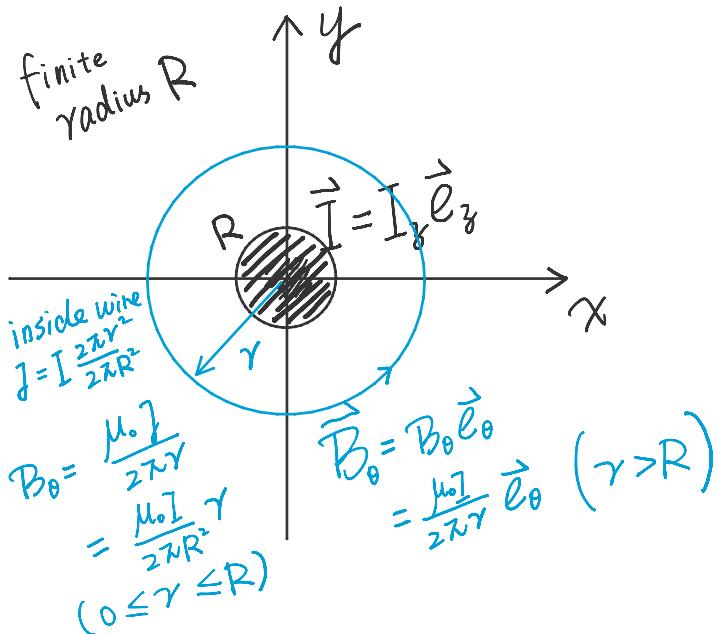


Infinitely long wire - finite radius

Thursday, September 17, 2020 3:01 PM



① Analytic Solution

i) Calc $B_\theta = \frac{\mu_0 I}{2\pi r}$ for $r > R$
 Singularity is avoided at $r=0$ $B_\theta = \frac{\mu_0 I}{2\pi R^2} r$ for $0 \leq r \leq R$

ii) Calc $\vec{e}_\theta = \text{rotate}(\vec{e}_r, \frac{\pi}{2})$

② Vector potential in Cartesian

$$\nabla^2 A_z = -\mu_0 J \quad J = \frac{\mu_0 I}{\pi R^2} \quad \text{Current uniform distribution}$$

$$\nabla^2 A_z = 0 \quad (r > R) \quad \vec{A}_z = A_z(x, y) \vec{e}_z$$

Vector potential always // current

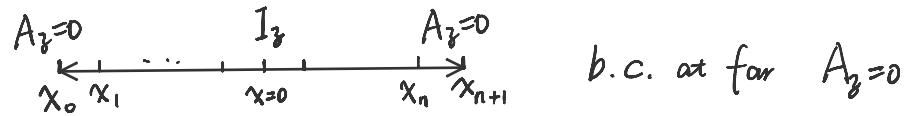
$$\vec{A} \parallel \vec{I}$$

$$\begin{cases} \nabla^2 A_z = 0 \quad (x^2 + y^2 \neq 0) \\ \nabla^2 A_z = I_z \quad (x = y = 0) \end{cases} \quad \text{Analytic Solution?}$$

Numerical Solution:

$$\nabla^2 A_z \rightarrow \frac{\partial^2}{\partial x^2} A_z(x, y) + \frac{\partial^2}{\partial y^2} A_z(x, y)$$

$$\nabla^2 A_3 \rightarrow \frac{\partial^2}{\partial x^2} A_3(x, y) + \frac{\partial^2}{\partial y^2} A_3(x, y)$$



for convinience, $A_3 \rightarrow A$

$$\frac{\partial^2}{\partial x^2} A = \frac{A_{i-1,j} - 2A_{i,j} + A_{i+1,j}}{\Delta x^2}$$

