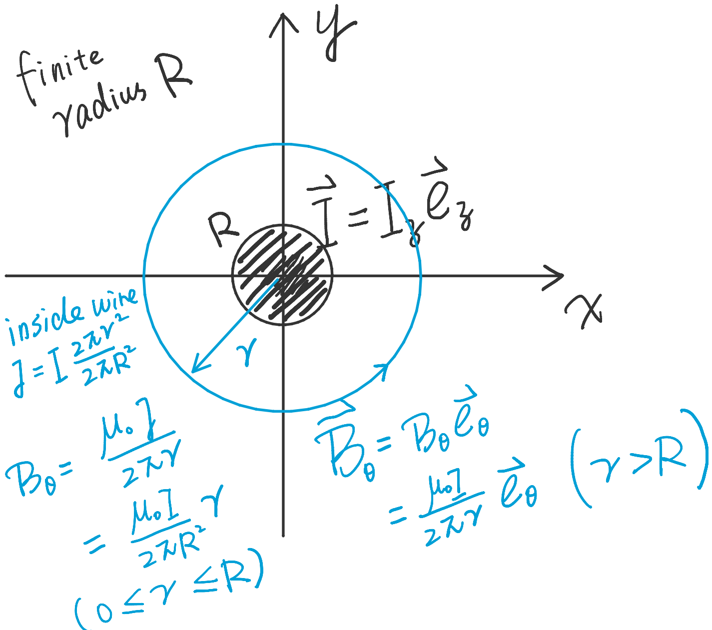
Infinitely long wire - finite radius

Thursday, September 17, 2020 3:01 PM



① Analytic Solution

i) Calc $B_\theta = \dfrac{\mu_0 I}{2\pi r}$ for $r > R$

Singularity is avoided at $r = 0$

$B_\theta = \dfrac{\mu_0 I}{2\pi R^2} r$ for $0 \le r \le R$

ii) Calc $\vec{e}_\theta = \text{rotate}(\vec{e}_r, \frac{\pi}{2})$

② Vector potential in Cartesian

$\nabla^2 A_z = -\mu_0 J \quad (r \le R)$ $\quad J = \dfrac{\mu_0 I}{\pi R^2}$ current uniform distribution

$\nabla^2 A_z = 0 \quad (r > R)$ $\quad \vec{A}_z = A_z(x, y)\, \vec{e}_z$

Vector potential always // current

$\vec{A} \parallel \vec{I}$

$$\begin{cases} \nabla^2 A_z = 0 \quad (x^2 + y^2 \ne 0) \\ \nabla^2 A_z = I_z \quad (x = y = 0) \end{cases}$$

Analytic Solution?

Numerical solution:

$$\nabla^2 A_z \longrightarrow \frac{\partial^2}{\partial x^2} A_z(x, y) + \frac{\partial}{\partial y^2} A_z(x, y)$$

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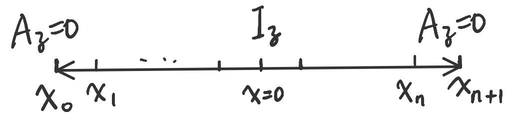


b.c. at far $A_z=0$

for convinience. $A_z \rightarrow A$

$A_{i,j} \neq A(x_i, j)$

$\downarrow$ Numerical $\downarrow$ real

$$\frac{\partial^2}{\partial x^2} A = \frac{A_{i-1,j} - 2A_{i,j} + A_{i+1,j}}{\Delta x^2}$$

$$\frac{\partial^2}{\partial y^2} A = \frac{A_{i,j-1} - 2A_{i,j} + A_{i,j+1}}{\Delta y^2}$$

for square uniform mesh, $\Delta x = \Delta y$

$$\nabla^2 A = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) A$$

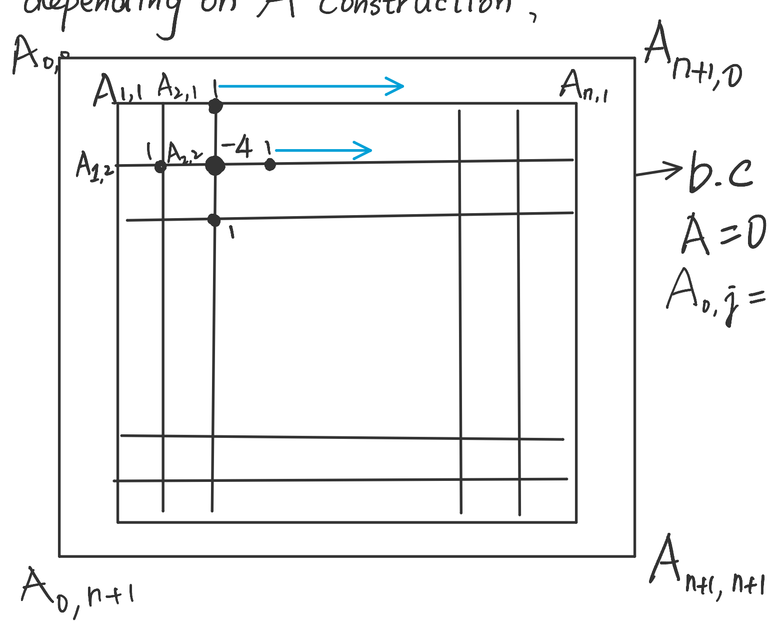
$$= \frac{1}{\Delta x^2}\left(A_{i,j-1} + A_{i-1,j} - 4A_{i,j} + A_{i+1,j} + A_{i,j+1}\right)$$

