Branching in Schubert calculus

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New Connections in Integrable Systems
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Grassmannians I

General setup: partial flag varieties

- G complex algebraic group, $T \subset B \subset G$, $W = N(T)/T$,
- For $B \subset P$ a parabolic, $(G/P)^T \cong W_P\backslash W \cong W/W_P$.

Study multiplication and restriction for $H_T^*(G/P)$ in a “nice” basis:

$$H_T^*(G/P) \otimes H_T^*(G/P) \to H_T^*(G/P)$$

$$H_S^*(G/P) \to H_S^*(H/Q)$$

where $H \leq G$ with compatible parabolic Q and torus S. Also,

$$H_T^*(G/P) \otimes H_T^*(G/R) \to H_T^*(G/(P \cap R))$$

For G of type $A_n/B_n/C_n/D_n$, P maximal, G/P is a **Grassmannian**.

$$\textit{E.g.}\ Gr(k;n) := GL_n/P_{k,n-k} \cong \{V \subseteq \mathbb{C}^n \mid \dim V = k\}$$

$$SpGr(k;2n) := Sp_{2n}/P^{Sp}_{k,2n-k} \cong \{V \subseteq \mathbb{C}^{2n} \mid \dim V = k, V \subseteq V^\perp\}$$

Schubert classes

Schubert classes For $\pi \in W_P \backslash W$, the corresp. **Schubert class** is

$$S_\pi := \left[\overline{B^- \pi^{-1} P/P}\right] \in H_T^*(G/P).$$

Then $\{S_\pi\}_{\pi \in W_P \backslash W}$ freely generate $H_T^*(G/P)$ as an $H_T^*(\text{pt})$–module.

Classical question: Determine the structure constants,

$$S_\lambda \cdot S_\mu = \sum_\nu c_{\lambda\mu}^\nu S_\nu$$

Note: if $G/P \cong Gr(k;n)$, then (in H^*, not H_T^*) $V_\lambda \otimes V_\mu = \bigoplus_\nu V_\nu^{\oplus c_{\lambda\mu}^\nu}$

$c_{\lambda\mu}^\nu$= *the Littlewood-Richardson coefficients for* GL_k

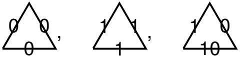
E.g. In $Gr(2;4)$, $(H_T^*(pt) \cong \mathbb{Z}[y_1, y_2, y_3, y_4])$:

$$S_{\square} \cdot S_{\square} = S_{\square\square} + S_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} + (y_2 - y_3) S_{\square} \qquad (\text{in } H_T^*)$$

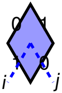
Grassmannian puzzles

Let $\lambda, \mu, \nu \in Gr(k;n)^T \cong 0^k 1^{n-k}$ (binary strings).
A **puzzle** P of type (λ, μ, ν) is a tiling of 

by the pieces:

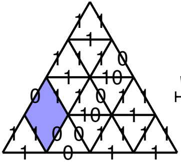
$\Bigg($ 

, their rotations $\Bigg) \overset{w}{\mapsto} 1$

$\Bigg($ 

(the equivariant piece) $\Bigg) \overset{w}{\mapsto} y_i - y_j$.

E.g.



$\overset{w}{\mapsto} w(P) = \prod_{\substack{p \in \text{ puzzle} \\ \text{pieces of } P}} w(p) = y_1 - y_2 \in \mathbb{Z}[y_1, \ldots, y_n]$

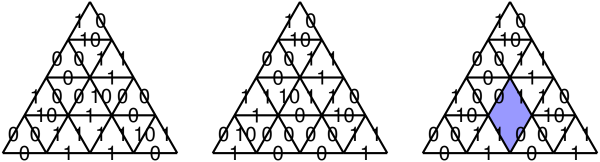
Schubert calculus via puzzles I

Theorem (Knutson-Tao '03, many extensions since)

For $\lambda, \mu \in 0^k 1^{n-k}$, the product of S_λ and S_μ in $H_T^*(Gr(k;n))$ is

$$S_\lambda \cdot S_\mu = \sum_\nu w\left(\underset{\nu}{{}^{\lambda}\triangle^{\mu}}\right) S_\nu, \quad \text{for } w\left(\underset{\nu}{{}^{\lambda}\triangle^{\mu}}\right) = \sum_{P \in (\lambda,\mu,\nu)} w(P) \in H_T^*(pt).$$