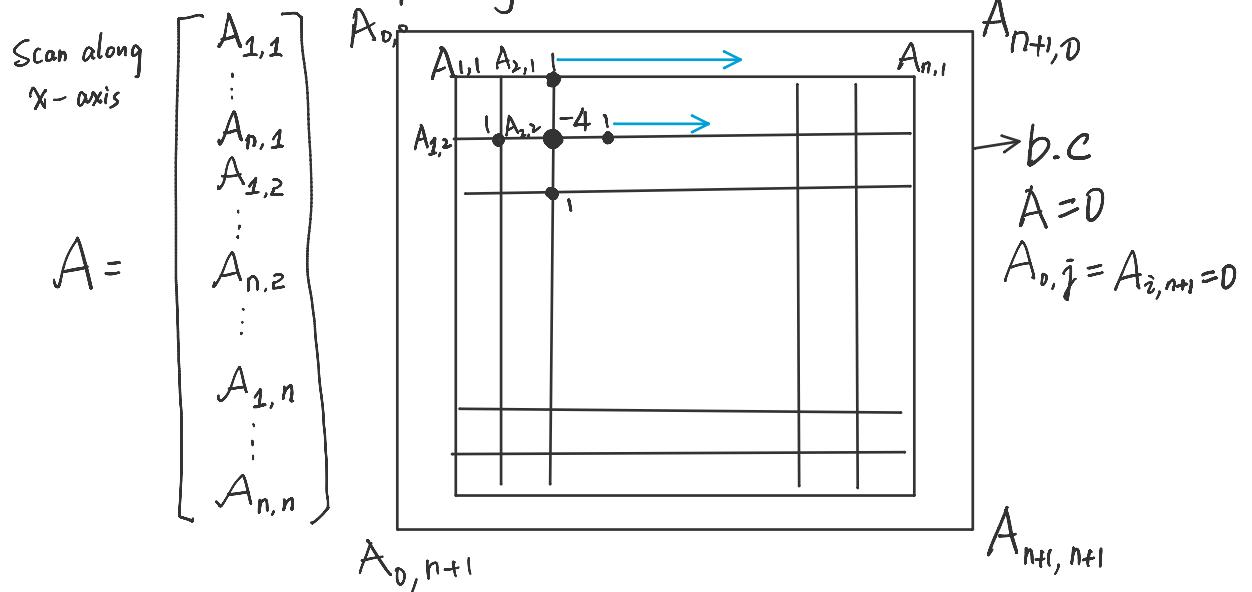
A_{i,j} ≠ A(x_{i,j})
↓ Numerical
↓ real

$$\frac{\partial^2}{\partial y^2} A = \frac{A_{i,j+1} - 2A_{i,j} + A_{i,j-1}}{\Delta y^2}$$

for square uniform mesh, $\Delta x = \Delta y$

$$\begin{aligned} \nabla^2 A &= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) A \\ &= \frac{1}{\Delta x^2} (A_{i,j-1} + A_{i-1,j} - 4A_{i,j} + A_{i+1,j} + A_{i,j+1}) \\ &= \frac{1}{\Delta x^2} MA \end{aligned}$$

M is constructed depending on A construction ,



This will result in an $mn \times mn$ linear system:

$$A\vec{u} = \vec{b}$$

where

$$A = \begin{bmatrix} D & -I & 0 & 0 & 0 & \dots & 0 \\ -I & D & -I & 0 & 0 & \dots & 0 \\ 0 & -I & D & -I & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & -I & D & -I & 0 \\ 0 & \dots & \dots & 0 & -I & D & -I \\ 0 & \dots & \dots & \dots & 0 & -I & D \end{bmatrix},$$

I is the $m \times m$ identity matrix, and D , also $m \times m$, is given by:

$$D = \begin{bmatrix} 4 & -1 & 0 & 0 & 0 & \dots & 0 \\ -1 & 4 & -1 & 0 & 0 & \dots & 0 \\ 0 & -1 & 4 & -1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & -1 & 4 & -1 & 0 \\ 0 & \dots & \dots & 0 & -1 & 4 & -1 \\ 0 & \dots & \dots & \dots & 0 & -1 & 4 \end{bmatrix},$$

[2] and \vec{b} is defined by

$$\vec{b} = -\Delta x^2 [g_{11}, g_{21}, \dots, g_{m1}, g_{12}, g_{22}, \dots, g_{m2}, \dots, g_{mn}]^T.$$

