SOME CONJECTURES CONCERNING NON-STATIONARY RUIJSENAARS FUNCTIONS

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1. Plan of my talk

Macdonald case (of type $\mathfrak{gl}_n$):

- Asymptotically free solutions to the Macdonald eigenvalue equations
- Laumon spaces
- Duality formulas
- Passages to the Macdonald/Schur symmetric polynomials
- Pieri formulas for the asymptotically free solutions
- Operator $\mathcal{T}^{\mathfrak{gl}_n} x(q,t)$

Affine generalization (type $\widehat{\mathfrak{gl}_n}$):

- Non-stationary Ruijsenaars functions
- Affine Laumon spaces
- Duality conjectures
- Passages to the irreducible affine characters (Shur limit)
- Conjecture concerning Ruijsenaars' eigenvalue problem (stationary limit)
- $\mathcal{T}^{\widehat{\mathfrak{gl}_n}} x(q,t)$ and a conjecture for the non-stationary Ruijsenaars functions

2. Notation

We use the standard notation [GR] for the q-shifted factorials and the double infinite products such as:

$$(u;q)_\infty = \prod_{i=0}^{\infty}(1-q^i u),$$

$$(u;q)_n = (u;q)_\infty/(q^n u;q)_\infty = (1-u)(1-qu)\cdots 1-q^{n-1}u) \qquad (n \in \mathbb{Z}_{\geq 0}),$$

$$(u;q,p)_\infty = \prod_{i,j=0}^{\infty}(1-q^i p^j u).$$

[GR] G. Gasper and M. Rahman, Basic hypergeometric series, 2nd ed., Encyclopedia of Mathematics and its Applications, vol. 96, Cambridge University Press, Cambridge, (2004).

For $\mathfrak{gl}_n$, denote the simple roots by $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ $(i = 1, 2, \ldots, n-1)$, the root lattice by Q, and the positive cone by Q_+. Write the formal exponentials as

$$e^{-\alpha_i} = e^{\varepsilon_{i+1}-\varepsilon_i} = x_{i+1}/x_i.$$

We will treat formal power series of the form

$$\sum_{\alpha\in Q_+} c_\alpha e^{-\alpha} = \sum_{i_1,i_2,\ldots,i_{n-1}\geq 0} c_{i_1,i_2,\ldots,i_{n-1}} \prod_{k=1}^{n-1}\Big(x_{i+1}/x_i\Big)^{i_k} \in \mathbb{F}[[e^{-\alpha_1}, e^{-\alpha_2}, \ldots, e^{-\alpha_{n-1}}]].$$

We do the same for the affine case $\widehat{\mathfrak{gl}}_n$.

3. Asymptotically free Macdonald functions

3.1. **Asymptotically free solutions.** We recall some facts about the Macdonald functions [S,NS,BFS].

Definition 3.1. *Let $D_x^{\mathfrak{gl}_n} = D_x^{\mathfrak{gl}_n}(q,t)$ be the Macdonald operator* [M] *of type $\mathfrak{gl}_n$*

$$D_x^{\mathfrak{gl}_n} = \sum_{i=1}^n \prod_{j\neq i} \frac{tx_i - x_j}{x_i - x_j} T_{q,x_i},$$

where T_{q,x_i} denotes the q-shift operator

$$T_{q,x_i} f(x_1,\ldots,x_i,\ldots,x_n) = f(x_1,\ldots,qx_i,\ldots,x_n).$$

Let $\mathsf{M}^{(n)}$ be the set of strictly upper triangular matrices with nonnegative integer entries: $\mathsf{M}^{(n)} = \{\theta = (\theta_{i,j})_{1\leq i,j\leq n} | \theta_{i,j} \in \mathbb{Z}_{\geq 0}, \theta_{i,j} = 0 \text{ if } i \geq j\}$. Define recursively $c_n(\theta; s; q,t) \in \mathbb{Q}(q,t,s_1,\cdots,s_n)$ by $c_1(-;s_1;q,t) = 1$, and

$$\begin{aligned} c_n(\theta \in \mathsf{M}^{(n)}; s_1,\cdots,s_n;q,t) &= c_{n-1}(\theta \in \mathsf{M}^{(n-1)}; q^{-\theta_{1,n}} s_1, \cdots, q^{-\theta_{n-1,n}} s_{n-1}; q,t) \\ &\times \prod_{1\leq i\leq j\leq n-1} \frac{(ts_{j+1}/s_i;q)_{\theta_{i,n}}}{(qs_{j+1}/s_i;q)_{\theta_{i,n}}} \frac{(q^{-\theta_{j,n}} qs_j/ts_i;q)_{\theta_{i,n}}}{(q^{-\theta_{j,n}} s_j/s_i;q)_{\theta_{i,n}}}. \end{aligned}$$

We have

$$\begin{aligned} & c_n(\theta; s_1,\cdots,s_n;q,t) \\ =& \prod_{k=2}^n \prod_{1\leq i\leq j\leq k-1} \frac{(q^{\sum_{a=k+1}^n (\theta_{i,a}-\theta_{j+1,a})} ts_{j+1}/s_i;q)_{\theta_{i,k}}}{(q^{\sum_{a=k+1}^n (\theta_{i,a}-\theta_{j+1,a})} qs_{j+1}/s_i;q)_{\theta_{i,k}}} \frac{(q^{-\theta_{j,k}+\sum_{a=k+1}^n (\theta_{i,a}-\theta_{j,a})} qs_j/ts_i;q)_{\theta_{i,k}}}{(q^{-\theta_{j,k}+\sum_{a=k+1}^n (\theta_{i,a}-\theta_{j,a})} s_j/s_i;q)_{\theta_{i,k}}}. \end{aligned}$$

Definition 3.2. *Define $f^{\mathfrak{gl}_n}(x|s|q,t) \in \mathbb{Q}(s,q,t)[[x_2/x_1,\ldots,x_n/x_{n-1}]]$ by*

$$f^{\mathfrak{gl}_n}(x|s|q,t) = \sum_{\theta\in\mathsf{M}^{(n)}} c_n(\theta;s;q,t) \prod_{1\leq i<j\leq n} (x_j/x_i)^{\theta_{i,j}}.$$

[M] I. G. Macdonald, Symmetric functions and Hall polynomials, 2nd ed., Oxford Mathematical Monographs, Oxford University Press, (1995).

