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POSTECH

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Quantum groups and relative Langlands

Lecture 3 (on other groups)

today:
- quantum affine groups
- quantum symmetric pairs
- (the Gaiotto conjectures)

Reminder: G reductive connected split / F local non-archimedean

$G \supset B = AU$, $\psi : U \to \mathbb{C}^*$

$\widetilde{G}$ metaplectic cover of G, depends on (Q, n)

$q \leftrightarrow v^2$ $\quad \zeta = \sqrt[2n]{1}$

Buciumas-Patnaik

$U_\zeta(\mathfrak{g}) =$ Lusztig's QG

$$C_c^{\infty,c}\left(K \backslash \widetilde{G} / K\right) =: \mathcal{H}(\widetilde{G}, K) \simeq \mathbb{C}[\operatorname{Rep} G^\vee_{(Q,n)}]$$

$$\circlearrowright * \qquad\qquad \circlearrowright - \otimes q^{Fr}(\ )$$

$$C^{\infty}\left((U,\psi) \backslash \widetilde{G}\right)^K =: \mathcal{W}(\widetilde{G}, K) \simeq \mathbb{C}_v[\operatorname{Rep} U_\zeta(\mathfrak{g}^\vee)]$$

moral: structure of $\mathcal{W}(\widetilde{G}, K)$ controlled by $U_\zeta(\mathfrak{g}^\vee)$

now: (SW) dual interpretation

Brubaker - Buciumas - Bump (BBB19) → Savin's cover of $\widetilde{GL_r}$ deg n

π_x unramified principal series of $\widetilde{GL_r}$ $\quad x \in T^\vee = (\mathbb{C}^*)^r$

$x = (x_1, \dots, x_r)$

$I_{s_i} : \pi_x \longrightarrow \pi_{s_i x}$ intertwiners

$$\mathcal{W}(\pi_x) = \{ f : \pi_x \to \mathbb{C},\ f(ug) = \psi(u) f(g) \}$$

$$\begin{array}{ccc}
\mathcal{W}(\pi_x) & \simeq & V_n(x_1) \otimes \dots \otimes V_n(x_i) \otimes V_n(x_{i+1}) \otimes \dots \otimes V_n(x_r) \\
\downarrow I_{s_i} & & \downarrow R_{i,i+1}(x_i x_{i+1}^{-1}) \\
\mathcal{W}(\pi_{s_i x}) & \simeq & \dots\dots V_n(x_{i+1}) \otimes V_n(x_i) \dots\dots
\end{array}$$

$V_n(x) \to$ evaluation module of $U_v(\widehat{\mathfrak{sl}_n})$

$R(z)_{i,i+1} \rightsquigarrow U_v(\widehat{\mathfrak{sl}_n})$ affine R-matrix

Gao-Gurevich-Karasiewicz GGK25 (slight reinterpretation)

$$\mathcal{H}(\widetilde{GL}_r, I) \simeq \underset{\overset{\Downarrow}{T_i}}{H_0} \otimes \underset{\theta_\lambda}{\mathbb{C}[\widetilde{\Lambda}^\vee]}$$

let $I_{s_i} = (v - v^{-1})\,\theta_{\alpha_i^\vee} + v^{-1}(1 - \theta_{\alpha_i^\vee})\,T_i \in \mathcal{H}(\widetilde{GL}_r, I)$

↳ intertwiner

$$\begin{array}{ccc} \mathcal{W}(\widetilde{GL}_r, I) & \overset{\text{v.s. iso}}{\simeq} & V_{\bar n}(z)^{\otimes r} \\ \downarrow I_{s_i} & & \downarrow R(z)_{i,i+1} \\ \mathcal{W}(\widetilde{GL}_r, I) & \simeq & V_{\bar n}(z)^{\otimes n} \end{array}$$

such that

$\rightsquigarrow$ affine R-matrix of $U_v(\widehat{\mathfrak{sl}}_n)$

$\rightsquigarrow$ induces $U_v(\widehat{\mathfrak{sl}}_n)$ action on $\mathcal{W}(\widetilde{GL}_r, I)$

Cor (GGK): $\mathrm{End}_{\mathcal{H}(\widetilde{GL}_r, I)} \mathcal{W}(\widetilde{GL}_r, I) \simeq$ quantum affine Schur alg.

$$
\begin{tikzcd}
 & C^\infty\left((U,\psi) \backslash \tilde{G} / TU \right) \arrow[ll, hook'] \arrow[d, phantom, "\wr"] \\
C^\infty\left((U,\psi) \backslash \tilde{G} / (U,\psi) \right) \arrow[r, hook] & C^\infty\left((U,\psi) \backslash \tilde{G} / I \right)
\end{tikzcd}
$$

