Bosonic lattice models and honeycombs for Grothendieck polynomials

P. Zinn-Justin

School of Mathematics and Statistics, the University of Melbourne

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Grothendieck polynomials

- Lascoux and Schützenberger introduced Grothendieck polynomials in connection with the K-theory of flag varieties.
- Grothendieck polynomials appear as representatives of classes of (structure sheaves of) Schubert varieties.
- If one considers Grassmannians instead of general flag varieties, one obtains a subset of Grothendieck polynomials which are symmetric.
- Recently, there has been a resurgence of interest in this family of polynomials, viewed as (one of the many) deformations of Schur polynomials.
- The underlying geometry automatically associates to these polynomials a solution of the Yang–Baxter equation, and therefore a quantum integrable/solvable lattice model.
- Today I want to focus on alternate solvable lattice models for these polynomials and their duals.

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K-theory of Grassmannians

Let $Gr(k,n) = \{k\text{-spaces in } \mathbb{C}^n\}$. It has a natural action of $GL(n)$ and in particular of its Cartan torus T. $K_T(Gr(k,n))$ is a commutative algebra over $K_T(\cdot) = \mathbb{Z}[t_1^{\pm}, \ldots, t_n^{\pm}]$.

It is a free module over $K_T(\cdot)$ with a privileged basis, the Schubert basis: S_λ where λ runs overs Young diagrams inside $k \times (n-k)$.

Weyl group action

There is a natural action of the Weyl group $W = N_T/T \cong \mathcal{S}_n$ on $K_T(Gr(k,n))$ (being careful that it acts on the base ring $K_T(\cdot)$ too).

Generators of W (elementary transpositions) acting on $K_T(Gr(k,n))$ are denoted $\check{R}_i$, $i = 1,\ldots,n-1$, and called R-matrices; they are $K_T(\cdot)$-valued matrices, and $\check{R}_i$ only depends on t_i/t_{i+1}. Collectively, they satisfy the Yang–Baxter equation:

$$\check{R}_i(t_{i+1}/t_{i+2})\check{R}_{i+1}(t_i/t_{i+2})\check{R}_i(t_i/t_{i+1}) \\ = \check{R}_{i+1}(t_i/t_{i+1})\check{R}_i(t_i/t_{i+2})\check{R}_{i+1}(t_{i+1}/t_{i+2})$$

Note: difference property

Nilhecke and 5-vertex solution of YBE

For general flag varieties we would obtain this way the solution of YBE associated with the nilHecke algebra. For Grassmannians, this takes the particular form of the 5-vertex model:

$$\check{R}_i(t_i/t_{i+1}) = 1 \otimes \cdots \otimes 1 \otimes \begin{pmatrix} 1 & & & \\ & t_i/t_{i+1} & 0 & \\ & 1 - t_i/t_{i+1} & 1 & \\ & & & 1 \end{pmatrix}_{i,i+1} \otimes 1 \otimes \cdots \otimes 1$$

acting on $\bigoplus_{k=0}^{n} K_T(Gr(k,n)) \cong (K_T(\cdot)^2)^{\otimes n}$.

Bethe Ansatz and pipe dreams

Bethe Ansatz provides formulae for Schubert classes, and therefore (via some stability) for double Grothendieck polynomials. For general flag varieties, one recovers pipe dreams [Billey Jockush Stanley Fomin Kirillov Knutson Miller].

In the case of Grassmannians, one finds formulae for symmetric Grothendieck polynomials G_λ in terms of "fermionic" lattice paths [Motegi Sakai '13].

