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POSTECH

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Quantum groups and relative Langlands

Lecture 1 (on the dual group)

today: - explain a result of Sakellaridis on a relative version of the Satake iso

- explain how to quantize it in a a special case

Let $\underline{G}$ be a split connected reductive group (say over $\mathbb{Z}$ or $\mathcal{O}$)

Fix a Borel $\underline{B}$ with maximal torus $\underline{A}$, $\underline{B} = \underline{A}\,\underline{U}$

$\rightsquigarrow$ based root datum $(X^*(A), \Phi_+, X_*(A), \Phi_+^\vee)$

$\rightsquigarrow$ dual root datum $(X_*(A), \Phi_+^\vee, X^*(A), \Phi_+)$

$$X_*(A) = \Lambda^\vee$$

$\rightsquigarrow$ dual group $\underline{G}^\vee$

with torus $\underline{A}^\vee \subset \underline{B}^\vee$

W: finite Weyl group of G (or $G^\vee$)

examples

G	$G^\vee$
GL_n	GL_n
SL_n	PGL_n
Sp_{2n}	SO_{2n+1}
SO_{2n}	SO_{2n}

From now on $G = \underline{G}(F)$ $B = \underline{B}(F)$ etc. $G^\vee = \underline{G}^\vee(\mathbb{C})$ etc.

here F is a local non-archimedean field

$\mathcal{O}$ its ring of integers, $\mathbb{F}_q$ residue fied, π uniformizer

$K = \underline{G}(\mathcal{O})$ maximal compact subgroup

$I = p^{-1}(B(\mathbb{F}_q))$ Iwahori, where $p: G(\mathcal{O}) \to G(\mathbb{F}_q)$

spherical Hecke algebra

$$\mathcal{H}(G,K) := C_c^\infty(K\backslash G / K) = \{ f : G \to \mathbb{C},\ f(k_1 g k_2) = f(g)$$

$$f \text{ locally constant, compact support} \}$$

- algebra via convolution
- Cartan decomposition $G = \coprod_{\lambda \in \Lambda^\vee_+} K \pi^\lambda K \rightsquigarrow \mathbb{1}_{K\pi^\lambda K}$ basis of $\mathcal{H}(G,K)$

Satake isomorphism

$$\mathcal{H}(G,K) \simeq \mathbb{C}[\Lambda^\vee]^W \simeq \mathbb{C}[A^\vee]^W \simeq \mathbb{C}[\operatorname{Rep} G^\vee]$$

here $\mathbb{C}[\Lambda^\vee]$ = group algebra of $\Lambda^\vee$, basis $\{e^{\lambda^\vee}\}_{\lambda^\vee \in \Lambda^\vee}$

$\mathbb{C}[A^\vee]$ = algebraic functions on $\underline{A^\vee}$ over $\mathbb{C}$

$\mathbb{C}[\operatorname{Rep} G^\vee] = \mathbb{C} \otimes_{\mathbb{Z}} K_0(\operatorname{Rep} G^\vee)$, multiplication given by $\otimes$

maps: left to middle   Satake transform

$$Sf(t) := \delta^{1/2}(t) \int_U f(tn)\,dn \in \mathbb{C}[A/A_0] \simeq \mathbb{C}[\Lambda^\vee]$$

right to middle: taking characters

Remark 1: in order to prove Satake, it is useful to understand $\mathcal{H}(G,I) \simeq H_0 \otimes \mathbb{C}[\Lambda^\vee]$

$\hookrightarrow$ basis θ_λ

$\hookrightarrow$ finite Hecke algebra

+ Bernstein relations

then $\mathbb{1}_K * \mathcal{H}(G,I) * \mathbb{1}_K \simeq \mathcal{H}(G,K)$

Remark 2: if $V \in \mathrm{Irr}(G)$ then $V \to V^K$ induces bijection

$$\left\{ \begin{array}{c} \text{unramified} \\ \text{objects in } \mathrm{Irr}(G) \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{irred f.d.} \\ \mathcal{H}(G,K)\text{-mod} \end{array} \right\}$$

$\rightsquigarrow$ implies a (very basic) instance of LLC.

