Infinitely long solenoid

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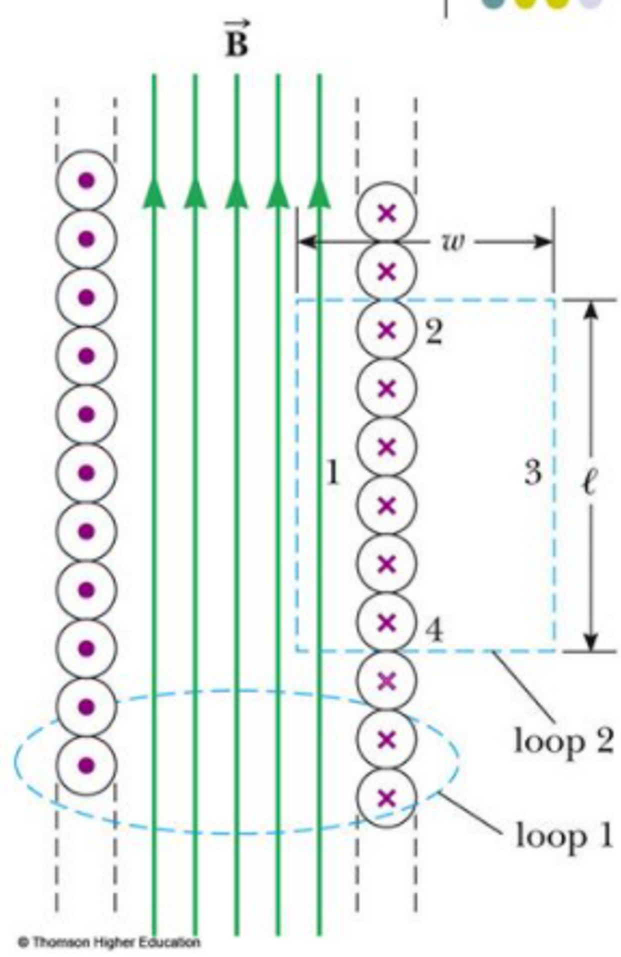
Ideal (infinitely long) Solenoid

- An *ideal solenoid* is approached when:
  - the turns are closely spaced
  - the length is much greater than the radius of the turns
- Apply Ampere's Law to loop 2:

$$\oint \vec{\mathbf{B}}\cdot d\vec{\mathbf{s}} = \int_{path\,1} \vec{\mathbf{B}}\cdot d\vec{\mathbf{s}} = B\int_{path\,1} ds = B\ell$$

$$\oint \vec{\mathbf{B}}\cdot d\vec{\mathbf{s}} = B\ell = \mu_o NI$$

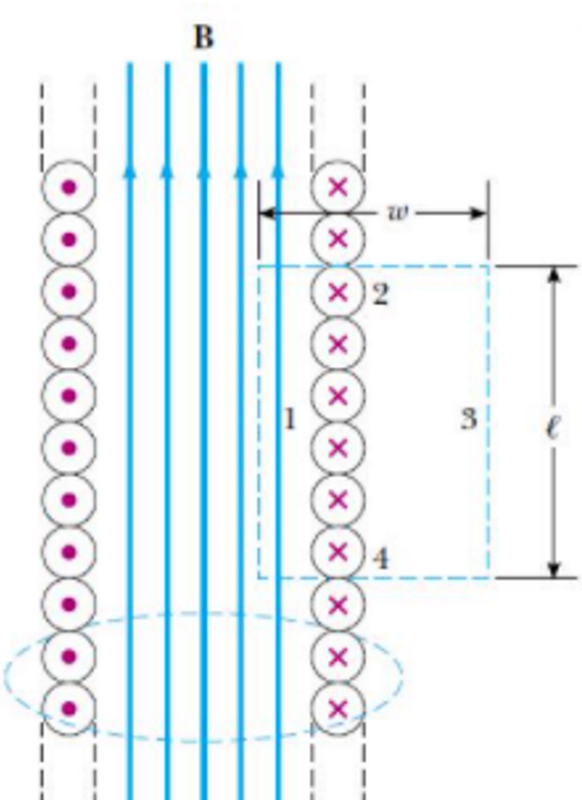
$$\boxed{B = \mu_o \frac{N}{\ell} I = \mu_o n I}$$



