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POSTECH

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Quantum groups and relative Langlands

Lecture 2 (on quantum groups)

today: - general intro to quantum groups
- explain in more detail result of BP25
- explain the appearance of quantum affine groups.

Let $\mathfrak{g}$ be a s.s. l.a. over $\mathbb{C}$

with $\underset{\text{roots}}{\Phi} \supset \underset{\text{pos. roots}}{\Phi^+} \supset \underset{\text{simple roots}}{\Delta}$, Λ = weight lattice

$\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$

Rep $\mathfrak{g}$ = cat. of f.d. complex reps of $\mathfrak{g}$

then Rep $\mathfrak{g}$ is a symmetric, monoidal, s.s. category

irreducibles $V\lambda$, $\lambda \in \Lambda_+$

A possible def of a quantum group is an object X

s.t. Rep X is braided monoidal

in technical terms X is a quasitriangular Hopf algebra

Main example: $U_q(\mathfrak{g}) \rightsquigarrow$ quantization of $U(\mathfrak{g})$.

$C =$ Cartan matrix, $D = \mathrm{diag}(d_\alpha)_{\alpha \in \Delta}$ s.t $d_\alpha \in \{1, 2, 3\}$

$D \cdot C =$ sym.

quantum integers $[n]_q = \dfrac{q^n - q^{-n}}{q - q^{-1}} \quad \left(\xrightarrow{q \to 1} n \right)$

$[n]_q! = [1]_q \cdots [n]_q \qquad \begin{bmatrix} n \\ m \end{bmatrix}_q = \dfrac{[n]_q!}{[m]_q! \, [n-m]_q!}$

if $q = \sqrt[\ell]{1} \qquad [\ell]_q = 0$

$U_q(\mathfrak{g}) := \mathbb{C}(q)\langle E_\alpha, F_\alpha, K_\lambda \rangle_{\substack{\alpha\in\Delta \\ \lambda\in\Lambda}}$ with relations *

$$K_\lambda K_\mu = K_{\lambda+\mu}$$

$$K_\lambda E_\alpha = q^{(\lambda,\alpha)} E_\alpha K_\lambda \qquad K_\lambda F_\alpha = q^{-(\lambda,\alpha)} F_\alpha K$$

$$E_\alpha F_\beta - F_\beta E_\alpha = \delta_{\alpha\beta} \frac{K_\alpha - K_\alpha^{-1}}{q-q^{-1}} \qquad \left(= \frac{q^{h_\alpha} - q^{-h_\alpha}}{q-q^{-1}} \xrightarrow[q\to 1]{} h_\alpha \right)$$

q Chevalley-Serre:

$$\sum_{k=0}^{1-\langle\alpha^\vee,\beta\rangle} (-1)^k \begin{bmatrix} 1-\langle\alpha^\vee,\beta\rangle \\ k \end{bmatrix}_q E_\alpha^{1-\langle\alpha^\vee,\beta\rangle-k} E_\beta E_\alpha^k = 0$$

+ same for F_α

replace $E_\alpha \to e_\alpha$, $F_\alpha \to f_\alpha$, $K_\alpha \to q^{h_\alpha}$ $\xrightarrow{q\to 1}$ $\rightsquigarrow$ get back relations for $\mathfrak{g}$.

integral form of $U_q(\mathfrak{g})$ is a $\mathbb{C}[q^{\pm}]$ subalgebra U of $U_q(\mathfrak{g})$

s.t. $U \otimes_{\mathbb{C}[q^{\pm}]} \mathbb{C}(q) \simeq U_q(\mathfrak{g})$ as $\mathbb{C}(q)$ algebras.