Proposition / Conjecture (Buciumas)

$\widetilde{GL}_r$ n-Savin cover

$q = v^2$,

$$\mathcal{H}(\widetilde{GL}_{r}, K) \simeq \mathbb{C}[\widetilde{\Lambda}^\vee]^{W} \simeq \mathbb{C}[\operatorname{Rep} GL_r^\vee]$$

Sp_{2r}

$$\circlearrowleft\, * \qquad \circlearrowleft \qquad \circlearrowleft\, - \otimes qF_n$$

$$\mathcal{W}(\widetilde{GL}_r, K) \overset{①}{\simeq} \Lambda_v^r\, V_n(z) \overset{②}{\simeq} \mathbb{C}_v[\operatorname{Rep} U_s(\mathfrak{gl}_n)]$$

SO_{2r+1}

$$\circlearrowleft \qquad \circlearrowleft \qquad \circlearrowleft\ (*)$$

$$U_v(\widehat{\mathfrak{sl}}_n) \qquad U_v(\widehat{\mathfrak{sl}}_n) \qquad \boxed{U_v(\widehat{\mathfrak{sl}}_n)}$$

$\to$ q-Fock space

(*) How to obtain (*) from Rep. th.

Endofunctors on Rep $U_\xi(\mathfrak{gl}_r)$:

$E : \quad V \otimes -$

$F : \quad V^* \otimes -$

E, F have natural endo with eigenvalues in $\mathbb{Z}$ $\rightsquigarrow$ $E = \bigoplus_{0 \le a \le n} E_a$

$F = \bigoplus_{0 \le a \le n} F_a$

at the level of $\mathbb{C}[\mathrm{Rep}(U_\xi(\mathfrak{gl}_r))]$ induce

$$\begin{array}{ccc} \langle E_a, F_a \rangle & \circlearrowright & \mathbb{C}[\mathrm{Rep}(U_\xi(\mathfrak{gl}_r))] \\ \wr\wr & & \wr\wr \\ U(\widehat{\mathfrak{gl}_n}) & \circlearrowright & \Lambda^r V_n(z) \end{array}$$

$\to$ shadow of categorical KM action
Chuang - Rouquier (see Riche - Williamson, Astérique)
for exposition

Conjecture: 2
if we keep track of grading on Rep $U_\xi(\mathfrak{gl}_r)$

$$U_v(\widehat{\mathfrak{gl}_n}) \curvearrowright \begin{array}{c} \mathbb{C}_v[\mathrm{Rep}\, U_\xi(\mathfrak{gl}_r)] \\ \wr\! \| \\ \Lambda^r_v V_n(z) \end{array}$$

def $\underline{\Lambda^r_v V_n(z)}$: $\quad V_n(z)^{\otimes r} \quad U_v(\widehat{\mathfrak{gl}_n})$

$t_i \leftarrow \boxed{q}, -q^{-1} \quad T_i \in H_{\mathrm{aff}}$

Motivation for conjecture:

Proposition 1

$$\mathcal{H}(\widetilde{GL}_r, K) \simeq \mathbb{C}[\widetilde{\Lambda}^\vee]^{W}$$

$$\circlearrowleft \qquad\qquad\qquad\qquad \circlearrowleft$$

$$\mathcal{W}(\widetilde{GL}_r, K) \simeq \Lambda_v^r V_{\bar{n}}(z)$$

$$\circlearrowright \qquad\qquad\qquad\qquad \circlearrowright$$

$$U_v(\widehat{\mathfrak{sl}}_n) \qquad\qquad U_v(\widehat{\mathfrak{sl}}_n)$$

$\rightsquigarrow$ q-Fock space of Kashiwara-Miwa-Stern, Leclerc-Thibon

it is known it contains info about KL pol appearing in Lusztig's conjecture

LT, Varagnolo-Vasserot.

Quantum symmetric pairs (iquantum groups)

ss la $\mathfrak{g} / \mathbb{C}$ $\rightsquigarrow$ $U_v(\mathfrak{g})$

quantum group $U_v(\mathfrak{g})$: algebra

coalgebra $\Delta : U_v(\mathfrak{g}) \to U_v(\mathfrak{g}) \otimes U_v(\mathfrak{g})$

$V, W \in \mathrm{Rep}\, U_v(\mathfrak{g})$ $\rightsquigarrow$ x acts on $V \otimes W$ as $\Delta(x)$

$x \in U_v$

symmetric pair $(\mathfrak{g}, \mathfrak{g}^\theta)$, involution $\theta : \mathfrak{g} \to \mathfrak{g}$

example: $(\mathfrak{gl}_{2m}, \mathfrak{gl}_m \times \mathfrak{gl}_m)$ type AIII

quantize symmetric pair

$$\begin{array}{cc} \mathfrak{gl}_{2m} & \mathfrak{gl}_m \times \mathfrak{gl}_m \\ & \downarrow \\ U_v(\mathfrak{gl}_{2m}) & U_v(\mathfrak{gl}_m) \otimes U_v(\mathfrak{gl}_m) \end{array}$$

