Bernstein components for Whittaker models and branching laws

Kei Yuen Chan

Shanghai Center for Mathematical Sciences, Fudan University

University of Alberta
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Outline

Bernstein decomposition

Let G be a p-adic reductive group. Let $\mathfrak{R}(G)$ be the category of smooth representations of G.
Bernstein decomposition:

$$\mathfrak{R}(G) = \prod_{\mathfrak{s}} \mathfrak{R}_{\mathfrak{s}}(G)$$

- $\mathfrak{s}$ runs for all inertial equivalence class $[P, \sigma]$
- P runs for all parabolic subgroup and σ is a cuspidal representation of the Levi M of P
- A representation π of M is cuspidal if π is not a composition factor of any proper parabolically induced modules.

Types

Definition (Bushnell-Kutzko)

1. K: a compact subgroup of G
2. τ: an irreducible repn. of K
3. $\mathfrak{R}^{\tau}(G)$ = full subcat. of $\mathfrak{R}(G)$ with objects generated by τ-isotypic components

We say that (K, τ) is a $\mathfrak{s}$-type for some $\mathfrak{s} \in \mathfrak{B}(G)$ if

$$\mathfrak{R}^{\tau}(G) = \mathfrak{R}_{\mathfrak{s}}(G).$$

Examples of types

Just to name few:

1. $[B, 1]$: unramified principal series (Iwahori component) due to Borel-Casselman. In this case, $R^{(I,1)}(G)$ is representations generated by I-fixed vectors (I denotes the Iwahori subgroup)
2. All Bernstein components of GL: Bushnell-Kutzko (93)
3. Depth zero Bernstein component: Morris

From types to Hecke algebras

1. For a type (K, τ), one associates a Hecke algebra:

$$\mathcal{H}(K,\tau) = \left\{ f : G \to \mathrm{End}(\tau^\vee) : \begin{array}{r} f(k_1 g k_2) = \tau^\vee(k_1) f(g) \tau^\vee(k_2) \\ \text{for } k_1, k_2 \in K \end{array} \right\}$$

2. We have a functor:

$$R^\tau(G) \to \text{category of } \mathcal{H}(K,\tau)\text{-modules}$$

$$\pi \mapsto (\tau^\vee \otimes \pi)^K$$

3. The action is given by:

$$\int_G f(g)(v) \otimes x \, dg$$

with $v \in \tau^\vee$ and $x \in \pi$.

Affine Hecke algebras

1. Extended affine Weyl group $W_{\mathrm{ex}} = N_G(T)/T(\mathcal{O})$
2. Length function $l : W_{\mathrm{ex}} \to \mathbb{N}_{\geq 0}$ defined by

$$q^{l(w)} = |IwI/I|$$

3. This gives an affine Hecke algebra:

Definition

The affine Hecke algebra $\mathcal{H}_{\mathrm{aff}}$ is generated by T_w $(w \in W_{\mathrm{ex}})$ given by

- (Braid relation) $T_{w_1} T_{w_2} = T_{w_1 w_2}$ if $l(w_1 w_2) = l(w_1) + l(w_2)$
- (Quadratic relation) $(T_s + 1)(T_s - q) = 0$ for $l(s) = 1$

4. Example: Iwahori case $H(I, 1) \cong \mathcal{H}_{\mathrm{aff}}$

Bernstein-Lusztig relation

Let $X^\vee = T/T(\mathcal{O})$ and $W = N_G(T)/T$ (Weyl group). And,

$$W_{\mathrm{ex}} = X^\vee \rtimes W$$

Proposition

$\mathcal{H}_{\mathrm{aff}}$ admits generators θ_x ($x \in X$) and T_w ($w \in W$) such that

1. (commutative) $\theta_x \theta_{x'} = \theta_{x'} \theta_x$
2. (Braid relation) $T_w T_{w'} = T_{ww'}$ and (quadratic relation) $(T_s + 1)(T_s - q) = 0$
3. $$\theta_x T_s - T_s \theta_{s(x)} = (q-1)\frac{\theta_x - \theta_{s(x)}}{1 - \theta_{-\alpha^\vee}}$$

Hecke algebra structure

- One expects that the structure $\mathcal{H}(K, \tau)$ is close to an affine Hecke algebra.
- For example, in GL_n, all the Bernstein component is isomorphic to the module category of a product of affine Hecke algebras of type A.
- For another construction from endomorphism algebras of projective generators, see M. Solleveld last week, and earlier by Heierman for classical groups.

