The kernel method and automated positive part extraction

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joint work with Manuel Kauers

Definition

Given a series

$$F = \sum_{\mathbf{i} \in \mathbb{Q}^n} f_{\mathbf{i}} x^{\mathbf{i}}, \quad f_{\mathbf{i}} \in \mathbb{C},$$

its positive part is

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Question

How can $[x^>]$ be applied to algebraic series when they are encoded by their minimal polynomials?

A motivating example from combinatorics

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and discarding non-positive powers in x results in

$$xF(x; t) = [x^{>}]\frac{x - \bar{x}}{1 - t(\bar{x} + x)}.$$

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Creative telescoping gives $U \in \mathbb{Q}(x,t)$ and $L \in \mathbb{Q}(t)\langle \partial_t \rangle$ with

$$L \cdot \left(\frac{\bar{x}}{1-\bar{x}} \frac{1-\bar{x}^2}{1-t(\bar{x}+x)} \right) = \partial_x \cdot U.$$

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Applying $[\bar{x}]$ then gives

$$L \cdot F(1;t) = 0.$$

The transformations of the functional equation are algebraic but in general they are not necessarily rational.

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Corollary

Let E/F be a separable field extension of finite degree. Then there is a polynomial $m \in F[X]$ such that

$$E \cong F[X]/\langle m \rangle.$$

Theorem (Shape Lemma)

Let K be a field with $\mathrm{char}(K) = 0$ and let $I \subseteq K[x_1, \ldots, x_n]$ be a 0-dimensional radical ideal in normal x_n-position. Then I has a Gröbner basis w.r.t. lex order of the form

$$\{x_1 - g_1, \ldots, x_{n-1} - g_{n-1}, g_n\}$$

for $g_1, \ldots, g_n \in K[x_n]$.

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$$p_1,p_2,p_3 \in \mathbb{Q}(x,y)[Z],$$

and the minimal polynomials

$$m_1,m_2,m_3,m \in \mathbb{Q}(x,y)[Z]$$

of $p_1(\alpha), p_2(\alpha), p_3(\alpha)$ and α, respectively?

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2) What are the series that correspond to α and the $p_i(\alpha)$'s?
3) What are their supports?

Another example

What is the positive part of the solution $F(x)$ to the equation

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The positive part depends on over which field of Laurent series,

$$\mathbb{Q}((x)) \quad \text{or} \quad \mathbb{Q}((\bar{x})),$$

the equation is solved.

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$$\mathbb{C}_{\preceq}((\mathbf{x})) := \left\{ \phi \;\middle|\; \phi \in \mathbb{C}^{\mathbb{Q}^n} \text{ and } \exists\ C \in \mathcal{C} \text{ s.t. } \operatorname{supp}(\phi) \subseteq C \right\}$$

is a field with respect to addition and multiplication.

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Proposition

Let $\preceq$ be a total order on $\mathbb{Q}^n$ and $q(\mathbf{x}) \in \mathbb{Q}[\mathbf{x}] \setminus \{0\}$, and let $c_{\mathbf{e}}\mathbf{x}^{\mathbf{e}}$ be its leading term w.r.t. $\preceq$.

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$$q(\mathbf{x})F(\mathbf{x}) - 1 = 0$$

in $\mathbb{Q}_{\preceq}((\mathbf{x}))$ is

$$F(\mathbf{x}) = \frac{1}{c_{\mathbf{e}}\mathbf{x}^{\mathbf{e}}} \sum_{k=0}^{\infty} \left(1 - \frac{q(\mathbf{x})}{c_{\mathbf{e}}\mathbf{x}^{\mathbf{e}}}\right)^k,$$

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and its support satisfies

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Furthermore, there is a bijection between the series solutions of the equation and the vertices of the convex hull of the support of $q(\mathbf{x})$.

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where $c_{\alpha_1} \mathbf{x}^{\alpha_1} + \cdots + c_{\alpha_N} \mathbf{x}^{\alpha_N}$ are the first N terms of a series solution ϕ ordered with respect to a total order $\preceq$ on $\mathbb{Q}^n$

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$$\operatorname{supp}(\phi) \subseteq \{\alpha_1, \ldots, \alpha_{N-1}\} \cup (\alpha_N + C) .$$

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Furthermore, the integer $N \geq k$ is minimal.

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Corollary

Given the minimal polynomial $m \in \mathbb{C}[\mathbf{x}][Z]$ of a series ϕ and a polynomial $p \in \mathbb{C}(\mathbf{x})[Z]$ we can determine all vectors $\mathbf{e}$ and all strictly convex minimal cones C in $\mathbb{Q}^n$ such that

$$\operatorname{lexp}_{\preceq}(p(\phi)) = \mathbf{e} \quad \text{and} \quad \operatorname{supp}_{\preceq}(p(\phi)) \subseteq e + C$$

for some total order $\preceq$ on $\mathbb{Q}^n$.

$$F(x, y; t) + \sum_{(p_1,p_2,p_3)} p_3(\phi)F(p_1(\phi), p_2(\phi); t)$$

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Question: How can we determine (minimal) cones $C \subseteq \mathbb{R}^3$ s.t.

$$\mathrm{supp}_{\preceq}\left(F(p_1(\phi), p_2(\phi); t)\right) \subseteq C?$$

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[Aparicio Monforte, Kauers] allows one to determine C based on a cone that contains $\operatorname{supp}(F)$ and pairs (e_1, C_1) and (e_2, C_2) such that

$$\operatorname{lexp}(p_i(\phi)) = e_i \quad \text{and} \quad \operatorname{supp}(p_i(\phi)) \subseteq e_i + C_i.$$

$$F(x, y; t) + \sum_{(p_1, p_2, p_3)} p_3(\alpha) F(p_1(\alpha), p_2(\alpha); t) = \mathrm{rat}(x, y, t)$$

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To summarize, we have an effective sufficient condition for

$$F(x,y;t) = [x^{\geq}y^{\geq}]\mathrm{rat}(x,y,t)$$

to hold.

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To summarize, we have an effective sufficient condition for

$$F(x, y; t) = [x^{\geq} y^{\geq}] \mathrm{rat}(x, y, t)$$

to hold. But in general the condition is not necessary.

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