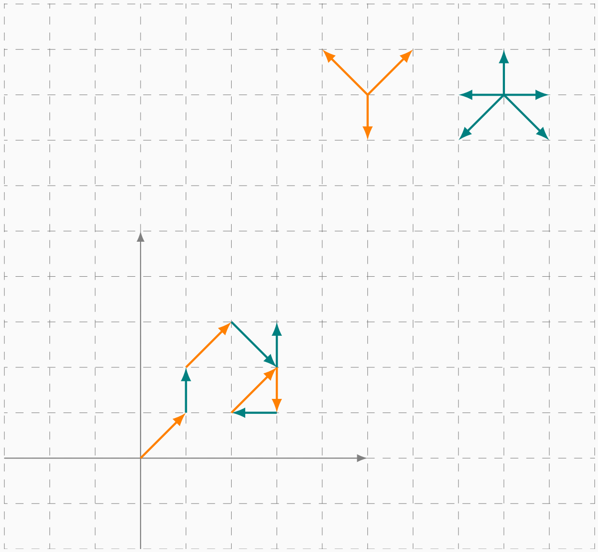


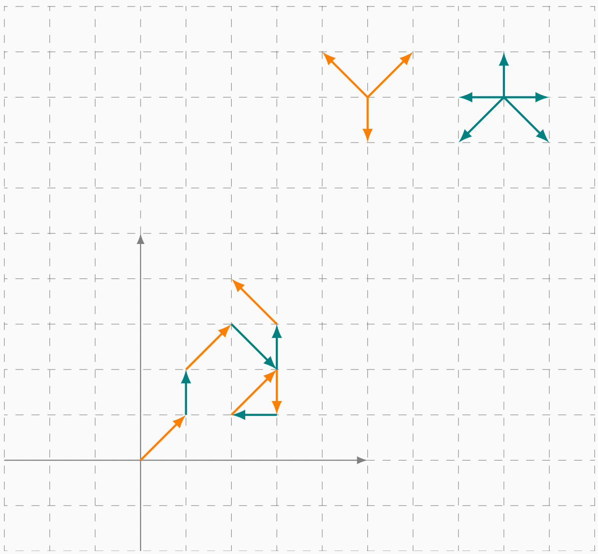


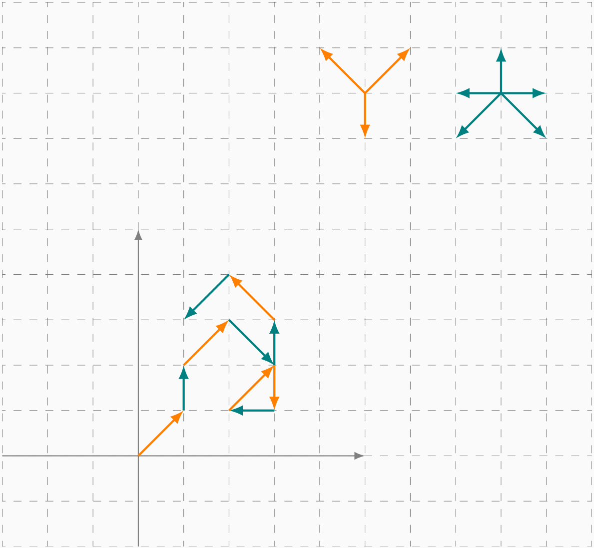


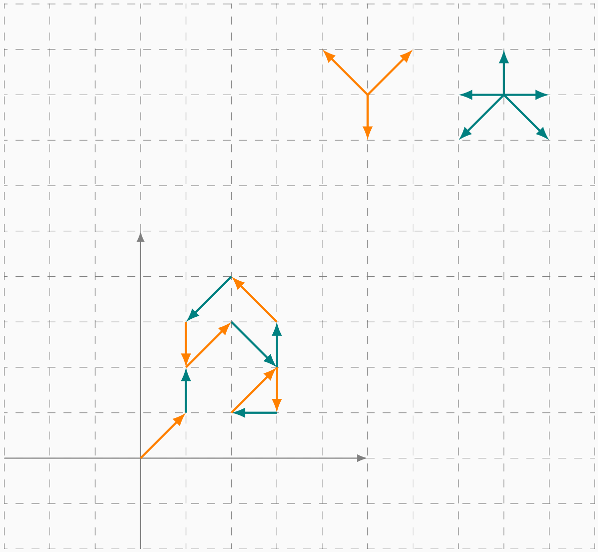


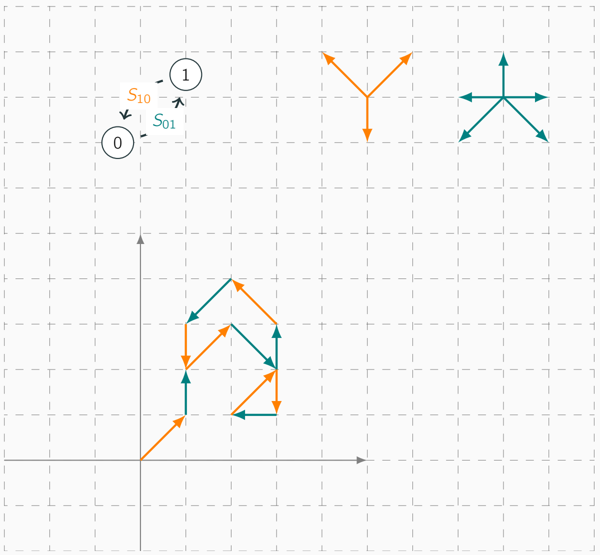




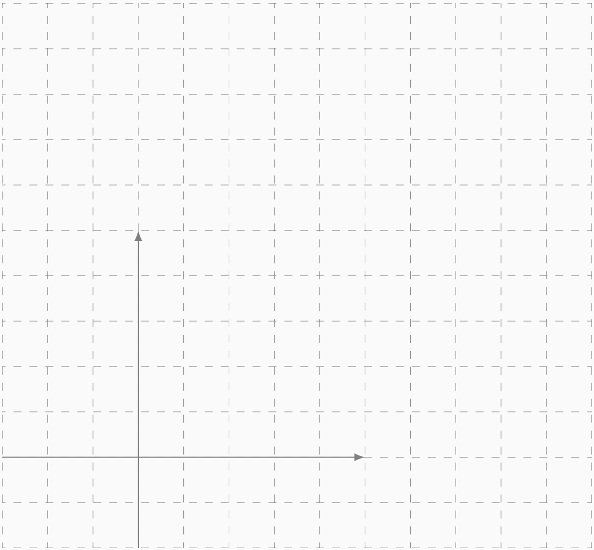


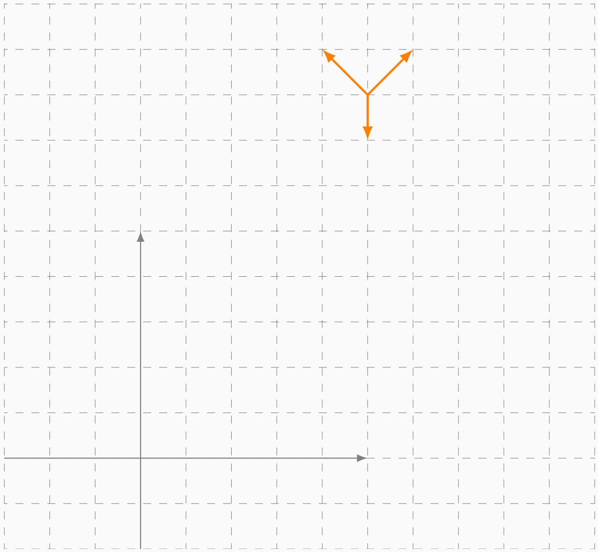


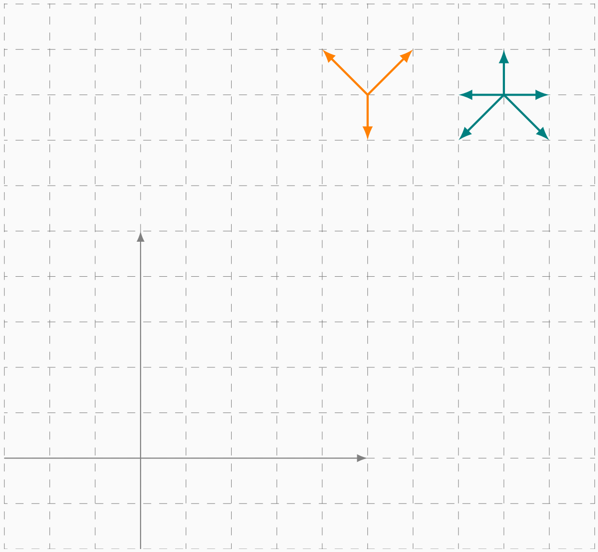


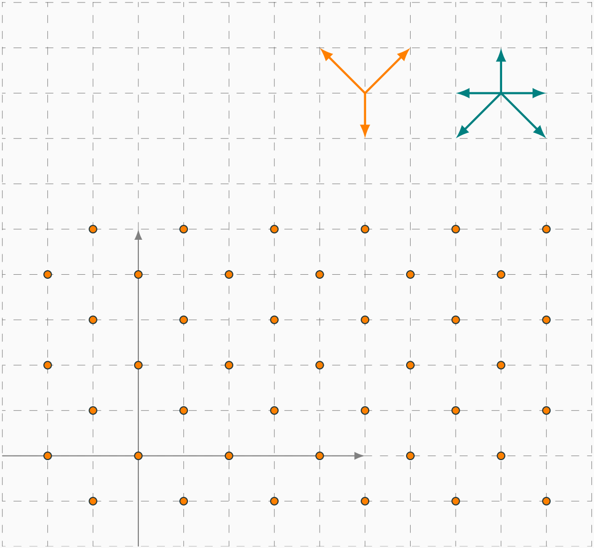


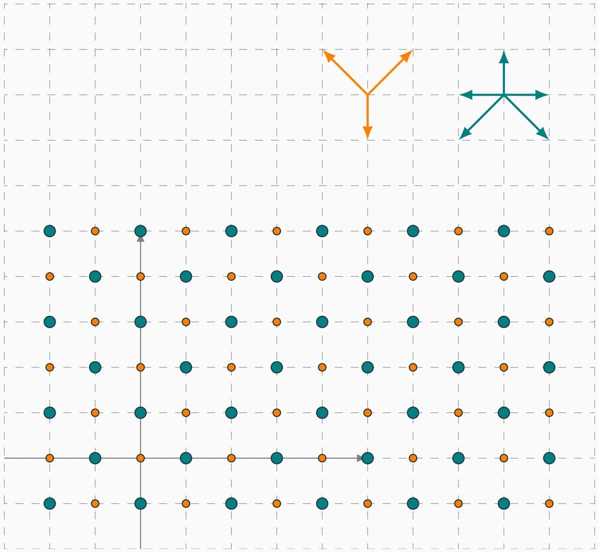
1
S_{10}
S_{01}
0

