Separated Variables on Plane Algebraic Curves

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Motivation and context

Lattice walks and
differential equations

Let $f(i;n)$ be the number of LW on $\mathbb{N}$ that start at 0, end at i and consist of (presicely) n steps taken from $\{-1,1\}$, and define

$$F(x) \equiv F(x;t) := \sum_{n\geq 0} \left(\sum_{i\geq 0} f(i;n) x^i \right) t^n.$$

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Then

$$F(x) = 1 + txF(x) + t\Delta_x F(x)$$

$$\Delta_x F := \left(F(x) - F(0)\right)/x.$$

A **DDE** is an equation of the form

$$F = P(x,y) + tQ\left(x, y, t, \ldots \Delta_x^i \Delta_y^j F \ldots\right),$$

where P, Q are polynomials, and F is a series.

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where P, Q are polynomials, and F is a series.

Is its solution algebraic, D-finite or D-algebraic?
What algebraic or differential equations does it satisfy?

Thm [Bousquet-Mélou, Jehanne]

The solution of an ordinary DDE is algebraic.

Let $S(x, y)$ be a Laurent polynomial such that

$$\mathrm{supp}(S) \subseteq \{-1, 0, 1\}^2$$

and define

$$K(x, y) := xy(1 - tS(x, y)).$$

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The nature of the solution of

$$\begin{aligned} K(x, y)F(x, y) &= xy + K(x, 0)F(x, 0) \\ &\quad + K(0, y)F(0, y) - K(0, 0)F(0, 0) \end{aligned}$$

depends on S and can be very diverse.

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has a (non-trivial) solution in $\mathbb{Q}(t)(x, y)$. It is algebraic iff it is D-finite and

$$xy + q(x, y)K = f(x) - g(y)$$

has solution. Furthermore, if it is not D-finite, then it is D-algebraic iff the latter equation has a solution.

Problem

Given $r \in \mathbb{C}(x, y)$ and $p \in \mathbb{C}[x, y]$, solve

$$r + q(x, y)p = f(x) - g(y)$$

for q, f and g in $\mathbb{C}(x, y)$.

Some field theoretic interpretations

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Intersections of fields

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$$f(x) \equiv g(y) \mod p$$

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$$r \in \mathbb{C}(x) \mod p$$

Sum-decomposition

$$r \in \mathbb{C}(x) + \mathbb{C}(y) \mod p$$

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- computer vision
- parameter identifcation in ODE models
- algebraic independence of solutions of ODEs
- designing diffractive optical systems

Define

$$\mathrm{F}(r, p) := \{(f, g) \in \mathbb{C}(x) \times \mathbb{C}(y) : f - g \in r + \langle p \rangle\},$$

and let (f, g) be any element of $\mathrm{F}(r, p)$.

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Then

$$F(r,p) = (f,g) + F(0,p)$$

and

$$F(p) \equiv F(0,p)$$

is a simple field.

Some examples

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is trivial as we will see in a moment.

Let

$$r = xy \quad \text{and} \quad p = xy - x - y - x^2y^2.$$

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The others can be computed from $F(p)$. We have

$$F(r,p) = \left(\frac{x-1}{x}, \frac{1}{y}\right) + F(p).$$

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2) determining any element of $\mathrm{F}(\mathrm{r}, \mathrm{p})$.

The non-linear problem of solving

$$r + qp = f - g$$

is reduced to a linear problem. The reduction is based on the computation of the poles of f and g and their multiplicities.

1) Finding a generator of $\mathrm{F}(\mathrm{p})$

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Poles

There is a generator $(f, g) \in F(p)$ such that

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If (s_1, s_2) is a root of p, then

$$f(s_1) = \infty \quad \text{iff} \quad g(s_2) = \infty.$$

Let $\sim$ be the smallest equivalence relation on the curve such that

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The equivalence class of (x_0, y_0) is called the orbit of (x_0, y_0).

Thm

The coordinates of the orbit of ∞ are poles of f and g, respectively. The orbit is finite, and it is exhaustive. If

$$F(p) \cong \mathbb{C},$$

it might however be infinite.

1) Finding a generator of $\mathrm{F}(\mathrm{p})$

Multiplicities

Any $\omega \in \mathbb{Z}^2$ induces a grading on $\mathbb{C}[x, y]$ by

$$\omega(ax^i y^j) = \omega_x i + \omega_y j.$$

The leading part of $p(x, y)$ with respect to ω is the sum of terms of maximal (weighted) degree $\omega(p)$. It is denoted by $\mathrm{lp}_\omega(p)$.

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Its leading part is

$$\mathrm{lp}_\omega(p) = -1 - x^2y.$$

Assume that $\omega \in \mathbb{N}^2$ is such that

$$\mathrm{lp}_\omega(q)\mathrm{lp}_\omega(p) = \mathrm{lp}_\omega(f) - \mathrm{lp}_\omega(g),$$

and let

$$F(\mathrm{lp}_\omega(p)) = \mathbb{C}\left((f_\omega, g_\omega)\right).$$

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Then there is some $k \in \mathbb{Z}$ such that

$$\mathsf{lp}_\omega(f) - \mathsf{lp}_\omega(g) = f_\omega^k - g_\omega^k.$$

In particular,

$$(\mathrm{m}(\infty, f), \mathrm{m}(\infty, g)) = k \cdot (\deg f_\omega, \deg g_\omega)\,.$$