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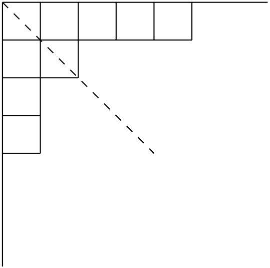

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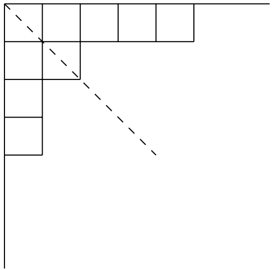

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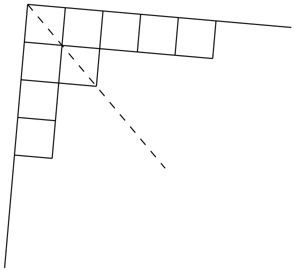

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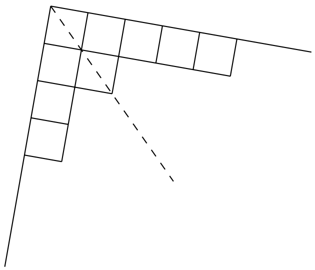

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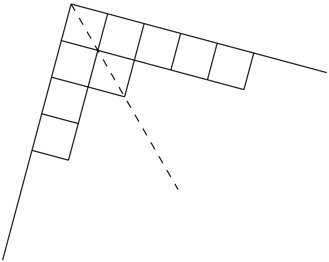

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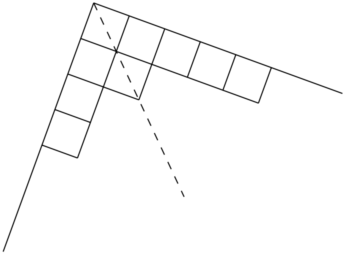

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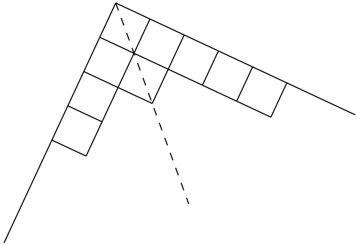

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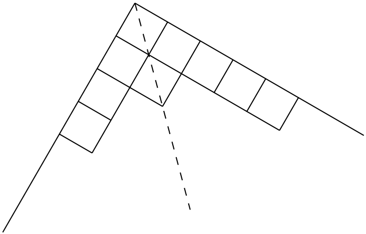

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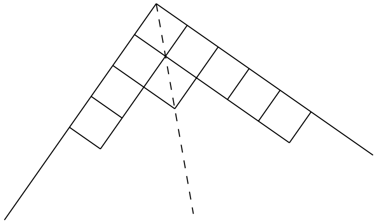

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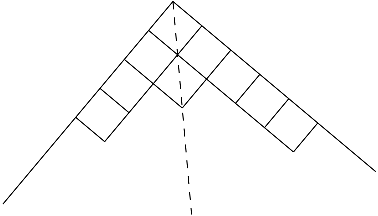

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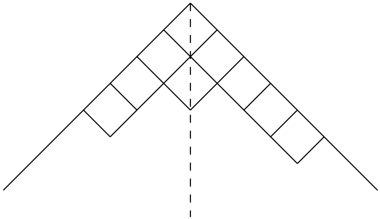

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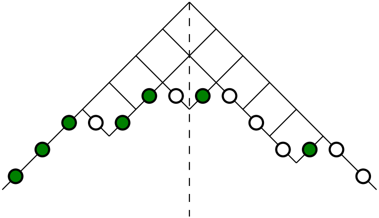

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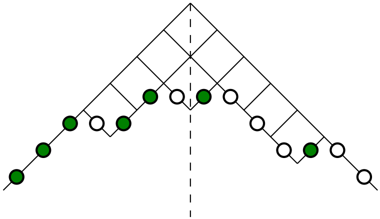

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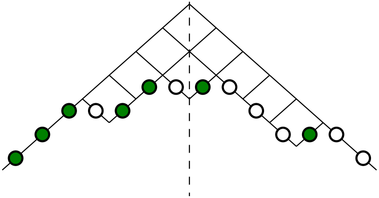

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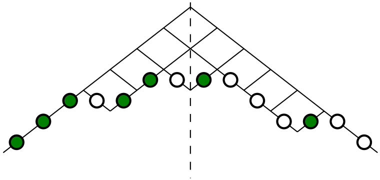