E.g. $S_{0101} \cdot S_{0101} = S_{0110} + S_{1001} + (y_2 - y_3) S_{0101}$

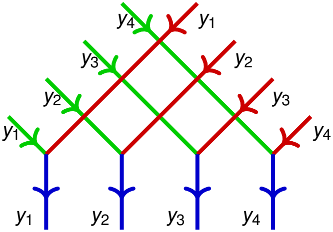


Scattering diagrams

[Zinn-Justin (ZJ) '09, Wheeler–ZJ '16, Knutson–ZJ '17]

- Reinterpret puzzles as (dual) scattering diagrams involving (rational) 5-vertex R-matrices and fusion. Upgrade to the 6-vertex model.

$w =$



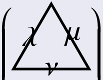
$: (\mathbb{C}^3)^{\otimes 4} \otimes (\mathbb{C}^3)^{\otimes 4} \to (\wedge^2 \mathbb{C}^3)^{\otimes 4}$

- Recast AJS/Billey formula for restriction to T-fixed points $S_\lambda|_\mu$.

Schubert calculus via puzzles II

Theorem (H–Knutson–Zinn-Justin '18)

Let $\lambda \in 0^j 1^{n-j}$, $\mu \in 0^k 1^{n-k}$, $\nu \in 0^j (10)^{k-j} 1^{n-k}$, defining equivariant Schubert classes S_λ, S_μ, S_ν on $Gr(j;n)$, $Gr(k;n)$, $Fl(j,k;n)$ respectively. The product in $H_T^*(Fl(j,k;n))$ can be computed as:

$$\pi_j^*(S_\lambda) \cdot \pi_k^*(S_\mu) = \sum_\nu w \Bigg($$


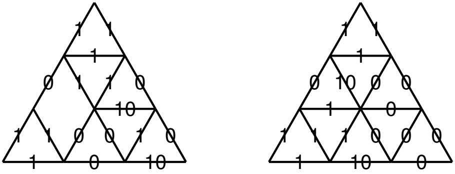
$$\Bigg) S_\nu$$

$$\begin{tikzcd} & Fl(j,k;n) \arrow[dl, "\pi_j"'] \arrow[dr, "\pi_k"] & \\ Gr(j;n) & & Gr(k;n) \end{tikzcd}$$

Example

For instance, for $Fl(1,2;3)$, $Gr(1;3)$, and $Gr(2;3)$:

$$
\begin{aligned}
\pi_1^*(S_{101}) \cdot \pi_2^*(S_{100}) &= S_{10,0,1} \cdot S_{1,0,10} \\
&= (y_1 - y_2) S_{1,0,10} + S_{1,10,0}
\end{aligned}
$$



Grassmann duality

Grassmann duality

There is a ring isomorphism (from a homeom. of Grassmannians):

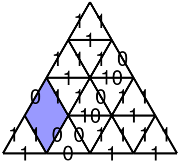
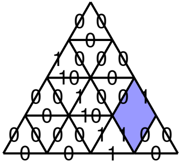
$$H_T^*(Gr(k,2n)) \cong H_T^*(Gr(2n-k,2n)), \quad S_\lambda \mapsto S_{\overline{\lambda}}$$

$$S_\lambda \cdot S_\mu \leftrightarrow S_{\overline{\mu}} \cdot S_{\overline{\lambda}} \qquad \overline{\lambda} = (\text{reverse } \lambda \text{ and switch } 0 \leftrightarrow 1)$$

$\overset{Dual}{\longleftrightarrow}$


:= reflect through vertical axis and swap 0 and 1

For instance,


$\overset{Dual}{\longleftrightarrow}$


Branching from A to C

We are interested in the cohomology pullback of the inclusion

$$SpGr(k;2n) \xhookrightarrow{\iota} Gr(k;2n).$$

Involution: $Sp_{2n} = GL_{2n}^{\sigma}$, for $J = \text{Antidiag}(-1,\dots,-1,1,\dots,1)$,

$$\sigma : GL_{2n} \to GL_{2n},\; X \mapsto J^{-1}(X^{-1})^{\text{tr}}J$$

Main question: $\boxed{\iota^*(S_\lambda) = \sum_\nu c_\nu^\lambda S_\nu}$ $\quad c_\nu^\lambda = ??$