$U_q^{Lus}(\mathfrak{g})$ is the $\mathbb{C}[q^{\pm}]$ subalgebra of $U_q(\mathfrak{g})$ gen. by

$$E_\alpha^{(m)}, F_\alpha^{(m)}, K_\lambda, \begin{bmatrix} K_\lambda ; m \\ n \end{bmatrix} \quad \text{where}$$

$$E_\alpha^{(m)} = \frac{E_\alpha^m}{[m]_q!} \qquad \begin{bmatrix} K_\lambda ; m \\ n \end{bmatrix} = \prod_{k=0}^{n} \frac{K_\alpha q^{(m+1-r)} - K_\alpha^{-1} q^{(-m-1+r)}}{q - q^{-1}}$$

$$F_\alpha^{(m)} = \frac{F_\alpha^m}{[m]_q!}$$

specialize $q \to \xi \in \mathbb{C}^*$ $\quad U_\xi^{lus}(\mathfrak{g}) := U_q^{lus}(\mathfrak{g}) \otimes_{\mathbb{C}[q^{\pm}]} \mathbb{C}$, $\; q \mapsto \xi$

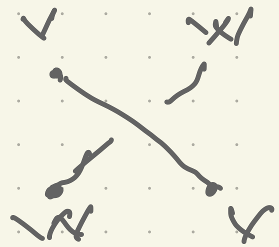
1] $\xi \neq$ root of unity,

Rep $U_\xi(\mathfrak{g})$ behaves like Rep $U(\mathfrak{g})$:

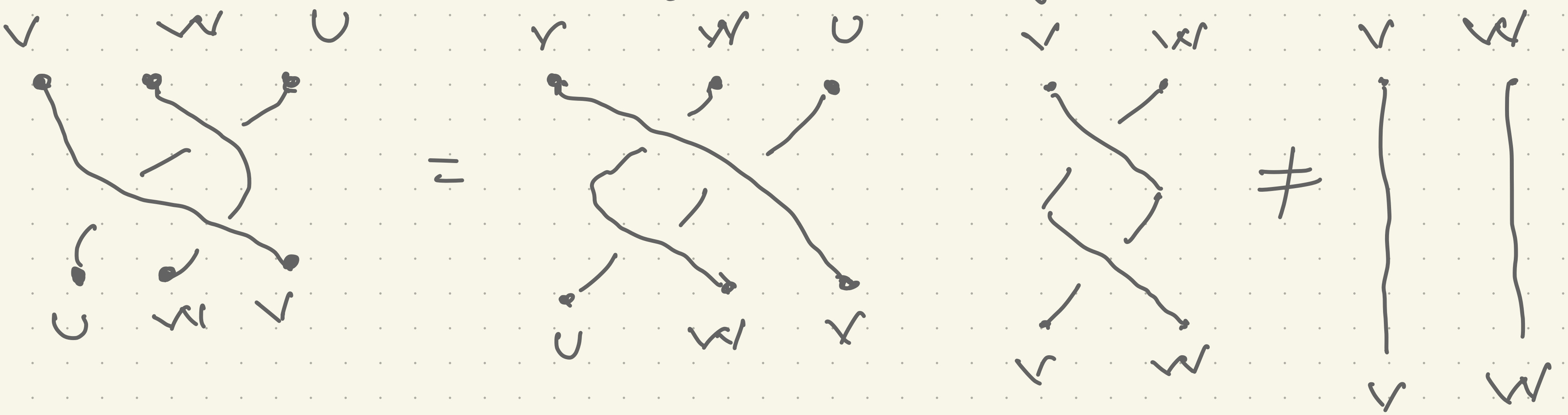
- you can use the K_λ's to define weight spaces, highest weight, etc. char
- rep is s.s., irred. indexed by $\lambda \in \Lambda^+$, same characters as when $q=1$, same $\otimes$ mult.

but Rep $U_\xi(\mathfrak{g})$ is now braided, not symmetric

i.e. nontrivial R-matrices $R_{V,W} : V \otimes W \to W \otimes V$



These R-matrices satisfy the Yang-Baxter equation



<u>Ex</u>: - $U_\xi(\mathfrak{sl}_2)$ has generators E, F, K, K^{-1}

- $U_\xi(\mathfrak{sl}_2) \curvearrowright V_2 = \mathbb{C}^2$ via $E \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $F \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

$K \mapsto \begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix}$ $(= q^h)$

