Computer-Aided Design 37 (2005) 791–797

COMPUTER-AIDED DESIGN

www.elsevier.com/locate/cad

Adaptive knot placement in B-spline curve approximation

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Received 28 June 2004; accepted 20 September 2004

Abstract

An adaptive knot placement algorithm for B-spline curve approximation to dense and noisy data points is presented in this paper. In this algorithm, the discrete curvature of the points is smoothed to expose the characteristics of the underlying curve. With respect to the integration of the smoothed curvature, knots are placed adaptively to satisfy a heuristic rule. Several experimental results are included to demonstrate the validity of this algorithm.

Keywords: Knot placement; Curve approximation; Filtering; Locally small deflection spline

1. Introduction

The choice of knots has considerable effect on the shape of a curve [1]. An unreasonable knot vector may introduce unpredictable and unacceptable shape. In curve interpolation, the placement of knots is straightforward. On the contrary, in curve approximation, it is difficult to determine the amount and distribution of knots. Generally, an error bound is specified as input together with the data points to be fitted. The amount and distribution of the knots, which are required to satisfy the bound, is both unknown a priori. Therefore, in theory, knot placement is a multivariate and multimodal nonlinear optimization problem [2].

In general, the measured data points in Reverse Engineering (RE) are often dense as well as noisy. RE starts with a physical model and its geometric model is reconstructed from the coordinate data acquired using a measuring system in order to create and/or refine the digital model. Advances in range image acquisition and other three-dimensional digitizing devices allow us to acquire very dense data points from physical objects. Although these devices are of high fidelity, measurement errors are unavoidable due to the surface attributes of the physical object, the impact of the environment and the uncertainty of the measuring system.

For the data points that are organized in the form of scan lines, a two step process, i.e. curve fitting and then surface lofting, is usually employed to reconstruct surfaces from the data. Generally, B-spline curve approximation instead of interpolation is preferred for dense and noisy data to create the curves. Although curve approximation is well understood, the knot placement problem has not been dealt with satisfactorily, especially when dense and noisy data points are to be approximated. In this case, smoothing and re-sampling are usually employed to pre-process the data in present applications in order to facilitate the placement of knots and improve the performance of curve approximation [3,4]. But the data smoothing and re-sampling operations are highly dependent on the intervention of the designers. The features are often blurred and the original design intent is lost if no special care is taken [5].

In most applications, curve approximation methods are iterative process. Generally speaking, these methods proceed in either ways [6]: (1) start with the minimum or a small number of knots and iteratively increase the amount of knots to satisfy the error bound; or (2) start with the maximum or many knots and iteratively reduce the amount of knots to satisfy the error bound. When the amount of points is very large, the methods become

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0010-4485//$ - see front matter
doi:10.1016/j.cad.2004.09.008

time-consuming especially if the initial knots are not well determined. Unfortunately, the problem of determining the initial knots is not well addressed in the literatures.

An exception to the above ways is the algorithm proposed by Razdan [7]. This algorithm is restricted to smooth data points. Similar algorithm was employed by Hölzle [8] to approximate a polygon to a curve. When the number of knots is given in advance and the points are reasonably distributed with regard to the curvature of the underlying curve, average knot method [9,10] can be used to determine the distribution of the knots. However, this method highly relies on the data pre-processing operations such as smoothing and re-sampling. Furthermore, if an error bound is to be satisfied, this method becomes a trial-and-error method.

In this paper, a new method of knot placement for B-spline curve approximation to dense and noisy data points is presented. In order to preserve the shape features obliterated by the noise and reduce the time consumption, the discrete curvature of the points, in contrast to the points themselves, is smoothed, and knots are automatically placed with respect to the integration of the smoothed curvature to satisfy a heuristic rule presented in this paper without iterative calculations of the approximating curves.

The organization of this paper is as follows. A brief introduction to B-spline curve approximation is given in Section 2. In Section 3, the discrete curvature of the points is stated, followed by Section 4, where the digital filter is described. The knot placement algorithm is presented in Section 5. Examples are shown in Section 6 to demonstrate the effectiveness of the presented algorithm, followed by Section 7 that closes the paper.

2. B-spline curve approximation

A kth-order B-spline curve is defined by

$$\mathbf{C}(u) = \sum_{i=0}^{n} N_{i,k}(u)\mathbf{P}_i, \quad u \in [t_{k-1}, t_{n+1}] \tag{1}$$

where $\{\mathbf{P}_i\}$ are the control points, and $\{N_{i,k}(u)\}$ are the kth-order B-spline basis functions defined on the knot vector $\mathbf{T} = \{t_j\} \quad j = 0, \ldots, n+k$.