- Pragacz '00: (building on work of Stembridge) positive tableau formulæ for $H^*(Gr(n;2n)) \to H^*(SpGr(n;2n))$
- Coşkun '11: positive geometric rule for $H^*(Gr(k;2n))$

A combinatorial branching rule

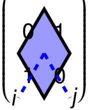
Theorem (H–Knutson–Zinn-Justin '18)

For $\lambda \in 0^k 1^{2n-k}$, $H_T^*(Gr(k;2n)) \xrightarrow{\iota^*} H_T^*(SpGr(k;2n))$ takes S_λ to

$\iota^*(S_\lambda) = \sum_\nu w\Big($  $\Big) S_\nu$

where $w\Big($  $\Big) \in H_T^*(pt) = \mathbb{Z}[y_1, \dots, y_n]$ is computed via R- and K-matrices from the 5–vertex model, and fusion.

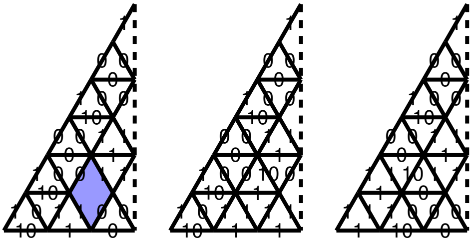
<u>*Note*</u>:  is half of a "self-dual" puzzle under Grassmann duality.

$w\Big($  $\Big) = \begin{cases} y_i - y_j, & j \le n \\ y_i + y_{2n+1-j}, & n < j \end{cases}$

$w\Big($  $\Big) = 1 \qquad (X, Y) = (0,1), (1,0)$

Example and goal

Example: $\iota^*(S_{110101}) = (y_2 - y_3) S_{10,1,0} + S_{10,1,1} + S_{1,10,0}$



Goal: generalize to the 6–vertex model, understand the underlying geometry, obtain a generalized puzzle rule.

Grassmannians II

Idea of proof:

I. Inclusion of T-fixed points:

$$\widetilde{\iota}(\nu) := (\nu\bar{\nu} \text{ with 10's turned into 1's}).$$

$$\begin{tikzcd}
(10)^{n-k}\{0,1\}^k \cong SpGr(k,2n)^T \arrow[r, hook, "f_2"] \arrow[d, hook, "\widetilde{\iota}"] & SpGr(k,2n) \arrow[d, hook, "\iota"] \\
0^k 1^{2n-k} \cong Gr(k,2n)^T \arrow[r, hook, "f_1"] & Gr(k,2n)
\end{tikzcd}$$

Note: We interchangeably consider binary strings $\pi \in 0^k 1^{2n-k}$ (i.e. in $W_P \backslash W$) and $\pi^{-1} \in W/W_P$.

Cohomology rings

In equivariant cohomology, we get:

$$\begin{tikzcd} H_T^*(SpGr(k;2n)^T) & H_T^*(SpGr(k;2n)) \arrow[l, "f_2^*"'] \\ H_T^*(Gr(k;2n)^T) \arrow[u, "(\bar{\iota})^*"] & H_T^*(Gr(k;2n)) \arrow[l, "f_1^*"'] \arrow[u, "\iota^*"'] \end{tikzcd}$$

- Since each f_i^* is injective (Kirwan), to understand ι^* we can instead compute in the left column.
- Use the Andersen-Jantzen-Soergel, Billey formula ('94,'97) for restriction to T-fixed points, $S_\lambda|_\mu$.

Scattering diagrams

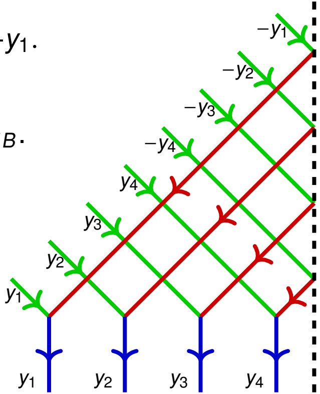
II. We get a **scattering diagram** with NW spectral parameters: $y_1, \ldots, y_n, -y_n, \ldots, -y_1$.