[S] J. Shiraishi, A conjecture about raising operators for Macdonald polynomials, Lett. Math. Phys. **73** (2005) 71–81.

[NS] M. Noumi and J. Shiraishi, A direct approach to the bispectral problem for the Ruijsenaars-Macdonald q-difference operators, `arXiv:1206.5364`.

[BFS] A. Braverman, M. Finkelberg and J. Shiraishi, Macdonald polynomials, Laumon spaces and perverse coherent sheaves, Perspectives in representation theory, 23–41, Contemp. Math., **610**, Amer. Math. Soc., Providence, RI, 2014.

Proposition 3.3. ([NS,BFS]) *Let* $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$, *and set* $s = t^\delta q^\lambda$ $(s_i = t^{n-i} q^{\lambda_i})$. *Then we have*

$$D_x^{\mathfrak{gl}_n} x^\lambda f^{\mathfrak{gl}_n}(x|s|q,t) = e_1(s)\ x^\lambda f^{\mathfrak{gl}_n}(x|s|q,t).$$

Laumon's quasiflags' space $\mathcal{Q}^\alpha$: the moduli space of the flags

$$\left\{0 \subset \mathcal{W}_1 \subset \mathcal{W}_2 \subset \cdots \subset \mathcal{W}_n = \mathcal{O}_{\mathbb{P}_1}^n\right\}$$

of locally free sheaves (where $\mathcal{W}_i$ a locally free sheaf on $\mathbb{P}_1$ of rank i, and $\deg \mathcal{W}_i = -\langle \alpha, \omega_i \rangle$). The based version is denoted by $\mathfrak{Q}^\alpha$. The torus $\mathbb{G}_m \times T$ acts $(q \in \mathbb{G}_m, s \in T)$.

Proposition 3.4. ([BFS]) *We have the geometric interpretation as the Euler characteristics of the de Rham complex of the Laumon spaces:*

$$f(x|s|q,q/t) = \sum_{\alpha \in Q_+} x^\alpha \mathfrak{J}_\alpha(q,t,s),$$
$$\mathfrak{J}_\alpha(q,t,s) = [H^\bullet(\mathfrak{Q}^\alpha, \Omega^\bullet_{\mathfrak{Q}^\alpha})] := \sum_{i,j} (-1)^{i+j} t^j [H^i(\mathfrak{Q}^\alpha, \Omega^j_{\mathfrak{Q}^\alpha})].$$

Lemma 3.5. *We have*

$$\lim_{\epsilon\to 0} f^{\mathfrak{gl}_n}(x|\epsilon^{-\delta}s|q,t) = \prod_{1\le i<j\le n} \frac{(qx_j/x_i;q)_\infty}{(qx_j/tx_i;q)_\infty}.$$

Proof. In the limit $\epsilon \to 0$, we have $\epsilon^{j-i}s_j/s_i \to 0$ for $1 \le i < j \le n$. Hence we have

$$\text{LHS} = \sum_{\theta\in\mathsf{M}^{(n)}} \prod_{1\le i<j\le n} \frac{(t;q)_{\theta_{i,j}}}{(q;q)_{\theta_{i,j}}} (qx_j/tx_i)^{\theta_{i,j}} = \text{RHS}.$$

□

Definition 3.6. *Let* $\varphi^{\mathfrak{gl}_n}(x|s|q,t) \in \mathbb{Q}(q,t)[[x_2/x_1,\ldots,x_n/x_{n-1},s_2/s_1,\ldots,s_n/s_{n-1}]]$ *be*

$$\varphi^{\mathfrak{gl}_n}(x|s|q,t) = \prod_{1\le i<j\le n} \frac{(qx_j/tx_i;q)_\infty}{(qx_j/x_i;q)_\infty} f^{\mathfrak{gl}_n}(x|s|q,t),$$

where $c_n(\theta;s;q,t)$*'s are expanded in* $\mathbb{Q}(q,t)[[s_2/s_1,\ldots,s_n/s_{n-1}]]$.

Proposition 3.7. ([NS]) *We have*

$$\varphi^{\mathfrak{gl}_n}(x|s|q,t) = \varphi^{\mathfrak{gl}_n}(s|x|q,t) \qquad (\textit{bispectral duality}),$$

$$\varphi^{\mathfrak{gl}_n}(x|s|q,t) = \varphi^{\mathfrak{gl}_n}(x|s|q,q/t) \qquad (\textit{Poincaré duality}).$$

4. Pieri fulmula

Set

$$e_1(x) = x_1 + x_2 + \cdots + x_n.$$

Remark 4.1. *Note that we have* $[D_x^{\mathfrak{gl}_n}(q,t), D_x^{\mathfrak{gl}_n}(q^{-1},t^{-1})] = 0$.

Definition 4.2. *Let* $\mathcal{D}^{\pm,\mathfrak{gl}_n}(x|s|q,t)$ *be the modified Macdonald operators defined by*

$$\begin{aligned}
&\mathcal{D}^{\pm,\mathfrak{gl}_n}(x|s|q,t) = x^{-\lambda} D_x^{\mathfrak{gl}_n}(q^{\pm 1}, t^{\pm 1}) x^{\lambda} \\
&= \sum_{i=1}^{n} s_i^{\pm 1} \prod_{j=1}^{i-1} \frac{1 - t^{\pm 1} x_i/x_j}{1 - x_i/x_j} \cdot \prod_{k=i+1}^{n} \frac{1 - t^{\mp 1} x_k/x_i}{1 - x_k/x_i} \cdot T_{q,x_i}^{\pm 1},
\end{aligned}$$