$$= \frac{1}{\Delta x^2} MA$$

M is constructed depending on A construction,

Scan along x-axis

$$A = \begin{bmatrix} A_{1,1} \\ \vdots \\ A_{n,1} \\ A_{1,2} \\ \vdots \\ A_{n,2} \\ \vdots \\ A_{1,n} \\ \vdots \\ A_{n,n} \end{bmatrix}$$



$\rightarrow$ b.c

$A=0$

$A_{0,j} = A_{i,n+1} = 0$

This will result in an *mn* × *mn* linear system:

$$A\vec{u} = \vec{b}$$

where

$$A = \begin{bmatrix} D & -I & 0 & 0 & 0 & \dots & 0 \\ -I & D & -I & 0 & 0 & \dots & 0 \\ 0 & -I & D & -I & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & -I & D & -I & 0 \\ 0 & \dots & \dots & 0 & -I & D & -I \\ 0 & \dots & \dots & \dots & 0 & -I & D \end{bmatrix},$$

I is the *m* × *m* identity matrix, and D, also *m* × *m*, is given by:

$$D = \begin{bmatrix} 4 & -1 & 0 & 0 & 0 & \dots & 0 \\ -1 & 4 & -1 & 0 & 0 & \dots & 0 \\ 0 & -1 & 4 & -1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & -1 & 4 & -1 & 0 \\ 0 & \dots & \dots & 0 & -1 & 4 & -1 \\ 0 & \dots & \dots & \dots & 0 & -1 & 4 \end{bmatrix},$$

[2] and $\vec{b}$ is defined by

$$\vec{b} = -\Delta x^2 [\, g_{11}, g_{21}, \dots, g_{m1}, g_{12}, g_{22}, \dots, g_{m2}, \dots, g_{mn} \,]^T.$$

After descretization,

$$MA = I \qquad I = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ I_{i,j} \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$ contains soure function and b.C.

$$A = M^{-1} I$$

$$\nabla \times \mathbf{F} = \begin{vmatrix} \boldsymbol{\hat{\imath}} & \boldsymbol{\hat{\jmath}} & \boldsymbol{\hat{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

where $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are the unit vectors for the x-, y-, and z-axes, respectively. This expands as follows:[7]

$$\nabla \times \mathbf{F} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\boldsymbol{\hat{\imath}} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)\boldsymbol{\hat{\jmath}} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\boldsymbol{\hat{k}} = \begin{bmatrix} \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \\ \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \\ \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \end{bmatrix}$$

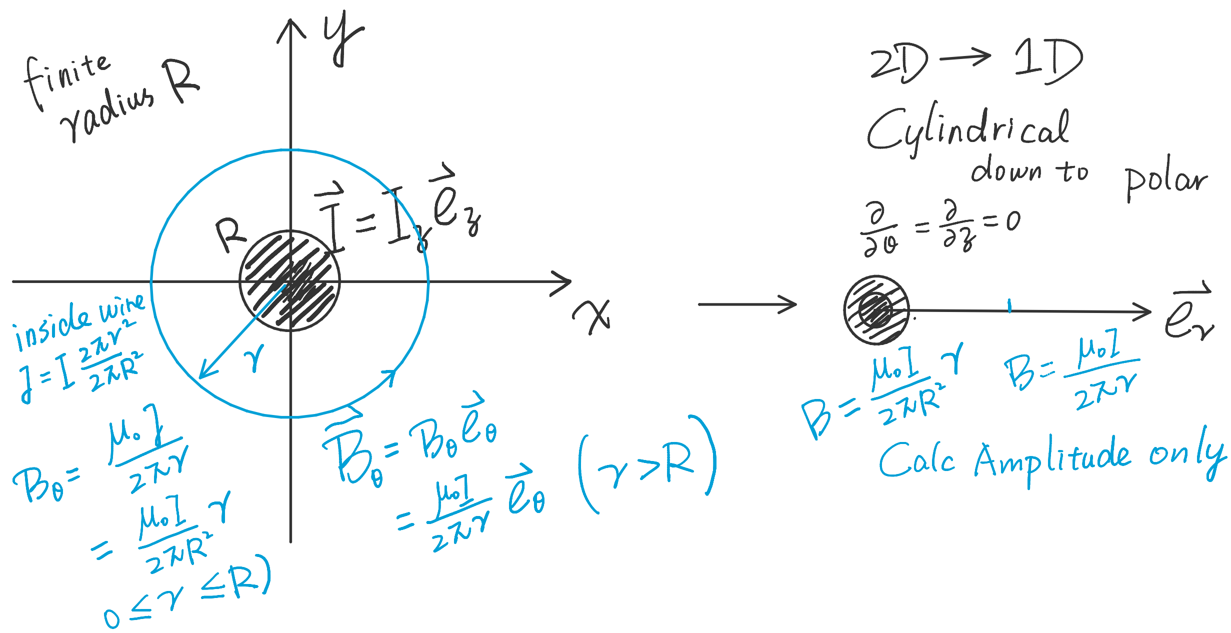
$\vec{B} = \nabla \times \vec{A} \qquad \vec{A} = A_z(x, y)$