Each coloured strand carries a copy of $\mathbb{C}^3_G, \mathbb{C}^3_R$, or $\wedge^2\mathbb{C}^3_B$, with basis $\{0, 10, 1\}_{R/G/B}$.

New ingredient:

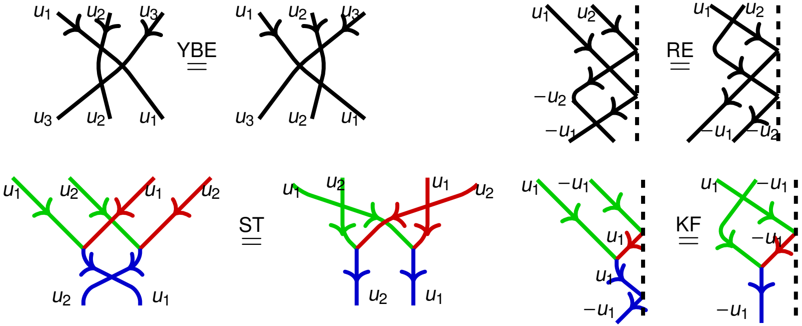
$K_C(a) =$ 

$: \mathbb{C}^3_C \to \mathbb{C}^3_D, \quad (a \mapsto -a)$



Relations

We ask that these maps satisfy the following identities:



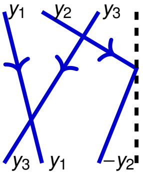
E.g. $K_B(u_1) \circ U_{GR}(u_1) \circ (\mathrm{Id} \otimes K_G(-u_1)) \overset{\mathrm{KF}}{=} U_{GR}(-u_1) \circ (\mathrm{Id} \otimes K_G(u_1)) \circ R_{GG}(2u_1)$

The AJS/Billey formula

Restriction to T-fixed points via scattering diagrams:
For $\lambda, \mu \in W_P \backslash W$, in the Grassmannian case, the AJS/Billey formula can be expressed as

$$S_\lambda|_\mu = w(\mu)^\lambda_{id} = \begin{array}{c}\text{the scattering diagram for } \mu \\ \text{evaluated at } (\lambda, id)\end{array}$$

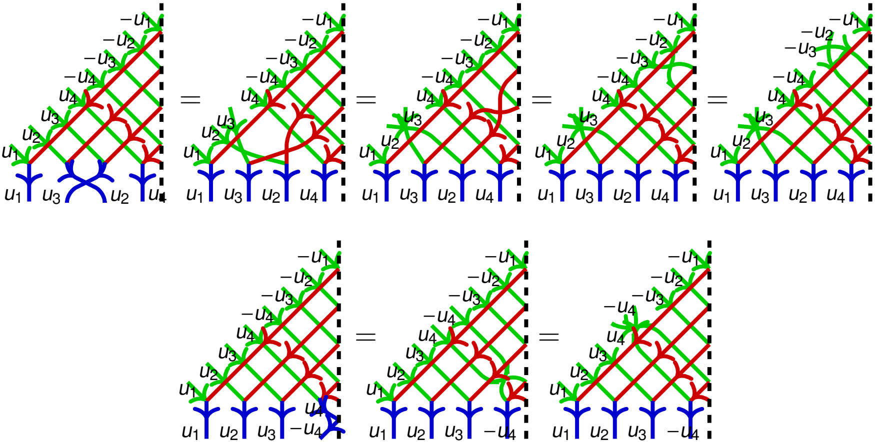
For $SpGr(1;6)$, $\mu = 10, 10, 0 \leftrightarrow s_2 s_3 s_1$, the scattering diagram is



$$(R_{BB}(y_1 - y_3) \otimes \mathrm{Id}) \circ (Id^{\otimes 2} \otimes K_B(y_2)) \circ (\mathrm{Id} \otimes R_{BB}(y_2 - y_3)) : (\wedge^2 \mathbb{C}^3_B)^{\otimes 3} \to (\wedge^2 \mathbb{C}^3_B)^{\otimes 3}$$