Lemma 4.3. *We have*

$$\begin{aligned}
&[e_1(x^{\mp 1}), \mathcal{D}^{\pm,\mathfrak{gl}_n}(x|s|q,t)] \\
&\qquad = (1 - q^{-1}) \sum_{i=1}^{n} \left(\frac{s_i}{x_i}\right)^{\pm 1} \prod_{j=1}^{i-1} \frac{1 - t^{\pm 1} x_i/x_j}{1 - x_i/x_j} \cdot \prod_{k=i+1}^{n} \frac{1 - t^{\mp 1} x_k/x_i}{1 - x_k/x_i} \cdot T_{q,x_i}^{\pm 1},
\end{aligned}$$

$$\begin{aligned}
&[e_1(s^{\mp 1}), \mathcal{D}^{\pm,\mathfrak{gl}_n}(s|x|q,t)] \\
&\qquad = (1 - q^{-1}) \sum_{i=1}^{n} \left(\frac{x_i}{s_i}\right)^{\pm 1} \prod_{j=1}^{i-1} \frac{1 - t^{\pm 1} s_i/s_j}{1 - s_i/s_j} \cdot \prod_{k=i+1}^{n} \frac{1 - t^{\mp 1} s_k/s_i}{1 - s_k/s_i} \cdot T_{q,s_i}^{\pm 1}.
\end{aligned}$$

Proposition 4.4. *We have*

$$\mathcal{D}^{\pm,\mathfrak{gl}_n}(x|s|q,t)\prod_{1\le i<j\le n}\frac{(qs_j/s_i;q)_\infty}{(ts_j/s_i;q)_\infty}f^{\mathfrak{gl}_n}(x|s|q,t)$$
$$=e_1(s^{\pm 1})\prod_{1\le i<j\le n}\frac{(qs_j/s_i;q)_\infty}{(ts_j/s_i;q)_\infty}f^{\mathfrak{gl}_n}(x|s|q,t),$$
$$\mathcal{D}^{\pm,\mathfrak{gl}_n}(s|x|q,q/t)\prod_{1\le i<j\le n}\frac{(qs_j/s_i;q)_\infty}{(ts_j/s_i;q)_\infty}f^{\mathfrak{gl}_n}(x|s|q,t)$$
$$=e_1(x^{\pm 1})\prod_{1\le i<j\le n}\frac{(qs_j/s_i;q)_\infty}{(ts_j/s_i;q)_\infty}f^{\mathfrak{gl}_n}(x|s|q,t).$$

Proposition 4.5. *We have*

$$[e_1(x^{\mp 1}),\mathcal{D}^{\pm,\mathfrak{gl}_n}(x|s|q,t)]\prod_{1\le i<j\le n}\frac{(qs_j/s_i;q)_\infty}{(ts_j/s_i;q)_\infty}f^{\mathfrak{gl}_n}(x|s|q,t)$$
$$=[e_1(s^{\pm 1}),\mathcal{D}^{\mp,\mathfrak{gl}_n}(s|x|q,q/t)]\prod_{1\le i<j\le n}\frac{(qs_j/s_i;q)_\infty}{(ts_j/s_i;q)_\infty}f^{\mathfrak{gl}_n}(x|s|q,t).$$

Proof. We have

$$\begin{aligned}
&[e_1(x^{\mp 1}),\mathcal{D}^{\pm,\mathfrak{gl}_n}(x|s|q,t)]\prod_{1\le i<j\le n}\frac{(qs_j/s_i;q)_\infty}{(ts_j/s_i;q)_\infty}f^{\mathfrak{gl}_n}(x|s|q,t)\\
=&\left(e_1(x^{\mp 1})e_1(s^{\pm 1})-\mathcal{D}^{\pm,\mathfrak{gl}_n}(x|s|q,t)\mathcal{D}^{\mp,\mathfrak{gl}_n}(s|x|q,q/t)\right)\prod_{1\le i<j\le n}\frac{(qs_j/s_i;q)_\infty}{(ts_j/s_i;q)_\infty}f^{\mathfrak{gl}_n}(x|s|q,t)\\
=&\left(e_1(s^{\pm 1})e_1(x^{\mp 1})-\mathcal{D}^{\mp,\mathfrak{gl}_n}(s|x|q,q/t)\mathcal{D}^{\pm,\mathfrak{gl}_n}(x|s|q,t)\right)\prod_{1\le i<j\le n}\frac{(qs_j/s_i;q)_\infty}{(ts_j/s_i;q)_\infty}f^{\mathfrak{gl}_n}(x|s|q,t)\\
=&[e_1(s^{\pm 1}),\mathcal{D}^{\mp,\mathfrak{gl}_n}(s|x|q,q/t)]\prod_{1\le i<j\le n}\frac{(qs_j/s_i;q)_\infty}{(ts_j/s_i;q)_\infty}f^{\mathfrak{gl}_n}(x|s|q,t).
\end{aligned}$$

□

5. The operator $\mathcal{T}_x^{\mathfrak{gl}_n}(q,t)$

Definition 5.1. *Set*

$$\Delta = \sum_{i=1}^{n} \vartheta_i^2,$$

where ϑ_i denotes the "shifted Euler operator" $\vartheta_i = x_i \frac{\partial}{\partial x_i} + (n-i)\beta$.

