On Riemann-Roch-Grothendieck theorem for punctured curves with hyperbolic singularities

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Plan of the talk

Riemann-Roch-Grothendieck theorem and curvature theorem

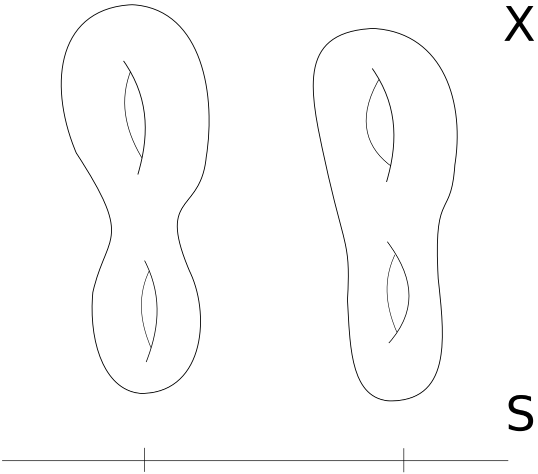
Family setting

$\pi : X \to S$ proper holomorphic submersion, relative dimension 1

$$\omega_{X/S} = (\Lambda^{\max} T^{*(1,0)} X) \otimes (\Lambda^{\max} T^{*(1,0)} S)^{-1}$$
the relative canonical line bundle of π

$$t \in S, X_t = \pi^{-1}(t)$$

A picture



Dolbeaut complex

ξ a holomorphic vector bundle over X

$$\Omega^{i,j}(X_t,\xi) = \mathscr{C}^{\infty}(X_t, T^{*(i,j)}X_t \otimes \xi), \quad i,j = 0,1$$

$$0 \to \Omega^{0,0}(X_t,\xi) \xrightarrow{\overline{\partial}} \Omega^{0,1}(X_t,\xi) \to 0$$

$$H^0(X_t,\xi) = \ker(\overline{\partial}), \qquad H^1(X_t,\xi) = \Omega^{0,1}(X_t,\xi)/\operatorname{Im}(\overline{\partial})$$

The determinant of the cohomology

$$\lambda(\xi)_t = (\Lambda^{\max} H^0(X_t, \xi|_{X_t}))^{-1} \otimes \Lambda^{\max} H^1(X_t, \xi|_{X_t}), \quad t \in S$$

family of complex lines over S

Grothendieck-Knudsen-Mumford :
$\lambda(\xi)_t$, $t \in S$ form a holomorphic line bundle $\lambda(\xi)$ over S

Theorem. (Riemann-Roch-Grothendieck, 1957)

The following identity holds in $H^\bullet(S, \mathbb{Q})$:

$$c_1(\lambda(\xi)) = -\int_\pi \Big[\mathrm{Td}(\omega_{X/S})\mathrm{ch}(\xi)\Big]^{[4]}$$

$$\mathrm{Td}(\xi) = 1 + \frac{c_1(\xi)}{2} + \frac{c_1(\xi)^2 + c_2(\xi)}{12} + \dots$$

$$\mathrm{ch}(\xi) = \mathrm{rk}(\xi) + c_1(\xi) + \frac{c_1(\xi)^2 - 2c_2(\xi)}{2} + \dots$$

Chern-Weil theory

- Y a complex manifold
  (E, h^E) a holomorphic Hermitian vector bundle over Y
  ∇^E the Chern connection on (E, h^E)
- $R^E = (\nabla^E)^2 \in \Omega^{1,1}(Y, \mathrm{End}(E))$
- $$\mathrm{ch}(E, h^E) = \mathrm{Tr}\Big[\exp\Big(-\frac{R^E}{2\pi\sqrt{-1}}\Big)\Big] \in \oplus_{p\in\mathbb{N}}\Omega^{p,p}(Y)$$
  $$\mathrm{Td}(E, h^E) = \det\Big[\frac{R^E}{\exp(R^E) - 1}\Big] \in \oplus_{p\in\mathbb{N}}\Omega^{p,p}(Y)$$
- $\mathrm{Td}(E, h^E), \mathrm{ch}(E, h^E)$ are closed forms
- Chern-Weil : $\big[\mathrm{ch}(E, h^E)\big]_{DR} = \mathrm{ch}(E) \in \oplus_{p\in\mathbb{N}} H^{2p}(Y, \mathbb{R})$
  $\big[\mathrm{Td}(E, h^E)\big]_{DR} = \mathrm{Td}(E) \in \oplus_{p\in\mathbb{N}} H^{2p}(Y, \mathbb{R})$