After discretization,

$$M A = I$$

$$A = M^{-1} I$$

$$I = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ I_{i,j} \\ \vdots \\ 0 \end{bmatrix}$$

Contains Source function
and b.c.

$$\nabla \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

where $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ are the unit vectors for the x -, y -, and z -axes, respectively. This expands as follows:^[7]

$$\nabla \times \mathbf{F} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) \hat{\mathbf{i}} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \right) \hat{\mathbf{j}} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \hat{\mathbf{k}} = \begin{bmatrix} \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \\ \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \\ \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \end{bmatrix}$$

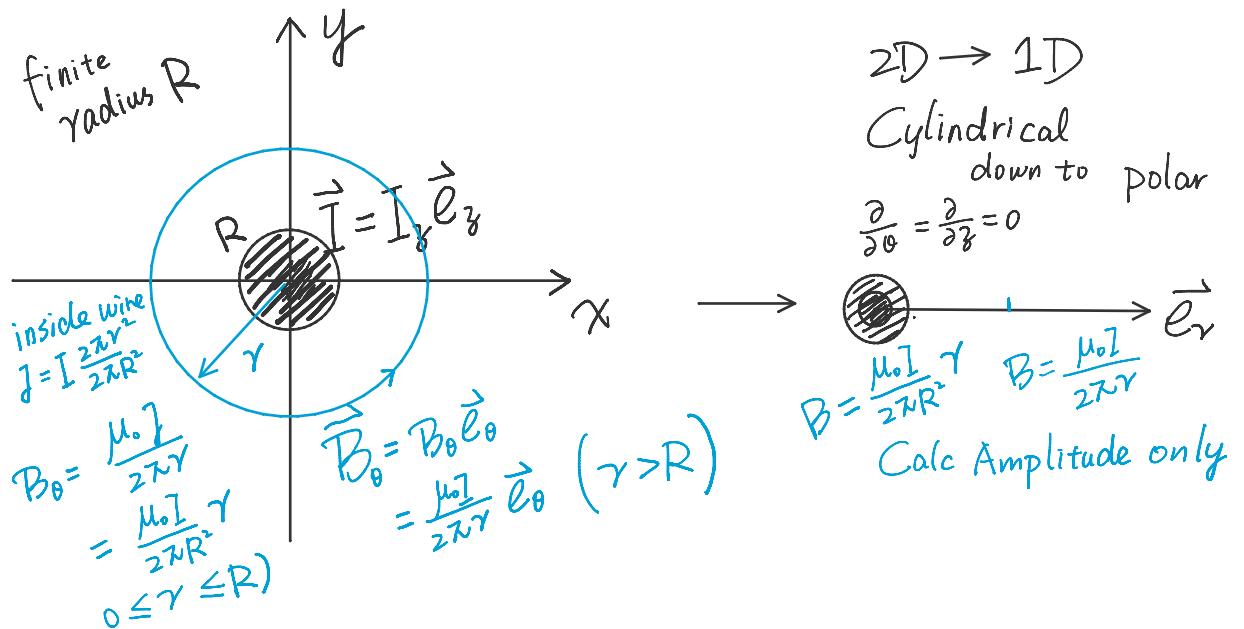
$$\vec{B} = \nabla \times \vec{A} \quad \vec{A} = A_z(x, y)$$

$$\vec{B} = \frac{\partial A}{\partial y} \vec{e}_x - \frac{\partial A}{\partial x} \vec{e}_y$$

use central differencing

$$\vec{B}_{i,j} = \frac{A_{i,j+1} - A_{i,j-1}}{2\Delta y} \vec{e}_x - \frac{A_{i+1,j} - A_{i-1,j}}{2\Delta x} \vec{e}_y$$