Definition 5.2. *Introduce the operator $\mathcal{T}_x^{\mathfrak{gl}_n} = \mathcal{T}_x^{\mathfrak{gl}_n}(q,t)$ as*

$$\mathcal{T}_x^{\mathfrak{gl}_n} = \sum_{\theta \in \mathsf{M}^{(n)}} \prod_{1 \le i < j \le n} (x_j/x_i)^{\theta_{i,j}} \cdot q^{\frac{1}{2}\Delta} \cdot c_N(\theta; x; q, t) \cdot \prod_{1 \le i < j \le n} \frac{(x_j/x_i;q)_\infty}{(tx_j/x_i;q)_\infty}.$$

Remark 5.3. *Note that we can rewrite $\mathcal{T}_x^{\mathfrak{gl}_n}(q,t)$ as*

$$\begin{aligned}\mathcal{T}_x^{\mathfrak{gl}_n} = &\prod_{1 \le i < j \le n} \frac{(qx_j/x_i;q)_\infty}{(qx_j/tx_i;q)_\infty} \\ &\times \sum_{\theta \in \mathsf{M}^{(n)}} c_N(\theta; x; q, q/t) \cdot q^{\frac{1}{2}\Delta} \cdot \prod_{1 \le i < j \le n} (x_j/x_i)^{\theta_{i,j}} \cdot \prod_{1 \le i < j \le n} (1 - x_j/x_i).\end{aligned}$$

Proposition 5.4. *We have the commutativity*

$$D_x^{\mathfrak{gl}_n}(q,t)\mathcal{T}_x^{\mathfrak{gl}_n}(q,t) = \mathcal{T}_x^{\mathfrak{gl}_n}(q,t)D_x^{\mathfrak{gl}_n}(q,t).$$

Hence we have

$$\mathcal{T}_x^{\mathfrak{gl}_n}(q,t)\cdot x^\lambda f^{\mathfrak{gl}_n}(x|s|q,t) = \varepsilon(\lambda)\cdot x^\lambda f^{\mathfrak{gl}_n}(x|s|q,t).$$

Proof. We show the commutativity $[D_x^{\mathfrak{gl}_n}(q,t), \mathcal{T}_x^{\mathfrak{gl}_n}(q,t)] = 0$ on the space

$$x^\lambda\mathbb{C}[[x_2/x_1,\dots,x_n/x_{n-1}]].$$

Let $\alpha = \sum_{i=1}^{n-1} l_i\alpha_i \in Q_+$. On the monomial $x^{\lambda-\alpha} = x^\lambda \prod_{i=1}^{n-1}(x_{i+1}/x_i)^{l_i}$ we have

$$\begin{aligned} &q^{\frac{1}{2}\Delta}x^{\lambda-\alpha} = \varepsilon(\lambda)\cdot\prod_{i=1}^{n-1}(s_{i+1}/s_i)^{l_i}\cdot q^{\sum_{i=1}^n l_i^2 - \sum_{i=1}^{n-1} l_i l_{i+1}}\cdot x^{\lambda-\alpha},\\ &\varepsilon(\lambda) = q^{\frac{1}{2}\sum_{i=1}^n(\lambda_i+(n-i)\beta)^2}. \end{aligned}$$

We denote the third Jacobi theta function as $\vartheta_3(z|q) = \sum_{n\in\mathbb{Z}} q^{n^2/2}z^n$. A simple calculation shows that we can represent the action of $q^{\frac{1}{2}\Delta}$ in terms of the constant term $[\cdots]_{1,y}$ in y as

$$\begin{aligned} &x^\lambda f(x) \in x^\lambda\mathbb{C}[[x_2/x_1,\dots,x_n/x_{n-1}]],\\ &q^{\frac{1}{2}\Delta}\cdot x^\lambda f(x) = \varepsilon(\lambda)\cdot x^\lambda\left[\prod_{i=1}^n \vartheta_3(s_ix_i/y_i|q)f(y)\right]_{1,y}. \end{aligned}$$

Hence we have the action of $\mathcal{T}_x^{\mathfrak{gl}_n}(q,t)$ on $x^\lambda f(x)$ as

$$\begin{aligned} \mathcal{T}_x^{\mathfrak{gl}_n}(q,t)\cdot x^\lambda f(x) =&\varepsilon(\lambda)\cdot x^\lambda\left[\prod_{i=1}^n\vartheta_3(s_ix_i/y_i|q)\cdot\prod_{1\le i<j\le n}\frac{(qy_j/y_i;q)_\infty}{(ty_j/y_i;q)_\infty}f^{\mathfrak{gl}_n}(x|y|q,t)\right.\\ &\left.\cdot\prod_{1\le i<j\le n}(1-y_j/y_i)\cdot f(y)\right]_{1,y}. \end{aligned}$$

Note that we have

$$\mathcal{D}^{+,\mathfrak{gl}_n}(x|s|q,t)\cdot\prod_{i=1}^n\vartheta_3(s_ix_i/y_i|q) = \frac{q^{-n/2}}{1-q^{-1}}\prod_{i=1}^n\vartheta_3(s_ix_i/y_i|q)\cdot[e_1(x^{-1}),\mathcal{D}^{+,\mathfrak{gl}_n}(x|y|q,t)],$$

$$\mathcal{D}^{-,\mathfrak{gl}_n}(y|s^{-1}|q,q/t)\cdot\prod_{i=1}^n\vartheta_3(s_ix_i/y_i|q) = \frac{q^{-n/2}}{1-q^{-1}}\prod_{i=1}^n\vartheta_3(s_ix_i/y_i|q)\cdot[e_1(y^{+1}),\mathcal{D}^{-,\mathfrak{gl}_n}(y|x|q,q/t)].$$

Now we can show the commutativity as follows:

$$
\begin{aligned}
&D_x^{\mathfrak{gl}_n}(q,t)\mathcal{T}_x^{\mathfrak{gl}_n}(q,t)\cdot x^\lambda f(x)\\
=&\varepsilon(\lambda)\cdot x^\lambda\left[\mathcal{D}^{+,\mathfrak{gl}_n}(x|s|q,t)\prod_{i=1}^{n}\vartheta_3(s_ix_i/y_i|q)\cdot f^{\mathfrak{gl}_n}(x|y|q,t)\cdot\prod_{1\le i<j\le n}\frac{(y_j/y_i;q)_\infty}{(ty_j/y_i;q)_\infty}f(y)\right]_{1,y}\\
=&\varepsilon(\lambda)\cdot x^\lambda\left[\left(\mathcal{D}^{-,\mathfrak{gl}_n}(y|s^{-1}|q,q/t)\prod_{i=1}^{n}\vartheta_3(s_ix_i/y_i|q)\cdot\prod_{1\le i<j\le n}\frac{(qy_j/y_i;q)_\infty}{(ty_j/y_i;q)_\infty}f^{\mathfrak{gl}_n}(x|y|q,t)\right)\right.\\
&\qquad\left.\cdot\prod_{1\le i<j\le n}(1-y_j/y_i)\cdot f(y)\right]_{1,y}\\
=&\varepsilon(\lambda)\cdot x^\lambda\left[\prod_{i=1}^{n}\vartheta_3(s_ix_i/y_i|q)\cdot\prod_{1\le i<j\le n}\frac{(qy_j/y_i;q)_\infty}{(ty_j/y_i;q)_\infty}f^{\mathfrak{gl}_n}(x|y|q,t)\right.\\
&\qquad\left.\cdot\prod_{1\le i<j\le n}(1-y_j/y_i)\cdot\left(\mathcal{D}^{+,\mathfrak{gl}_n}(y|s|q,t)f(y)\right)\right]_{1,y}\\
=&\mathcal{T}_x^{\mathfrak{gl}_n}(q,t)\cdot D_x^{\mathfrak{gl}_n}(q,t)\cdot x^\lambda f(x).
\end{aligned}
$$