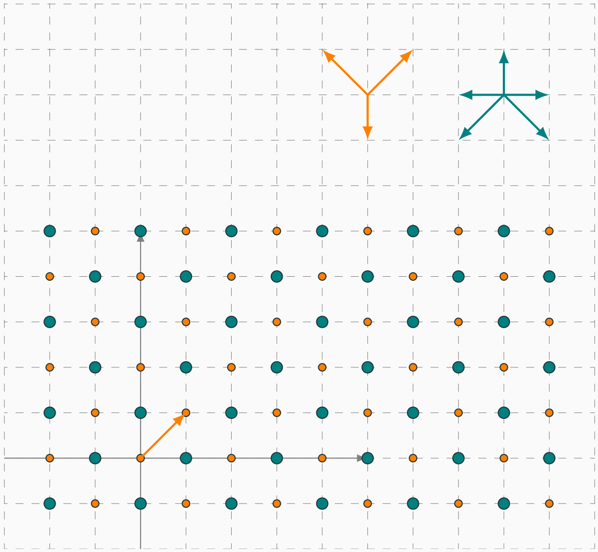


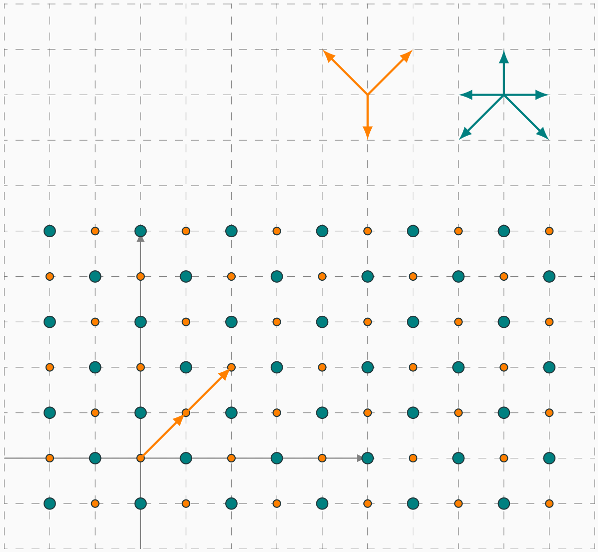


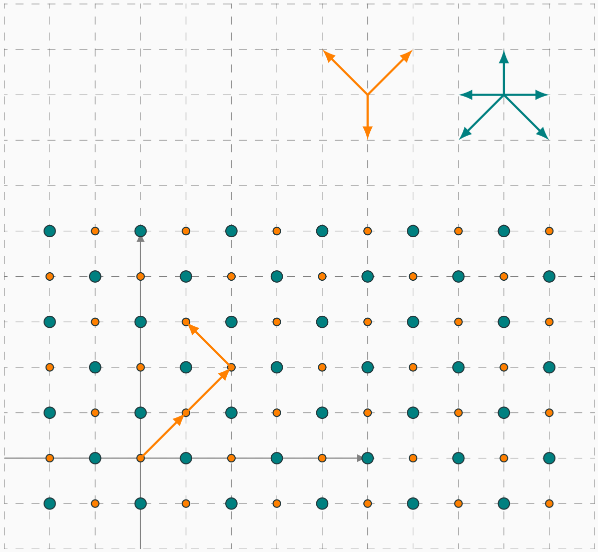


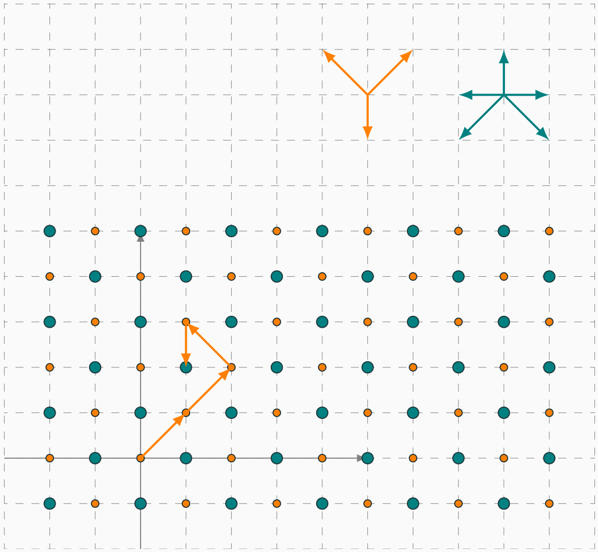


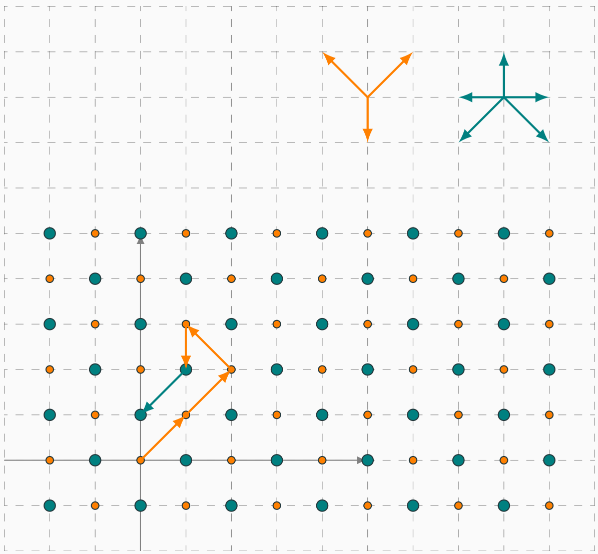


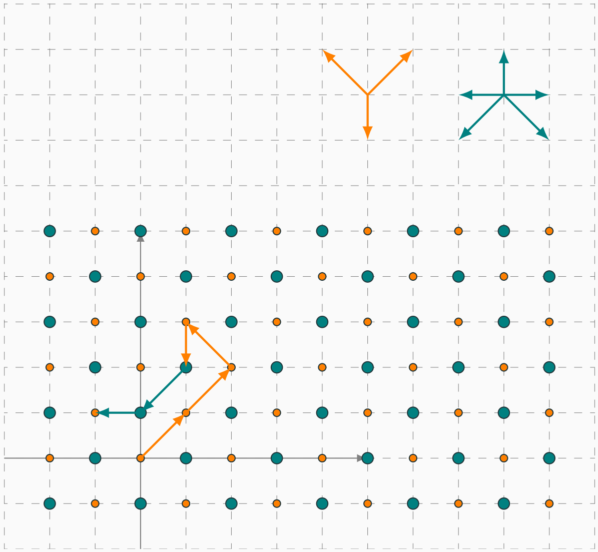


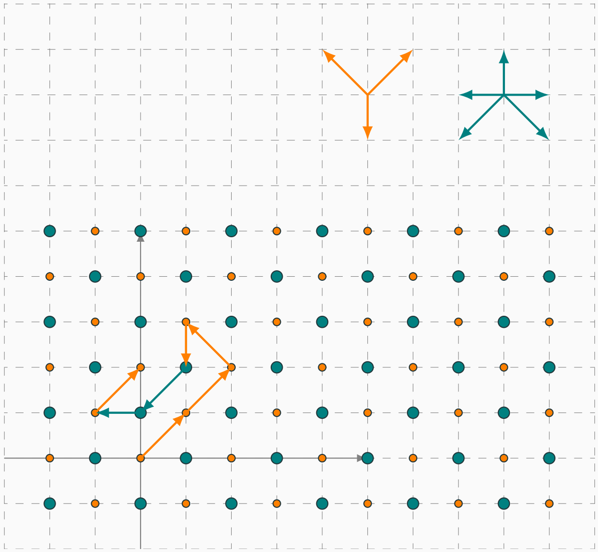


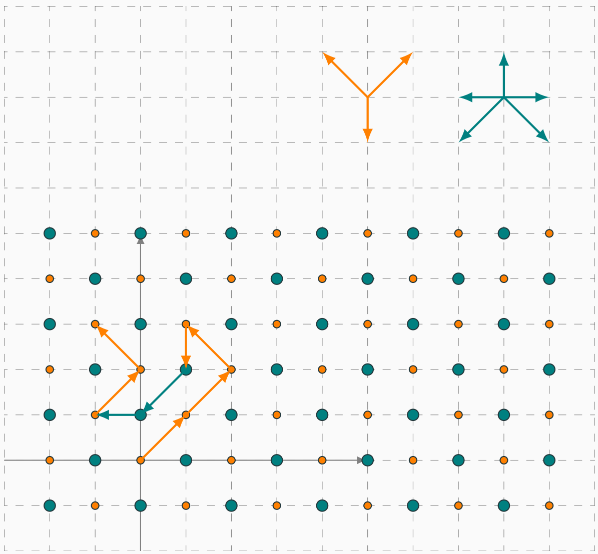


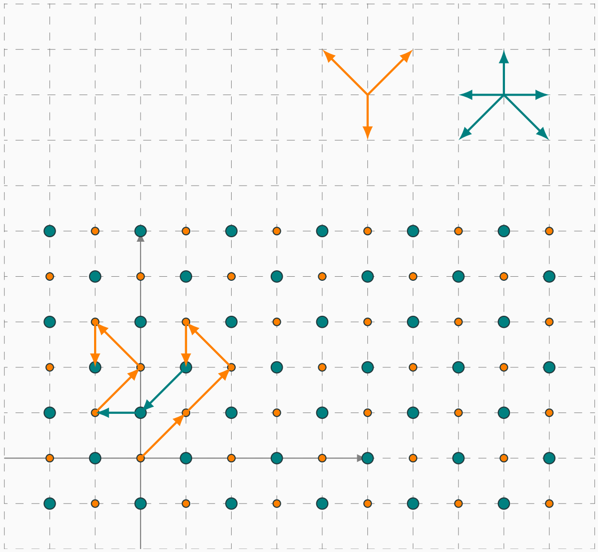


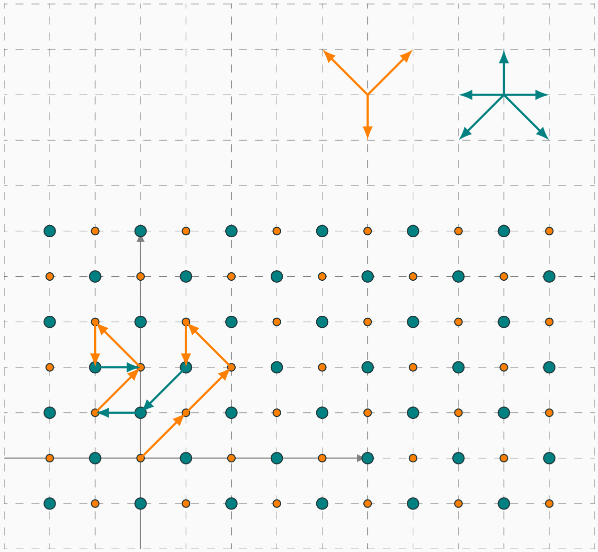


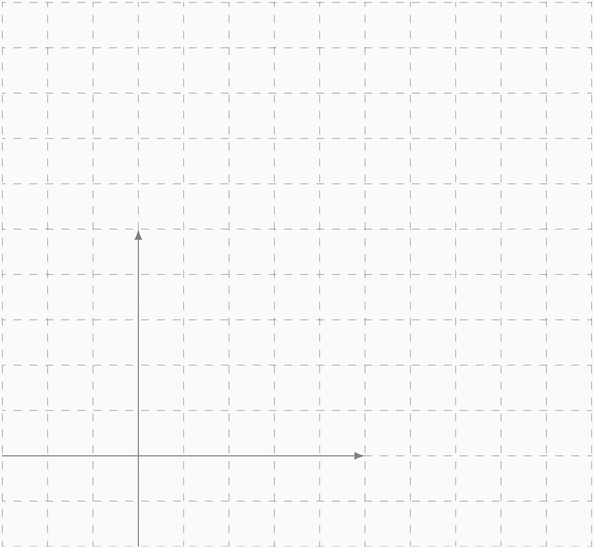