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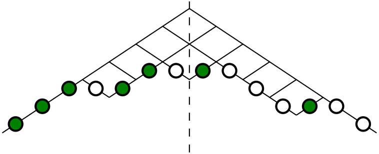

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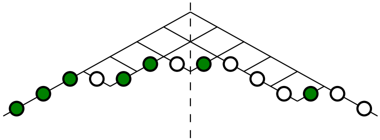

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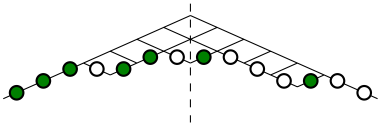

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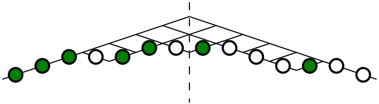

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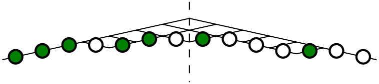

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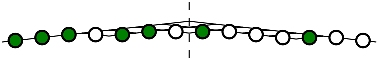

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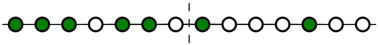
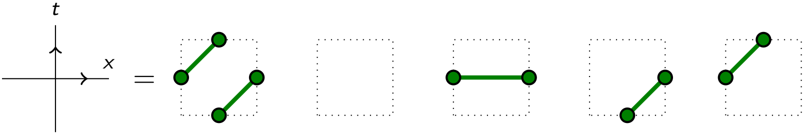


Grothendieck polynomials from 5-vertex model



$1 \qquad 1 \qquad 1 - x/t \qquad x/t \qquad 1$

$G_\lambda = \sum_{\text{paths}}$

$\lambda =$

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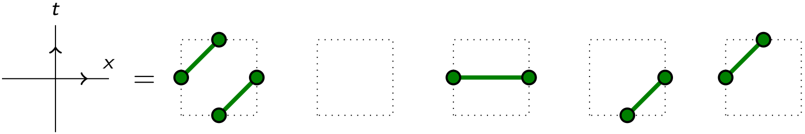

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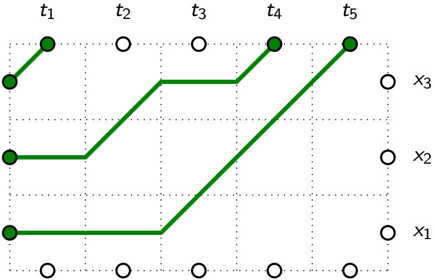

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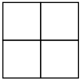
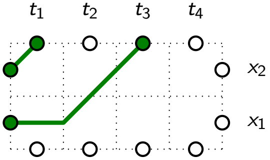


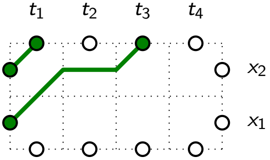
Grothendieck polynomials from 5-vertex model cont'd

Example.

$G_\square =$



$+$



$$\overset{t_i=1}{=} x_2(1-x_1)+1-x_2$$

$$= 1 - x_1 x_2$$

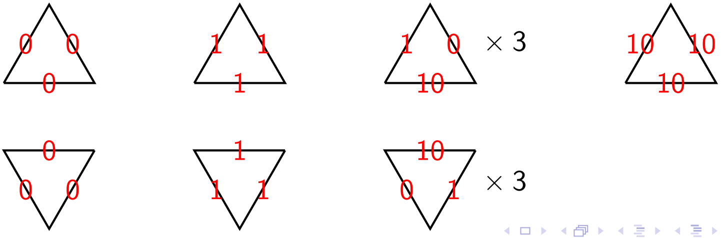
Puzzles

One can also investigate the product structure of $K_T(Gr(k,n))$ using integrability, according to the general framework of [Knutson Z-J '17].