□

6. Non-stationary Ruijsenaars function

[**LNS**] E. Langmann, M. Noumi and J. Shiraishi, Basic properties of non-stationary Ruijsenaars functions, `arXiv:2006.07171`.

[**S**] J. Shiraishi, Affine Screening Operators, Affine Laumon Spaces, and Conjectures Concerning Non-Stationary Ruijsenaars Functions, J. of Int. Systems **4** (2019), xyz010.

Let $n \in \mathbb{Z}_{\geq 2}$. Introduce the collections of independent indeterminates

$$(x,p) = (x_1, x_2, \dots, x_n, p), \qquad (s,\kappa) = (s_1, s_2, \dots, s_n, \kappa).$$

Extend the indices of x and s to $\mathbb{Z}$, assuming the cyclic identifications $x_{i+n} = x_i$ and $s_{i+n} = s_i$. Let ω be the permutation acting on (x,p) and (s,κ) by $\omega x_i = x_{i+1}, \omega p = p, \omega s_i = s_{i+1}, \omega\kappa = \kappa$. Denote bt P the set of partitions.

Definition 6.1. *For $k \in \mathbb{Z}/n\mathbb{Z}$, and $\lambda, \mu \in \mathsf{P}$, set*

$$\mathsf{N}^{(k|n)}_{\lambda,\mu}(u|q,\kappa) = \mathsf{N}^{(k)}_{\lambda,\mu}(u|q,\kappa)$$
$$= \prod_{\substack{j \geq i \geq 1 \\ j-i \equiv k \pmod n}} (uq^{-\mu_i+\lambda_{j+1}}\kappa^{-i+j};q)_{\lambda_j-\lambda_{j+1}} \cdot \prod_{\substack{\beta \geq \alpha \geq 1 \\ \beta-\alpha \equiv -k-1 \pmod n}} (uq^{\lambda_\alpha-\mu_\beta}\kappa^{\alpha-\beta-1};q)_{\mu_\beta-\mu_{\beta+1}}.$$

Note that the ordinary K-theoretic Nekrasov factor reads

$$\mathsf{N}_{\lambda,\mu}(u|q,\kappa) = \prod_{(i,j)\in\lambda}(1-uq^{-\mu_i+j-1}\kappa^{\lambda'_j-i}) \cdot \prod_{(k,l)\in\mu}(1-uq^{\lambda_k-l}\kappa^{-\mu'_l+k-1}),$$

or equivalently

$$\mathsf{N}_{\lambda,\mu}(u|q,\kappa) = \prod_{j\geq i\geq 1}(uq^{-\mu_i+\lambda_{j+1}}\kappa^{-i+j};q)_{\lambda_j-\lambda_{j+1}} \cdot \prod_{\beta\geq\alpha\geq 1}(uq^{\lambda_\alpha-\mu_\beta}\kappa^{\alpha-\beta-1};q)_{\mu_\beta-\mu_{\beta+1}}.$$

We have the factorization $\mathsf{N}_{\lambda,\mu}(u|q,\kappa) = \prod_{k=1}^n \mathsf{N}^{(k|n)}_{\lambda,\mu}(u|q,\kappa)$.

Definition 6.2. *Let $f^{\widehat{\mathfrak{gl}}_n}(x,p|s,\kappa|q,t)$ be the formal power series*

$$f^{\widehat{\mathfrak{gl}}_n}(x,p|s,\kappa|q,t) \in \mathbb{Q}(s,\kappa,q,t)[[px_2/x_1,\dots,px_n/x_{n-1},px_1/x_n]],$$

$$f^{\widehat{\mathfrak{gl}}_n}(x,p|s,\kappa|q,t) = \sum_{\lambda^{(1)},\dots,\lambda^{(n)}\in\mathsf{P}} \prod_{i,j=1}^n \frac{\mathsf{N}^{(j-i|n)}_{\lambda^{(i)},\lambda^{(j)}}(ts_j/s_i|q,\kappa)}{\mathsf{N}^{(j-i|n)}_{\lambda^{(i)},\lambda^{(j)}}(s_j/s_i|q,\kappa)} \cdot \prod_{\beta=1}^n\prod_{\alpha\geq 1}(px_{\alpha+\beta}/tx_{\alpha+\beta-1})^{\lambda^{(\beta)}_\alpha}.$$

We call $f^{\widehat{\mathfrak{gl}}_n}(x,p|s,\kappa|q,t)$ the non-stationary Ruijsenaars function.

Note that we have $\omega f^{\widehat{\mathfrak{gl}}_n}(x,p|s,\kappa|q,t) = f^{\widehat{\mathfrak{gl}}_n}(x,p|s,\kappa|q,t)$.

A simple calculation using the q-binomial formula [?] gives us the following factorization formula.