Given data points $\{\mathbf{d}_i\}$, and associating parameters $\{u_i\}$, $i = 0, \ldots, m$. The approximating curve $\mathbf{C}(u)$ in the least squares sense is defined by [1]

$$\text{minimize} \sum_{i=0}^{m} \|\mathbf{d}_i - \mathbf{C}(u_i)\|^2 \tag{2}$$

For B-spline curves, the normal equation is

$$\mathbf{MP} = \mathbf{Q} \tag{3}$$

where

$$\mathbf{M} = \left[\sum_{i=0}^{m} N_{l,k}(u_i) N_{j,k}(u_i)\right]$$

$\mathbf{P} = [\mathbf{p}_j]$ and

$$\mathbf{Q} = \left[\sum_{i=0}^{m} \mathbf{d}_i N_{l,k}(u_i)\right] \quad l = 0, \ldots, n, \ j = 0, \ldots, n.$$

Constraints can be incorporated into Eq. (2) using Lagrange multipliers [6].

The matrix $\mathbf{M}$ is singular if and only if there is a span $[t_j, t_{j+k}]$, $j = 0, \ldots, n$ that contains no u_i. This fact is known as Schoenberg–Whitney condition for unconstrained curve fitting [1]. Therefore, if there is no span $[t_j, t_{j+k}]$, $j = 0, \ldots, n$ that contains no u_i, i.e. $\mathbf{M}$ is not singular, the control points can be obtained by solving Eq. (3).

Parameterization of the points is well discussed and many methods have been proposed [1,6]. Chord length parametrization method is employed in this paper. The remaining problem, which is to determine a reasonable knot vector, will be discussed in the following sections.

3. Discrete curvature of data points

Generally, the scan lines used in curve approximation are ordered and distributed in two-dimension, or they can be sorted to get an ordered point set according to certain criteria.

For an ordered point set $\{\mathbf{p}_i, (i = 0, \ldots, n)\}$, the discrete curvature k_i at point $\mathbf{p}_i$ $(i = 1, \ldots, n-1)$ is defined as the inverse of the radius r_i of the circle passing through the three points $\mathbf{p}_{i-1}$, $\mathbf{p}_i$ and $\mathbf{p}_{i+1}$, as illustrated in Fig. 1.

The signed discrete curvature can be expressed as [5]

$$k_i = \frac{2\Delta\mathbf{p}_{i-1}\mathbf{p}_i\mathbf{p}_{i+1}}{L_i L_{i+1} Q_i} = \operatorname{sign}(\Delta\mathbf{p}_{i-1}\mathbf{p}_i\mathbf{p}_{i+1}) \frac{2\sin(a_i)}{Q_i} \tag{4}$$

where $\mathbf{Q}_i = \mathbf{p}_{i+1} - \mathbf{p}_{i-1}$, $Q_i = \|\mathbf{Q}_i\|$, $\mathbf{L}_i = \mathbf{p}_i - \mathbf{p}_{i-1}$, $L_i = \|\mathbf{L}_i\|$, $\Delta\mathbf{p}_{i-1}\mathbf{p}_i\mathbf{p}_{i+1} = \det(\mathbf{L}_i, \mathbf{L}_{i+1})$, $\cos(a_i) = (\mathbf{L}_i \mathbf{L}_{i+1})/(L_i L_{i+1})$.

Hamann and Chen [11] presented a more complex scheme to estimate the discrete curvature by computing a locally interpolating quadratic polynomial. However, due to the noise in the measured data points, the discrete curvature

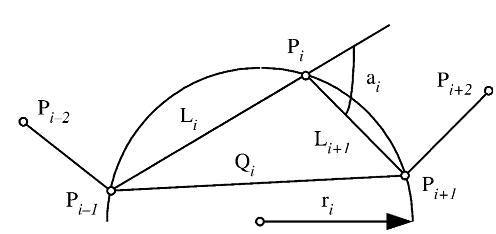

Fig. 1. Discrete curvature of ordered points.

changes very frequently. Consequently, this quadratic polynomial scheme has no advantage over the circle scheme in terms of exposing the curvature characteristic of the underlying curve.