- $R_\xi = R_{V_2, V_2} = \begin{pmatrix} \xi & & & \\ & \xi - \xi^{-1} & 1 & \\ & 1 & 0 & \\ & & & \xi \end{pmatrix}$

Note $(R_\xi - \xi)(R_\xi + \xi^{-1}) = 0$

then $T_i \mapsto (\xi R_\xi)_{i,i+1}$ gives a representation of the finite Hecke algebra

$U_\xi(\mathfrak{sl}_n) \circlearrowright V_n^{\otimes d} \circlearrowleft H_d(\xi^2)$ double commuting action $\rightsquigarrow$ SW duality

$\langle T_i,\ 1 \leq i \leq d-1 \dots \rangle$

$$\rightsquigarrow \mathcal{F}^q_{SW} : \operatorname{Rep} \underset{\mathbb{C}[S_d]}{H_d(\xi^2)} \to \operatorname{Rep} \underset{U(\mathfrak{sl}_n)}{U_\xi(\mathfrak{sl}_n)} \quad \text{(Jimbo)}$$

$$M \mapsto V_n^{\otimes d} \otimes_{H_d(\xi^2)} M$$

$\mathcal{F}^q_{SW}$ is: - $\otimes$ functor: maps $\operatorname{Ind}^{H^{d+d'}}_{H^d \otimes H^{d'}} \to \otimes$

- equivalence of cat. if $n \geq d$ + restrict

- quantizes $\mathcal{F}_{SW}: \mathrm{Rep}\, \mathbb{C}[S_d] \to \mathrm{Rep}\, U(\mathfrak{sl}_n)$

$\boxed{2}$ ζ = root of unity $\sqrt[\ell]{1}$, assume ℓ = odd

$U_\zeta^L(\mathfrak{g})$
$\shortparallel$
$U_\zeta(\mathfrak{g})$

Rep $U_\zeta(\mathfrak{g})$ resembles Rep G in char p.
(if $p = \ell$)

$\hookrightarrow$ not s.s.

$\boxed{2.1}$ for each $\lambda \in \Lambda_+$, $\exists$ reps

$\Delta_\lambda \rightsquigarrow$ standard (Weyl) module of h.w. λ

$\nabla_\lambda \rightsquigarrow$ costandard module of h.w. λ

Δ_λ has unique irr quotient L_λ

$$\Delta_\lambda \twoheadrightarrow L_\lambda \hookrightarrow \nabla_\lambda$$

$$\operatorname{char} \Delta_\lambda = \operatorname{char} \nabla_\lambda = \text{Weyl char form.}$$

Important problem: understand $\operatorname{char} L_\lambda$.

2.2 block decomposition

$$\mathrm{Rep}\, U_\xi(\mathfrak{g}) = \bigoplus_{\lambda \in \Lambda^+/(W_\ell, \bullet)} \left(\mathrm{Rep}\, U_\xi(\mathfrak{g})\right)_\lambda$$

$$W_\ell = W \ltimes \Lambda \qquad = \{ w t_\mu ,\ w \in W, \mu \in \Lambda \}$$

$$W_\ell \curvearrowright \Lambda \qquad w t_\mu \bullet \lambda = w \bullet \lambda + \ell \mu$$
$$= w(\lambda + \rho) - \rho + \ell \mu$$

$(\mathrm{Rep}\, U_\xi(\mathfrak{g}))_\lambda \ni V$ if all comp. fact. in V have h.w. in $W_\ell \bullet \lambda$.