$n = N / \ell$ is the number of turns per unit length

Ampere's Law: Example, Infinitely long solenoid

Cross sectional view:



$$\oint \vec{B}\cdot d\vec{s} = \overbrace{\int_{①} \vec{B}\cdot d\vec{s}}^{\vec{B}\parallel ds,\ B\ell} + \overbrace{\int_{②} \vec{B}\cdot d\vec{s}}^{\vec{B}\perp d\vec{s},\ 0} + \underbrace{\int_{③} \vec{B}\cdot d\vec{s}}_{B=0 \text{ outside}} + \underbrace{\int_{④} \vec{B}\cdot d\vec{s}}_{\vec{B}\perp d\vec{s},\ 0}$$

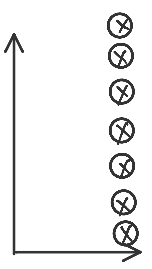
$I_{enc} = \frac{N}{L}\ell I$, N = # of turns
L = total length of Solenoid
I = Current in wire

$B\ell = \mu_0\left(\frac{N}{L}\ell I\right) \Rightarrow B = \mu_0 n I$, RHR
where n = # of turns per length

B is uniform inside solenoid!!

Magnetic field outside the solenoidIt was just proved that the magnetic field outside the solenoid is constant, independent of the distance to the solenoid. Applying the physically plausible boundary condition that the field must go to zero for large distances, one sees that the field is equal to zero everywhere outside the solenoid.

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)A_\theta - \frac{A_\theta}{r^2} + \frac{\partial}{\partial z^2}A_\theta = -\mu_0\left(J_{coil} + J_{plasma}\right)$$



infinitely long solenoid

A_θ = Const in $\vec{e}_z$

$A_\theta = A_\theta(r)$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)A_\theta - \frac{A_\theta}{r^2} = -\mu_0 J_{coil}$$

$$\boxed{\frac{\partial^2}{\partial r^2}A_\theta} + \boxed{\frac{1}{r}\frac{\partial}{\partial r}A_\theta} - \boxed{\frac{A_\theta}{r^2}} = -\mu_0 J_{coil}$$

$A_\theta = 0$ $\quad r_1 \quad r_2 \quad \ldots \quad \ldots \quad r_{n-2} \quad r_{n-1} \quad r_n \quad$ $A_\theta = 0$

$r = 0$ $\qquad r = r_{coil}$ $\qquad r = r_{max}$

$$\frac{\partial^2}{\partial r^2}A_\theta = \frac{A_\theta(r_{i+1}) - 2A_\theta(r_i) + A_\theta(r_{i-1})}{\Delta r^2} \rightarrow \frac{1}{\Delta r^2}\begin{bmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & & \ddots & & \\ & & 1 & -2 & 1 \\ & & & 1 & -2 \end{bmatrix}\begin{bmatrix} A_\theta(r_1) \\ A_\theta \\ A_\theta(r_n) \end{bmatrix} = \frac{1}{\Delta r^2}M_1 A_\theta$$

$$\frac{1}{r}\frac{\partial}{\partial r}A_\theta = \frac{A_\theta(r_i) - A_\theta(r_{i-1})}{r_i \Delta r}$$

$$= \frac{A_\theta(r_i) - A_\theta(r_{i-1})}{i\Delta r^2} \longrightarrow \frac{1}{\Delta r^2}\begin{bmatrix} \frac{1}{1} & & & & \\ -\frac{1}{2} & \frac{1}{2} & & & \\ & -\frac{1}{3} & \frac{1}{3} & & \\ & & \ddots & & \\ & & -\frac{1}{n-1} & \frac{1}{n-1} & \\ & & & -\frac{1}{n} & \frac{1}{n} \end{bmatrix}\begin{bmatrix} A_\theta(r_1) \\ A_\theta \\ A_\theta(r_n) \end{bmatrix} = \frac{1}{\Delta r^2}M_2 A_\theta$$

$$-\frac{A_\theta}{r^2} = -\frac{A_\theta(r_i)}{r_i^2} = -\frac{A_\theta(r_i)}{i^2\Delta r^2} \longrightarrow \frac{1}{\Delta r^2}\begin{bmatrix} -\frac{1}{1^2} & & & & & \\ & -\frac{1}{2^2} & & & & \\ & & -\frac{1}{3^2} & & & \\ & & & \ddots & & \\ & & & & -\frac{1}{(n-1)^2} & \\ & & & & & -\frac{1}{n^2} \end{bmatrix}\begin{bmatrix} A_\theta(r_1) \\ A_\theta \\ A_\theta(r_n) \end{bmatrix} = \frac{1}{\Delta r^2} M_3 A_\theta$$

