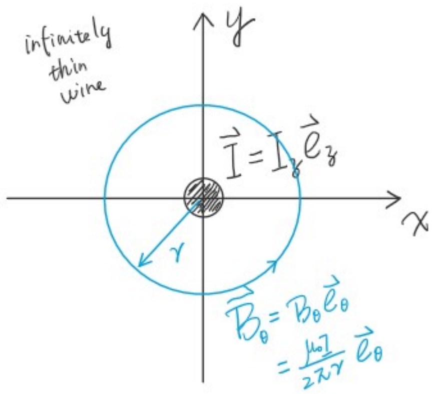
Infinitely long wire - infinitely thin

Thursday, September 17, 2020 3:01 PM



- ODE associated to R is a Euler Equation

$$r^2 R'' + rR' - \lambda R = 0, \quad 0 < r < L.$$

$$R(r) = \begin{cases} C_0 + D_0 \ln r & \text{for } \lambda = 0; \\ C_n r^n + D_n r^{-n}, & \text{for } \lambda_n \neq 0, n = 1, 2, \ldots. \end{cases}$$

- Regularity condition: $|u|$ is finite in $0 < r < L$ requires that $D_0 = D_n = 0$, because $\ln r \to -\infty$ and $r^{-n} \to \infty$ as $r \to 0$.
- A general solution $0 < r < L$ and $0 \leq \phi < 2\pi$ is

$$u(r, \phi) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n r^n \cos n\phi + b_n r^n \sin n\phi), \; n = 1, 2, \ldots$$

① Analytic Solution

i) Calc $B_\theta = \frac{\mu_0 I}{2\pi r}$ for $r > 0$

ii) Calc $\vec{e}_\theta = \text{rotate}(\vec{e}_r, \frac{\pi}{2})$

② Vector potential in Cartesian

$\nabla^2 A_z = -\mu_0 I_z \delta(x = y = 0)$ $\quad \vec{A}_z = A_z(x, y)\vec{e}_z$

Vector potential always // current

$\vec{A} \parallel \vec{I}$

$$\begin{cases} \nabla^2 A_z = 0 \; (x^2 + y^2 \neq 0) \\ \nabla^2 A_z = -\mu_0 I_z \; (x = y = 0) \end{cases}$$

Analytic Solution?

Numerical Solution:

$$\nabla^2 A_z \longrightarrow \frac{\partial^2}{\partial x^2} A_z(x, y) + \frac{\partial}{\partial y^2} A_z(x, y)$$

$A_z = 0$ $\quad$ I_z $\quad$ $A_z = 0$

$x_0 \; x_1 \; \ldots \; x=0 \; \ldots \; x_n \; x_{n+1}$ $\quad$ b.c. at far $A_z = 0$

for convinience, $A_z \to A$

$A_{i,j} \neq A(x_i, j)$

Numerical $\quad$ real

$$\frac{\partial^2}{\partial x^2} A = \frac{A_{i-1,j} - 2A_{i,j} + A_{i+1,j}}{\Delta x^2}$$

$$\frac{\partial^2}{\partial y^2} A = \frac{A_{i,j+1} - 2A_{i,j} + A_{i,j+1}}{\Delta y^2}$$

for square uniform mesh, $\Delta x = \Delta y$

$$\nabla^2 A = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) A$$

$$= \frac{1}{\Delta x^2}\left(A_{i,j-1} + A_{i-1,j} - 4A_{i,j} + A \ldots + A \ldots\right)$$