Remark: think of W_ℓ as $W_{\mathrm{aff}}(\mathcal{D}(Q, e))$

$\boxed{2.3}$ quantum Frobenius:

if ℓ = even

$\exists$ hom. $qFr: U_\xi(\mathfrak{g}) \twoheadrightarrow U(\mathfrak{g})$

$$E_\alpha^{(n)}, F_\alpha^{(n)} \mapsto \begin{cases} \frac{e_\alpha^{n/\ell}}{(n/\ell)!}, \frac{f_\alpha^{n/\ell}}{(n/\ell)!} & \text{if } \ell \mid n \\ 0 & \text{else} \end{cases}$$

$$K_\lambda \to 1$$

$\rightsquigarrow qFr: \operatorname{Rep} U(\mathfrak{g}) \to \operatorname{Rep} U_\xi(\mathfrak{g})$

$$\rightsquigarrow \quad \begin{matrix} \operatorname{Rep} U(\mathfrak{g}) \\ \downarrow \\ \operatorname{Rep} U_\xi(\mathfrak{g}) \end{matrix} \qquad \begin{matrix} (\rightsquigarrow \operatorname{Rep} G(\mathbb{C})) \\ \\ \operatorname{Rep}(U_\xi(\mathfrak{g})) \end{matrix}$$

Def: $\lambda \in \Lambda$ is restricted if $0 \leq \langle \lambda, \alpha^\vee \rangle < \ell$, $\alpha \in \Delta$

we say $\lambda \in \square_\ell$

$\boxed{2.4}$ Steinberg-Lusztig:

$$\lambda = \lambda_0 + \ell \lambda_1, \quad \lambda_0 \in \square, \quad \lambda_1 \in \Lambda^+$$

$$L_\lambda \simeq L_{\lambda_0} \otimes qFr(V_{\lambda_1}).$$ V_{λ_1} cmod of $U(\mathfrak{g})$.

$$qFr(V_\mu) \otimes L_\lambda = \underbrace{qFr(V_\mu) \otimes qFr(V_{\lambda_1})}_{\text{in } Rep(U\mathfrak{g})} \otimes L_{\lambda_0}$$

$\boxed{2.5}$ Lusztig's conjecture (now thm: KL, KT, AJS)

principal block : $\lambda \in A_{\bar{e}}$, $s_i \cdot \lambda \neq \lambda$

$$\operatorname{ch} L(x \cdot \lambda) = \sum_{\substack{y \in W_e \\ y \cdot \lambda \in \Lambda^+}} m_{y,x}(-1) \operatorname{ch} \Delta(y \cdot \lambda)$$

here $m_{y,x}$ are affine KL polynomials.

graded version: $\quad m_{yx}(v) = \sum_{i \geq 0} \dim \operatorname{Ext}^i\left(L(x \cdot \lambda), \nabla(y \cdot \lambda)\right) v^i$

similar version for singular blocks.

Lecture 1: main result in BP25

$$\underset{c_\lambda}{\mathcal{H}(\tilde{G}, K)} \simeq \mathbb{C}[\overset{[V_\lambda]}{\mathrm{Rep}\, U_q(\check{\mathfrak{g}})}(\mathbb{Q}, n)] \rightsquigarrow \text{Satake}$$

$$\circlearrowright * \qquad\qquad \circlearrowright -\otimes q^{Fr}$$

$$\mathcal{W}(\tilde{G}, K) \simeq \mathbb{C}_v[\mathrm{Rep}\, U_\zeta(\mathfrak{g}^\vee)]$$

$$J_\mu \qquad\qquad [L_\mu], [\Delta_\mu], [\nabla_\mu]$$

$$q = v^2$$

action easy on basis $[L_\mu]$ (Steinberg-Lusztig)

but $J_\mu \longleftrightarrow [\nabla_\mu]$ (modulo Gauss sums)

$$g(a) \qquad (\mathbb{Q}, n) \qquad g(a)\, g(n-a) = q$$

so $\quad c_\lambda * J_\mu = \sum_\nu c_{\lambda\mu}^{\nu} J_\nu \quad$ should be same

$(\rightsquigarrow)$ $\quad [{}^{q}\mathrm{Fr}(V_\lambda) \otimes \nabla_\mu] = \sum_\nu d_{\lambda\mu}^{\nu} [\nabla_\nu]$

in $\mathbb{C}_v[\mathrm{Rep}\, U_\xi(\mathfrak{g}^\vee)]$ $\rightsquigarrow$ Cline-Parshall-Scott this means

$$d_{\lambda\mu}^{\nu} = \sum_i \dim \mathrm{Ext}^i({}^{q}\mathrm{Fr}(V_\lambda) \otimes \nabla_\mu, \Delta_\nu)\, v^i$$

Proof in BP25 :

1] - understand $\mathcal{W}(\tilde{G},K) \overset{*}{\curvearrowleft} \mathcal{H}(\tilde{G},K)$

in terms of Chinta-Gunnells-Puskás action.