A natural question

$\pi : X \to S$ proper holomorphic submersion, relative dimension 1

$\|\cdot\|^{\omega}_{X/S}$ a Hermitian norm on $\omega_{X/S}$

(ξ, h^{ξ}) a holomorphic Hermitian vector bundle over X

$$c_1(\lambda(\xi), \textbf{?}) = -\int_{\pi} \left[\mathrm{Td}(\omega_{X/S}, (\|\cdot\|^{\omega}_{X/S})^2)\mathrm{ch}(\xi, h^{\xi}) \right]^{[4]}$$

- L^2-Hermitian product. Let $\alpha, \alpha' \in \Omega^{0,\bullet}(X_t, \xi)$
  $\langle \alpha, \alpha' \rangle_{L^2} = \int_{X_t} \langle \alpha(x), \alpha'(x) \rangle_h dv_{X_t}(x)$,
  $\langle \cdot, \cdot \rangle_h$ the pointwise Hermitian product induced by $h^\xi, \|\cdot\|^\omega_{X/S}$.

- $0 \to \Omega^{0,0}(X_t, \xi) \xrightarrow{\overline{\partial}} \Omega^{0,1}(X_t, \xi) \to 0$,
  $\square^\xi_t = \overline{\partial}\, \overline{\partial}^* + \overline{\partial}^* \overline{\partial}$
- $\langle \square^\xi_t \alpha, \alpha \rangle_{L^2} = \langle \overline{\partial}\alpha, \overline{\partial}\alpha \rangle + \langle \overline{\partial}^* \alpha, \overline{\partial}^* \alpha \rangle$,
  $\ker(\square^\xi|_{\Omega^{0,\bullet}(X_t,\xi)}) = \{ s \in \Omega^{0,\bullet}(X_t, \xi) \mid \overline{\partial} s = 0, \ \overline{\partial}^* s = 0 \}$

$$\ker(\square^\xi|_{\Omega^{0,\bullet}(X_t,\xi)}) \to H^\bullet(X_t, \xi) \quad \ker(\square^\xi|_{\Omega^{0,\bullet}(X_t,\xi)}) \simeq H^\bullet(X_t, \xi)$$

Hodge theory

- induces the L^2-norm $\|\cdot\|_{L^2}\left(g^{TX_t}, h^\xi\right)$ over
  $\lambda(\xi)_t = (\Lambda^{\max} H^0(X_t, \xi|_{X_t}))^{-1} \otimes \Lambda^{\max} H^1(X_t, \xi|_{X_t})$

Infinite product

From now on $\Box_t^\xi := \Box^\xi|_{\Omega^{0,0}(X_{t,\xi})}$

$\Box_t^\xi$ essentially self-adjoint

$\mathrm{Spec}(\Box_t^\xi) = \{\lambda_{1,t}, \lambda_{2,t}, \ldots\}$, $\lambda_{i,t}$ non decreasing, $\lambda_{i,t} \to \infty$

$$\det{}'\Box_t^\xi = \prod_{\lambda_{i,t}\neq 0}^{\infty} \lambda_{i,t}.$$

Problem : Need to make sense of the infinite product...

Weyl's law : $\lambda_{i,t}$ increase asymptotically linearly with i

$$\zeta_{\xi,t}(s) = \sum_{\lambda_{i,t} \neq 0}^{\infty} \frac{1}{(\lambda_{i,t})^s}, \text{ for } \mathrm{Re}(s) > 1$$

Definition of the determinant. (Ray-Singer, 1973)

$$\det{}' \square_t^{\xi} = \exp\Big(- \zeta'_{\xi,t}(0) \Big)$$

Quillen norm

Hermitian norm on $\lambda(\xi)$, given by

$$\|\cdot\|^{Q}\left(g^{TX_t}, h^{\xi}\right) = \left(\det{}'\square_t^{\xi}\right)^{1/2} \cdot \|\cdot\|_{L^2}\left(g^{TX_t}, h^{\xi}\right)$$

Curvature theorem. (Bismut-Gillet-Soulé, 1988)