$$\vec{B} = \frac{\partial A}{\partial y}\vec{e}_x - \frac{\partial A}{\partial x}\vec{e}_y$$

use central differencing

$$\vec{B}_{i,j} = \frac{A_{i,j+1} - A_{i,j-1}}{2\Delta y}\vec{e}_x - \frac{A_{i+1,j} - A_{i-1,j}}{2\Delta x}\vec{e}_y$$

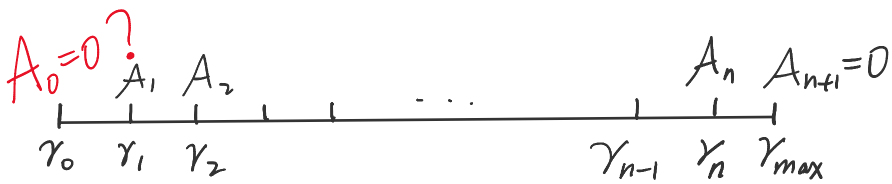
③ Vector potential in Cylindrical



$$\begin{cases} \nabla^2 A = -\mu_0 J \quad J = \dfrac{\mu_0 I}{\pi R^2} \quad (0 \le r \le R) \\ \nabla^2 A = 0 \quad \vec{A} = A(r)\vec{e}_z \quad (r > R) \end{cases}$$ Continous at $r=R$

Vector potential always // current $\vec{A} \parallel \vec{I}$

$\nabla^2 A = 0$ with b.c. $\nabla^2 A \big|_{r=0} = I_z \big|_{r=0}$, $A(r=r_{max})=0$



$\nabla^2 A = 0$ in cylindrical coordinate

$$\boxed{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)A - \frac{1}{r^2}A = 0 \quad (r>0)} \longrightarrow \text{When } A = A_\theta$$

$$\nabla^2 \vec{A} = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial f}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \varphi^2} + \frac{\partial^2 f}{\partial z^2}$$

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$$\left(\nabla^2 A_\rho - \frac{A_\rho}{\rho^2} - \frac{2}{\rho^2}\frac{\partial A_\varphi}{\partial \varphi}\right)\hat{\boldsymbol{\rho}} + \left(\nabla^2 A_\varphi - \frac{A_\varphi}{\rho^2} + \frac{2}{\rho^2}\frac{\partial A_\rho}{\partial \varphi}\right)\hat{\boldsymbol{\varphi}} + \nabla^2 A_z \hat{\mathbf{z}}$$

When $\vec{A} = A_z(r)\vec{e}_z$, $\nabla^2 A_z = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)A_z$

Analytic solution:

Reverse Derivation $B_\theta = \dfrac{\mu_0 I}{2\pi r}$ $\vec{B} = \nabla \times \vec{A}$, $\vec{B} = B_\theta \vec{e}_\theta$, $\vec{A} = A_z \vec{e}_z$

$$\nabla \times \vec{A} = \left(\frac{1}{\rho}\frac{\partial A_z}{\partial \varphi} - \frac{\partial A_\varphi}{\partial z}\right)\hat{\boldsymbol{\rho}} + \left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho}\right)\hat{\boldsymbol{\varphi}} + \frac{1}{\rho}\left(\frac{\partial(\rho A_\varphi)}{\partial \rho} - \frac{\partial A_\rho}{\partial \varphi}\right)\hat{\mathbf{z}}$$

$B_\theta = -\frac{\partial A_z}{\partial r}$, $A_z = -\int B_\theta \, dr$

i) $0 \le r \le R$, $B_\theta = \frac{\mu_0 I}{2\pi R^2} r$ $A_z = -\int B_\theta \, dr = -\frac{\mu_0 I}{4\pi R^2} r^2 + C$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial(ar^2+C)}{\partial r}\right) = \frac{1}{r}\frac{\partial}{\partial r}(r \cdot 2ar) = 4a = -\frac{\mu_0 I}{\pi R^2}$$