Outline

1. Bernstein components and types
2. Gelfand-Graev representations
3. Branching laws

Gelfand-Graev representations

Let ψ be a non-degenerate character on U i.e. dense B-orbits on ψ. The Gelfand-Graev representation is

$$\mathrm{ind}_U^G \psi,$$

space of compactly supported smooth functions satisfying

$$f(ug) = \psi(u) f(g)$$

Remarkable properties:

1. The smooth dual of $\mathrm{ind}_U^G \psi$ is the so-called Whittaker functional space.
2. (Shalika) Each $\pi \in \mathrm{Irr}(G)$ appears with at multiplicity one in the quotient of $\mathrm{ind}_U^G \psi$.
3. (C.-Savin) $\mathrm{ind}_U^G \psi$ is projective

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Bernstein components of GG

Some remarkable Hecke algebra structure of GG:

1. (Bushnell-Henniart 00's) Each Bernstein component of $\mathrm{ind}_U^G \psi$ is finitely-generated.
2. (Bushnell-Henniart 00's)

$$\mathrm{End}_G((\mathrm{ind}_U^G \psi)_{\mathfrak{s}}) \cong \mathfrak{Z}_{\mathfrak{s}},$$

   the Bernstein center of $\mathfrak{R}_{\mathfrak{s}}(G)$.
3. The above implies $(\mathrm{ind}_U^G \psi)_{\mathfrak{s}}$ is indecomposable.
4. (Barbasch-Moy, Reeder 90's) As $\mathcal{H}_W$-representation, any generic representation (i.e. with Whittaker model) has sign representation sgn i.e. T_w acts by $(-1)^{l(w)}$.
5. (C.-Savin 18,19) Explicit structure...

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Iwahori component of GG

Theorem (C.-Savin 2019)

Let G be a split p-adic group. Fix a Whittaker character

$$\psi(\prod_{\alpha\in\Pi} x_\alpha(t_\alpha)) = \prod_{\alpha\in\Pi} \bar{\psi}(t_\alpha),$$

where $\psi : F \to \mathbb{C}$ *non-degenerate and depth zero. Then the Iwahori component of the Gelfand-Graev representation*

$$(\mathrm{ind}_U^G \psi)^I$$

is isomorphic to $\mathcal{H}_{\mathrm{aff}} \otimes_{\mathcal{H}_W} \mathrm{sgn}$.

From different viewpoint, Brubaker-Buciumas-Bump-Friedberg (Selecta Math.) independently obtained similar result.

Related work

1. All Bernstein components of GL_n: C.-Savin (19)
2. Iwahori component for Bessel model: C.-Savin (00)
3. Principal series components (with some conditions): Misrah-Pattanayak
4. Bernstein components of SO_n and Sp_n: Bakic-Savin

Outline

Restriction problems

- Gan-Gross-Prasad problems: Let (G, H) be pairs of

  $$(\mathrm{GL}_{n+1}, \mathrm{GL}_n), (\mathrm{SO}_{n+1}, \mathrm{SO}_n), (\mathrm{U}_{n+1}, \mathrm{U}_n)$$

  One determines

  $$\mathrm{Hom}_H(\pi|_H, \tau) = ?$$

  for $\pi \in \mathrm{Irr}(G)$ and $\tau \in \mathrm{Irr}(H)$.
- Many progress due to many people. (Non-tempered GGP conjecture, C. 2020 preprint)

A classical example of restriction

Let $G_n = \mathrm{GL}_n$. Let $\pi \in \mathrm{Irr}(G_{n+1})$

1. Let π be cuspidal. Then

$$\pi|_{G_n} \cong \mathrm{ind}_U^{G_n} \psi$$

   In particular, $\pi|_{G_n}$ is projective.

2. When π is generic,

$$\mathrm{ind}_U^{G_n} \psi \hookrightarrow \pi \hookrightarrow \mathrm{Ind}_U^{G_n} \psi$$

3. When one has $\pi \cong \mathrm{ind}_U^{G_n} \psi$ in general?

Projectivity criteria

Theorem (C.-Savin 21, C.21)

Let $\pi \in \mathrm{Irr}(G_{n+1})$. Then the following conditions are equivalent:

- $\pi|_{G_n}$ *is projective.*
- π *is generic (as G_{n+1}-representation) and any G_n-quotient of π is generic.*
- $\pi \cong \mathrm{ind}_U^{G_n} \psi$.

Classification of relatively projective representations

Theorem (C. 21)

Let $\pi \in \mathrm{Irr}(G_{n+1})$. *Then* $\pi|_{G_n}$ *is projective if and only if* π *is isomorphic to one of the followings:*

- π *is essentially square integrable; or*
- $(n+1)$ *is even and* $\pi \cong \sigma_1 \times \sigma_2$ *for cuspidal* $\sigma_1, \sigma_2 \in \mathrm{Irr}(G_{(n+1)/2})$.

But, what we can talk for Bernstein components of arbitrary $\pi|_{G_n}$?

Bernstein components

Theorem (C. 21)

Let $\pi \in \mathrm{Irr}(G_{n+1})$. Then each Bernstein component of $\pi|_{G_n}$ is indecomposable!