The discrete curvature of a noisy point set is also flawed by noise. As a matter of fact, noise in the curvature is more severe than that in the data themselves. In this paper, the discrete curvature is considered as equally spaced digital signal and processed using digital signal processing methods to extract the tendency and characteristic of the curvature.

4. Smoothing of discrete curvature

Like many other situations, when the discrete curvature is considered as digital signal, noise all but obliterates the signal of interest. So a lowpass filter is employed to smooth the discrete curvature.

Filters can be grouped into two categories [12]: finite impulse response (FIR) filters and infinite impulse response (IIR) filters. Comparing their performance, a FIR filter is used in this paper. Suppose that the sequence of numbers $\{v_n\}$ is such a set of equally spaced measurements of some quantity $v(t)$, where n is an integer and t is a continuous variable. Typically, t represents time, and $v_n = v(n)$. FIR filters are defined by [12]

$$y_n = \sum_{k=-\infty}^{\infty} c_k v_{n-k} \tag{5}$$

The coefficients c_k are the constants of the filter, the v_{n-k} are the input data, and the y_n are the outputs. In practice, the number of products must be finite, and the length of the run of non-zero coefficients c_k is shorter than the run of data y_n. A lowpass filter means that the low frequencies pass through and the high frequencies are stopped, with a transition zone between the passband and stopband of frequencies. Hence, lowpass filters are employed to smooth out the noise in the signal. In this case, we assign the reciprocal of the number of the data points that could define the circle with the largest curvature of the data to the passband. The width of the transition zone is set to be 5–10% of the passband.

When the signal is masked by a large amount of noise, any small peaks left in the spectrum of the signal after filtering out the noise might be either from the original signal or from ripples in the transfer function used in the filtering process. Although, careful analysis could separate the two, the problem can also be avoided if a class of filters that 'vary smoothly' rather than ripple is used. By vary smoothly, we mean that the filters are monotone over long intervals of the frequency band.

Due to the nature of knot placement, the resulting sequence of the filtering should have precise zero-phase distortion. In this paper, this is implemented by processing the input data in both forward and reverse directions. After filtered in the forward direction, the sequence is reversed and run back through the filter again.

5. Knot placement

Knot vector plays an effective role in retrieving the underlying curve of the data points. In particular, given data points with considerable curvature variance of the underlying curve, the reconstructed curve may be significantly different from the underlying curve if the knots are not distributed reasonably in accordance with the varying of the curvature.

In this section, we will give a heuristic rule for knot placement, and subsequently, the knot placement algorithm for B-spline curve approximation to dense and noisy data.

5.1. A heuristic rule for knot placement

Su and Liu [13] demonstrated that, given points $\mathbf{p}_i$ $(i=0,\ldots,n)$ (Fig. 2), the interpolating spline curve is of locally small deflection if $a_i \le \pi/6$, which means that a cubic spline curve could be used to interpolate the given points. Here, a_i is the angle between vectors $\mathbf{L}_{i+1}$ and $\mathbf{L}_i$, as shown in Fig. 2 with $\mathbf{L}_i = \mathbf{p}_i - \mathbf{p}_{i-1}$ $(i=1,\ldots,n)$.

If the given points satisfy the above condition, the interpolating curve will approximate the underlying curve. Then a heuristic rule for knot placement in curve approximation is deduced from the above condition, i.e. if the points corresponding to the knots can satisfy this condition, the reconstructed curve will be a good approximation to the given data points. Hence, the problem of knot placement becomes how best to select points and place the knots from the given data points to fulfill the above rule.

5.2. The algorithm

Unlike the problem of piecewise linear curve approximation to freeform curves, the feature points, such as curvature extrema and inflection points, are critical to define the underlying curves of the data points. These kinds of points play a basic role in constraining the overall shape of the reconstructed curves. They also tend to impact the quality of the reconstructed curves. However, the detection of curvature extrema is relied on the removal of noise in

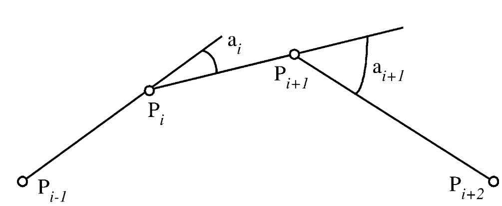

Fig. 2. Point distribution for locally small defleciton spline interpolation.