It is time for an example

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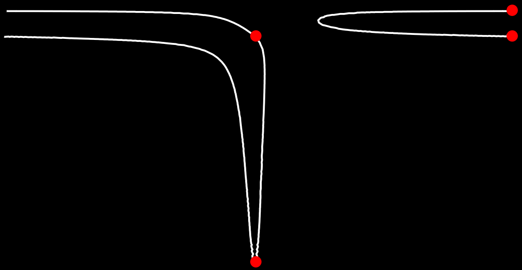

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Since

$$F(\mathrm{lp}_\omega(p)) = \mathbb{C}((x^2, -y^{-1})),$$

there is a $k \in \mathbb{N}$ such that

$$(\mathrm{m}(\infty, f), \mathrm{m}(0, g)) = k \cdot (2, 1).$$

The analysis of the other poles is done analogously. It results in the following 1-parameter family for their multiplicities.

f		g	
∞	$2k$	∞	k
0	$2k$	0	k
		-1	$2k$

Making the ansatz

$$f = \frac{f_0 + f_1 x + \cdots + f_4 x^4}{x^2} \quad \text{and} \quad g = \frac{g_0 + g_1 y + \cdots + g_4 y^4}{y(1+y)^2}$$

and

$$q = \frac{q_{0,0} + q_{1,0} x + \cdots + q_{2,1} x^2 y}{x^2 y (1+y)^2}$$

we find that

$$F(p) = \mathbb{C}\left(\left(\frac{(1-x)^2(1+x+x^2)}{x^2}, -\frac{(1+y+y^2)^2}{y(1+y)^2}\right)\right).$$

Thm

If $F(p)$ is non-trivial, then choosing $k = 1$ results in the multiplicities of the poles of a generator.

Negative results

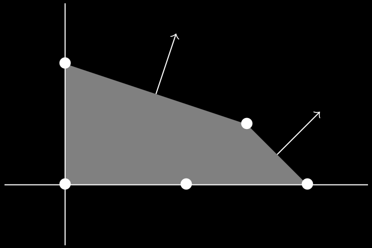
Prop

Let $\omega_1, \omega_2 \in \mathbb{Z}^2$ be two non-zero weights for which the leading parts of p are different and involve at least two terms. If

$$\mathsf{sgn}(w_1) = \mathsf{sgn}(w_2),$$

then $\mathrm{F}(\mathrm{p}) \cong \mathbb{C}$.

The polynomial $1 + x^2 + x^4 + x^3y + y^2$ does not have a separated multiple.



Assume that $\mathrm{lp}_\omega(\mathrm{p})$ is not a single term.

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If $\mathrm{F}(\mathrm{lp}_\omega(\mathrm{p}))$ is trivial, then so is $\mathrm{F}(\mathrm{p})$.

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Problem

Compute an upper bound on the size of a finite orbit.

Questions?

2) Determining an element of $\mathrm{F}(\mathrm{r}, \mathrm{p})$

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Poles

We restrict

$$r + qp = f - g$$

to the curve defined by p and relate the poles of r, f and g.

Let (s_1, s_2) be a root of p such that

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If

$$r(s_1, s_2) < \infty, \quad \text{then} \quad f(s_1) = \infty \quad \text{iff} \quad g(s_2) = \infty.$$

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It is also necessary to consider the orbits of its roots if $F(p)$ is non-trivial, and the finite orbits if $F(p)$ is trivial.

Prop [Bell, Moosa, Topaz, Bellaïche]

If $F(p)$ is trivial, then the number of finite orbits is finite.

2) Determining an element of $\mathrm{F}(\mathrm{r}, \mathrm{p})$

Multiplicities

Let $\varphi \in \mathbb{C}\{\{x^{-1}\}\}$ be a root of $p \in \mathbb{C}(x)[y]$ such that

$$\deg \varphi > 0 \quad \text{and} \quad \deg r(x, \varphi) > 0.$$

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Then

$$r(x,\varphi) = f(x) - g(\varphi),$$

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The inequality might be strict!

One can determine upper bounds for

$$\omega(f),\ \ \omega(g)$$

for $\omega = (1, \varphi)$ by analyzing the corresponding orbit and the behavior of r and p thereon.

How do we compute $\omega(f)$ when $\mathrm{lp}_\omega(f-g)$ is unknown?

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If $F(p)$ is trivial, and the orbit is infinite, then there is one element one of whose coordinates is not a pole of f and g, respectively.

If $\mathrm{F}(\mathrm{p})$ is trivial, but the orbit is finite, and we assume that

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and there is a k such that

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We believe that k is minimal such that

$$\omega(r) < k \cdot \deg f_\omega.$$

Sketch of why we consider orbits of roots when $\mathrm{F}(\mathfrak{p})$ is non-trivial.

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$$\deg r(x, \varphi) < \omega(\mathrm{lp}_\omega(r)) = \omega(r).$$

Sketch of why we consider orbits of roots when $F(p)$ is non-trivial.

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If

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then

$$\deg r(x, \varphi) < \omega(\mathrm{lp}_\omega(r)) = \omega(r).$$

Hence

$$\mathrm{lp}_\omega(r)(x, \overline{\varphi}) = 0 \quad \text{and} \quad r(\infty, \infty) = 0.$$

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Problem

How do the semi-algorithms generalize?

References

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