This naturally leads to puzzles:

- First introduced by Knutson and Tao for $H_T(Gr(k,n))$ ('03)
- generalized to $K(Gr(k,n))$ by Buch ('02) and Vakil ('16)
- and to $K_T(Gr(k,n))$ by Pechenik Yong and Wheeler Z-J ('16)

Example. $G_{\square}^2 = G_{\square\square} + G_{\substack{\square\\\square}} - G_{\substack{\square\square\\\square}}$



Motivation

- K-theory has a natural scalar product $\to$ dual basis. The same integrable system describes them (rotate pictures 180 degrees!)
- From an "algebraic combinatorics" point of view, there is another, more natural scalar product on symmetric functions (Hall inner product) which switches product and coproduct. (here we're thinking of symmetric Grothendieck polynomials as elements of some completion of Λ)
- In $H_*(Gr)$, where Grothendieck $\to$ Schur, these two scalar products are closely related. Not so in $K(Gr)$.
- Is there a natural quantum integrable framework which incorporates both Grothendieck polynomials and their duals, preserving the difference property?
- Similarly, what is the analogue of puzzles for coproduct structure constants?

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Parameterization

In order to define dual Grothendieck polynomials, we need to move away from the geometrically natural parameterization.
Requiring $G_\lambda = s_\lambda +$ higher order terms leaves one nontrivial parameter, though it is convenient to keep a trivial scaling parameter as well:

$$G_\lambda^{(\alpha,\beta)}(x_1,\ldots,x_n) = (-(\alpha+\beta))^{-|\lambda|} G_\lambda \left(\frac{1+\beta x_1}{1-\alpha x_1}, \ldots, \frac{1+\beta x_n}{1-\alpha x_n} \right)$$

It was noticed in [Yeliussizov '17] that

$$\omega(G_\lambda^{(\alpha,\beta)}) = G_{\lambda'}^{(\beta,\alpha)}$$

where ω is the involution that sends s_λ to $s_{\lambda'}$.

Remark. Reparameterization affects neither product nor coproduct.

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Dual Grothendieck polynomials

Let $g_\lambda^{(\alpha,\beta)}$ be the dual basis of the $G_\lambda^{(-\alpha,-\beta)}$:

$$\left\langle g_\lambda^{(\alpha,\beta)}, G_\mu^{(-\alpha,-\beta)} \right\rangle = \delta_{\lambda,\mu}$$

for the Hall inner product $\langle s_\lambda, s_\mu \rangle = \delta_{\lambda,\mu}$. $g_\lambda^{(\alpha,\beta)} = s_\lambda +$ lower order terms.

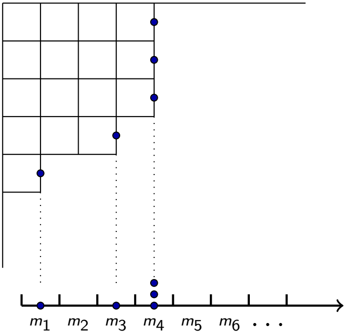
Examples. Starting from $G_\square = 1 - \prod_i x_i$, one can deduce

$$G_\square^{(\alpha,\beta)} = \frac{1}{-(\alpha+\beta)} \left(1 - \prod_i \frac{1+\beta x_i}{1-\alpha x_i} \right) = \sum_{k,\ell \geq 0} \alpha^k \beta^\ell s_{\begin{smallmatrix} \square & \overbrace{\square\square\square}^{k} \\ \ell\left\{\begin{smallmatrix}\square\\\square\end{smallmatrix}\right. & \end{smallmatrix}}$$

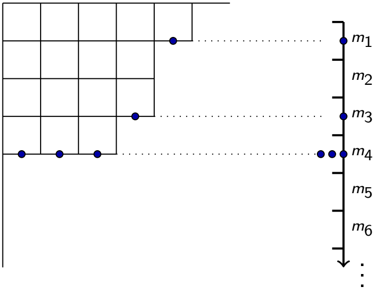
$$g_\square^{(\alpha,\beta)} = \sum_i x_i = s_\square$$

$$g_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}^{(\alpha,\beta)} = s_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} + \beta s_\square \qquad g_{\square\square}^{(\alpha,\beta)} = s_{\square\square} + \alpha s_\square$$