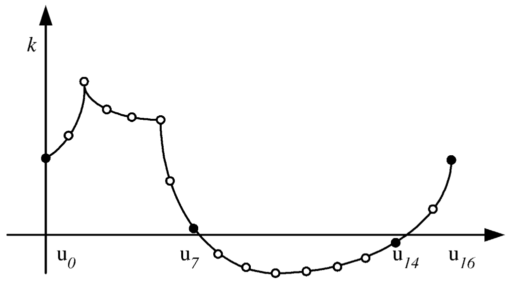

Fig. 3. Initial placement of knots.

the curvature, which is difficult to achieve perfectly. In this paper, only inflection points, which correspond to zero crossing points on the curvature plot, are defined as feature points. In the following, $k(u_i)$ is abbreviated to k_i.

It is easy to demonstrate that there is a zero-crossing point in $[u_i, u_{i+1}]$ if $k_i k_{i+1} \leq 0$. In this case, while $|k_i| < |k_{i+1}|$, u_i is set to be knot; otherwise, u_{i+1} is set to be knot. In the curvature plot, this kind of points correspond to the points nearest to the zero-crossing points, such as (u_7, k_7) and (u_{14}, k_{14}) illustrated in Fig. 3. The parameters of the two end points are also set to be knots. In this figure, the circles indicate the curvature of the data points $\mathbf{p}_i$ $(i = 0, \ldots, 16)$, and the positions corresponding to the knots are showed with filled circles. These initial knots segment the curvature plot and divide the initial parameter-curvature set into several subsets.

Then the absolute value of the curvature is integrated and the curvature integration is used as a function for knot placement in succession. As the data are quite dense, Newton-Cotes formula is employed to calculate the integration, i.e.

$$\mathbf{K} = \int_{u_0}^{u_m} |k| \mathrm{d}u = \sum_{i=1}^{m} (|k_i| + |k_{i-1}|)(u_i - u_{i-1})/2$$

As integration of any function is a smoothing process, the impact of the small peaks left in the signal after filtering will be eliminated.

At first, each segment is bisected iteratively with respect to the integration of the curvature and the parameters nearest to the bisecting positions are set to be knots to make all internal data points corresponding to the knots satisfy the heuristic rule of knot placement. This process is shown in Fig. 4(a). The horizontal axis indicates the parameters of the points, and the vertical axis indicates the discrete curvature of the points. Here, t'_0 and t'_1 are the knots inserted into $[t_i, t_{i+1}]$ and the subscript variables indicate the order of the insertion of the knots.

Then, the feature points are tested to verify if they satisfy the heuristic rule. More knots are inserted into the two adjacent intervals that joint at a feature point if the rule is not satisfied. A knot is inserted into the interval, where the absolute value of the curvature integration difference of the endpoints is bigger than that of the other. As illustrated in

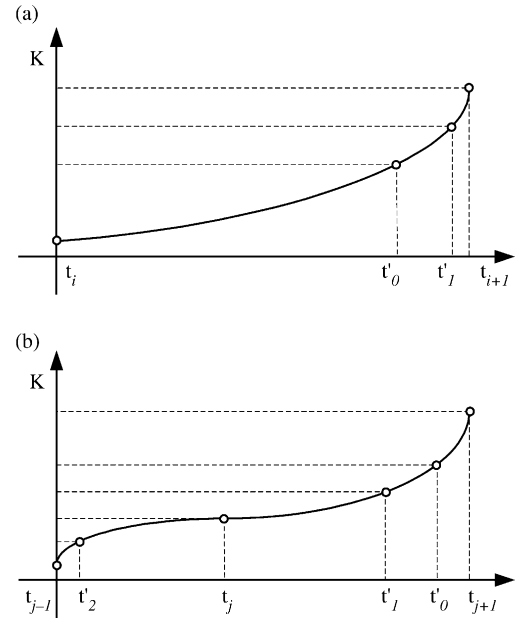

Fig. 4. Knot placement: (a) interior of the segment; (b) the feature point.

Fig. 4(b), at the beginning,

$$\int_{t_0}^{t_{j+1}} |k| \mathrm{d}u - \int_{t_0}^{t_j} |k| \mathrm{d}u > \int_{t_0}^{t_j} |k| \mathrm{d}u - \int_{t_0}^{t_{j-1}} |k| \mathrm{d}u,$$

t'_0 is placed at the point whose curvature is nearest to

$$\left(\int_{t_0}^{t_{j+1}} |k| \mathrm{d}u + \int_{t_0}^{t_j} |k| \mathrm{d}u \right) /2$$

Then, more knots are inserted iteratively in the same way to make t_j satisfy the heuristic rule. The subscript variables also indicate the order of the insertion of the knots.