- reinterpret as $\mathcal{W}(\tilde{G},K) \simeq \bigoplus_{\nu \in A\text{-}\ell} \mathcal{W}(\tilde{G},K)(\nu)$ $\quad \varepsilon^+ \mathrm{Haff}\, \varepsilon^+$

where $\mathcal{W}(\tilde{G},K)(\nu)$ is something like $\varepsilon_J^- \underline{\mathrm{Haff}}\, \varepsilon^+$.
$J \subset A$

2] if we want to match $J_\mu \longleftrightarrow [\nabla_\mu]$, we need to understand how to go from

$c_\lambda * J_\mu \longleftrightarrow c_\lambda * \mathcal{L}_\mu$ $\qquad V_\lambda \cdot L_\mu$

$q^{Fr}(V_\lambda) \otimes \nabla_\mu \longleftrightarrow q^{Fr}(V_\lambda) \otimes L_\mu$ $\qquad \downarrow$

$\qquad V_\lambda \cdot \nabla_\mu$

3] keep track of Gauss sums. $\qquad c_\lambda * J_\mu \quad \mathcal{L}_\mu$

Rmk: integral form $E_\alpha\ E_\alpha^{(n)}\ F_\alpha\ F_\alpha^{(n)} \rightsquigarrow U_\xi^{Lus}$

$E,\ E_\alpha^{(n)},\ F_\alpha \rightsquigarrow U_\xi^{mxd}$

$E_\alpha \quad F_\alpha \rightsquigarrow U_\xi^{DK}$

Conj (Gaitsgory geometrically)

at the Iwahori

$$\mathcal{W}(\tilde{G}, I) \overset{\sim}{\longleftrightarrow} \operatorname{Rep}^{\heartsuit} U_\xi^{mxd}$$

$$\circlearrowleft *$$

$G\ G\ K$

$$\mathcal{H}(\tilde{G}, I)$$

1) cat $\heartsuit$

2) Losev, Situ, Chen - Fu

Rmk Gauss sums

Quantum affine groups

let $\widehat{sl_n}$ be affine Lie algebra $sl_n \otimes \mathbb{C}[t,t^{-1}] \oplus \mathbb{C}c$

$\rightsquigarrow$ quantization $U_v(\widehat{sl_n})$, v generic $(=\sqrt{q})$

let $\underline{V_n}$ be the n-dim def. rep of $U_v(sl_n)$
(highest weight $(1,0,..,0)$)

for $x \in \mathbb{C}^*$ induces reps $V_n(x)$ of $U_v(\widehat{sl_n})$ n dim

z ind $V_n(z)$ $n\mathbb{Z}$ dim

$v_i z^k$

think about them as reps where t act by $\begin{matrix} x \\ z \end{matrix}$

c by 0

$\mathrm{Rep}\, U_v(\widehat{\mathfrak{sl}}_n)$ finite dim, type 1, reps of $U_v(\widehat{\mathfrak{sl}}_n)$

↳ having some type of wt. decomp.

↳ (meromorphically) braided

p-adic connection via Schur-Weyl dualities

idea: $q = v^2$ (1)

$$\left\{ \begin{array}{l} V \in \mathrm{Rep}\, GL_r(F) \\ V \text{ gen by } V^I \end{array} \right\} \underset{V \mapsto V^I}{\overset{\text{Borel-Casselman}}{\underset{\longleftarrow}{\cong}}} \left\{ \begin{array}{l} \text{f.d. reps of} \\ \mathcal{H}(G,I) = H_{aff} \end{array} \right\} \underset{\text{Chari-Pressley}}{\overset{\mathcal{F}_{SW}}{\underset{T}{\longrightarrow}}} \left\{ \begin{array}{l} \text{f.d. type 1} \\ \text{of } U_v(\widehat{\mathfrak{sl}}_n) \end{array} \right\}$$

parabolic induction $\rightsquigarrow$ induction $\rightsquigarrow$ $\otimes$ (***)

$\mathcal{F}_{SW}$ equivalence if $n > r$ and restrict r.h.s.