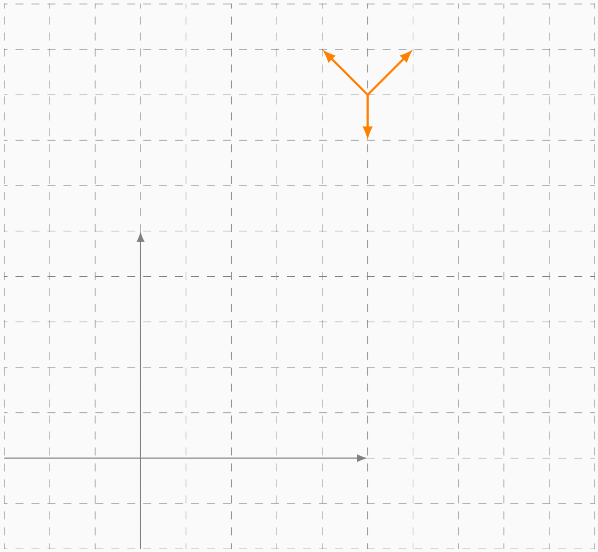


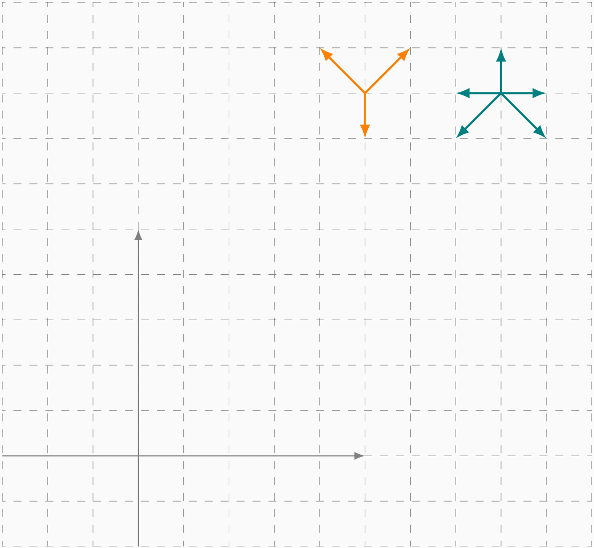


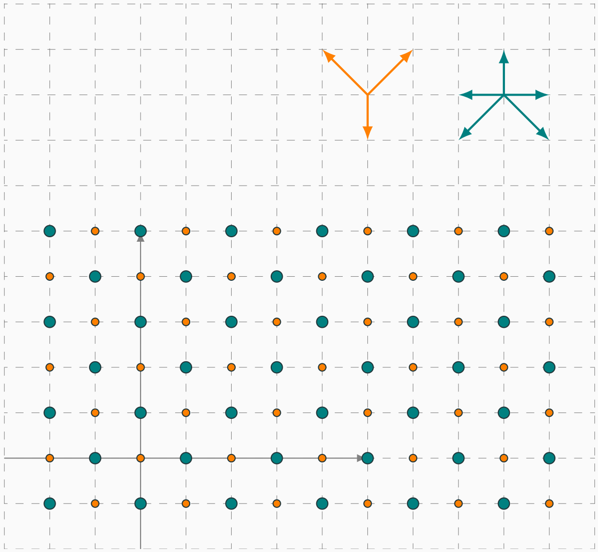


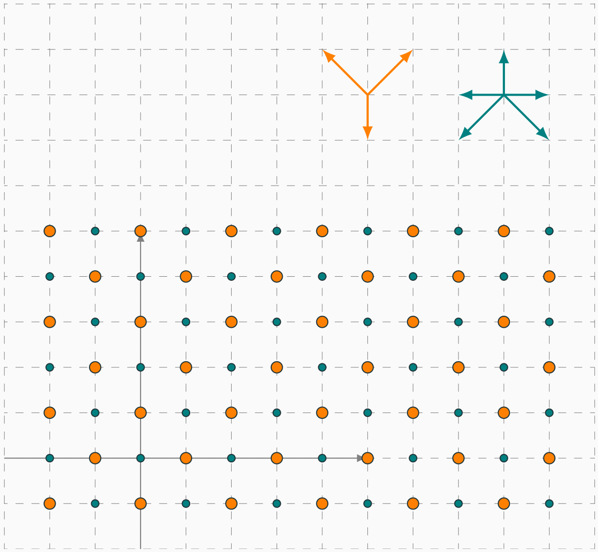


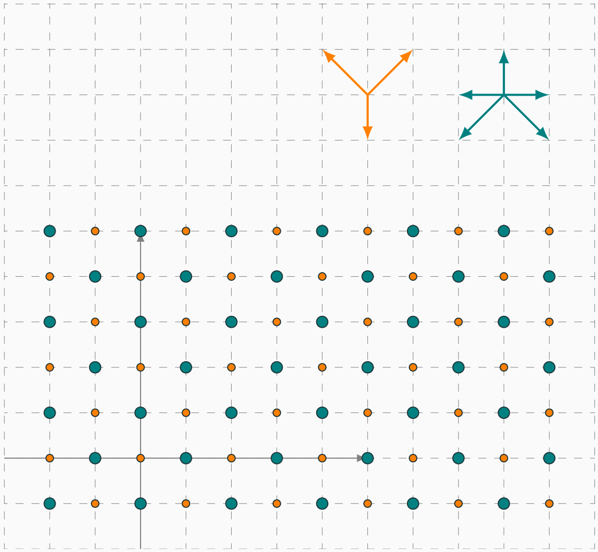


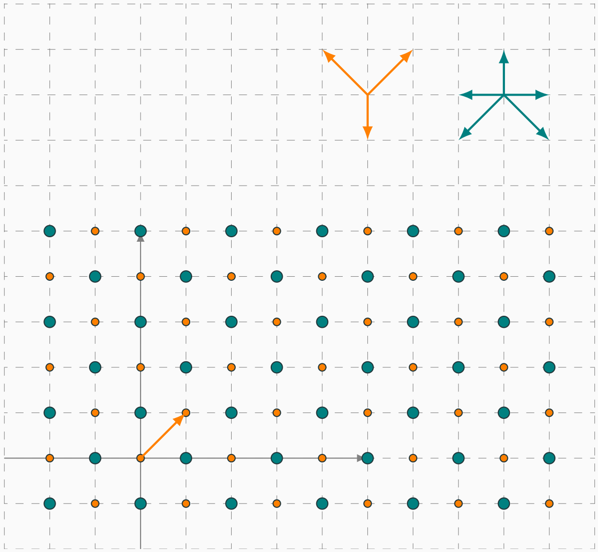


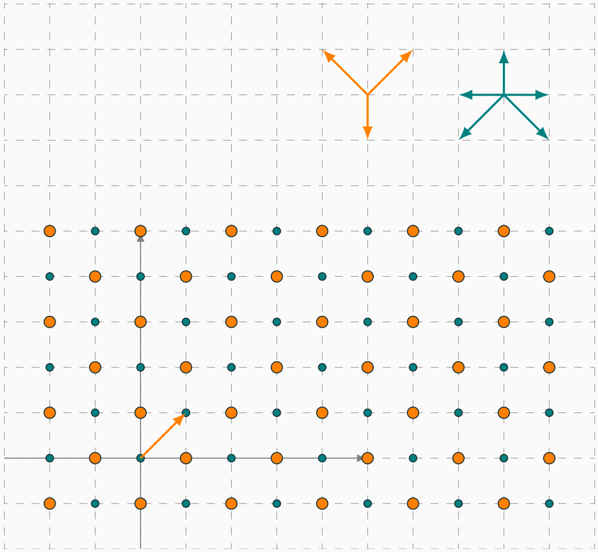


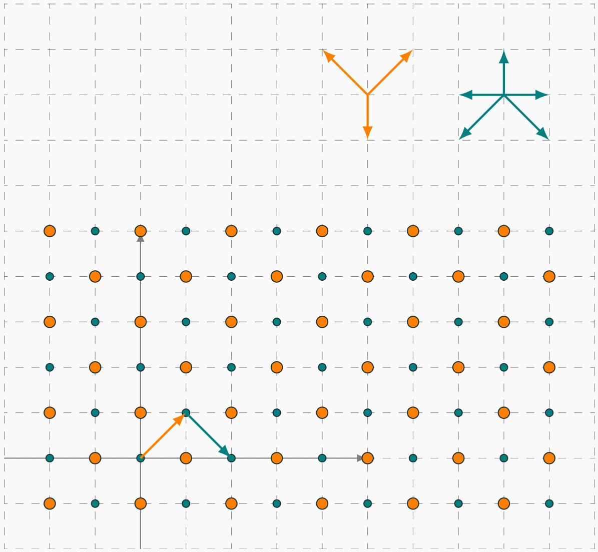


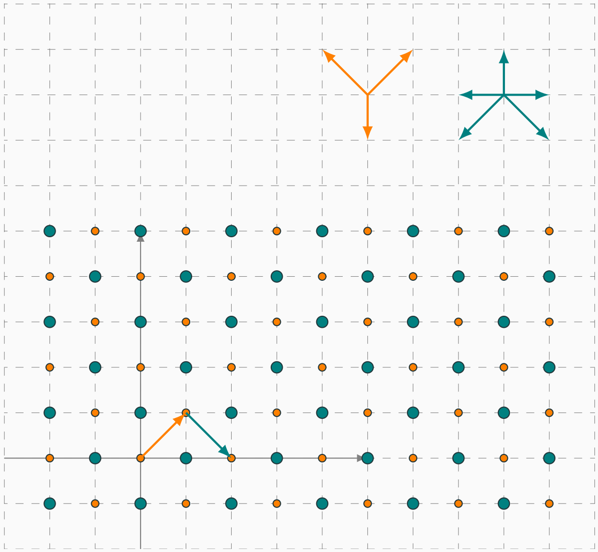


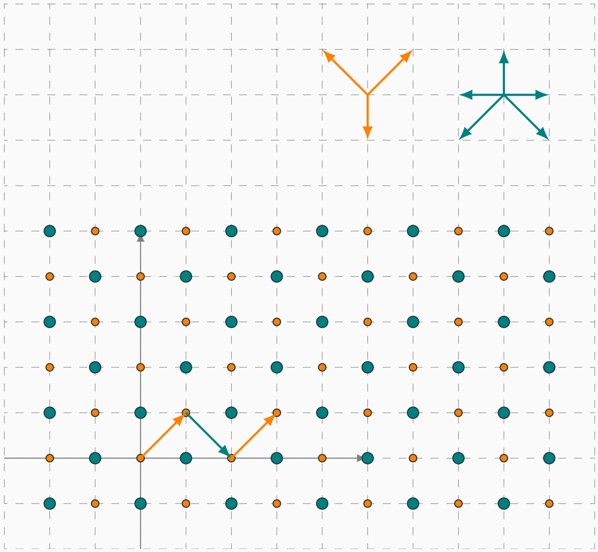