Proposition 6.3. *Setting $\kappa = 0$, we have*

$$f^{\widehat{\mathfrak{gl}}_n}(x,p|s,0|q,t) = \prod_{1\le i<j\le n} \frac{(p^{j-i}qx_j/x_i;q,p^n)_\infty}{(p^{j-i}tx_j/x_i;q,p^n)_\infty} \cdot \prod_{1\le i\le j\le n} \frac{(p^{n-j+i}qx_i/x_j;q,p^n)_\infty}{(p^{n-j+i}tx_i/x_j;q,p^n)_\infty}.$$

Dividing $f^{\widehat{\mathfrak{gl}}_n}(x,p|s,\kappa|q,t)$ by $f^{\widehat{\mathfrak{gl}}_n}(x,p|s,0|q,t)$, we introduce the normalized version $\varphi^{\widehat{\mathfrak{gl}}_n}(x,p|s,\kappa|q,t)$ as follows.

Definition 6.4. *Let $\varphi^{\widehat{\mathfrak{gl}}_n}(x,p|s,\kappa|q,t)$ be the formal power series*

$$\varphi^{\widehat{\mathfrak{gl}}_n}(x,p|s,\kappa|q,t) \in \mathbb{Q}(q,t)[[px_2/x_1,\dots,px_n/x_{n-1},px_1/x_n, \kappa s_2/s_1,\dots,\kappa s_n/s_{n-1},\kappa s_1/s_n]],$$

$$\varphi^{\widehat{\mathfrak{gl}}_n}(x,p|s,\kappa|q,t) = \prod_{1\le i<j\le n} \frac{(p^{j-i}tx_j/x_i;q,p^n)_\infty}{(p^{j-i}qx_j/x_i;q,p^n)_\infty} \cdot \prod_{1\le i\le j\le n} \frac{(p^{n-j+i}tx_i/x_j;q,p^n)_\infty}{(p^{n-j+i}qx_i/x_j;q,p^n)_\infty} \cdot f^{\widehat{\mathfrak{gl}}_n}(x,p|s,\kappa|q,t),$$

where the coefficients $\prod_{i,j=1}^n \mathsf{N}^{(j-i|n)}_{\lambda^{(i)},\lambda^{(j)}}(ts_j/s_i|q,\kappa)/\mathsf{N}^{(j-i|n)}_{\lambda^{(i)},\lambda^{(j)}}(s_j/s_i|q,\kappa)$ in $f^{\widehat{\mathfrak{gl}}_n}(x,p|s,\kappa|q,t)$ are Taylor expanded in κ at $\kappa = 0$.

We have $\omega\varphi^{\widehat{\mathfrak{gl}}_n}(x,p|s,\kappa|q,t) = \varphi^{\widehat{\mathfrak{gl}}_n}(x,p|s,\kappa|q,t)$.

Conjecture 6.5. *We have the duality properties*

$$\varphi^{\widehat{\mathfrak{gl}}_n}(x,p|s,\kappa|q,t) = \varphi^{\widehat{\mathfrak{gl}}_n}(s,\kappa|x,p|q,t) \qquad \text{(bispectral duality)},$$

$$\varphi^{\widehat{\mathfrak{gl}}_n}(x,p|s,\kappa|q,t) = \varphi^{\widehat{\mathfrak{gl}}_n}(x,p|s,\kappa|q,q/t) \qquad \text{(Poincaré duality)}.$$

The affine Laumon space [FFNR] is the moduli space $\mathcal{P}_{\underline{d}}$ of parabolic sheaves (or infinite flag of torsion free coherent sheaves of rank n)

$$\cdots \subset \mathcal{F}_{-1} \subset \mathcal{F}_0 \subset \mathcal{F}_1 \cdots$$

with certain prescribes conditions.

Proposition 6.6. *The Euler characteristic* $\mathfrak{J}_{\underline{d}}(s,\kappa|q,t) := [H^\bullet(\mathcal{P}_{\underline{d}}, \Omega^\bullet_{\mathcal{P}_{\underline{d}}})]$ *of the de Rham complex on* $\mathcal{P}_{\underline{d}}$ *is given via the Atiyah-Bott-Lefschetz localization technique as*

$$\mathfrak{J}_{\underline{d}}(s,\kappa|q,t) = \sum_{i,j} (-1)^{i+j} t^j [H^i(\mathcal{P}_{\underline{d}}, \Omega^j_{\mathcal{P}_{\underline{d}}})] = \sum_{\substack{\boldsymbol{\lambda} \\ \underline{d}=\underline{d}(\boldsymbol{\lambda})}} \prod_{i,j=1}^{n} \frac{\mathsf{N}^{(j-i|n)}_{\lambda^{(i)},\lambda^{(j)}}(s_j/ts_i|q,\kappa)}{\mathsf{N}^{(j-i|n)}_{\lambda^{(i)},\lambda^{(j)}}(s_j/s_i|q,\kappa)}.$$

Hence, the non-stationary Ruijsenaars function is the generating function for the Euler characteristics of the affine Laumon spaces

$$f^{\widehat{\mathfrak{gl}}_N}(x,p|s,\kappa|q,1/t) = \sum_{\underline{d}} \mathfrak{J}_{\underline{d}}(s,\kappa|q,t) \prod_{i=1}^{N} (ptx_{i+1}/x_i)^{d_i}.$$

[FFNR] B. Feigin, M. Finkelberg, A. Negut and L. Rybnikov, Yangians and cohomology ring of Laumon spaces, Sel. Math. New. Ser. (2011) **17**:573-607, DOI 10.1007/s00029-011-0059-x.

7. IRREDUCIBLE AFFINE CHARACTERS (SHUR LIMIT $q = t$)

The Schur polynomials are obtained from the Macdonald polynomials by taking the limit $t \to q$. In the same manner, we have the $\widehat{\mathfrak{sl}}_n$ dominant integrable characters (up to the character of $\widehat{\mathfrak{gl}}_1$) from $f^{\widehat{\mathfrak{gl}}_n}(x,p|s,\kappa|q,q/t)$ by considering the limit $t \to q$. Set $\delta = (n-1, n-2, \ldots, 1, 0)$. Here and hereafter, we use the standard notation as $t^\delta s = (t^{n-1}s_1, t^{n-2}s_2, \ldots, ts_{n-1}, s_n)$.