Bosonic encoding of Young diagrams

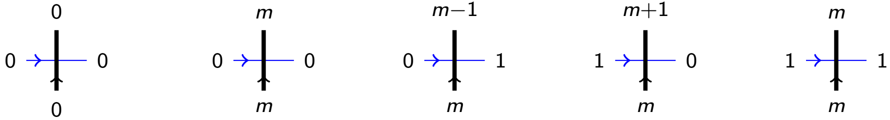


or



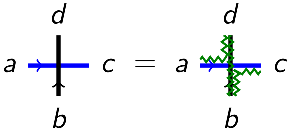
Lattice models for $G^{(\alpha,\beta)}$

Row model:



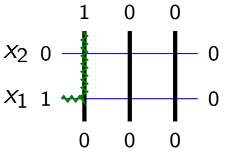
$$1 \qquad \frac{1+\beta x}{1-\alpha x} \qquad \frac{1+\beta x}{1-\alpha x} \qquad \frac{x}{1-\alpha x} \qquad \frac{x}{1-\alpha x}$$

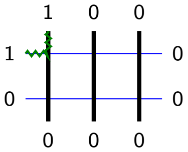
Column model:



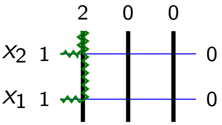
$$= \delta_{a+b,c+d} \begin{cases} \left(\frac{x}{1-\alpha x}\right)^a & b=c \\ \left(\frac{x}{1-\alpha x}\right)^a \left(\frac{1+\beta x}{1-\alpha x}\right) & b>c \\ 0 & b<c \end{cases}$$

Examples for $G^{(0,-1)}$

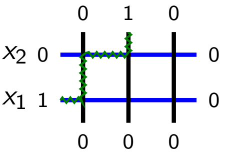
$G_{\square}(x_1, x_2) =$ 

$+$ 

$= x_1(1 - x_2) + x_2$
$= 1 - (1 - x_1)(1 - x_2)$

$G_{\substack{\square\\\square}}(x_1, x_2) =$ 

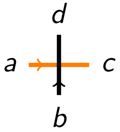
$= x_1 x_2$

$=$ 

$= x_1 x_2$

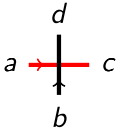
Lattice models for $g^{(\alpha,\beta)}$

Row model:



$$= \delta_{a+b,c+d} \begin{cases} (\alpha+\beta)^{a-d-1}(x+\alpha)\beta^d & a > d \\ \beta^{a-1}x & 0 < a \leq d \\ 1 & a = 0 \end{cases}$$

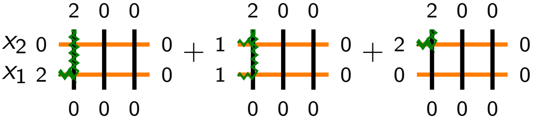
Column model:



$$= \delta_{a+b,c+d} \begin{cases} (\alpha+\beta)^{a-d-1}\beta(x+\alpha)^d & a > d \\ x(x+\alpha)^{a-1} & 0 < a \leq d \\ 1 & a = 0 \end{cases}$$

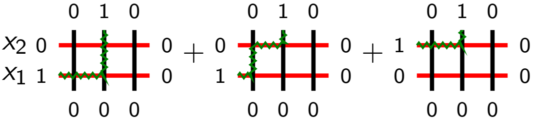
Examples for $g^{(\alpha,\beta)}$

$$g_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}(x_1, x_2) =$$



$$= \beta x_1 + x_1 x_2 + \beta x_2$$

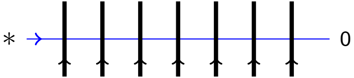
$$=$$



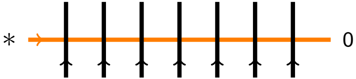
Commutation relations and Cauchy identity