(***) very useful: one may study induction on l.h.s. via $\otimes$ on r.h.s.

work in this style by Lapid–Minguez, M. Gurevich and others (p-adic results)

based on / inspired by Hernandez, Leclerc, Kang-Kashiwara-Kim-Oh etc.

can be generalized to

1) outside of tamely ramified block for $GL_r(F)$

2) tamely ramified block of $\widetilde{GL}_r^{\,n}(F)$ (at least Savin cover)

basic example of application

Thm (Hernandez): V_i are irreps in Rep $U_v(\widehat{\mathfrak{sl}_n})$.

Then $V_1 \otimes .. \otimes V_k$ = irred $\Leftrightarrow$ $V_i \otimes V_j$ = irred $\forall\ i \neq j$.

$\Rightarrow$ for $GL_r(F)$: $\mathrm{Ind}_{GL_{a_1} \times .. \times GL_{a_k}}^{GL_{a_1 + .. + a_k}} \pi_1 \times .. \times \pi_k$ = irrep iff

$\mathrm{Ind}_{GL_{a_i} \times GL_{a_j}}^{GL_{a_i + a_j}} \pi_i \times \pi_j$ = irrep $\forall\ i \neq j$

Thm (Buciumas – unpublished (~)
Gao-Gurevich-Karasiewicz GGK25)

$\widetilde{GL}_r^{(n)}$ $\rightsquigarrow$ Savin's cover $\qquad \bar{n} = \frac{n}{(n,2)}$

$\pi \in \mathrm{Irr}\, \widetilde{GL}_r(F)$, $\pi^I \neq 0$.

$U_v(\widehat{\mathfrak{sl}_{\bar{n}}})$ $\circlearrowright$

$\pi \rightsquigarrow \mathcal{F}_{BC}(\pi) \rightsquigarrow \mathcal{F}_{SW}\,\mathcal{F}_{BC}(\pi) = \sigma$

π = irred $\boxed{\dim(\mathrm{Wh}_\psi\, \pi) = \dim \sigma}$ (*)

where $\mathrm{Wh}_\psi(\pi) = \mathrm{Hom}_G\left(\mathrm{ind}_U^{\widetilde{GL}_r} \psi, \pi\right)$

Cor: if $\bar{n} > r \rightsquigarrow \dim(\mathrm{Wh}_\psi(\pi)) \geq 1 \;\; \forall \pi$.

"Cor": $\boxed{n=1}$ $\quad \underline{\pi}$ $\qquad U_v(\widehat{\mathfrak{sl}_1})$ = commutative

$\underline{\sigma} \rightsquigarrow$ 1 dim
0 dim

Fact: $q = 1$ (n)

1) understand how to index irreducibles

$$\left\{ \begin{array}{l} V \in \operatorname{Rep} GL_n(F) \\ V \text{ gen by } V^I \end{array} \right\} \overset{\text{Zelevinsky multisegments}}{} \underset{\longleftrightarrow}{\overset{BC}{\simeq}} \left\{ \begin{array}{l} \text{f.d. reps of} \\ \mathcal{H}(G,I) = H_{\text{aff}} \end{array} \right\} \underset{CP}{\xrightarrow{\text{Fsur}}} \left\{ \begin{array}{l} \text{f.d. type 1} \\ \text{of } U_v(\widehat{sl}_n) \end{array} \right\}$$

$q = v^2$ Drinfeld pol.

and how to go from one to the other

2) theory of q characters on r.h.s. + formulas to compute them for $\underline{irr}$ (based on Nakajima's work)

3) in principle, Cor gives us all dim of Whittaker models.