Borel-Casselman: $V \to V^I$ induces equivalence of cat

$$\left\{ \text{reps } V \text{ of } G \text{ gen by } V^I \right\} \longleftrightarrow \left\{ \text{f.d. reps of } \mathcal{H}(G,I) \right\}$$

let $\underline{X} = \underline{H} \backslash \underline{G}$ be a homogeneous spherical variety

$X = \underline{X}(F)$, $C_c^\infty(X)$ = smooth c.s. functions on X.

$\rightsquigarrow$ $C_c^\infty(X)^K$ is a module of $\mathcal{H}(G,K)$.

relative version of Satake

(for a nice class of X) there is a dual group

$G_X^\vee \subset G^\vee$ with torus $A_X^\vee$, Weyl group W_X s.t.

$$\begin{array}{ccccc} \mathcal{H}(G,K) & \simeq & \mathbb{C}[A^\vee]^W & \simeq & \mathbb{C}[\mathrm{Rep}(G^\vee)] \\ \circlearrowright * & & \circlearrowright & & \circlearrowright \otimes \\ C_c^\infty(X)^K & \simeq & \mathbb{C}[\delta_X^{1/2} A_X^\vee]^{W_X} & \simeq & \mathbb{C}[\mathrm{Rep}\, G_X^\vee] \\ \mathbb{1}_{X(\mathcal{O})} & \sim & & \rightsquigarrow & \mathrm{triv} \end{array}$$

Remark: the statement above is due to Sakellaridis [S13], while the existence of (a version) of $\check{G}_X$ and a geometric analogue appeared before in Gaitsgory-Nadler [GN10].

A lot of interesting work in the local p-adic setting by Sakellaridis [S08], [S12], [S13], [S18], also [SV17]

A lot of important work in geometry by Brion, Knop and others.

Remark: proof needs I level computations (as in the Satake or CS cases)

Remark: if $\underline{H}$ contains $\underline{U}_P$ for some proper parabolic $\underline{P}$, and we have $\theta : U_P \to \mathbb{G}_m$ fixed under $\underline{H}$ conj,

$\psi_0 : F \to \mathbb{C}^*$ character

we can also work with $C^\infty(X, \mathcal{L}_{\psi_0 \circ \theta})$

$\hookrightarrow$ complex line bundle defined by $\psi_0 \circ \theta$

typical example: $C^\infty({}_{(U,\psi)}\backslash G)$ for ψ nondegenerate char of U

constructed from $\psi_0 : F \to \mathbb{C}^*$ (of conductor $\mathcal{O}$)

$$C^\infty({}_{(U,\psi)}\backslash G) := \{ f : G \to \mathbb{C} \quad \ldots \quad f(ug) = \psi(u) f(g) \}$$

A basic problem in relative (local) Langlands:

* understand H distinguished reps of G,

  i.e. reps V of G that appear as quotient of $C_c^\infty(X)$.

recall - unramified irreps of G $\longleftrightarrow$ $A^\vee / W$

- any unramified irrep of G can be realized as the (sub) quotient of a UPS $I(\chi) = \mathrm{Ind}_B^G (\chi\, \delta^{1/2})$ (for $\chi \in A^\vee$)

Thm [S08] (under assumptions)

- a nec. condition for $I(\chi)$ to be H-distinguished is $\chi \in {}^{w}(\delta_X^{-1/2} A_X^\vee)$ for $w \in W/X$.