Yang–Baxter equation (RLL relations) ⇒ commutation relations for transfer matrices ⇒ Cauchy identities:

$T(x) =$



$t(x) =$



$$t(y)\,T^*(x) = \frac{1}{1-xy}\,T^*(x)t(y)$$

$$\sum_\lambda G_\lambda^{(-\alpha,-\beta)}(x_1,\ldots,x_m)g_\lambda^{(\alpha,\beta)}(y_1,\ldots,y_n)$$

$$= \langle 0|t(y_1)\ldots t(y_n)\,T^*(x_1)\ldots\,T^*(x_m)|0\rangle = \prod_{i,j}\frac{1}{1-x_iy_j}$$

Inversion relations

Transfer matrices for row and column models are related by inversion relations:

$$T(-x)\,\widetilde{T}\left(\frac{x}{1+(\alpha-\beta)x}\right)=1$$

and for $\alpha=1,\beta=0$:

$$t(-x)\tilde{t}(x)=1$$

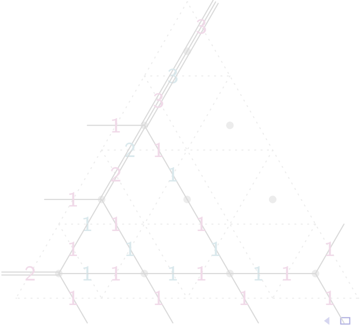
They realize the involution ω at the level of the lattice models:

$$t(y)\,T^*(x)=\frac{1}{1-xy}\,T^*(x)t(y)\quad\Longleftrightarrow\quad\tilde{t}(y)\,T^*(x)=(1+xy)\,T^*(x)\tilde{t}(y)$$

Honeycombs as bosonic puzzles

According to the general philosophy of [Knutson Z-J '17], to a family of polynomials based on bosonic (infinite spin Verma module) representations of $\mathfrak{sl}(2)$ should be associated a product rule based on bosonic (parabolic Verma module) representations of $\mathfrak{sl}(3)$.

This program was successfully realized in [Z-J '20] for Hall–Littlewood polynomials; the bosonic analogue of puzzles turns out to be honeycombs:



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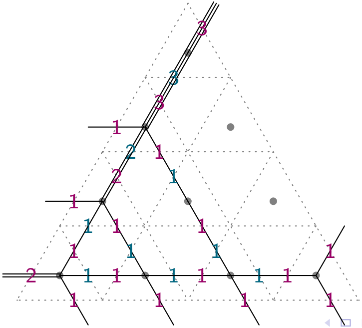
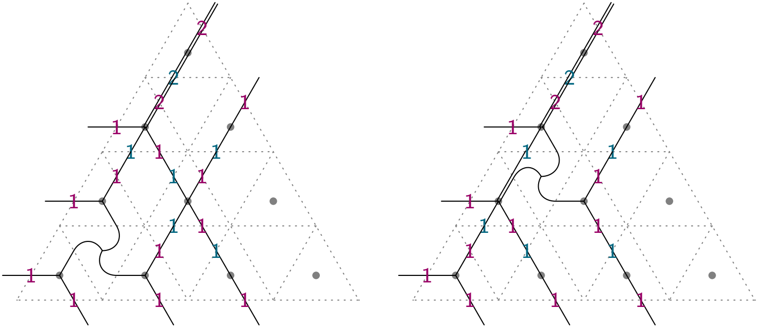


Generalised honeycombs

Honeycombs are not the most general model based on parabolic Verma modules of $\mathfrak{sl}(3)$, even in the infinite spin limit $\to$ generalised honeycombs:

$$g_{\yng(2,1)}\, g_{\yng(1)} = g_{\yng(3,1)} + g_{\yng(2,2)} + g_{\yng(2,1,1)} - 2g_{\yng(2,1)}$$



Remark. Bijection with (not quite local) puzzles of [Pylyavskyy Yang]?