6. Examples

In this section, we present and discuss the performance of the presented knot placement algorithm in the context of three numerical experiments. The three examples, which are commonly found in reverse engineering, are to place knots for approximating curves to a turbine blade section (Fig. 6), a car hook section (Fig. 7) and a section of the caudal region of a car (Fig. 8). All examples have small features that are important in defining the shape of the underlying curves. Using traditional data smoothing and reduction methods, it is very difficult to preserve them. It may be time consuming to process these data in practice in order to reconstruct the intended curve.

The lowpass filter, like the one shown in Fig. 5, is used to smooth the discrete curvature of the data points. As discussed in Section 2, the magnitude response of this filter varies smoothly in order to reduce the impact of filter on the output. This filter is also used in both forward and reverse directions to prevent phase distortion. Program is provided

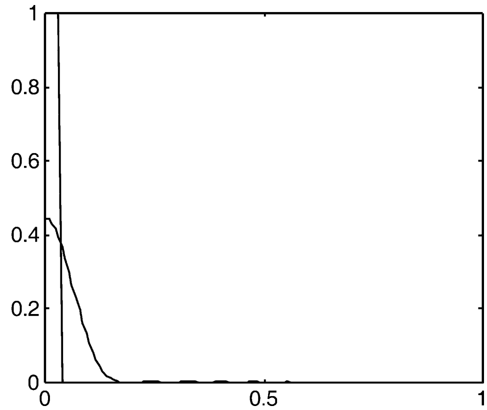

Fig. 5. Lowpass filter.

in [14] for the implementation of digital filter in one direction.

The graphical output of this section consists of four kinds of figures, namely the data point figures, the curve figures, the original and smoothed discrete curvature plots of the data points. In the curve figures, the circles indicate the points corresponding to the knots determined using the presented method.

6.1. Blade section example

The data of this example originate from a section of a turbine blade (Fig. 6(a)). The curve is sampled evenly with respect to the parameter, and the obtained data points are disturbed with magnitude less than 0.1. The original curvature plot of the data is shown in Fig. 6(b), and the smoothed curvature is shown in Fig. 6(c). The curve approximated with the knots determined using the presented method is showed in Fig. 6(d). The amount and distribution of the knots cohere with the variance of the curvature very well, and subsequently the small features of the curve are retrieved successfully. The maximum and average errors are 0.16416, 0.04849, respectively.

6.2. Car hook example

This example is from the RE application in automotive industry. The data of this example originate from a section of the hood of a car (Fig. 7(a)). The points are not evenly distributed, but they are dense in most regions. From the discrete curvature plot shown in Fig. 7(b), we can observe that the noise is very severe, and it is difficult to find out the design intent from the curvature plot. The smoothed curvature shown in Fig. 7(c) mostly coincides with the design intent of the data. The variance intention of curvature is roughly exposed. The distribution of the knots in Fig. 7(d) coheres with the intended variance of the curvature of the underlying curve in most portions. The small features are also retrieved successfully. The maximum and average errors are 0.09272, 0.02677, respectively.