- almost all unramified H-dist irreps π are $\simeq \mathrm{Ind}_{P(X)}^{G} \chi \delta^{1/2}$ for $P(X)$ associated to X, $\chi \in (\delta_X^{-1/2} A_X^\vee)$

- under more assumptions on X : $\mathrm{Hom}(C_c^\infty(X), \pi) \leq 1$ [S13]

multiplicity 1 property

for the last part: - $\mathbb{C}[A^\vee]^W \to \mathbb{C}[\delta_{(X)}^{1/2} A_X^\vee]^{W}$ needs to be surjective

the dual group

we wish to define $G_X^\vee \hookrightarrow G^\vee$ (or better $G_X^\vee \times SL_2 \to G^\vee$)

Let $\underline{X}$ be a spherical variety for $\underline{G}$, which means - $\underline{X}$ is a normal variety with a $\underline{G}$ action

- such that $\underline{B}$ has a open orbit $\underline{X}^\circ$

- assume $\underline{X}$ is homogeneous, quasi affine
- moreover assume $\underline{X}$ is wavefront (no type N roots)

there is a way to define parabolic $B \subset P_X \subset G$, Levi L_X, torus A with $A \twoheadrightarrow A_X$, root system Φ_X,

$\Leftrightarrow A_X^\vee \hookrightarrow A^\vee$ Weyl group W_X

and ultimately distinguished morphisms

$G_X^\vee \to G^\vee$ which naturally extends $A_X^\vee \hookrightarrow A^\vee$

$SL_2 \times G_X^\vee \to G^\vee$

s.t. restr. to $G_X^\vee$ is distinguished

restr. to SL_2 is principal: $SL_2 \to L_X^\vee \subset G^\vee$

with weight $2\rho_{L_X} : \mathbb{G}_m \to G^\vee$

Read more: [SV17] sections 2 & 3 + references

Gan Bourbaki seminar 2025

LAWRGE 2024 notes (Day 3 Lect 3) (***)

Leslie Symmetric varieties..., Appendix B
for X symmetric

Recall thm [S 13]

$$\begin{array}{ccccc}
\mathcal{H}(G,K) & \simeq & \mathbb{C}[A^\vee]^W & \simeq & \mathbb{C}[\mathrm{Rep}(G^\vee)] \\
\circlearrowright * & & & & \circlearrowright \otimes \\
C_c^\infty(X)^K & \simeq & \mathbb{C}[\delta_X^{1/2} A_X^\vee]^{W_X} & \simeq & \mathbb{C}[\mathrm{Rep}\, G_X^\vee] \\
1_{X(\mathcal{O})} & \mapsto & & \mapsto & \mathrm{triv}
\end{array}$$

right action:

$i : G_X^\vee \to G^\vee$ induces action $\mathrm{Rep}\, G^\vee \circlearrowright \mathrm{Rep}\, G_X^\vee$

not the correct one

action needs to be twisted by $SL_2 \to G^\vee$ (how?)

$\rightsquigarrow$ responsible for $\mathbb{C}[\delta_X^{1/2} A_X^\vee]$.

Examples

1) group case $G = H \times H$, $\mathcal{H} = H$, $X = H$

$$h \cdot (h_1, h_2) = h_1^{-1} h h_2$$

then $W_X = \{(w,w), w \in W_H\} \simeq W_H$

$$\begin{aligned} A_H^\vee = A_X^\vee &\to A^\vee = A_H^\vee \times A_H^\vee \\ t &\mapsto (t^{-1}, t) \end{aligned}$$

$$G_X^\vee = H^\vee \hookrightarrow H^\vee \times H^\vee$$

$$C_c^\infty(H\backslash G)^{K_G} \simeq C_c^\infty\left(H \backslash H \times H / K_H \times K_H\right) \simeq C_c^\infty\left(K \backslash H / K\right) \simeq \mathcal{H}(G, K)$$

we recover Satake

$$\mathcal{H}(H, K_H) \simeq \mathbb{C}[\operatorname{Rep} H^\vee]$$

2) (U,ψ) case: $G = G$, $X = (U,\psi)\backslash G$ $\quad G_X^\vee = G^\vee$

we recover geometric Casselman-Shalika [FGKV00]
Frenkel Gaitsgory Kazhdan Vilonen

$$\mathcal{H}(G,K) \simeq \mathbb{C}[\operatorname{Rep} G^\vee]$$
$$\circlearrowleft * \qquad\qquad \circlearrowleft \otimes$$
$$\mathbb{C}_c^\infty(X)^K \simeq \mathbb{C}[\operatorname{Rep} G^\vee]$$