1. A main tool of the proof uses left-right derivatives. For example, the Steinberg representation St in G_2 gives *two* filtrations:

$$0 \to C_c^\infty(F^\times) \to \mathrm{St}|_{G_1} \to \nu \to 0$$

$$0 \to C_c^\infty(F^\times) \to \mathrm{St}|_{G_1} \to \nu^{-1} \to 0$$

2. The two filtrations come from restricting to two mirabolic subgroups $M = \left\{ \begin{pmatrix} * & * \\ & 1 \end{pmatrix} \right\}$ and M^t.

Bernstein variety

An inertial equivalence class of G_n takes the form

$$[M = G_{n_1} \times \ldots \times G_{n_k}, \sigma_1 \boxtimes \ldots \boxtimes \sigma_k].$$

When all $G_{n_i} \cong F^\times$ and $\sigma_i \cong 1$, it gives the Iwahori component.
Let $W_\mathfrak{s} = W(\mathfrak{s})/M$. The Bernstein variety is defined as:

$$\mathfrak{X}_\mathfrak{s} = (\mathbb{C}^\times)^k / W_\mathfrak{s}$$

and the Bernstein center

$$\mathfrak{Z}_\mathfrak{s} \cong \mathbb{C}[\mathfrak{X}_\mathfrak{s}].$$

Bernstein spectrum

1. Via the isomorphism
$$\mathfrak{Z}_{\mathfrak{s}} \cong \mathbb{C}[\mathfrak{X}_{\mathfrak{s}}],$$
each point in $\mathfrak{X}_{\mathfrak{s}}$ corresponds to a maximal idea of $\mathfrak{Z}_{\mathfrak{s}}$.
2. The Bernstein spectrum $\mathcal{B}(\pi)$ of a module π in $\mathfrak{R}_{\mathfrak{s}}(G_n)$ is the set of points σ in $\mathfrak{X}_{\mathfrak{s}}$ whose associated maximal ideal $\mathcal{J}_\sigma$ gives $\pi/(\mathcal{J}_\sigma^k \pi) \neq 0$ for some large k.
3. Example: If π is irreducible, then $\mathcal{B}(\pi)$ is a point.

Geometric interpretation

Roughly, one may think:

- 'Irreducibility of a variety' relates to 'indecomposability of a module'

In particular, we have:

Corollary

Let $\pi \in \mathrm{Irr}(G_{n+1})$*. For each* G_n*-Bernstein component* $\pi_{\mathfrak{s}}$*, the Bernstein spectrum of* $\pi_{\mathfrak{s}}$ *is a closed irreducible subvariety.*

Explicit description of Bernstein spectrum

We need Langlands-Zelevinsky classification of irreducible modules:

- Zelevinsky segment $\Delta = [\nu^a\rho, \nu^b\rho]$, where $b - a \in \mathbb{Z}_{\geq 0}$ and ρ is cuspidal
- (Zelevinsky) Square-integrable representations $\mathrm{St}([\nu^a\rho, \nu^b\rho])$: the unique irreducible quotient of

$$\nu^a\rho \times \ldots \times \nu^b\rho$$

- (Classification) A multisegment is a multiset of segments.

Theorem (Zelevinsky, Langlands)

There is a one-to-one correspondence between collection of multisegments and irreducible G_n-modules.

Explicit description

We describe the Iwahori component for a representation in a unramified principal series.

1. $\mathfrak{m} = \{\Delta_1, \ldots, \Delta_k\}$ with each $\Delta_i = [\nu^{a_i}, \nu^{b_i}]$
2. Irreducible module $\langle \mathfrak{m} \rangle$
3. The highest derivative $\langle \mathfrak{m} \rangle^-$ is a G_{n+1-k}-module
$$\langle \mathfrak{m} \rangle^- = \langle \{\Delta_1^-, \ldots, \Delta_k^-\} \rangle$$
and
$$\Delta_i^- = [\nu^{a_i}, \nu^{b_i - 1}]$$
4. The spectrum of $\pi|_{G_n}$ is given by the S_n orbits on the points:
$$(\nu^{a_1}, \ldots, \nu^{b_1-1}, \ldots, \nu^{a_k}, \ldots, \nu^{b_k-1}, \nu^{x_1}, \ldots, \nu^{x_r}),$$
where $x_1, \ldots, x_r$ are any points.

References

Bernstein components for Whittaker models:

1. (Joint with G. Savin) *Iwahori components for Gelfand-Graev representations*, Math. Z. (2018)
2. (Joint with G. Savin) *Bernstein-Zelevinsky derivatives: a Hecke algebra approach*, IMRN (2019)

Bernstein components for branching laws:

1. (Joint with G. Savin)*A vanishing Ext-branching theorem for* $(\mathrm{GL}_{n+1}(F), \mathrm{GL}_n(F))$, to appear in Duke Math. J.
2. *Homological branching law for* $(\mathrm{GL}_{+1}(F), \mathrm{GL}_n(F))$: projectivity and indecomposability, to appear in Invent. Math.
3. *Restriction for general linear groups: the local non-tempered Gan-Gross-Prasad conjecture*, arXiv

Thank you!