$$-\mu_0 J_{coil} = -\mu_0 J_{coil}\,\delta(r = r_{coil}) \longrightarrow \begin{bmatrix} 0 \\ \vdots \\ -\mu_0 J_{coil} \\ \vdots \\ 0 \end{bmatrix} \rightarrow r_j = r_{coil}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)A_\theta - \frac{A_\theta}{r^2} = -\mu_0 J_{coil} \longrightarrow \boxed{\frac{1}{\Delta r^2}(M_1 + M_2 + M_3)A_\theta = -\mu_0 J_{coil}}$$

$$\vec{B} = \nabla \times \vec{A}$$

$$\nabla \times \vec{A} = \begin{matrix} \left(\frac{1}{\rho}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right)\hat{\boldsymbol{\rho}} + \\ \left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho}\right)\hat{\boldsymbol{\phi}} + \\ \frac{1}{\rho}\left(\frac{\partial(\rho A_\phi)}{\partial \rho} - \frac{\partial A_\rho}{\partial \phi}\right)\hat{\boldsymbol{z}} \end{matrix}$$

$$\vec{B} = B_z(r)\vec{e}_z = \frac{1}{r}\left(\frac{\partial(rA_\theta)}{\partial r} - \cancel{\frac{\partial A_r}{\partial \theta}}\right)\vec{e}_z$$

$$= \frac{1}{r}\frac{\partial(rA_\theta)}{\partial r}\vec{e}_z$$

$$= \left(\frac{\partial A_\theta}{\partial r} + \frac{A_\theta}{r}\right)\vec{e}_z$$

$$B_z(r) = \frac{\partial A_\theta}{\partial r} + \frac{A_\theta}{r}$$

$$B_z(r_i) = \frac{A_\theta(r_i) - A_\theta(r_{i-1})}{\Delta r} + \frac{A_\theta(r_i)}{i\Delta r} = \frac{1}{\Delta r}\left(1 + \frac{1}{i}\right)A_\theta(r_i) - \frac{1}{\Delta r}A_\theta(r_{i-1})$$

$$\begin{bmatrix} \\ B_z \\ \\ \end{bmatrix} = \frac{1}{\Delta \gamma} \begin{bmatrix} 1+\frac{1}{1} & & & & & \\ -1 & 1+\frac{1}{2} & & & & \\ & -1 & 1+\frac{1}{3} & & & \\ & & & \ddots & & \\ & & & -1 & 1+\frac{1}{n-1} & \\ & & & & -1 & 1+\frac{1}{n} \end{bmatrix} \begin{bmatrix} A_\theta(\gamma_1) \\ \\ A_\theta \\ \\ A_\theta(\gamma_n) \end{bmatrix}$$

$$B(\gamma_1) = 2A_\theta(\gamma_1)$$

$$B(\gamma_2) = \left(1+\frac{1}{2}\right)A_\theta(\gamma_2) - A_\theta(\gamma_1)$$

$$B(\gamma_2) = B(\gamma_1) \Rightarrow \quad 1.5A_\theta(\gamma_2) - A_\theta(\gamma_1) = 2A_\theta(\gamma_1)$$

$$1.5A_\theta(\gamma_2) = 3A_\theta(\gamma_1)$$

$$A_\theta(\gamma_2) = 2A_\theta(\gamma_1)$$

$$B(\gamma_3) = B(\gamma_1) \Rightarrow \quad \left(1+\frac{1}{3}\right)A_\theta(\gamma_3) - A_\theta(\gamma_2) = 2A_\theta(\gamma_1)$$

$$\left(1+\frac{1}{3}\right)A_\theta(\gamma_3) - 2A_\theta(\gamma_1) = 2A_\theta(\gamma_1)$$

$$\frac{4}{3}A_\theta(\gamma_3) = 4A_\theta(\gamma_1)$$

$$A_\theta(\gamma_3) = 3A_\theta(\gamma_1)$$

$$\Downarrow$$

$$A_\theta(\gamma_i) = i\,A_\theta(\gamma_1)$$