Note: 1) [FGKV00] don't have the $\delta_X^{1/2}$ factor, so in matching to [S13] one has to be careful.
2) $g^{CS} \simeq cCS$ computing values $W_X(\pi^\lambda)$, where $W_X \in C^\infty(X)^K$, eigenfct of $\mathcal{H}(G,K)$ with eigenvalue $x \in A^\vee$

Exercise: Prove 2)

3) $X = GL_n \times GL_n \backslash GL_{2n} \rightsquigarrow G_X^\vee = Sp_{2n}$

4) $X = Sp_{2n} \backslash GL_{2n} \rightsquigarrow G_X^\vee = GL_n$

Exercise (important, I think, see (***)):

For 1), 2), 3), 4) (or any other X you might like) write down explicitly the construction of P_X, L_X, A_X, W_X, $A_X^\vee \hookrightarrow A^\vee$, $G_X^\vee \hookrightarrow G^\vee$ and the action $\mathrm{Rep}\, G^\vee \curvearrowright \mathrm{Rep}\, G_X^\vee$.

Metaplectic groups

in order to quantise, we need metaplectic groups.

* fix $n \geq 1$ s.t. $q \equiv 1 \, (n)$ (which implies $\mu_n \hookrightarrow F^*$)
* Fix a W invariant bilinear form $B : \Lambda^\vee \times \Lambda^\vee \to \mathbb{Z}$

or alternatively $Q : \Lambda^\vee \to \mathbb{Z}$ where $Q(\gamma) = B(\gamma, \gamma)/2$

to the data (B, n) (or (Q, n)) and some arithmetic data (n-Hilbert symbol $(\, , \,) : F^* \times F^* \to \mu_n$)

assign $1 \to \mu_n \to \tilde{G} \xrightarrow{p} G \to 1$

based on the work of Matsumoto, Kubota, Brylinski-Deligne so as a set $\widetilde{G} = G \times \mu_n$ and

$$(g_1, \varepsilon_1)(g_2, \varepsilon_2) = (g_1 g_2, \varepsilon_1 \varepsilon_2 \sigma(g_1, g_2))$$

for some $\sigma \in H^2(G, \mu_n)$.

important difference between G and $\widetilde{G}$:

$\widetilde{T}$ is not commutative

for $x, y \in F$, $\lambda, \mu \in \Lambda^\vee$, $[x^{\lambda^\vee}, y^{\mu^\vee}] = (x, y)^{B(\lambda, \mu)} \in \mu_n$

this leads to difficulties.

Examples of B: fix $n \geq 1$.

1) if G is simply-connected, there is essentialy one primitive $B : \Lambda^\vee \times \Lambda^\vee \to \mathbb{Z}$ s.t. $Q = 1$ on short coroots.

2) $G = GL_r$, then $\Lambda^\vee \simeq \mathbb{Z}^r$ basis $\{e_i\}_{1 \leq i \leq r}$

$$\rightsquigarrow \quad B(e_i, e_j) = \begin{cases} 2\bar{p} & \text{if } i = j \\ \bar{q} & \text{if } i \neq j \end{cases} \qquad \bar{p}, \bar{q} \in \mathbb{Z}$$

- if $2\bar{p} - \bar{q} = 1 \rightsquigarrow$ Kazhdan Patterson cover
- if $\bar{p} = 1, \bar{q} = 0 \rightsquigarrow$ Savin's cover

a subgroup H of G is split by the extension if $\exists$ iso $p^{-1}(H) \simeq H \times \mu_n$ that commutes with $p: \tilde{G} \to G$

Fact: U is split (canonically) by central ext.

Assumption: K is split by central ext.

Assumption holds - if G is simply connected
- in many other cases
- not always

counter example already for PGL_2, $n=2$

Gan-Gao Asterique 4.6.