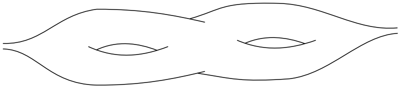
- Hermitian norm $\|\cdot\|^{Q}\left(g^{TX_t}, h^{\xi}\right)$ is smooth over S

$$c_1\Big(\lambda(\xi), \left(\|\cdot\|^{Q}\left(g^{TX_t}, h^{\xi}\right)\right)^2\Big) = -\int_{\pi}\left[\mathrm{Td}(\omega_{X/S}, (\|\cdot\|^{\omega}_{X/S})^2)\mathrm{ch}(\xi, h^{\xi})\right]^{[4]}$$

Motivation

Motivation

We want to extend the theory of Quillen metrics to surfaces with hyperbolic cusps and degenerating families with singular fibers



$\overline{M}$ a compact Riemann surface

$D_M = \{P_1, P_2, \dots, P_m\} \subset \overline{M}$, $M = \overline{M} \setminus D_M$

g^{TM} is a Kähler metric on M

$z_1, \dots, z_m$ local holomorphic coordinates, $z_i(0) = \{P_i\}$

Suppose g^{TM} over $\{|z_i| < \epsilon\}$ is induced by

$$\frac{\sqrt{-1} dz_i d\overline{z}_i}{\left| z_i \log |z_i| \right|^2}.$$

We call $(\overline{M}, D_M, g^{TM})$ a surface with cusps

Suppose $2g(\overline{M}) - 2 + \#D_M > 0$, i.e. $(\overline{M}, D_M)$ is stable

By uniformization theorem, there is exactly one csc -1 complete metric g^{TM}_{hyp} of finite volume on $M = \overline{M} \setminus D_M$

The triple $(\overline{M}, D_M, g^{TM}_{\mathrm{hyp}})$ is a surface with cusps

We want to extend the theory of Quillen metrics to surfaces with hyperbolic cusps and degenerating families with singular fibers

Why ?

- Problem on its own.
- Universal curve $\pi : \mathscr{C}_{g,m} \to \mathscr{M}_{g,m}$ with csc -1 metric $\|\cdot\|_{X/S}^{\omega,\mathrm{hyp}}$

  On $\mathscr{M}_{g,m}$, we have $\int_\pi \left[\mathrm{Td}(\omega_{X/S}, (\|\cdot\|_{X/S}^{\omega,\mathrm{hyp}})^2)\right]^{[4]} =^* \omega_{WP}$.

  As we expect $c_1(\lambda, (\|\cdot\|^Q)^2) = -\int_\pi \left[\mathrm{Td}(\omega_{X/S}, (\|\cdot\|_{X/S}^{\omega,\mathrm{hyp}})^2)\right]^{[4]}$

$$\text{Regularity of } \|\cdot\|^Q \text{ near } \partial\mathscr{M}_{g,m}$$
$$\Downarrow$$
$$\text{Regularity of } \omega_{WP} \text{ near } \partial\mathscr{M}_{g,m}.$$

- Curvature theorem of Takhtajan-Zograf (csc -1).
- Arithmetic Riemann-Roch theorem for pointed stable curves relation to Freixas, Freixas-von Pippich, Dutour.

Definition of Quillen metric for surfaces with cusps

The L^2-norm

$$\|\cdot\|^{Q} = \left(\det{}'\Box\right)^{1/2} \cdot \|\cdot\|_{L^2}$$

The L^2-norm

- Let $(\overline{M}, D_M, g^{TM})$ be a surface with cusps
  $\|\cdot\|^{\omega}_{M}$ the induced Hermitian norm on $\omega_{\overline{M}}$ over M
- $\omega_M(D) = \omega_{\overline{M}} \otimes \mathscr{O}_{\overline{M}}(D_M)$ the twisted canonical line bundle
  $\omega_M(D) \simeq \omega_{\overline{M}},$ over M
  induces the Hermitian norm $\|\cdot\|_M$ on $\omega_M(D)$ over M

  This norm has $\log$ singularity $\left\| dz_i \otimes s_{D_M}/z_i \right\|_M = |\log |z_i||$
- (ξ, h^{ξ}) a holomorphic Hermitian vector bundle over $\overline{M}$

$$E^{\xi}_n = \xi \otimes \omega_M(D)^n, \qquad h^{E^{\xi}_n} = h^{\xi} \otimes (\|\cdot\|_M)^{2n}$$