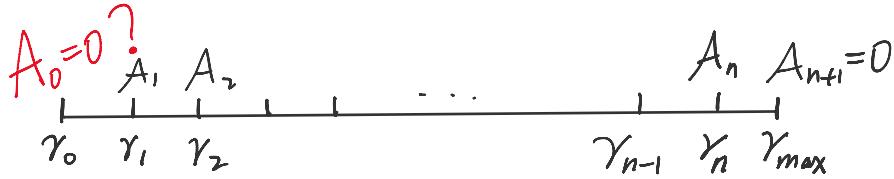
③ Vector potential in Cylindrical



$$\begin{cases} \nabla^2 A = -\mu_0 J \quad J = \frac{\mu_0 I}{\pi R^2} \quad (0 \leq r \leq R) \\ \nabla^2 A = 0 \quad \vec{A} = A(r) \hat{e}_z \quad (r > R) \end{cases} \text{ continuous at } r=R$$

Vector potential always // current $\vec{A} \parallel \vec{I}$

$$\nabla^2 A = 0 \text{ with b.c. } \nabla^2 A \Big|_{r=0} = I_z \Big|_{r=0}, \quad A(r=r_{\max}) = 0$$



$\nabla^2 A = 0$ in cylindrical coordinate

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial A}{\partial r} \right) - \frac{1}{r^2} A = 0 \quad (r > 0) \quad \rightarrow \text{When } A = A_\theta$$

$$\nabla^2 \vec{A} = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial f}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 f}{\partial \varphi^2} + \frac{\partial^2 f}{\partial z^2}$$

— View by clicking [show] —

[hide]

$$\begin{aligned} & \left(\nabla^2 A_\rho - \frac{A_\rho}{\rho^2} - \frac{2}{\rho^2} \frac{\partial A_\varphi}{\partial \varphi} \right) \hat{\rho} \\ & + \left(\nabla^2 A_\varphi - \frac{A_\varphi}{\rho^2} + \frac{2}{\rho^2} \frac{\partial A_\rho}{\partial \varphi} \right) \hat{\varphi} \\ & \quad + \nabla^2 A_z \hat{z} \end{aligned}$$

$$\text{When } \vec{A} = A_z(r) \hat{e}_z, \quad \nabla^2 A_z = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial A_z}{\partial r} \right) A_z$$

Analytic solution:

$$\text{Reverse Derivation} \quad B_\theta = \frac{\mu_0 I}{2\pi r} \quad \vec{B} = \nabla \times \vec{A}, \quad \vec{B} = B_\theta \hat{e}_\theta, \quad \vec{A} = A_z \hat{e}_z$$

$$\nabla \times \vec{A} = \left(\frac{1}{\rho} \frac{\partial A_z}{\partial \varphi} - \frac{\partial A_\varphi}{\partial z} \right) \hat{\rho} + \left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho} \right) \hat{\varphi} + \frac{1}{\rho} \left(\frac{\partial (\rho A_\varphi)}{\partial \rho} - \frac{\partial A_\rho}{\partial \varphi} \right) \hat{z}$$

$$B_\theta = - \frac{\partial A_z}{\partial r}, \quad A_z = \int B_\theta dr$$

$$i) 0 \leq r \leq R, \quad B_\theta = \frac{\mu_0 I}{2\pi r^2} r \quad A_\theta = - \int B_\theta dr = - \frac{\mu_0 I}{4\pi R^2} r^2 + C$$

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial (ar^2 + C)}{\partial r} \right) = \frac{1}{r} \frac{\partial}{\partial r} (r \cdot 2ar) = 4a = - \frac{\mu_0 I}{\pi R^2}$$

$$a = - \frac{\mu_0 I}{4\pi R^2} \quad \text{Match!} \quad \text{assume } C=0 \text{ for vector potential}$$

$$A = - \frac{\mu_0 I}{4\pi R^2} r^2 \quad (0 \leq r \leq R)$$

$$ii) r > R, \quad B_\theta = \frac{\mu_0 I}{2\pi r} \quad A_z = \int B_\theta dr = - \int \frac{\mu_0 I}{2\pi r} dr = - \frac{\mu_0 I}{2\pi} \ln(r) + C$$