$\rightsquigarrow$ U, K subgroup of $\tilde{G}$.

metaplectic Hecke algebras

fix embedding $\underline{\varepsilon}: \mu_n \hookrightarrow \mathbb{C}^*$

spherical $\mathcal{H}^{\varepsilon}(\widetilde{G}, K)$: only consists of genuine functions $f(\eta g) = \underline{\varepsilon(\eta)}\, f(g)$

Fact: support of $f \in \mathcal{H}^{\varepsilon}(\widetilde{G}, K)$ is $\bigsqcup_{\lambda^\vee \in \widetilde{\Lambda}^\vee} \mu_n K \pi^{\lambda^\vee} K$ $\cancel{\lambda^\vee \in \Lambda^\vee}$

where $\widetilde{\Lambda}^\vee := \{ \lambda^\vee \in \Lambda^\vee ,\ B(\lambda^\vee, \mu^\vee) \equiv 0\ (n)\ \forall \mu^\vee \in \Lambda^\vee \} \subset \Lambda^\vee$

Fact: support of $f \in \mathcal{H}^{\varepsilon}(\widetilde{G}, K)$ is $\bigsqcup_{\lambda^\vee \in \widetilde{\Lambda}^\vee} \mu_n K \pi^{\lambda^\vee} K$

where $\widetilde{\Lambda}^\vee := \{ \lambda^\vee \in \Lambda^\vee, \ B(\lambda^\vee, \mu^\vee) \equiv 0 \ (n) \ \forall \mu^\vee \in \Lambda^\vee \}$

"Proof": one way: if $\lambda^\vee \notin \widetilde{\Lambda}^\vee \rightsquigarrow \exists \ t \in T(\mathcal{O}) \subset K \subset \widetilde{G}$ s.t. $[t, \pi^{\lambda^\vee}] \neq 1$. if f is supported on $\mu_n K \pi^{\lambda^\vee} K$

$\rightsquigarrow 0 \neq f(\pi^{\lambda^\vee}) = f(t\pi^{\lambda^\vee}) = \varepsilon(n) f(\pi^{\lambda^\vee} t) = \varepsilon(n) f(\pi^{\lambda^\vee})$.

Exercise: given explicit Hilbert symbol in (2.2) [McNamara 16] find such a t for any $\pi^{\lambda^\vee}$, $\lambda^\vee \notin \widetilde{\Lambda}^\vee$

Exercise: find root datum $\mathcal{D}$ and (Q, n) $n > 1$ such that $\widetilde{\Lambda}^\vee = \Lambda^\vee$.

dual group of $\tilde{G}$

if G has associated root datum $\mathcal{D} = \{\Lambda, \Delta, \Lambda^\vee, \Delta^\vee\}$

then define:

1) for $\alpha_i^\vee \in \Delta^\vee$, $\tilde{\alpha}_i^\vee = n(\alpha_i)\,\alpha_i^\vee$

$n(\alpha_i) =$ smallest >0 integer s.t. $n(\alpha_i)\, Q(\alpha_i) = 0\ (n)$

$\tilde{\Delta}^\vee = \{\tilde{\alpha}_i^\vee\}$

2) $\tilde{\Lambda}^\vee := \{ \lambda^\vee \in \Lambda^\vee$ s.t. $B(\lambda^\vee, \mu^\vee) \equiv 0\ (n)\ \forall \mu^\vee \in \Lambda^\vee \}$

3) for $\alpha_i \in \Delta$, $\tilde{\alpha}_i = \frac{1}{n(\alpha_i)}\alpha_i$ $\quad \tilde{\Delta} = \{\tilde{\alpha}_i\}$

4) $\tilde{\Lambda} = \{\lambda \in \Lambda \otimes \mathbb{Q}\ \ \langle \mu^\vee, \lambda \rangle \in \mathbb{Z}\ \forall \mu^\vee \in \tilde{\Lambda}^\vee\}$

Fact: $\tilde{\mathcal{D}} = (\tilde{\Lambda}, \tilde{\Delta}, \tilde{\Lambda}^\vee, \tilde{\Delta}^\vee)$ is a root datum.