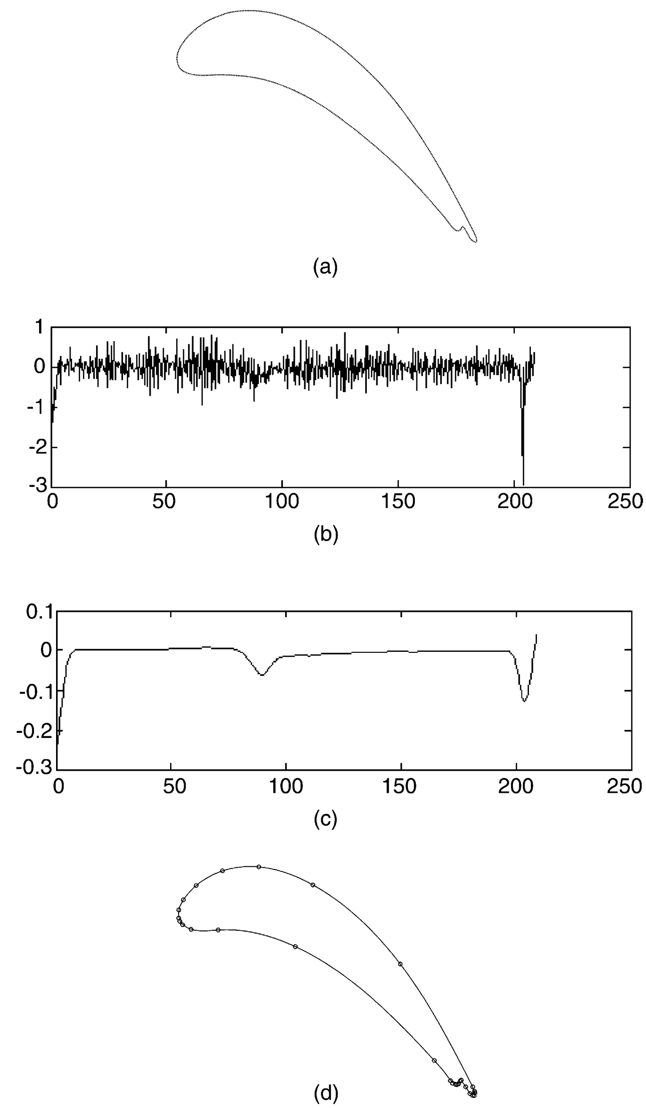

Fig. 6. Blade section: (a) point data; (b) discrete curvature; (c) smoothed discrete curvature; (d) approximated curve.

6.3. Car caudal region example

This example is also from the RE application in automotive industry. The data of this example originate from a section of the caudal region of a car (Fig. 8(a)). As shown in Fig. 8(b), the noise is not severe. The knots placed according to the initial curvature is showed in Fig. 8(c). The maximum and average errors are 0.12302, 0.02629, respectively. We also tried to filter the curvature (Fig. 8(d)) and place the knots according to the smoothed curvature (Fig. 8(e)). As the curvature is over smoothed, the knots are not as good as that showed in Fig. 8(c). The maximum and average errors are 0.22727, 0.05306, respectively. This example demonstrates that the small peak left in the curvature plot will not affect the distribution of the knots as a result of the integration of the curvature.

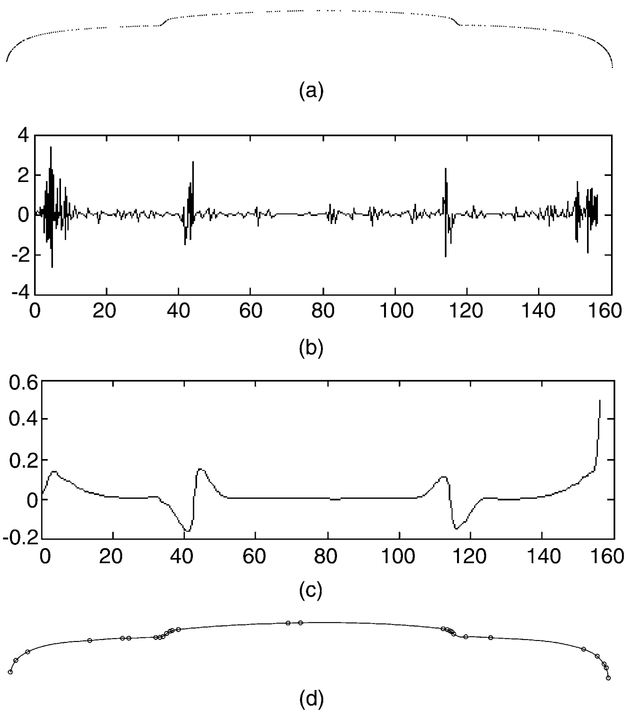

Fig. 7. Car hood section: (a) point data; (b) discrete curvature; (c) smoothed discrete curvature; (d) approximated curve.

In comparison, if the curvature is over smoothed, the distribution of the knots will get worse.

The examples showed above demonstrate that the presented knot placement method gives rise to good approximation to the given data. The approximation error of the examples is quite acceptable for RE applications. Meanwhile, the knots determined using our algorithm always satisfy the Schoenberg–Whitney condition. Other techniques, such as parameter correction and curve fairing [1,3], can be used subsequently to refine the curves for the applications, where the requirements are very strict.

The data points need not be pre-processed in the knot placement procedure except for the interactive delection of outliers and dropouts when it is necessary. From the viewpoint of accuracy, pre-processing of the data, such as smoothing and re-sampling, may introduce uncertainty into the data. The curves reconstructed using the presented algorithm are freed from this kind of uncertainty and conform to the given data.