$a = -\frac{\mu_0 I}{4\pi R^2}$ Match! assume $C=0$ for vector potential

$$\boxed{A = -\frac{\mu_0 I}{4\pi R^2} r^2 \quad (0 \le r \le R)}$$

ii) $r > R$, $B_\theta = \frac{\mu_0 I}{2\pi r}$ $A_z = -\int B_\theta \, dr = -\int \frac{\mu_0 I}{2\pi r} dr = -\frac{\mu_0 I}{2\pi} \ln(r) + C$

Analytic Solution

$$\boxed{A_z = a \ln(r) + C} \longrightarrow \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial}{\partial r}\right) A = 0$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial(a\ln(r)+C)}{\partial r}\right) = \frac{1}{r}\frac{\partial}{\partial r}\left(r \cdot \frac{a}{r}\right) = 0$$ Correct!

assume $C=0$ for vector potential

$$\boxed{A_z = a \ln(r)}$$

All B.C. are for B instead of A

B is continuous at $r=R$

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$$\left.\frac{\partial A_z}{\partial r}\right|_{r=R} = \left.\frac{\partial A_z}{\partial r}\right|_{r=R} \qquad a = -\frac{\mu_0 I}{2\pi}$$

$$\boxed{\begin{aligned} A_z(r) &= -\frac{\mu_0 I}{4\pi R^2} r^2 \quad \text{for } 0 \le r \le R \\ A_z(r) &= -\frac{\mu_0 I}{2\pi} \ln(r) \quad \text{for } r \ge R \end{aligned}}$$

Analytic Solution

$$\underbrace{r\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) A}_{\text{Direct descretization}} = 0$$

$$r\frac{\partial}{\partial r}\left(\frac{(r_{i+1}+r_i)}{2}\frac{(A_{i+1}-A_i)}{\Delta r}\right) = 0$$

too much

$$\frac{r_i}{4\Delta r}\frac{\partial}{\partial r}\left(\frac{(r_{i+1}+r_i)(A_{i+1}-A_i) - (r_i+r_{i-1})(A_i-A_{i-1})}{\Delta r}\right) = 0$$

$$r\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) = r^2\frac{\partial^2}{\partial r^2} + r\frac{\partial}{\partial r}$$

$$r^2\frac{\partial^2}{\partial r^2}A + r\frac{\partial}{\partial r}A = 0$$

$$r_i^2\,\frac{A_{i+1}-2A_i+A_{i-1}}{\Delta r^2} + r_i\frac{A_{i+1}-A_{i-1}}{2\Delta r} = 0$$

According to the 1D mesh, $r_i = i\Delta r$

$$i^2(A_{i+1}-2A_i+A_{i-1}) + \frac{i}{2}(A_{i+1}-A_{i-1}) = 0$$

$$\left(i^2+\frac{i}{2}\right)A_{i+1} - 2i^2A_i + \left(i^2-\frac{i}{2}\right) = 0$$

$$\left(i^2+\frac{i}{2}\right)A_{i+1} - 2i^2 A_i + \left(i^2-\frac{i}{2}\right) = 0$$

$$\left(i^2-\frac{i}{2}\right)A_{i-1} - 2i^2 A_i + \left(i^2+\frac{i}{2}\right) = 0$$

$$i=1, \quad -\frac{1}{2}A_0 - 2A_1 + \frac{3}{2}A_2 = 0$$

Assumed as known, worked as b.c.

$A_{n+1}=0$

$$-2A_1 + \frac{3}{2}A_2 = \frac{1}{2}A_0$$

$$i=n, \quad n\left(n-\frac{1}{2}\right)A_{i-1} - 2n^2 A_i + n\left(n+\frac{n}{2}\right)A_{n+1} = 0$$

$$n\left(n-\frac{1}{2}\right)A_{i-1} - 2n^2 A_i = 0$$

Src as b.c.

how to solve for A_0?

$$MA = I$$

$$\begin{bmatrix} -2\times 1^2 & 1\left(1+\frac{1}{2}\right) & & & & \\ 2\left(2-\frac{1}{2}\right) & -2\times 2^2 & 2\left(2+\frac{1}{2}\right) & & & \\ & & \ddots & & & \\ & & i\left(i-\frac{1}{2}\right) & -2i^2 & i\left(i+\frac{1}{2}\right) & \\ & & & & \ddots & \\ & & & & n\left(n-\frac{1}{2}\right) & -2n^2 \end{bmatrix} \begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ \vdots \\ A_n \end{pmatrix} = \begin{pmatrix} \frac{1}{2}A_0 \\ 0 \\ \vdots \\ \\ 0 \end{pmatrix}$$