$$\nabla^2 A = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) A$$

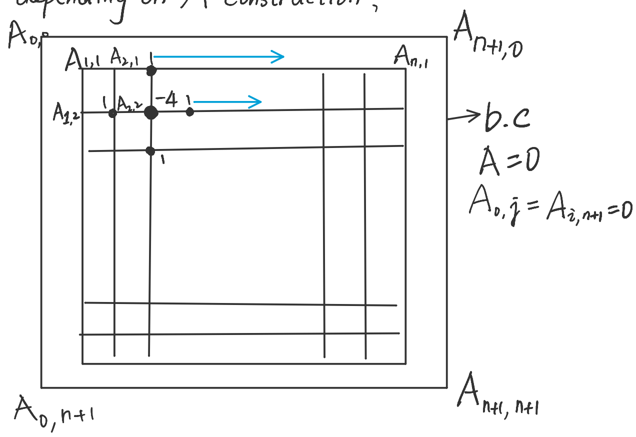
$$= \frac{1}{\Delta x^2}\left(A_{i,j-1} + A_{i-1,j} - 4A_{i,j} + A_{i+1,j} + A_{i,j+1}\right)$$

$$= \frac{1}{\Delta x^2} MA$$

M is constructed depending on A construction,

Scan along x-axis

$$A = \begin{bmatrix} A_{1,1} \\ \vdots \\ A_{n,1} \\ A_{1,2} \\ \vdots \\ A_{n,2} \\ \vdots \\ A_{1,n} \\ \vdots \\ A_{n,n} \end{bmatrix}$$



This will result in an $mn \times mn$ linear system:

$$A\vec{u} = \vec{b}$$

where

$$A = \begin{bmatrix} D & -I & 0 & 0 & 0 & \ldots & 0 \\ -I & D & -I & 0 & 0 & \ldots & 0 \\ 0 & -I & D & -I & 0 & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & -I & D & -I & 0 \\ 0 & \ldots & \ldots & 0 & -I & D & -I \\ 0 & \ldots & \ldots & \ldots & 0 & -I & D \end{bmatrix},$$

I is the $m \times m$ identity matrix, and D, also $m \times m$, is given by:

$$D = \begin{bmatrix} 4 & -1 & 0 & 0 & 0 & \ldots & 0 \\ -1 & 4 & -1 & 0 & 0 & \ldots & 0 \\ 0 & -1 & 4 & -1 & 0 & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & -1 & 4 & -1 & 0 \\ 0 & \ldots & \ldots & 0 & -1 & 4 & -1 \\ 0 & \ldots & \ldots & \ldots & 0 & -1 & 4 \end{bmatrix},$$

[2] and $\vec{b}$ is defined by

$$\vec{b} = -\Delta x^2 [g_{11}, g_{21}, \ldots, g_{m1}, g_{12}, g_{22}, \ldots, g_{m2}, \ldots, g_{mn}]^T.$$

After descretization,

$$MA = I$$

$$A = M^{-1} I$$

$$I = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ I_{i,j} \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Contains soure function and b.c.

$$\nabla \times \mathbf{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\[4pt] \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\[8pt] F_x & F_y & F_z \end{vmatrix}$$

where **i**, **j**, and **k** are the unit vectors for the x-, y-, and z-axes, respectively. This expands as follows:[7]

$$\nabla \times \mathbf{F} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\hat{\imath} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)\hat{\jmath} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\hat{k} = \begin{bmatrix} \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \\ \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \\ \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \end{bmatrix}$$

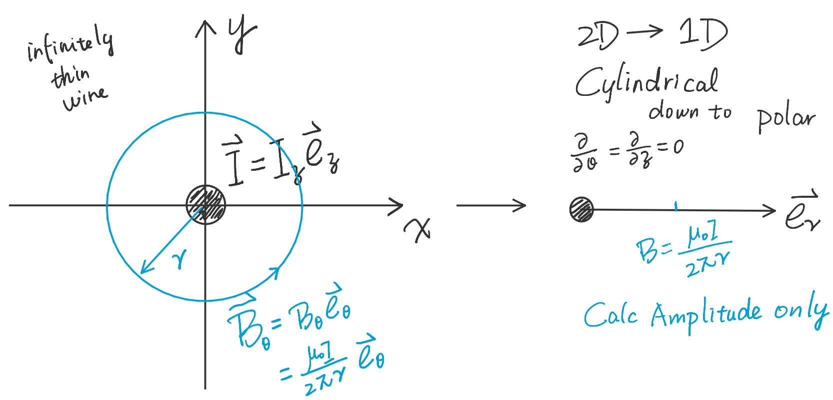
$\vec{B} = \nabla \times \vec{A}$ $\quad$ $\vec{A} = A_z(x, y)$

$$\vec{B} = \frac{\partial A}{\partial y}\vec{e}_x - \frac{\partial A}{\partial x}\vec{e}_y$$

use central differencing

$$\vec{B}_{i,j} = \frac{A_{i,j+1} - A_{i,j-1}}{2\Delta y}\vec{e}_x - \frac{A_{i+1,j} - A_{i-1,j}}{2\Delta x}\vec{e}_y$$