4) in practice, it's hard to compute explicitly.

What can try to do?

prove / reprove conjectures of Fan Gao
on dim. of Whittaker models.

Rem: see Jiandi Zou 25 for results outside
the Iwahori block

end day 2 $\square$

statement in GGK25 (reintepreted by Buciumas)

$$\mathcal{H}(\widetilde{GL}_r, I) \simeq \underset{\substack{\Downarrow \\ T_i}}{H_0} \otimes \underset{\theta_\lambda}{\mathbb{C}[\widetilde{\Lambda}^\vee]}$$

let $I_{s_i} = (v - v^{-1})\,\theta_{\alpha_i^\vee} + v^{-1}(1 - \theta_{\alpha_i^\vee})\,T_i \in \mathcal{H}(\widetilde{GL}_r, I)$

$$\begin{array}{ccc} \mathcal{W}(\widetilde{GL}_r, I) & \overset{\text{v.s. iso}}{\simeq} & V_{\bar{n}}(z)^{\otimes r} \\ \big\downarrow I_{s_i} & & \big\downarrow R(z)_{i,i+1} \\ \mathcal{W}(\widetilde{GL}_r, I) & \simeq & V_{\bar{n}}(z)^{\otimes r} \end{array}$$

such that $R(z)_{i,i+1} \rightsquigarrow$ affine R-matrix of $U_v(\widehat{sl}_n)$

$\rightsquigarrow$ induces $U_v(\widehat{sl}_n)$ action on $\mathcal{W}(\widetilde{GL}_r, I)$

Thm/conj (Buciumas, in progress) $\widetilde{GL}_r$ n-Savin cover, $q = v^2$,

$$
\begin{array}{ccccc}
\mathcal{H}(\widetilde{GL}_r, K) & \simeq & \mathbb{C}[\widetilde{\Lambda}^\vee]^W & \simeq & \mathbb{C}[\mathrm{Rep}\, GL_r^\vee] \\
\circlearrowright & & \circlearrowright & & \circlearrowright \; -\otimes q F_r \\
\mathcal{W}(\widetilde{GL}_r, K) & \simeq & \Lambda^r_q V_{\bar{n}}(z) & \simeq & \mathbb{C}_v[\mathrm{Rep}\, U_s(\mathfrak{gl}_n)] \\
\circlearrowleft & & \circlearrowleft & & \circlearrowleft \;(*) \\
U_v(\widehat{\mathfrak{sl}}_n) & & U_v(\widehat{\mathfrak{sl}}_n) & & U_v(\widehat{\mathfrak{sl}}_n)
\end{array}
$$

How to obtain (*) from Rep.th.

Endofunctors on $\operatorname{Rep} U_\xi(\mathfrak{gl}_r)$:

$E : \quad V \otimes -$

$F : \quad V^* \otimes -$

which should decompose $E = \bigoplus_{0 \le a \le \bar{n}} E_a$

$F = \bigoplus_{0 \le a \le \bar{n}} F_a$

at the level of $\mathbb{C}[\operatorname{Rep}(U_\xi(\mathfrak{gl}_r))]$ induce

$$\begin{array}{ccc} \langle E_a , F_a \rangle & \circlearrowright & \mathbb{C}[\operatorname{Rep}(U_\xi(\mathfrak{gl}_r))] \\ \wr\wr & & \wr\wr \\ U(\widehat{\mathfrak{gl}_{\bar{n}}}) & \circlearrowright & \Lambda^r V_{\bar{n}}(\mathbb{Z}) \end{array}$$

$\rightarrow$ shadow of categorical KM action
Chuang - Rouquier (see Riche - Williamson Astérique)

if we keep track of grading on Rep $U_\xi(\mathfrak{gl}_n)$

$\rightsquigarrow$ *

Remark: BBB $\widehat{\mathfrak{sl}_n}$

Remark B B B $\widehat{gl}(n|1)$