- For $n \leq 0$, by Hodge theory*
  $\langle \cdot, \cdot \rangle_{L^2}$ induces the L^2-norm $\|\cdot\|_{L^2}$ on
  $\lambda(E^{\xi}_n) = (\Lambda^{\max} H^0(\overline{M}, E^{\xi}_n))^{-1} \otimes \Lambda^{\max} H^1(\overline{M}, E^{\xi}_n)$

The determinant

$$\left\|\cdot\right\|^{Q} = \left(\det{}'\Box\right)^{1/2} \cdot \left\|\cdot\right\|_{L^2}$$

Problem with the determinant

$$\square^{E_n^{\xi}} : \Omega^{0,0}(M, E_n^{\xi}) \to \Omega^{0,0}(M, E_n^{\xi})$$

It is again essentially self-adjoint by the same reason

As M is non-compact, in general $\mathrm{Spec}(\square^{E_n^{\xi}})$ is not discrete

$$\det{}' \square^{E_n^{\xi}} \neq \prod_{\lambda_i \neq 0}^{\infty} \lambda_i .$$

$$\{ \text{ Length of closed geodesics } \} \leftrightarrow \mathrm{Spec}(\square^{E_n^\xi})$$

Suppose (ξ, h^ξ) trivial, $(M, D_M, g^{TM}_{\mathrm{hyp}})$ has csc -1
then the set of simple closed geodesics is discrete

$$Z_{(\overline{M}, D_M)}(s) = \prod_\gamma \prod_{k=0}^{\infty} (1 - e^{-(s+k)l(\gamma)})$$

γ simple closed geodesics on M ; $l(\gamma)$ is the length of γ.

Takhtajan-Zograf definition using Selberg zeta-function, 1991

$$\det{}'_{TZ}\square^{E_n^\xi} = \begin{cases} Z'_{(\overline{M},D_M)}(1), & \text{for } n = 0, \\ Z_{(\overline{M},D_M)}(-n+1), & \text{for } n < 0. \end{cases}$$

Motivated by a theorem of D'Hoker-Phong, 1986, which says that when $m = 0$, two sides of the previous equation coincide*

Limitations of this approach

- Restriction on the topology $2g(\overline{M}) - 2 + \#D_M > 0$.
- Complex structure predefines the Kähler metric.
- No liberty in choosing (ξ, h^ξ).

- $$\lambda^{-s} = \frac{1}{\Gamma(s)} \int_0^{+\infty} \exp(-\lambda t) t^{s-1} dt$$
- If M is compact, i.e. $m = 0$

$$\zeta_{E_n^\xi}(s) = \sum_{\lambda \in \mathrm{Spec}(\Box^{E_n^\xi}) \setminus \{0\}} \lambda^{-s} \qquad (\star)$$

$$= \frac{1}{\Gamma(s)} \int_0^{+\infty} \mathrm{Tr}\Big[\exp^{\perp}(-t\Box^{E_n^\xi})\Big] t^{s-1} dt \qquad (\star\star)$$

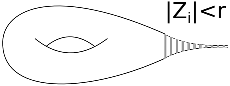
- For $m > 0$?
  Idea : define $\zeta_{E_n^\xi}(s)$ for $m > 0$ using $(\star\star)$ and not $(\star)$
- Problem : $\exp^{\perp}(-t\Box^{E_n^\xi})$ is not of trace class for $m > 0$

Regularizing trace, I

- The operator $\exp(-t\square^{E_n^\xi})$ has a smooth Schwartz kernel
$$\exp(-t\square^{E_n^\xi})(x,y) \in (E_n^\xi)_x \otimes (E_n^\xi)_y^*, \qquad x, y \in M$$
$$\exp(-t\square^{E_n^\xi})s = \int_M \Big\langle \exp(-t\square^{E_n^\xi})(x,y), s(y) \Big\rangle dv_M(y).$$

- If $m = 0$, $\mathrm{Tr}\Big[\exp(-t\square^{E_n^\xi})\Big] = \int_{\overline{M}} \mathrm{Tr}\Big[\exp(-t\square^{E_n^\xi})(x,x)\Big] dv_M(x)$.