Thm (Savin, McNamara, Finkelberg-Lysenko)

$$\mathcal{H}^{\varepsilon}(\widetilde{G}, K) \simeq \mathbb{C}[\widetilde{\Lambda}^{\vee}]^{W} \simeq \mathbb{C}[\operatorname{Rep} G^{\vee}_{(Q,n)}]$$

$G^{\vee}_{(Q,n)}$ is the group corresponding to $\widetilde{\mathcal{D}}^{\vee}$

Cor (Local Shimura correspondence)

$$\mathcal{H}^{\varepsilon}(\widetilde{G}, K) \simeq \mathcal{H}(G_{(Q,n)}, K_{(Q,n)})$$

dual group of X "=" ${}_{(U,\psi)}\backslash \widetilde{G}$

consider $\mathcal{W}(\widetilde{G},K) := C^{\infty,\varepsilon}(X)^K$

there is no drop in support now!

in particular, $\mathcal{W}(\widetilde{G},K)$ is a free rank $\underbrace{|\Lambda^\vee/\widetilde{\Lambda}^\vee|}_{\geq 1}$ $\mathcal{H}(\widetilde{G},K)$ module

$$\dim \mathrm{Hom}\left(\pi, {}_{(U,\psi)}\backslash\widetilde{G}\right) \text{ can be } > 1.$$

$G \quad (\mathbb{Q}, n)$

$$\mathcal{H}(\widetilde{G}, K) \simeq \mathbb{C}[\operatorname{Rep} G^\vee_{(\mathbb{Q},n)}]$$

$$\begin{array}{ccc} \cap & & \cap \\ \mathcal{W}(\widetilde{G}, K) & \simeq & \boxed{?} \end{array}$$

Conjecture (Lurie–Gaitsgory G07)

$$\boxed{?} = \operatorname{Rep} U_\xi(\mathfrak{g}^\vee)$$

Lusztig's quantum group at $\xi = \sqrt[2n]{1}$

G07: conjecture at the geometric level
(+ proof for ξ generic)

Campbell - Dhillon - Raskin 21
proof of a geometric statement (D-module setting)

Buciumas - Patnaik BP25 $\rightsquigarrow$ today's focus
proof in algebraic setting

étale setting work in progress

Remarks on BP 25

$$\begin{tikzcd}
\mathcal{H}(\widetilde{G}, K) \arrow[loop left, "*"] \arrow[r, phantom, "\simeq"] & \mathbb{C}[\operatorname{Rep} U_q(\mathfrak{g}^\vee_{(\mathbb{Q},n)})] \arrow[loop right, "-\otimes qFr"] \\
\mathcal{V}(\widetilde{G}, K) \arrow[r, phantom, "\simeq"] & \mathbb{C}_v[\operatorname{Rep} U_\xi(\mathfrak{g}^\vee)]
\end{tikzcd}$$

- we should work with v-graded version of $\operatorname{Rep} U_\xi(\mathfrak{g}^\vee)$ when we identify with the left side $v^2 \to q$. (geometrically we should be working at the derived level)

- action on left is convolution
right: $qFr : U_\xi(\mathfrak{g}^\vee) \to U\mathfrak{g}^\vee_{(\mathbb{Q},n)}$ replaces $SL_2 \times G^\vee_X \to G^\vee$

Highlights

$$\mathcal{W}(\tilde{G}, K) \overset{*}{\curvearrowleft} \mathcal{H}(\tilde{G}, K)$$

$$J_\mu \qquad\qquad\qquad c_\lambda$$

J_μ is supported on $U\pi^\mu K$, $J_\mu(\pi^\mu) = 1$.

c_λ geometric basis (corresponds to $V_\lambda \in \operatorname{Rep} G^\vee_{(Q,n)}$)

$$J_\mu * c_\lambda$$

1) we give very precise formulas for expansion

$$c_\lambda * J_\mu \longrightarrow J_\eta \qquad \text{in terms of}$$

certain quantum Littlewood-Richardson coeff.

hard from POV of p-adic groups, already understood by Lascoux-Leclerc-Thibon (implicitly) on q-side.