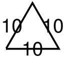
Theorem proof (sketch)

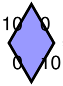
We show $\iota^*(S_\lambda)|_\mu = (\tilde{\iota})^*(S_\lambda|_{\tilde{\iota}(\mu)}) = \sum_\nu w\left(\overset{\lambda}{\underset{\nu}{\triangle}}\right) S_\nu|_\mu$ using:

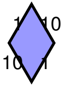


6-vertex upgrade

- Non-compact, symplectic resolution upgrade: We upgrade the Grassmannians G/P to their cotangent bundles T^*G/P.
- Additional puzzle pieces and $R_{GR}(a)$:



and its rotation,


,


, (equivariant pieces).

$$
\begin{array}{cc}
 & \begin{array}{ccccccccc} 0\vee0 & 0\vee10 & 0\vee1 & 10\vee0 & 10\vee10 & 10\vee1 & 1\vee0 & 1\vee10 & 1\vee1 \end{array} \\
\begin{array}{c} 0\wedge0 \\ 0\wedge10 \\ 0\wedge1 \\ 10\wedge0 \\ 10\wedge10 \\ 10\wedge1 \\ 1\wedge0 \\ 1\wedge10 \\ 1\wedge1 \end{array} &
\left[\begin{array}{ccc|ccc|ccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{h}{h-a} & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \frac{h}{h-a} \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{a}{h-a} & 0 & 0 \\
\hline
0 & \frac{a}{h-a} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{h}{h-a} & 0 & 1 & 0 & 0 & 0 & 0 \\
\frac{h}{h-a} & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
\hline
0 & 0 & 1 & 0 & \frac{h}{h-a} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{a}{h-a} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{h}{h-a} & 0 & 0 & 0 & 0 & 1
\end{array}\right]
\end{array}
$$

Maulik–Okounkov classes

For a regular circle action $S \curvearrowright T^*G/P$ and a fixed pt. $\lambda \in W/W_P$, the Maulik–Okounkov stable envelope construction produces a cycle

$$MO_\lambda = \overline{BB}_\lambda + \sum_{\mu \le \lambda} a_{\lambda,\mu} \overline{BB}_\mu, \quad a_{\lambda,\mu} \in \mathbb{Z}_{\ge 0}$$

$BB_\lambda = Attr(\lambda) = CX_\lambda^o :=$ conormal bundle of the Bruhat cell X_λ^o.

This in turn gives a class $[MO_\lambda] \in H^*_{T\times\mathbb{C}^\times}(T^*G/P) \cong H^*_T(G/P)[\hbar]$.

Segre–Schwartz–MacPherson:

$$SSM_\lambda = \frac{[MO_\lambda]}{[\text{zero section}]} \in \widetilde{H}^0_{T\times\mathbb{C}^\times}(T^*G/P)$$

$$\Rightarrow SSM_\lambda = \hbar^{-\ell(\lambda)} S_\lambda + \text{l.o.t}(\hbar) \quad \Rightarrow S_\lambda = \lim_{\hbar\to\infty} (SSM_\lambda \cdot \hbar^{\ell(\lambda)})$$

Structure constants: $c^\nu_{\lambda\mu} = \lim_{\hbar\to\infty} ((c')^\nu_{\lambda\mu} \cdot \hbar^{\ell(\lambda)+\ell(\mu)-\ell(\nu)})$

Geometric interpretation

A **Lagrangian correspondence** L between two symplectic manifolds A and B, $A \overset{L}{\leftrightarrow} B$, is:

A Lagrangian cycle L in $(-A) \times B$
(equivalently L in $A \times (-B)$).

If $T \curvearrowright A, B$ and L is T–invariant, then

$$H_T^*(A) \xrightarrow{(\pi_A)^*} H_T^*(A \times B) \xrightarrow{\cup[L]} H_T^*(A \times B) \xrightarrow{(\pi_B)_*} H_T^*(B) \cong H_T^*(B)$$

Note: In our setting, will work with T^*G/P.