③ Vector potential in Cylindrical

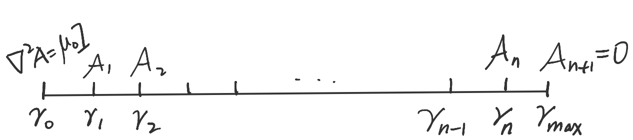


$\nabla^2 A = -\mu_0 I\delta(r=0)$ $\quad$ $\vec{A} = A(r)\vec{e}_z$

Vector potential always // current

$$\vec{A} \parallel \vec{I}$$

$\nabla^2 A = 0$ with b.c. $\nabla^2 A\big|_{r=0} = I_z\big|_{r=0}$, $A(r = r_{max}) = 0$



$\nabla^2 A = 0$ in cylindrical coordinate

$$\boxed{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)A - \frac{1}{r^2}A = 0 \quad (r > 0)} \rightarrow \text{When } A = A_\theta$$

$$\nabla^2 \vec{A} = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial f}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial\varphi^2} + \frac{\partial^2 f}{\partial z^2}$$

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$$\left(\nabla^2 A_\rho - \frac{A_\rho}{\rho^2} - \frac{2}{\rho^2}\frac{\partial A_\varphi}{\partial\varphi}\right)\hat{\boldsymbol{\rho}} + \left(\nabla^2 A_\varphi - \frac{A_\varphi}{\rho^2} + \frac{2}{\rho^2}\frac{\partial A_\rho}{\partial\varphi}\right)\hat{\boldsymbol{\varphi}} + \nabla^2 A_z \hat{\mathbf{z}}$$

When $\vec{A} = A_z(r)\vec{e}_z$, $\nabla^2 A_z = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)A_z$

Analytic solution:

Reverse Derivation $B_\theta = \frac{\mu_0 I}{2\pi r}$ $\vec{B} = \nabla\times\vec{A}$, $\vec{B} = B_\theta \vec{e}_\theta$, $\vec{A} = A_z\vec{e}_z$

$$\nabla\times\vec{A} = \left(\frac{1}{\rho}\frac{\partial A_z}{\partial\varphi} - \frac{\partial A_\varphi}{\partial z}\right)\hat{\boldsymbol{\rho}} + \left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial\rho}\right)\hat{\boldsymbol{\varphi}} + \frac{1}{\rho}\left(\frac{\partial(\rho A_\varphi)}{\partial\rho} - \frac{\partial A_\rho}{\partial\varphi}\right)\hat{\mathbf{z}}$$

$B_\theta = -\frac{\partial A_z}{\partial r}$, $A_z = -\int B_\theta \, dr = -\int \frac{\mu_0 I}{2\pi r} dr = -\frac{\mu_0 I}{2\pi}\ln(r) + C$

Analytic Solution $\boxed{A_z = a\ln(r) + C} \longrightarrow \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)A = 0$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\left(a\ln(r)+C\right)}{\partial r}\right) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\cdot\frac{a}{r}\right) = 0$$ Correct!

a & C in $A_z = a\ln(r) + C$ are determined by b.c.

However $A_z \longrightarrow \pm\infty$ as $r \to 0, +\infty$

How to apply B.C.