+ Kashiwara-Miwa-Stern, Grojnowski-Haiman

Example: $G = GL_n$, $n > 1$ Savin's cover.

$\Lambda^\vee \simeq \mathbb{Z}^2$ $\quad \tilde{\Lambda}^\vee = (\bar{n}\mathbb{Z})^2$ $\quad \bar{n} = n/(n,2)$

$\lambda = (\bar{n}\lambda_1, \bar{n}\lambda_2) \in \tilde{\Lambda}^\vee_+$, $\mu \in (\mu_1, \mu_2) \in \Lambda^\vee_+$

$\bar{\lambda} = (\lambda_1, \lambda_2)$, $\lambda_1 \geq \lambda_2$

to compute $c_\lambda * J_\mu = c_{\lambda\mu}^{\eta} J_\eta$

ex: $(\lambda_1, \lambda_2) = (3, 1)$

step 1: write Schur pol

$$s_{\bar{\lambda}}(\underline{Y^{\bar{n}}}) = Y_1^{3\bar{n}} Y_2^{\bar{n}} + Y_1^{2\bar{n}} Y_2^{2\bar{n}} + Y_1^{\bar{n}} Y_2^{3\bar{n}}$$

step 2: $s_\lambda(Y^{\bar{n}}) \cdot \boxed{Y^\mu}$, expand in $\underline{Y}$

step 3: for any Y^ν, ν not dominant,
use straightening relations to make it ≥ 0

$$s \cdot (m,\ell) = (\ell-1, m+1)$$

straightning relations

$$\nu = (m,\ell)$$

if $m \geq \ell$, $m+1-\ell = 0\ (\bar{n})$ $\qquad Y^{\nu} = -Y^{s\cdot\nu}$

if $0 < m+1-\ell < \bar{n}$ $\qquad Y^{\nu} = v\, Y^{s\cdot\nu}$

if $m+1-\ell > \bar{n}$, $m+1-\ell = j\ (\bar{n})$ $\qquad \nu^{(1)} = \nu - (j,-j)$

$$Y^{\nu} = v\, Y^{s\cdot\nu} + v\, Y^{\nu^{(1)}} - Y^{s\cdot\nu^{(1)}}$$

once we have only dominant coweights

coeff. of Y^{η} is $c^{\eta}_{\lambda\mu}$.*

(*) only true up to $\prod$ of Gauss sums.
we can keep track of Gauss sums!!!

2) explains many mysterious phenomena

2.1) Chinta Gunnells action $\longleftrightarrow$ $\bullet$ action

2.2) subspaces of $\mathcal{W}(\widetilde{G}, K)$ behaving like

$\mathcal{W}(G(\mathbb{Q}_m), K(\mathbb{Q}_m)) \rightsquigarrow$ most singular block

in $\operatorname{Rep} U_\zeta(\mathfrak{g}^\vee)$

$\mathcal{W}(G(\mathbb{Q}_m), I(\mathbb{Q}_m)) \rightsquigarrow$ principal block

Cor (local Shimura):

$$\begin{array}{ccc} \mathcal{H}(\widetilde{G}, K) & \simeq & \mathcal{H}(G(\mathbb{Q}_m), K(\mathbb{Q}_m)) \\ \wr & & \wr \\ \mathcal{W}(\widetilde{G}, K)_{(-\zeta)} & \simeq & \mathcal{W}(G(\mathbb{Q}_m), K(\mathbb{Q}_m)) \end{array}$$

Proof (very vague, more tomorrow)

1) need to understand $\mathcal{H}(\tilde{G}, I) \circlearrowright W(\tilde{G}, I)$

Gao - Gurevich - Karasiewicz GGK26

very interesting proof

(another proof: BP25)

2) need to average to understand $\mathcal{H}(\tilde{G}, K) \circlearrowright W(\tilde{G}, K)$

3) need to match to quantum side.