Examples

1. *Symplectic reduction*
For $T \subseteq G \curvearrowright X$ Hamiltonian action, have a moment map $X \xrightarrow{\mu} \mathfrak{g}^*$. Take a regular point a for μ s.t. $a \in (\mathfrak{g}^*)^G$ Let $Z = \mu^{-1}(a),\ Y = \mu^{-1}(a)//G$. Then $X \hookleftarrow Z \twoheadrightarrow Y$.
[Marsden-Weinstein '74] ∃! symplectic structure on Y s.t. $Z \subseteq (-X) \times Y$ is Lagrangian.

2. *Maulik–Okounkov stable envelopes*
Suppose $S \curvearrowright X$ is a sympl. res. with a circle action.
Let C be a fixed point component.
The **stable envelope construction** produces a certain Lagrangian cycle $L = \overline{Attr(C)} + \ldots$ in $(-C) \times X$.

Correspondences from graphs

General setting

Let $A \xrightarrow{f} B$ be a morphism of oriented manifolds. $\Gamma(f)$=graph of f.
$\Gamma(f)^{tr} \subseteq B \times A$ is a correspondence inducing $f^* : H^*(B) \to H^*(A)$.

Examples:

- Diagonal inclusion $M \xhookrightarrow{\Delta} M \times M$. Then $\Gamma(\Delta)^{tr}$ induces
$$H^*(M) \otimes H^*(M) \xrightarrow{m} H^*(M).$$
- The graph of the inclusion $Fl(j,k;n) \hookrightarrow Gr(j;n) \times Gr(k;n)$ induces multiplication
$$H^*(Gr(j;n)) \otimes H^*(Gr(k;n)) \xrightarrow{m} H^*(Fl(j,k;n)).$$
- The graph of $SpGr(k;2n) \xhookrightarrow{\iota} Gr(k;2n)$ induces the restriction
$$H^*(Gr(k;2n)) \to H^*(SpGr(k;2n)).$$

Lifting to cotangent bundles

Assume we have a torus action $T \curvearrowright A, B$. We have the following commutative diagram of correspondences. It allows us to study the bottom row in cohomology via the symplectic setting of the top row.

$$\begin{tikzcd}
T^*B \arrow[r, "C(\Gamma(f))^{tr}"] & T^*A \\
B \arrow[u, "\Gamma(\iota_B)"] \arrow[r, "\Gamma(f)^{tr}"] & A \arrow[u, "\Gamma(\iota_A)"']
\end{tikzcd}$$

$$\begin{tikzcd}
\widetilde{H}^*_{T\times\mathbb{C}^\times}(T^*B) \arrow[r] & \widetilde{H}^*_{T\times\mathbb{C}^\times}(T^*A) & \beta \arrow[r, mapsto] & \alpha \\
\widetilde{H}^*_{T\times\mathbb{C}^\times}(B) \arrow[u] \arrow[r, "f^*"] & \widetilde{H}^*_{T\times\mathbb{C}^\times}(A) \arrow[u] & \frac{\beta}{[B\subseteq T^*B]} \arrow[u, mapsto] \arrow[r, mapsto] & \frac{\alpha}{[A\subseteq T^*A]} \arrow[u, mapsto]
\end{tikzcd}$$

The Sp_{2n} case

Theorem in progress (H–Knutson–Zinn-Justin '20)

There are Lagrangian correspondences

$$\lambda \xleftrightarrow{L_1} T^*Gr(k,2n) \xleftrightarrow{L_2} T^*OGr(k,4n) \xleftrightarrow{L_3} T^*SpGr(k,2n)$$

that compute the restriction of SSM classes, and together with the 6-vertex R- and K-matrices and fusion realize a puzzle rule.

- $L_1 = MO_\lambda$ is the stable envelope for the circle action

$$S_1 \cong Diag(t, t^2, \ldots, t^{2n}).$$

- $L_2 = Attr(T^*Gr(k,2n))$ is the stable envelope for the circle

$$S_2 \cong Diag(t, \ldots, t, t^{-1}, \ldots, t^{-1}).$$

- L_3 is obtained by symplectic reduction.

The end

Thank you!