- Idea : define $\mathrm{Tr}^{\boldsymbol{r}}\Big[\exp(-t\square^{E_n^\xi})\Big]$ by taking the finite part of
$$\int_{M_r} \mathrm{Tr}\Big[\exp(-t\square^{E_n^\xi})(x,x)\Big] dv_M(x)$$
as $r \to 0$, where M_r is the non-striped region



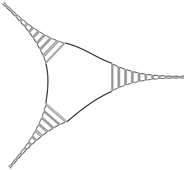
$$P = \mathbb{C}P^1 \setminus \{0, 1, \infty\},$$

g^{TP} hyperbolic metric csc -1 over P

We fix $n \leq 0$

$$g_n(r, t) = \frac{1}{3} \int_{P_r} \exp(-t\Box^{\omega_P(D)^n})(x, x) dv_P(x), \tag{4.1}$$

where P_r is the non-striped region



Theorem. (-, 2018)

For any $(\overline{M}, D_M, g^{TM})$, (ξ, h^{ξ}), $t > 0$, the function

$$\mathbb{R}_{>0} \ni r \mapsto \int_{M_r} \mathrm{Tr}\Big[\exp(-t\Box^{E_n^{\xi}})(x,x) \Big] dv_M(x) - \mathrm{rk}(\xi) \cdot m \cdot g_n(r,t)$$

extends continuously over $r = 0$.

Regularized heat trace

$$\mathrm{Tr}^{\boldsymbol{r}}\Big[\exp(-t\Box^{E_n^\xi})\Big] = \lim_{r\to 0}\Big(\int_{M_r}\mathrm{Tr}\Big[\exp(-t\Box^{E_n^\xi})(x,x)\Big]dv_M(x) - \mathrm{rk}(\xi)\cdot m\cdot g_n(r,t)\Big).$$

$$\zeta_{E_n^\xi}(s) = \frac{1}{\Gamma(s)} \int_0^{+\infty} \mathrm{Tr}^{\boldsymbol{r}} \left[\exp^{\perp}(-t \Box^{E_n^\xi}) \right] t^{s-1} dt.$$

Theorem. (-, 2018)

- $\zeta_{E_n^\xi}(s)$ is well-defined and extends meromorphically to $\mathbb{C}$
- $0 \in \mathbb{C}$ is a holomorphic point of $\zeta_{E_n^\xi}(s)$

Finally, the determinant

Definition of the determinant

$$\det{}' \square^{E_n^\xi} = \exp\Big(- \zeta'_{E_n^\xi}(0) \Big).$$

Theorem. (-, 2019)

Suppose $(M, D_M, g^{TM}_{\mathrm{hyp}})$ has csc -1, (ξ, h^ξ) trivial. Then for any $m \geq 0$, $n \leq 0$, we have

$$\det{}' \square^{E_n^\xi} =^* \det{}'_{TZ} \square^{E_n^\xi}.$$

$=^*$ means up to some computed universal constant

$m = 0$, D'Hoker-Phong, 1986

Finally, the Quillen norm

Quillen norm

Hermitian norm on $\lambda(E_n^\xi)$, given by

$$\left\|\cdot\right\|^{Q}\left(g^{TM}, h^{E_n^\xi}\right) = \left(\det{}'\Box^{E_n^\xi}\right)^{1/2} \cdot \left\|\cdot\right\|_{L^2}\left(g^{TM}, h^{E_n^\xi}\right)$$

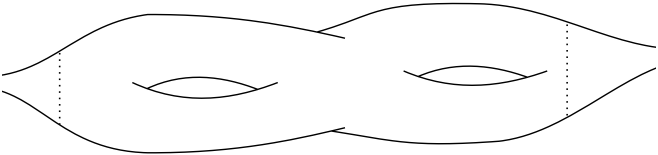
A question

How to compute the Quillen norm ?

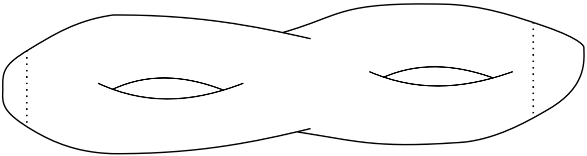
Relative compact perturbation theorem

A notion of flattening

Flattening of a metric with cusps g^{TM}



is a Kähler metric g_{f}^{TM} on $\overline{M}$ such that



The same for $\|\cdot\|_M$

Let $(\overline{M}, D_M, g^{TM})$ be a surface with cusps,

(ξ, h^ξ) Hermitian vector bundle over $\overline{M}$

$g^{TM}_{\mathrm{f}}, \|\cdot\|^{\mathrm{f}}_M$ the flattenings of $g^{TM}, \|\cdot\|_M$

Relative compact perturbation theorem calculates

$$\frac{\|\cdot\|_Q\left(g^{TM}, h^\xi \otimes \|\cdot\|_M^{2n}\right)}{\|\cdot\|_Q\left(g^{TM}_{\mathrm{f}}, h^\xi \otimes (\|\cdot\|^{\mathrm{f}}_M)^{2n}\right)}$$

In other words : it answers
How Quillen metric changes under compact perturbation.