7. Conclusions

An adaptive knot placement algorithm has been described in this paper for B-spline curve approximation to dense and noisy data points. In this algorithm, the discrete curvature of the data points is smoothed using a lowpass digital filter to expose the curvature charateristic of the underlying curve of the data. Then knots are automatically placed to make the curve, which passes the corresponding points and have locally small deflection. The knots determined using this approach are sensitive to the variation of the curvature, which means knots are concentrated in the regions, where the function

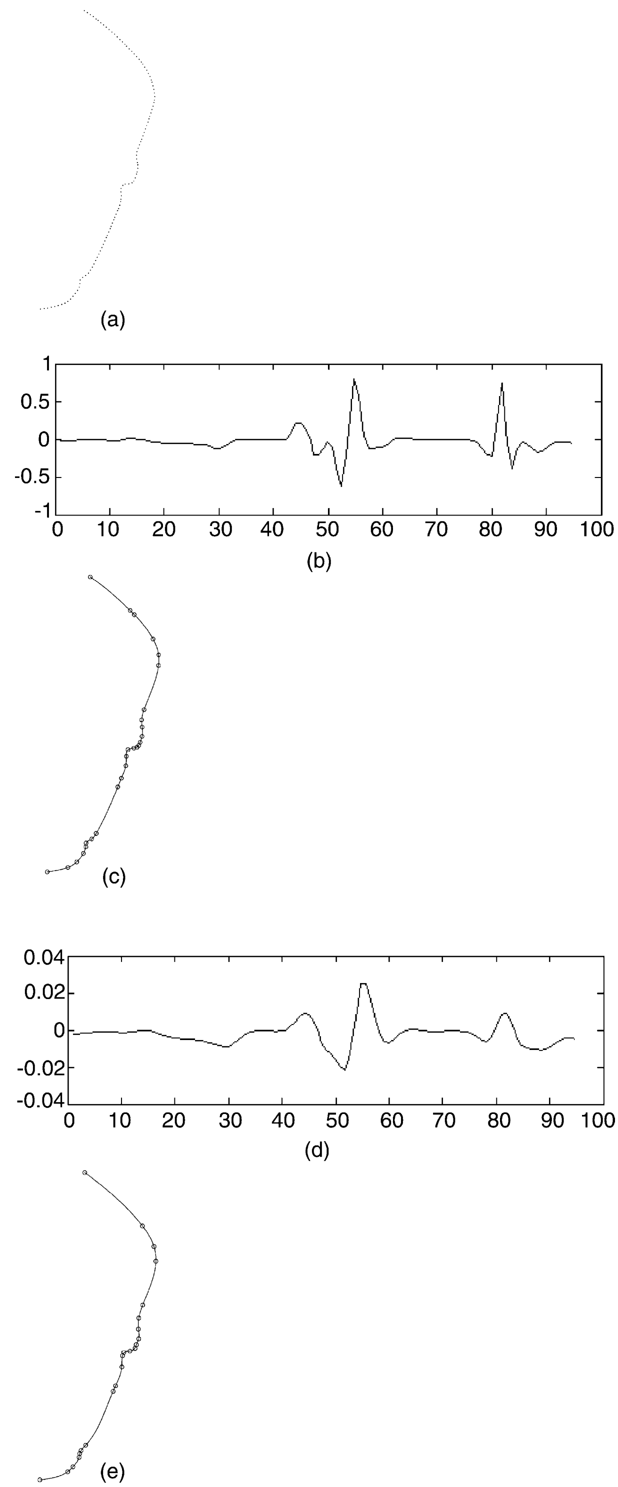

Fig. 8. Car caudal region section: (a) point data; (b) discrete curvature; (c) curve approximated with respect to the original curvature; (d) smoothed discrete curvature; (e) curve approximated with respect to the smoothed curvature.

underlying the data is more 'severe'. The heuristic rule for knot placement can also be used in B-spline curve approximation to smooth data points.

This algorithm can be used in estimating the initial knot vector for error bounded curve approximation. As this knot vector estimation reflects the shape of the data points very well, the curve approximation iterations could converge faster than those in traditional methods. Furthermore, the curves reconstructed using the presented algorithm are freed from the uncertainty introduced during the pre-processing of the data points, and conform to the given data points. It also has the potential to eliminate the amounts of knots and control points in curve approximation.

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