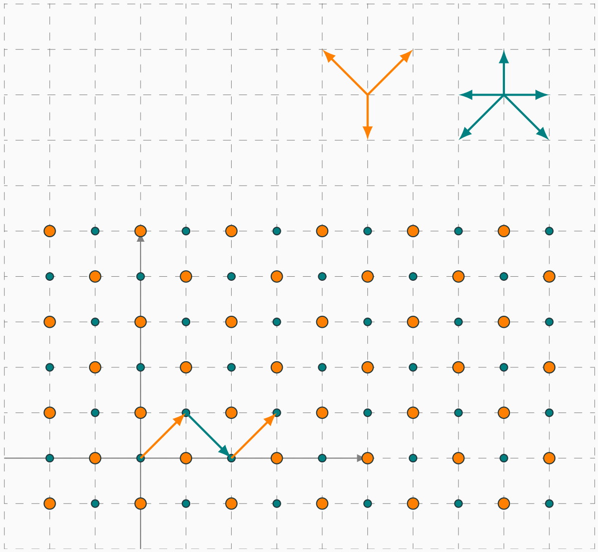


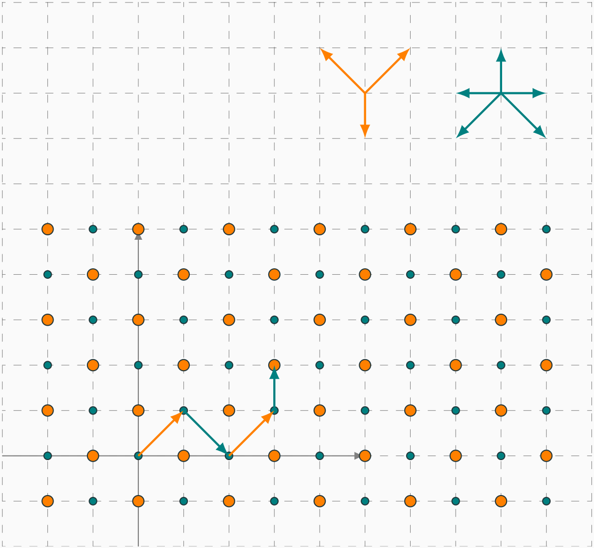


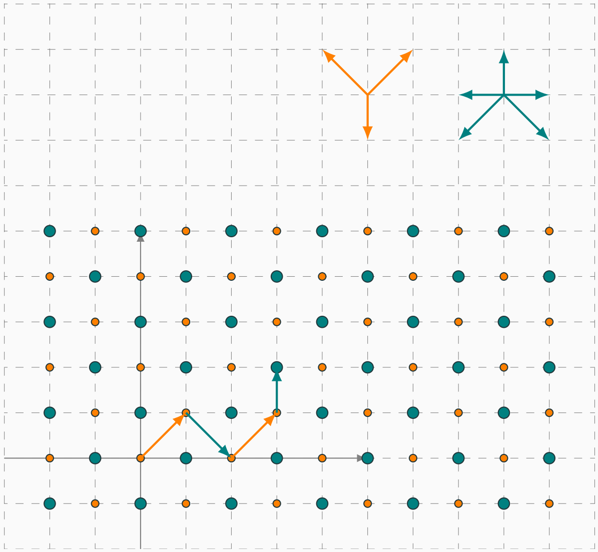


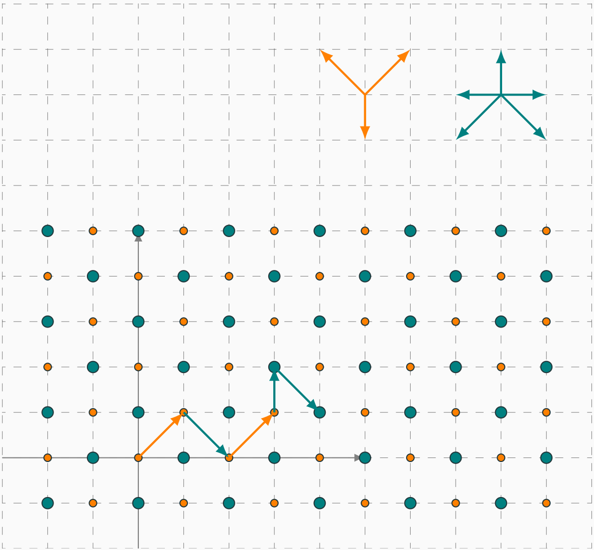


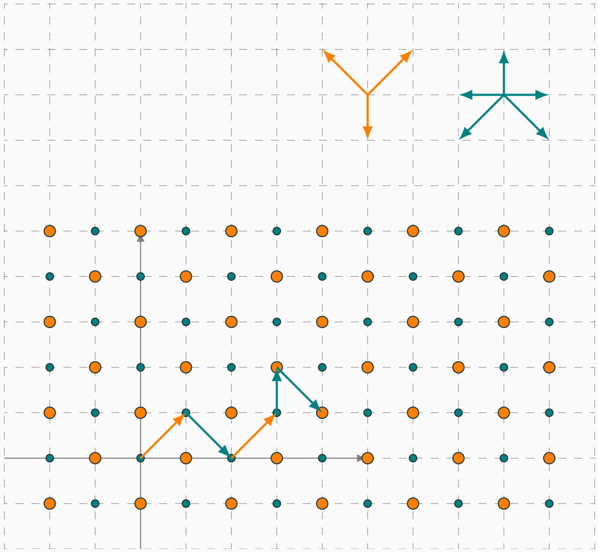


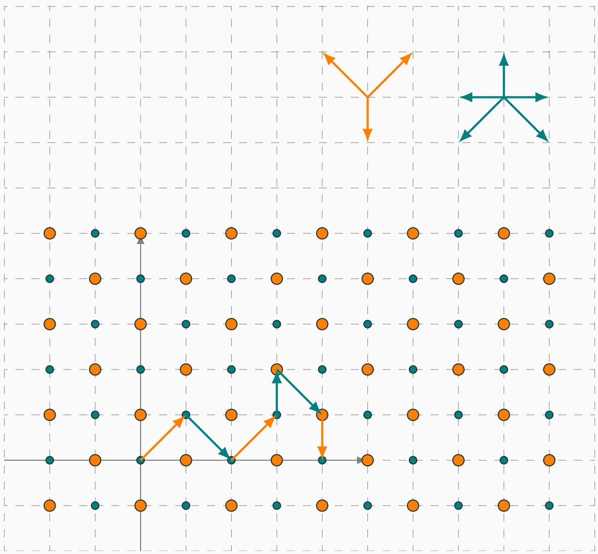












A first step in finding exact or asymptotic formulas for the number of lattice walks is to decide the **nature** of the generating function and to find an expression for it.