There is NO b.c. directly related to A_z

imp analysis
- when $r \to +\infty$, for sure $B \to 0$, $\frac{\partial A}{\partial r} \to 0$, Satisfied
- a & C are determined by $r \to 0$
- $r = 0$ is a singularity, so an integral form is used instead

$\oint B_\theta(r_0)\vec{e}_\theta \cdot d\vec{s} = \mu_0 I \longrightarrow -\frac{\partial A}{\partial r}\Big| \quad \oint \vec{e}_\theta \cdot d\vec{s}\Big| = \mu_0 I$

注 $r = 0$ is a singularity, so an integral form is used instead

$$\oint B_\theta(r_0)\vec{e}_\theta \cdot d\vec{s} = \mu_0 I \longrightarrow -\left.\frac{\partial A}{\partial r}\right|_{r=r_0} \left.\oint \vec{e}_\theta \cdot d\vec{s}\right|_{r=r_0} = \mu_0 I$$

$$-\frac{a}{r_0} \cdot 2\pi r_0 = \mu_0 I \longrightarrow a = -\frac{\mu_0 I}{2\pi}$$

C cannot be determined, since A is potential

C is chosen as 0 (reference point)

$$\boxed{A_z(r) = -\frac{\mu_0 I}{2\pi}\ln(r) \quad \text{for } r > 0}$$

Analytic Solution

$$\underbrace{r\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)}_{\text{Direct descretization}} A = 0$$

$$r\frac{\partial}{\partial r}\left(\frac{(r_{i+1}+r_i)}{2}\frac{(A_{i+1}-A_i)}{\Delta r}\right) = 0$$

too much

$$\frac{r_i}{4\Delta r}\frac{\partial}{\partial r}\left(\frac{(r_{i+1}+r_i)(A_{i+1}-A_i) - (r_i+r_{i-1})(A_i - A_{i-1})}{\Delta r}\right) = 0$$

$$r\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) = r^2\frac{\partial^2}{\partial r^2} + r\frac{\partial}{\partial r}$$

$$r^2\frac{\partial^2}{\partial r^2}A + r\frac{\partial}{\partial r}A = 0$$

$$r_i^2\,\frac{A_{i+1} - 2A_i + A_{i-1}}{\Delta r^2} + r_i\frac{A_{i+1}-A_{i-1}}{2\Delta r} = 0$$

According to the 1D mesh, $r_i = i\Delta r$

$$i^2(A_{i+1} - 2A_i + A_{i-1}) + \frac{i}{2}(A_{i+1} - A_{i-1}) = 0$$

$$\left(i^2+\frac{i}{2}\right)A_{i+1} - 2i^2A_i + \left(i^2 - \frac{i}{2}\right) = 0$$

$$\left(i^2-\frac{i}{2}\right)A_{i-1} - 2i^2A_i + \left(i^2 + \frac{i}{2}\right) = 0$$

$i = 1$, $-\frac{1}{2}A_0 - 2A_1 + \frac{3}{2}A_2 = 0$

Assumed as known, worked as b.C.

$A_{n+1} = 0$

$$-2A_1 + \frac{3}{2}A_2 = \frac{1}{2}A_0$$

$i = n$, $n\left(n-\frac{1}{2}\right)A_{i-1} - 2n^2A_i + n\left(n+\frac{n}{2}\right)A_{n+1} = 0$

$$n\left(n-\frac{1}{2}\right)A_{i-1} - 2n^2A_i = 0$$

Src as b.c.

how to solve for A_0?

$$MA = I$$

$$\Big[\; -2\times 1^2 \quad 1\left(1+\tfrac{1}{2}\right) \qquad\qquad \Big(A_1 \;\; A \qquad \Big(\tfrac{1}{2}A_0$$

$$
\begin{bmatrix}
-2\times 1^2 & 1(1+\frac{1}{2}) & & & \\
2(2-\frac{1}{2}) & -2\times 2^2 & 2(2+\frac{1}{2}) & & \\
 & \ddots & & & \\
 & i(i-\frac{1}{2}) & -2i^2 & i(i+\frac{1}{2}) & \\
 & & & \ddots & \\
 & & & n(n-\frac{1}{2}) & -2n^2
\end{bmatrix}
\begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_n \end{pmatrix}
=
\begin{pmatrix} \frac{1}{2}A_0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}
$$