Definition 7.1. *Let K be a nonnegative integer. We call K the level. Let $\mu = (\mu_1, \ldots, \mu_n)$ be a partition satisfying the condition $K + \mu_n - \mu_1 \geq 0$. Then set*

$$s = (\kappa t)^\delta q^\mu = q^{-K\delta/n+\mu}, \qquad \kappa = q^{-K/n}t^{-1}.$$

i.e. for s, we set $s_i = q^{-K(n-i)/n+\mu_i}$ $(1 \leq i \leq N)$.

For such K and μ, we have the level K dominant integrable weight $\Lambda(K,\mu) = (K + \mu_n - \mu_1)\Lambda_0 + \sum_{i=1}^{n-1}(\mu_i - \mu_{i+1})\Lambda_i$, and the dominant integrable representation $L(\Lambda(K,\mu))$ of $\widehat{\mathfrak{sl}}_n$, where $\Lambda_0, \ldots, \Lambda_{n-1}$ denote the fundamental weights. Denote by $\mathrm{ch}^{\widehat{\mathfrak{sl}}_n}_{L(\Lambda(K,\mu))}$ the character of $L(\Lambda(K,\mu))$ associated with the principal gradation.

Theorem 7.2. *Let K, μ, s, κ be fixed as above. We have*

$$\lim_{t\to q} x^\mu f^{\widehat{\mathfrak{gl}}_n}(x,p|q^{-K\delta/n+\mu}, q^{-K/n}t^{-1}|q,q/t) = \frac{1}{(p^n;p^n)_\infty} \cdot \mathrm{ch}^{\widehat{\mathfrak{sl}}_n}_{L(\Lambda(K,\mu))}.$$

Note that the factor $1/(p^n;p^n)_\infty$ is interpreted as the $\widehat{\mathfrak{gl}}_1$ character. A proof of this is based on the affine Gelfand-Tsetlin pattern obtained in [FFNR], which we can regard as Tingley's $\widehat{\mathfrak{sl}}_N$-crystal [T].

[T] P. Tingley, Three Combinatorial Models for $\widehat{sl}_n$ Crystals, with Applications to Cylindric Plane Partitions, Int. Math. Res. Not. Article ID rnm 143, 41 pages, doi: 10.1093/imrn/rnm143.

8. Ruijsenaars' eigenvalue equation

Now, we turn to the eigenvalue problem associated with the elliptic Ruijsenaars operator [[?],]R], from the point of view of the series $f^{\widehat{\mathfrak{gl}}_n}(x,p|s,\kappa|q,t)$. We use the multiplicative notation for the elliptic theta function as $\Theta_p(z) = (z;p)_\infty (p/z;p)_\infty (p;p)_\infty$.

Definition 8.1. *Let $D_x(p) = D_x(p|q,t)$ denotes the Ruijsenaars operator*

$$D_x(p) = \sum_{i=1}^{n} \prod_{j \neq i} \frac{\Theta_p(tx_i/x_j)}{\Theta_p(x_i/x_j)} T_{q,x_i},$$

where T_{q,x_i} is the q-shift operator $q^{x_i \partial/\partial x_i}$.

[R] R.N.M. Ruijsenaars, Complete inegrability of relativistic Calogero-Moser systems and elliptic function identities, Commun. Math. Phys. **110** (1987) 191-213.

Naively speaking, we take the "stationary limit $\kappa \to 1$ of $f^{\widehat{\mathfrak{gl}}_n}(x,p|s,\kappa|q,t)$". Such a limit, however, does not exists. It seems that we need to normalize $f^{\widehat{\mathfrak{gl}}_n}$, before taking the limit $\kappa \to 1$. The simplest way might be to divide $f^{\widehat{\mathfrak{gl}}_n}$ by its constant term in x.

We closely follow the method developed in Atai and Langmann for the non-stationary Heun and Lamé equations. Let $\boldsymbol{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(n)})$ be an N-tuple of partitions. Set

$$|\boldsymbol{\lambda}| = \sum_{i=1}^{n} |\lambda^{(i)}|, \qquad m_i = m_i(\boldsymbol{\lambda}) = \sum_{\beta=1}^{n} \sum_{\substack{\alpha \geq 1 \\ \alpha+\beta \equiv i \pmod{n}}} \lambda_\alpha^{(\beta)} - \lambda_\alpha^{(\beta+1)}.$$

Then we have $\prod_{\beta=1}^{n} \prod_{\alpha \geq 1} (px_{\alpha+\beta}/tx_{\alpha+\beta-1})^{\lambda_\alpha^{(\beta)}} = (p/t)^{|\boldsymbol{\lambda}|} \prod_{i=1}^{n} x_i^{m_i}$. Note that when $m_1 = \cdots = m_N = 0$, we have $|\boldsymbol{\lambda}| \equiv 0 \pmod{n}$.

Definition 8.2. *Let $\alpha(p|s,\kappa|q,t) = \sum_{d \geq 0} p^{nd} \alpha_d(s,\kappa|q,t)$ be the constant term of the series $f^{\widehat{\mathfrak{gl}}_n}(x,p|s,\kappa|q,t)$ with respect to x_i's. Namely,*

$$\alpha(p|s,\kappa|q,t) = \sum_{\substack{\lambda^{(1)},\ldots,\lambda^{(n)} \in \mathsf{P} \\ m_1 = \cdots = m_N = 0}} (p/t)^{|\boldsymbol{\lambda}|} \prod_{i,j=1}^{n} \frac{\mathsf{N}^{(j-i|n)}_{\lambda^{(i)},\lambda^{(j)}}(ts_j/s_i|q,\kappa)}{\mathsf{N}^{(j-i|n)}_{\lambda^{(i)},\lambda^{(j)}}(s_j/s_i|q,\kappa)}.$$

Conjecture 8.3. *We have the properties:*

(1) *The series* $f^{\widehat{\mathfrak{gl}}_n}(x,p|s,\kappa|q,t)$ *is convergent on a certain domain. With respect to* κ*, it is regular on a certain punctured disk* $\{\kappa\in\mathbb{C}||\kappa-1|<r,\kappa\neq 1\}$.