Generating functions of **unrestricted** lattice walks are **rational**.

Let F be the generating function of walks that start at the origin, counted by length and endpoint, and let F_q be the one of those associated with paths in the automaton that end at final state q.

Then

$$F = \sum_{q \text{ final}} F_q$$

and the F_q's uniquely solve a **linear system** of functional equations of the form

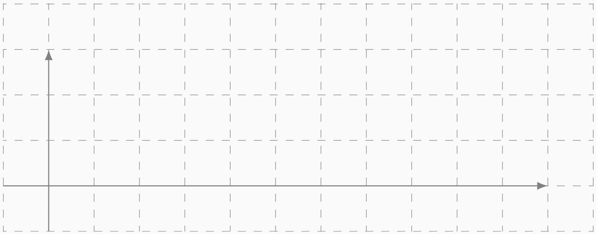
$$F_q = [q = q_0] + t \sum_p S_{pq} F_p,$$

where S_{pq} is the step polynomial of the step set $\mathbf{S}_{pq}$.

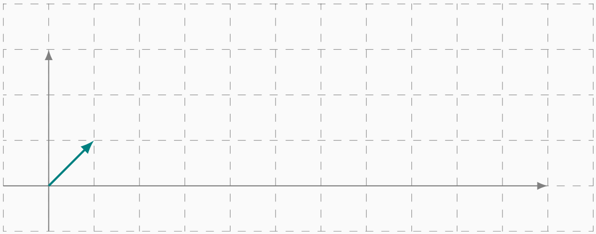
Generating functions of lattice walks restricted to the half-space $\mathbb{Z}^{d-1} \times \mathbb{Z}_{\geq 0}$ are **algebraic**.

Example

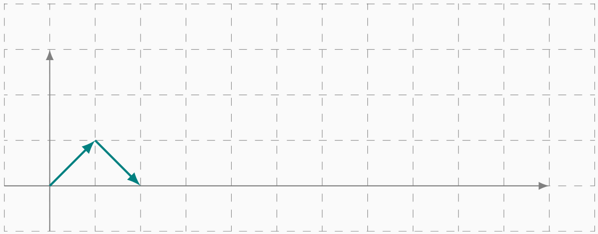
Walks in the half-plane $\mathbb{Z} \times \mathbb{Z}_{\geq 0}$ which start at the origin and take its steps from $\mathbf{S} = \{(1,1), (1,-1)\}$.