Six dualities

Jimbo: $U_q(\mathfrak{gl}_{2m}) \curvearrowright V_{2m}^{\otimes d} \curvearrowleft \mathcal{H}_d^A = \langle T_{s_1}, \dots, T_{s_{d-1}} \rangle$

T_i acts as $R_{i,i+1}$

$$\cup \qquad\qquad \| \qquad\qquad \cap$$

Bao-Wang: $U_q^c(\mathfrak{gl}_{2m}) \curvearrowright V_{2m}^{\otimes d} \curvearrowleft \mathcal{H}_d^C = \langle T_{s_0}, T_{s_1}, \dots, T_{s_{d-1}} \rangle$

T_i act as $R_{i,i+1}$

T_0 acts as K_1

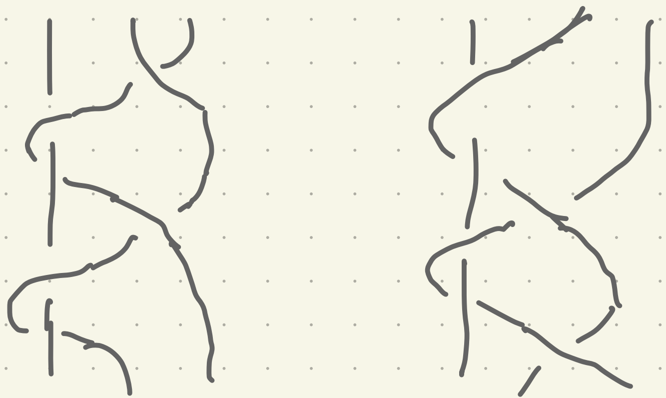
R: quantizes $v_i \otimes v_j \to v_j \otimes v_i$

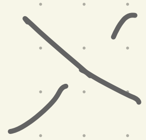
$K: V_n \to V_n$ quantizes $v_i \mapsto v_{n-i}$

K 

$RK_1RK_1 = K_1RK_1R$ (braid relation of type C)

reflection equation



R 

$U_q^C(\mathfrak{gl}_n) \subset U_q(\mathfrak{gl}_n)$ is subalgebra, not sub coalgebra

$$\Delta : U_q^C(\mathfrak{gl}_n) \to U_q^C(\mathfrak{gl}_n) \otimes U_q(\mathfrak{gl}_n)$$

coideal subalgebra

* right structure : in SW duality

$U_v^C(\mathfrak{gl}_n)$

$$\mathrm{ind}_{\mathcal{H}_d^C \otimes \mathcal{H}_e^A}^{\mathcal{H}_{d+e}^C} M \otimes N \to \mathcal{F}_{SW}^C(M) \otimes \mathcal{F}_{SW}^A(N)$$

$U_v^C(\mathfrak{gl}_n) \qquad U_v(\mathfrak{gl}_n)$

* QSP : defined for $(\mathfrak{g}, \mathfrak{g}^\theta)$, rich structure : universal K-matrix, canonical basis etc

Letzter, Kolb, Bao, Wang, Ehrig, Stroppel etc.

Bucienas:

π_χ unramified principal series of $\cancel{\widetilde{GL}_r}$ $\widetilde{Sp}_{2r}$ $\chi \in T^\vee \simeq (\mathbb{C}^*)^r$

$\chi = (x_1, \dots, x_r)$

$I_{s_i} : \pi_\chi \longrightarrow \pi_{s_i \chi}$ intertwiners

$i = 0, \dots, r-1$

n cover

$$\mathcal{W}(\pi_\chi) = \{ f : \pi_\chi \to \mathbb{C},\ f(ug) = \psi(u) f(g) \}$$

$$\begin{array}{ccc}
\mathcal{W}(\pi_\chi) & \simeq & V_n(x_1) \otimes \dots \otimes V_n(x_i) \otimes V_n(x_{i+1}) \otimes \dots \otimes V_n(x_r) \\
I_{s_0} \downarrow I_{s_i} & & K(x_1) \downarrow \qquad \downarrow R_{i,i+1}(x_i x_{i+1}^{-1}) \text{ of type } A \\
\mathcal{W}(\pi_{s_i \chi}) & \simeq & V_n(x_1^{-1}) \dots\dots V_n(x_{i+1}) \otimes V_n(x_i) \dots\dots
\end{array}$$