(2) *The* $f^{\widehat{\mathfrak{gl}}_n}(x,p|s,\kappa|q,t)$ *and* $\alpha(p|s,\kappa|q,t)$ *are essential singular at* $\kappa=1$*. (The coefficient* $\alpha_d(s,\kappa|q,t)$ *has a pole of degree* d *in* κ *at* $\kappa=1$*.)*

(3) *The ratio* $f^{\widehat{\mathfrak{gl}}_n}(x,p|s,\kappa|q,t)/\alpha(p|s,\kappa|q,t)$ *is regular at* $\kappa=1$*.*

Definition 8.4. *Assuming the above conjecture, set*

$$f^{\mathrm{st}.\widehat{\mathfrak{gl}}_n}(x,p|s|q,t)=\left.\frac{f^{\widehat{\mathfrak{gl}}_n}(x,p|s,\kappa|q,t)}{\alpha(p|s,\kappa|q,t)}\right|_{\kappa=1}.$$

We call $f^{\mathrm{st}.\widehat{\mathfrak{gl}}_n}(x,p|s|q,t)$ *the stationary Ruijsenaars function.*

Conjecture 8.5. *Let* $s=t^{\delta}q^{\lambda}$ $(s_i=t^{n-i}q^{\lambda_i})$*. Denote by* $p^{\delta/n}x$ *the collection of the shifted coordinates* $p^{(n-i)/n}x_i$*. The stationary Ruijsenaars function* $x^{\lambda}f^{\mathrm{st}.\widehat{\mathfrak{gl}}_N}(p^{\delta/N}x,p^{1/N}|s|q,q/t)$ *is an eigenfunction of the Ruijsenaars operator:*

$$D_x(p)\,x^{\lambda}f^{\mathrm{st}.\widehat{\mathfrak{gl}}_n}(p^{\delta/n}x,p^{1/n}|s|q,q/t)=\varepsilon(p|s|q,t)\,x^{\lambda}f^{\mathrm{st}.\widehat{\mathfrak{gl}}_n}(p^{\delta/n}x,p^{1/n}|s|q,q/t),$$

$$\varepsilon(p|s|q,t)=\sum_{i=1}^{n}s_i+\sum_{d>0}\varepsilon_d(s|q,t)p^d.$$

9. Operator $\mathcal{T}_x^{\widehat{\mathfrak{gl}}_n}$

Definition 9.1. *Introduce the operator* $\mathcal{T}_x^{\widehat{\mathfrak{gl}}_n} = \mathcal{T}_x^{\widehat{\mathfrak{gl}}_n}(q,t)$ *as*

$$\mathcal{T}_x^{\widehat{\mathfrak{gl}}_n} = \sum_{\lambda^{(1)},\dots,\lambda^{(n)}\in\mathsf{P}} \prod_{\beta=1}^{n}\prod_{\alpha\geq 1}(ptx_{\alpha+\beta}/qx_{\alpha+\beta-1})^{\lambda_\alpha^{(\beta)}} \cdot q^{\frac{1}{2}\Delta}T_{\kappa,p}\cdot \prod_{i,j=1}^{n}\frac{\mathsf{N}^{(j-i|n)}_{\lambda^{(i)},\lambda^{(j)}}(qx_j/tx_i|q,p)}{\mathsf{N}^{(j-i|n)}_{\lambda^{(i)},\lambda^{(j)}}(x_j/x_i|q,p)}$$
$$\cdot \prod_{1\leq i<j\leq n}\frac{(p^{j-i}x_j/x_i;q,p^n)_\infty}{(p^{j-i}tx_j/x_i;q,p^n)_\infty}\cdot \prod_{1\leq i\leq j\leq n}\frac{(p^{n-j+i}x_i/x_j;q,p^n)_\infty}{(p^{n-j+i}tx_i/x_j;q,p^n)_\infty}.$$

Remark 9.2. *Note that the duality conjecture implies that we can rewrite* $\mathcal{T}_x^{\widehat{\mathfrak{gl}}_n}(q,t)$ *as*

$$\mathcal{T}_x^{\widehat{\mathfrak{gl}}_n} = \prod_{1\leq i<j\leq n}\frac{(p^{j-i}qx_j/x_i;q,p^n)_\infty}{(p^{j-i}qx_j/tx_i;q,p^n)_\infty}\cdot \prod_{1\leq i\leq j\leq n}\frac{(p^{n-j+i}qx_i/x_j;q,p^n)_\infty}{(p^{n-j+i}qx_i/tx_j;q,p^n)_\infty}$$
$$\cdot \sum_{\lambda^{(1)},\dots,\lambda^{(n)}\in\mathsf{P}} \prod_{i,j=1}^{n}\frac{\mathsf{N}^{(j-i|n)}_{\lambda^{(i)},\lambda^{(j)}}(tx_j/x_i|q,p)}{\mathsf{N}^{(j-i|n)}_{\lambda^{(i)},\lambda^{(j)}}(x_j/x_i|q,p)}\cdot q^{\frac{1}{2}\Delta}T_{\kappa,p}\cdot \prod_{\beta=1}^{n}\prod_{\alpha\geq 1}(px_{\alpha+\beta}/tx_{\alpha+\beta-1})^{\lambda_\alpha^{(\beta)}}$$
$$\cdot \prod_{1\leq i<j\leq n}(p^{j-i}x_j/x_i;p^n)_\infty\cdot \prod_{1\leq i\leq j\leq n}(p^{n-j+i}x_i/x_j;p^n)_\infty.$$

Conjecture 9.3. *We have*

$$\mathcal{T}_x^{\widehat{\mathfrak{gl}}_n}\cdot x^\lambda f^{\widehat{\mathfrak{gl}}_n}(x,p|s,\kappa|q,q/t) = \varepsilon(\lambda)\cdot x^\lambda f^{\widehat{\mathfrak{gl}}_n}(x,p|s,\kappa|q,q/t).$$