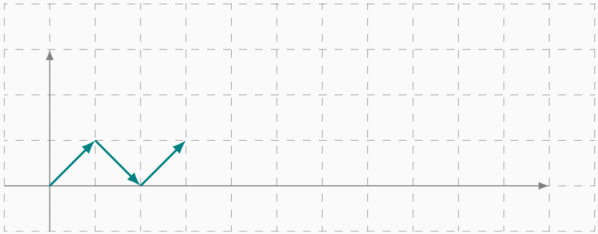
Walks in the half-plane $\mathbb{Z} \times \mathbb{Z}_{\geq 0}$ which start at the origin and take its steps from $\mathbf{S} = \{(1,1),(1,-1)\}$.



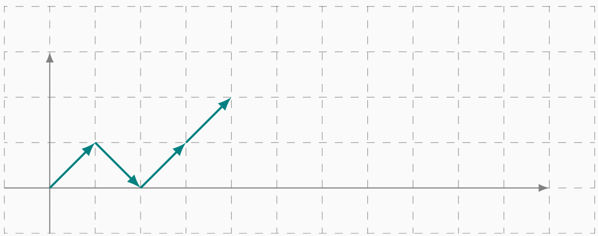
Walks in the half-plane $\mathbb{Z} \times \mathbb{Z}_{\geq 0}$ which start at the origin and take its steps from $\mathbf{S} = \{(1,1), (1,-1)\}$.



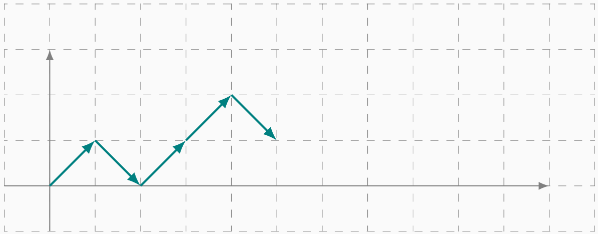
Walks in the half-plane $\mathbb{Z} \times \mathbb{Z}_{\geq 0}$ which start at the origin and take its steps from $\mathbf{S} = \{(1,1),(1,-1)\}$.



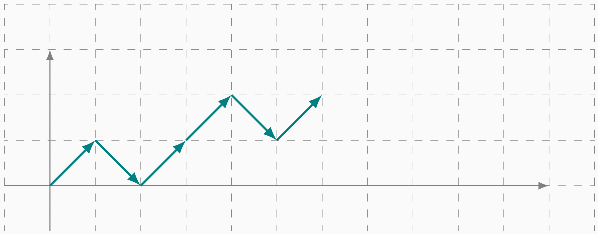
Walks in the half-plane $\mathbb{Z} \times \mathbb{Z}_{\geq 0}$ which start at the origin and take its steps from $\mathbf{S} = \{(1,1),(1,-1)\}$.



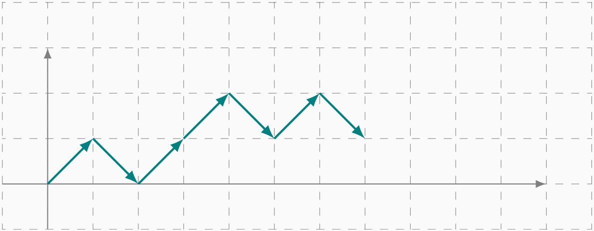
Walks in the half-plane $\mathbb{Z} \times \mathbb{Z}_{\geq 0}$ which start at the origin and take its steps from $\mathbf{S} = \{(1, 1), (1, -1)\}$.



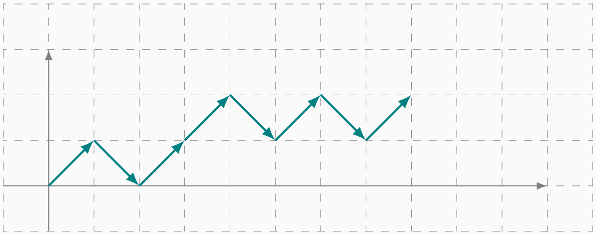
Walks in the half-plane $\mathbb{Z} \times \mathbb{Z}_{\geq 0}$ which start at the origin and take its steps from $\mathbf{S} = \{(1,1),(1,-1)\}$.



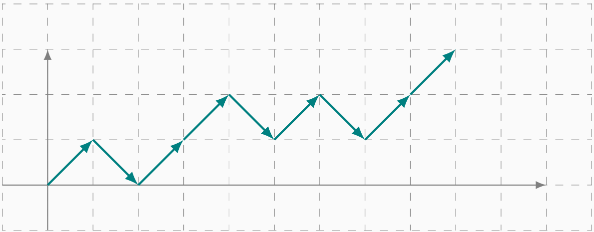
Walks in the half-plane $\mathbb{Z} \times \mathbb{Z}_{\geq 0}$ which start at the origin and take its steps from $\mathbf{S} = \{(1,1),(1,-1)\}$.



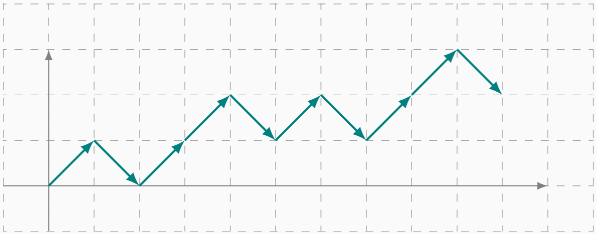
Walks in the half-plane $\mathbb{Z} \times \mathbb{Z}_{\geq 0}$ which start at the origin and take its steps from $\mathbf{S} = \{(1,1),(1,-1)\}$.



Walks in the half-plane $\mathbb{Z} \times \mathbb{Z}_{\geq 0}$ which start at the origin and take its steps from $\mathbf{S} = \{(1,1), (1,-1)\}$.



Walks in the half-plane $\mathbb{Z} \times \mathbb{Z}_{\geq 0}$ which start at the origin and take its steps from $\mathbf{S} = \{(1,1),(1,-1)\}$.



The functional equation for the corresponding generating function

$$F(x,t) = 1 + t(\bar{x} + x)F(x,t) - t\bar{x}F(0,t)$$

rewrites into

$$(1 - t(\bar{x} + x))F(x,t) = 1 - t\bar{x}F(0,t)$$

and simplifies to

$$0 = 1 - t\bar{x}(t)F(0,t)$$

when evaluated at the power series root $x(t)$ of $1 - t(\bar{x} + x)$.

Hence

$$F(0,t) = x(t)/t \quad \text{and} \quad F(x,t) = \frac{1 - \bar{x}x(t)}{1 - t(\bar{x} + x)}.$$

Theorem

Let $\mathbb{K}$ be a field of characteristic zero, and let Δ be given by

$$\Delta : \mathbb{K}[x][[t]]^n \to \mathbb{K}[x][[t]]^n$$
$$\Delta f(x,t) = (f(x,t) - f(0,t))/x.$$

Then, for $a \in \mathbb{K}[x,t]^n$ and $B_i \in \mathbb{K}[x,t]^{n \times n}$,

$$f = a + t \sum_{i=0}^{k} B_i \Delta^i f$$

has a **unique** solution f in $\mathbb{K}[x][[t]]^n$, and its components are **algebraic** over $\mathbb{K}[x,t]$.