$V_n(x) \to$ ev. module of $U_v(\widehat{\mathfrak{sl}}_n)$

$R(z), K(z)$ affine R-matrix / K-matrix

$U_v^c(\widehat{\mathfrak{sl}}_n)$ affine QSP type AIII (Chen-Guay-Ma, Appel-Przeździecki)

type C picture

$\widetilde{Sp}_{2n}$ n cover

$q = v^2$, assume $n = \text{odd}$

$$
\begin{array}{ccccc}
\mathcal{H}(\widetilde{Sp}_{2r}, K) & \simeq & \mathbb{C}[\widetilde{\Lambda}^\vee]^W & \simeq & \mathbb{C}[\mathrm{Rep}(SO_{2r+1})] \\
\circlearrowleft & & \circlearrowleft & & \circlearrowleft\, -\otimes q\mathrm{Fr} \\
\mathcal{W}(\widetilde{Sp}_{2r}, K) & \simeq & \Lambda^r_v V_n(z) & \simeq & \mathbb{C}_v[\mathrm{Rep}\, U_\zeta(\mathfrak{so}_{2r+1})] \\
\circlearrowright & & \circlearrowright & & \circlearrowright\ (*) \\
U^c_v(\widehat{\mathfrak{sl}}_n) & & U^c_v(\widehat{\mathfrak{sl}}_n) & & U^c_v(\widehat{\mathfrak{sl}}_n)
\end{array}
$$

Λ^r_v is now exterior power wrt action of $\underset{R}{T_i}$ and $\underset{K}{T_0}$

matrix

* one endofunctor $-\otimes V \rightsquigarrow U_v^c(\widehat{sl_n})$

Remark BBB19/ BBB+Gustafsson 21

$\widetilde{GL}_r$ Savin n-cover

π_χ = unramified princical series

$\mathcal{W}(\pi_\chi)$ = $\bar{n}^r$-dim, pick basis f_μ, $\mu \in \tilde{\Lambda}/\tilde{\Lambda}^\vee$.

π_χ^K = 1dim, basis v_K

π_χ^I = $n!$ dim, basis v_w, $w \in W$

explicit combinatorial formulas for

$f_\mu(\pi^\lambda v_K)$ in terms of lattice models with $U_v(\widehat{gl}(n|1))$ symmetry

$f_\mu(\pi^\lambda v_w)$ $U_v(\widehat{gl}(n|r))$

we "understand" the $\widehat{\mathfrak{gl}}_n$ part of $U_v(\widehat{\mathfrak{gl}}(n|1))$

what about the whole $\widehat{\mathfrak{gl}}(n|1)$?

Gaiotto conjectures (Braverman–Finkelberg–Ginzburg–Travkin–Yang)

so far $C_c^\infty(K\backslash \tilde{G})^K \rightsquigarrow G^\vee(\mathbb{Q},m)$

$$C_c^\infty((U,\psi)\backslash \tilde{G})^K \rightsquigarrow U_\xi(\mathfrak{g}^\vee)$$

$$(H,\psi)\backslash \tilde{G} \rightsquigarrow U_v\,\underline{\mathfrak{gl}}(M|N)$$

Gaiotto conjecture aims to interpolate $K \leftrightarrow (U,\psi)$

type A: $G = GL_N(F) \rightsquigarrow GL_M(\mathcal{O})$

$$H = \begin{array}{c|cc} & M & N-M \\ \hline M & \begin{pmatrix} * & * & * \\ * & * & * \end{pmatrix} & \begin{pmatrix} 0 & * & * \\ 0 & * & * \end{pmatrix} \\ N-M & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix} \end{array}, \psi \qquad \rightsquigarrow \underline{\mathfrak{gl}}(M|N)$$

U_{N-M}

$$GL_N \rightsquigarrow \widetilde{GL}_N \,/\, \underline{\mathfrak{gl}}(M|N) \to U_v(\mathfrak{gl}(M|N))$$

More about this and relation to full BZSV program in Finkelberg's talk (I think)

$$C_c^\infty\left(K_1\backslash G/K\right)$$

$$\wr$$

$$C_c^\infty\left((H,\psi)\backslash G/K\right) \simeq \mathbb{C}_v[\mathrm{Rep}\,\underline{\mathfrak{gl}}(M|N)]$$

BBB: $\widehat{\mathfrak{gl}}(1|1)$ $(n=1)$

$\mathrm{Rep}\, \mathfrak{gl}(M|N)$ $\quad V_{m|n}$

Brundan $\quad U(\mathfrak{gl}_\infty)$

$E = \ -\otimes V_{m|n}$

$F \quad \otimes (V_{m|n})^*$

$E = \bigoplus_{a \in \mathbb{Z}} E_a$

$F = \bigoplus_{a \in \mathbb{Z}} F_a$