Anomaly formula

The Wolpert norm

$(\overline{M}, D_M, g^{TM})$, $D_M = \{P_1, \dots, P_m\}$ surface with cusps
$z_1, \dots, z_m$ local holomorphic coordinates, $z_i(0) = \{P_i\}$
g^{TM} over $\{|z_i| < \epsilon\}$ is induced by

$$\frac{\sqrt{-1} dz_i d\overline{z}_i}{\left| z_i \log |z_i| \right|^2}$$

Wolpert norm

$\|\cdot\|^W$ on $\otimes_{i=1}^m \omega_{\overline{M}}|_{P_i}$ is defined by

$$\left\| \otimes_i dz_i|_{P_i} \right\|^W = 1.$$

$$on \quad D^* \qquad \frac{\sqrt{-1} dz d\overline{z}}{\left| z \log |z| \right|^2} \quad \rightsquigarrow \quad \left\| dz|_0 \right\|^W = 1 on \quad D^* \qquad \frac{\sqrt{-1} dz d\overline{z}}{\left| z \log |2z| \right|^2}$$

Wolpert norm is related to the “constant term"
of the conformal transformation at cusp

Anomaly formula, setting

$(\overline{M}, D_M)$ a pointed Riemann surface
g^{TM}, g_0^{TM} metrics with cusps at D_M

$\|\cdot\|_M, \|\cdot\|_M^0$ the norms induced by g^{TM}, g_0^{TM} on $\omega_M(D)$

$\|\cdot\|^W, \|\cdot\|_0^W$ the associated Wolpert norms on $\otimes_{P \in D_M} \omega_{\overline{M}}|_P$

ξ holomorphic vector bundle on $\overline{M}$
h^ξ, h_0^ξ Hermitian metrics on ξ over $\overline{M}$

Theorem. (-, 2018)

$$2\log\left(\|\cdot\|_Q\left(g_0^{TM}, h_0^{\xi}\otimes(\|\cdot\|_M^0)^{2n}\right)\Big/\|\cdot\|_Q\left(g^{TM}, h^{\xi}\otimes\|\cdot\|_M^{2n}\right)\right)$$
$$=\int_M\Big[\text{Bott-Chern terms, analogic to the anomaly}$$
$$\text{for compact manifolds of Bismut-Gillet-Soulé}\Big]$$
$$-\frac{\mathrm{rk}(\xi)}{6}\log\left(\|\cdot\|^W/\|\cdot\|_0^W\right)+\sum\log\left(\det(h^{\xi}/h_0^{\xi})|_{P_i}\right).$$

What is a family of curves with cusps ?

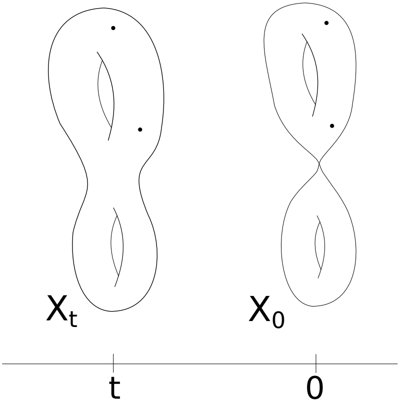
Family of curves with cusps

- $\pi : X \to S$ proper holomorphic of relative dimension 1, $t \in S$, $X_t = \pi^{-1}(t)$ has at most double-point singularities (i.e. those of the form $\{z_0 z_1 = 0\}$)