Sketch of Proof

1. Rewrite the equation in the form

$$\left(x^k I_n - t\sum_{i=0}^{k} x^{k-i} B_i\right) f(x,t) = x^k a - t\sum_{j=0}^{k-1}\left(\sum_{i=j+1}^{k}\frac{x^{k+j-i}}{j!}B_i\right) f^{(j)}(0,t).$$

2. Eliminate $f(x,t)$ by
   a) replacing x by a root $x(t)$ of $\det\bigl(x^k I_n - t\sum_{i=0}^{k} x^{k-i} B_i\bigr)$, and
   b) multiplying the equation by elements of the co-kernel of the matrix.
3. Solve the resulting linear systems for the $f^{(j)}(0,t)$'s and for $f(x,t)$.

To avoid difficulties arising from the linear system possibly being **singular** one solves a **perturbation** of the original equation:

$$\tilde{f} = a(x, t^2) + \epsilon t E \Delta^k \tilde{f} + t^2 \sum_{i=0}^{k} B_i(x, t^2) \Delta^i \tilde{f}.$$

Algebraicity of $\tilde{f}$ then implies algebraicity of f, since

$$f(x, t^2) = [\epsilon^0] \tilde{f}(x, t).$$

Generating functions of lattice walks restricted to $\mathbb{Z}^d_{\geq 0}$ are more **diverse** when $d \geq 2$, and so are the **methods** to study their nature.

The **dimension** of a model

Decomposition into and **projection** onto lower dimensional models

D-finiteness and the **algebraic kernel method**

Let $\mathbf{S} \subseteq \{-1, 0, 1\}^2 \setminus \{(0, 0)\}$, and write

$$\begin{aligned} S(x, y) &= A_{-1}(x)\bar{y} + A_0(x) + A_1(x)y \\ &= B_{-1}(y)\bar{x} + B_0(y) + B_1(y)x \end{aligned}$$

for the corresponding step polynomial.

Assume that

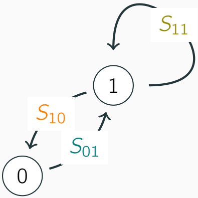
$$A_{-1}, A_1, B_{-1}, B_1 \neq 0,$$

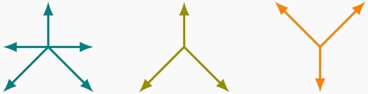
and let $G(\mathbf{S})$ be the group generated by the transformations

$$\phi : (x, y) \mapsto \left(\bar{x}\frac{B_{-1}(y)}{B_1(y)}, y\right) \quad \text{and} \quad \psi : (x, y) \mapsto \left(x, \bar{y}\frac{A_{-1}(x)}{A_1(x)}\right)$$

which leave $S(x, y)$ invariant.

Example

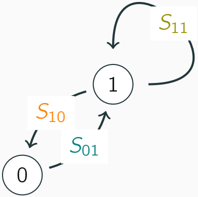


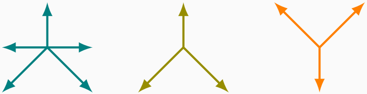


$$G(\mathbf{S}_{01}) = G(\mathbf{S}_{11}) = \{(x, y), (\bar{x}, y), (\bar{x}, \bar{y}(x + \bar{x})), (x, \bar{y}(x + \bar{x}))\}$$

$$G(\mathbf{S}_{10}) = \{(x, y), (\bar{x}, y), (\bar{x}, \bar{y}\frac{1}{x + \bar{x}}), (x, \bar{y}\frac{1}{x + \bar{x}})\}$$

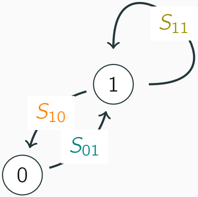
Note: replacing y by $\bar{y}(x + \bar{x})$ maps $G(\mathbf{S}_{10})$ to $G(\mathbf{S}_{01}) = G(\mathbf{S}_{11})$.

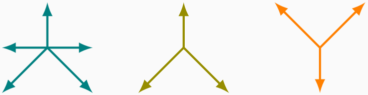




$$F_0 = 1 + tS_{10}F_1 - \bar{y}F_1(x,0) - \bar{x}yF_1(0,y)$$

$$F_1 = tS_{01}F_0 + tS_{11}F_1 + \dots$$





$$xyF_0 = xy + tS_{10}xyF_1 - xF_1(x,0) - y^2F_1(0,y)$$

$$xyF_1 = tS_{01}xyF_0 + tS_{11}xyF_1 + \ldots$$

$$\sum_{g\in G(\mathsf{S}_{10})} \operatorname{sgn}(g) g(xyF_0) =$$

$$\sum_{g\in G(\mathsf{S}_{10})} \operatorname{sgn}(g) g(xy) + tS_{10} \sum_{g\in G(\mathsf{S}_{10})} \operatorname{sgn}(g) g(xyF_1)$$

$$\sum_{g\in G(\mathsf{S}_{01})} \operatorname{sgn}(g) g(xyF_1) =$$

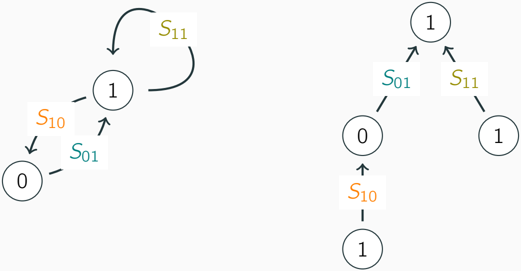
$$tS_{01} \sum_{g\in G(\mathsf{S}_{01})} \operatorname{sgn}(g) g(xyF_0) + tS_{11} \sum_{g\in G(\mathsf{S}_{01})} \operatorname{sgn}(g) g(xyF_1)$$

$$\sum_{g \in G(\mathsf{S}_{01})} \operatorname{sgn}(g) g(xyF_1) =$$

$$\frac{tS_{01}}{1 - tS_{11} - t^2 S_{10}(x, \bar{y}(x+\bar{x}))S_{01}} \sum_{g \in G(\mathsf{S}_{01})} \operatorname{sgn}(g) g(xy)$$

$$xyF_1 = [x^{>0}y^{>0}]\frac{tS_{01}}{1 - tS_{11} - t^2 S_{10}(x, \bar{y}(x + \bar{x}))S_{01}} \sum_{g \in G(\mathsf{S}_{01})} \operatorname{sgn}(g) g(xy)$$

As the positive part of a rational function is D-finite, so is F_1. Closure properties of D-finite functions then show that F_0 and $F = F_0 + F_1$ are D-finite as well.



S_{11}
1
S_{10}
S_{01}
0
1
S_{01}
S_{11}
0
1
S_{10}
1

Non-D-finiteness

Thank you for your attention.

References

Manfred Buchacher and Manuel Kauers. *"Inhomogeneous restricted lattice walks"*. In: *Proceedings of FPSAC'19*. 2019.