Analytic Solution

$$A_z = a \ln(r) + C \longrightarrow \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial (a \ln(r) + C)}{\partial r} \right) A = 0$$

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial (a \ln(r) + C)}{\partial r} \right) = \frac{1}{r} \frac{\partial}{\partial r} \left(r \cdot \frac{a}{r} \right) = 0 \quad \text{Correct!}$$

assume $C=0$ for vector potential

$$A_z = a \ln(r)$$

All B.C. are for B instead of A

B is continuous at $r=R$

B is continuous at $r=R$

$$\frac{\partial A_3}{\partial r} \Big|_{r=R} = \frac{\partial A_3}{\partial r} \Big|_{r=R} \quad a = -\frac{\mu_0 I}{2\pi}$$

$$A_3(r) = -\frac{\mu_0 I}{4\pi R^2} r^2 \quad \text{for } 0 \leq r \leq R$$

$$A_3(r) = -\frac{\mu_0 I}{2\pi} \ln(r) \quad \text{for } r \geq R$$

Analytic Solution

$$\underbrace{r \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \right) A}_{} = 0$$

Direct discretization

$$r \frac{\partial}{\partial r} \left(\frac{(\gamma_{i+1} + \gamma_i)(A_{i+1} - A_i)}{2 \Delta r} \right) = 0 \quad \text{too much}$$

$$\frac{\gamma_i}{4 \Delta r} \frac{\partial}{\partial r} \left(\frac{(\gamma_{i+1} + \gamma_i)(A_{i+1} - A_i) - (\gamma_i + \gamma_{i-1})(A_i - A_{i-1})}{\Delta r} \right) = 0$$

$$\rightarrow r \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \right) = r^2 \frac{\partial^2}{\partial r^2} + r \frac{\partial}{\partial r}$$

$$r^2 \frac{\partial^2}{\partial r^2} A + r \frac{\partial}{\partial r} A = 0$$

$$\gamma_i^2 \frac{A_{i+1} - 2A_i + A_{i-1}}{\Delta r^2} + \gamma_i \frac{A_{i+1} - A_{i-1}}{2 \Delta r} = 0$$

According to the 1D mesh, $\gamma_i = i \Delta r$

$$i^2 (A_{i+1} - 2A_i + A_{i-1}) + \frac{i}{2} (A_{i+1} - A_{i-1}) = 0$$

$$\left(i^2 + \frac{i}{2} \right) A_{i+1} - 2i^2 A_i + \left(i^2 - \frac{i}{2} \right) = 0$$

$$\left(i^2 + \frac{i}{2}\right) A_{i+1} - 2i^2 A_i + \left(i^2 - \frac{i}{2}\right) = 0$$

$$\left(i^2 - \frac{i}{2}\right) A_{i-1} - 2i^2 A_i + \left(i^2 + \frac{i}{2}\right) = 0$$

$$i=1, -\frac{1}{2}A_0 - 2A_1 + \frac{3}{2}A_2 = 0$$

Assumed as known, worked as b.c.

$$-2A_1 + \frac{3}{2}A_2 = \frac{1}{2}A_0 \quad A_{n+1}=0$$

$$i=n, n(n-\frac{1}{2})A_{i-1} - 2n^2 A_i + n(n+\frac{n}{2})A_{n+1} = 0$$

$$n(n-\frac{1}{2})A_{i-1} - 2n^2 A_i = 0$$

$$MA = I$$

SRC as b.c.
↑ how to solve
for A_0 ?

$$\begin{bmatrix} -2 \times 1^2 & 1(1+\frac{1}{2}) \\ 2(2-\frac{1}{2}) & -2 \times 2^2 & 2(2+\frac{1}{2}) \\ & \ddots & & \\ i(i-\frac{1}{2}) & -2i^2 & i(i+\frac{1}{2}) \\ & \ddots & & \\ n(n-\frac{1}{2}) & -2n^2 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_n \end{bmatrix} = \begin{bmatrix} \frac{1}{2}A_0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$