  $\Sigma_{X/S}$ singular points of the fibers, $\Delta = \pi_*(\Sigma_{X/S})$
- $\sigma_1, \dots, \sigma_m : S \to X \setminus \Sigma_{X/S}$ hol. non intersect. sections
  $D_{X/S} = \mathrm{Im}(\sigma_1) + \cdots + \mathrm{Im}(\sigma_m)$
- $\|\cdot\|^{\omega}_{X/S}$ Herm. norm on $\omega_{X/S}$ over $X \setminus (|D_{X/S}| \cup \pi^{-1}(|\Delta|))$
  $\|\cdot\|^{\omega}_{X/S}|_{X_t}$ induces metric g^{TX_t} on $X_t \setminus |D_{X/S}|$, $t \in S \setminus |\Delta|$
  So that $(X_t, \{\sigma_1(t), \dots, \sigma_m(t)\}, g^{TX_t})$ is a surface with cusps

  $(\pi : X \to S, D_{X/S}, \|\cdot\|^{\omega}_{X/S})$ a family of curves with cusps

A picture



Curvature theorem for family of curves with cusps

$(\pi : X \to S, D_{X/S}, \|\cdot\|^{\omega}_{X/S})$ a family of curves with cusps

$\omega_{X/S}(D) = \omega_{X/S} \otimes \mathscr{O}_X(D_{X/S})$, $\|\cdot\|_{X/S}$

twisted relative canonical line bundle on X

(ξ, h^{ξ}) a holomorphic Hermitian vector bundle over X

$$E^{\xi}_n = \xi \otimes \omega_{X/S}(D)^n$$

$$\lambda(E^{\xi}_n)_t = (\Lambda^{\max} H^0(X_t, E^{\xi}_n|_{X_t}))^{-1} \otimes \Lambda^{\max} H^1(X_t, E^{\xi}_n|_{X_t})$$

Quillen norm

We define the Quillen norm on $\lambda(\xi \otimes \omega_{X/S}(D)^n)$ by

$$\|\cdot\|^Q \left(g^{TX_t}, h^\xi \otimes \|\cdot\|^{2n}_{X/S}\right) = \left(\det{}'\Box_t^{E_n^\xi}\right)^{1/2} \cdot \|\cdot\|_{L^2} \left(g^{TX_t}, h^\xi \otimes \|\cdot\|^{2n}_{X/S}\right).$$

Wolpert norm

We define the Wolpert norm $\|\cdot\|^W$ on $\otimes_i \sigma_i^*(\omega_{X/S})$ over S by gluing the Wolpert norms $\|\cdot\|_t^W$ on $\otimes_i \omega_{X/S}|_{\sigma_i(t)}$ induced by g^{TX_t}.

Riemann-Roch-Grothendieck theorem in the presence of cusps

$$\mathscr{L}_n = \lambda(E_n^{\xi})^{12} \otimes (\otimes_i \sigma_i^* \omega_{X/S})^{-\mathrm{rk}(\xi)} \otimes \mathscr{O}_S(\Delta)^{\mathrm{rk}(\xi)} \otimes (\otimes_i \sigma_i^* \det \xi)^6$$

Canonical singular norm

s_Δ the canonical holomorphic section of $\mathscr{O}_S(\Delta)$
$\|\cdot\|_\Delta^{\mathrm{div}}$ on $\mathscr{O}_S(\Delta)$ is defined by $\|s_\Delta\|_\Delta^{\mathrm{div}}(x) = 1, \qquad x \in S \setminus |\Delta|$

$$\|\cdot\|^{\mathscr{L}_n} = \left(\|\cdot\|^Q \left(g^{TX_t}, h^{\xi} \otimes \|\cdot\|_{X/S}^{2n}\right)\right)^{12} \otimes \left(\|\cdot\|^W\right)^{-\mathrm{rk}(\xi)} \otimes \left(\|\cdot\|_\Delta^{\mathrm{div}}\right)^{\mathrm{rk}(\xi)} \otimes \left(\otimes_i \sigma_i^* h^{\det \xi}\right)^3$$

Theorem. (-, 2018)

Under mild degenerating assumptions on $\|\cdot\|_{X/S}$, the norm $\|\cdot\|^{\mathscr{L}_n}$ extends continuously* over $|\Delta|$, smooth* over $S \setminus |\Delta|$, and on the level of currents over S :

$$c_1\left(\mathscr{L}_n, (\|\cdot\|^{\mathscr{L}_n})^2\right) = -12 \int_{\pi} \left[\mathrm{Td}(\omega_{X/S}(D), \|\cdot\|_{X/S}^2)\mathrm{ch}(\xi, h^{\xi})\mathrm{ch}(\omega_{X/S}(D), \|\cdot\|_{X/S}^{2n})\right]^{[4]}$$

Thank you !