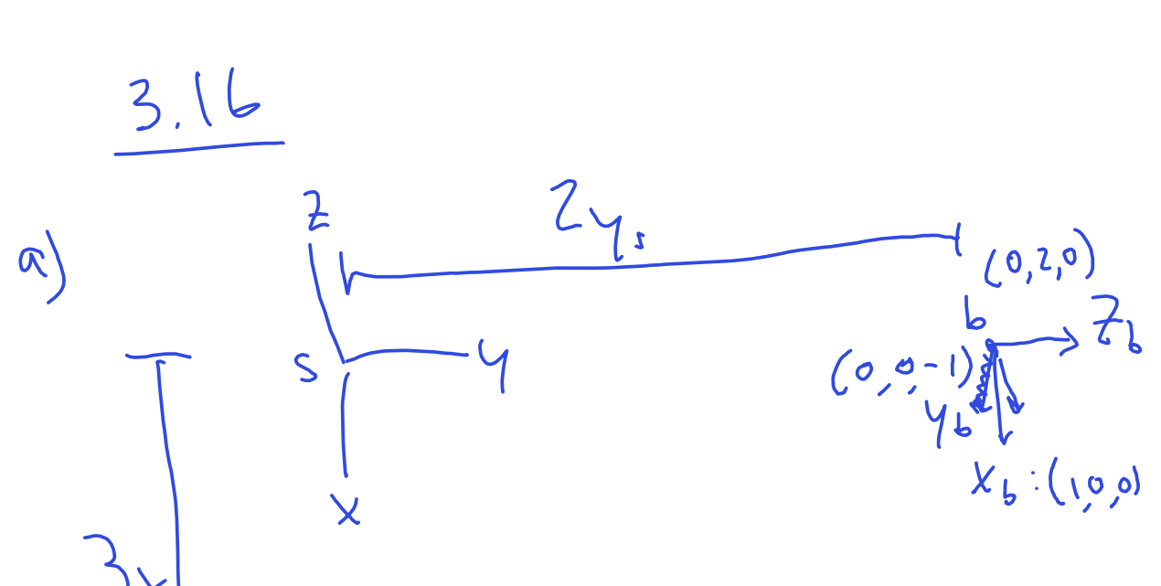
Asst 2

3.16

a)



b) $R_{sa} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}$ $R_{sb} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$

$T_{sa} = \begin{bmatrix} 0 & -1 & 0 & 3 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $T_{sb} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

c) $T_{sb}^{-1} = \begin{bmatrix} R^T & -R^T p \\ 0 \; 0 \; 0 & 1 \end{bmatrix}$

$R_{sb}^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \to -\begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix}$ ✓

$\hookrightarrow T_{sb}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

d) $T_{ab} = T_{sa}^{-1} T_{sb} = \begin{bmatrix} R_{sa}^T & -R_{sa}^T p \\ 0 & 1 \end{bmatrix} T_{sb}$

$= \begin{bmatrix} 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 3 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 3 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

e) $T = T_{sb} = \text{rot}(\hat{x}, -90^\circ) + \text{trans}(\hat{y}, 2)$

$T_1 = T_{sa} T = \begin{bmatrix} 0 & -1 & 0 & 3 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$T_1 = \begin{bmatrix} 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

T_1 should correspond to a transformation of T_{sa} w.r.t. $\hat{x}_a$ & $\hat{y}_a$ because it's a right-multiplication

$T_2 = T T_{sa} =$

$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 & 3 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 0 & -1 & 0 & 3 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

rotation in the space frame

f) $p_s = T_{sb} p_b = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix}$

$= \begin{bmatrix} 1 \\ 5 \\ -2 \\ 1 \end{bmatrix} \to (1, 5, -2)^T$

g) $p_s = (1, 2, 3)^T$, $p' = T_{sb} p_s$

$p'' = T_{sb}^{-1} p_s$

p_s' from part f)

$\boxed{p_s' = (1, 5, -2)^T} \to$ it's a movement of p in {s}

$p'' = T_{sb}^{-1} p_s = T_{bs} p_s = p_b \to$ it's changing coords from {s} to {b} frame

$p'' = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 0 \\ 1 \end{bmatrix} \to \boxed{(1, -3, 0)^T}$

h) $\mathcal{V}_s = \begin{bmatrix} 3 \\ 2 \\ 1 \\ -1 \\ -2 \\ -3 \end{bmatrix}$ ω_s, v_s $\boxed{\mathcal{V}_a = [Ad_{T_{as}}] \mathcal{V}_s}$ eq. 3.83

$T_{as} = (R, p)$

$T_{sa} = \begin{bmatrix} 0 & -1 & 0 & 3 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $[Ad_{T_{as}}] = \begin{bmatrix} R_{as} & 0 \\ [p]R_{as} & R_{as} \end{bmatrix}$

$\to T_{as} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & -3 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ (R_{as}, $-R^T p$) $\to [Ad_{T_{as}}] = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 3 & 0 & -1 & 0 \end{bmatrix}$

$[p] = \begin{bmatrix} 0 & -p_3 & p_2 \\ p_3 & 0 & -p_1 \\ -p_2 & p_1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -3 \\ 0 & 0 & 0 \\ 3 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ (R_{as})

$[Ad_{T_{as}}] \mathcal{V}_s = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 3 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \\ -1 \\ -2 \\ -3 \end{bmatrix} = \boxed{\begin{bmatrix} 1 \\ -3 \\ -2 \\ 3 \\ 1 \\ 6 \end{bmatrix} = \mathcal{V}_b}$

i) $[S]\theta$ of T_{sa} "log"$(T_{sa}) = [S]\theta \in se(3)$

following the alg. $R \neq I \to$ first find "log"(R_{sa})

$T_{sa} = \begin{bmatrix} 0 & -1 & 0 & 3 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ (R_{sa}, p_{sa})

$\text{tr}(R) = 0$ eq 3.61

$\Rightarrow \boxed{\theta = a\cos\left(\frac{1}{2}(-1)\right) = \frac{2\pi}{3}}$

$\to [\omega] = \frac{1}{2\sin\theta}(R - R^T)$ ✓

$= \frac{1}{2 \cdot \frac{\sqrt{3}}{2}} \begin{bmatrix} 0 & -1 & -1 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix} \to \hat{\omega} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \frac{1}{\sqrt{3}}$

eq 3.91 $v = G^{-1}(\theta) p$

eq 3.92 $G^{-1}(\theta) = \frac{3}{2\pi} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \frac{1}{2}\frac{1}{\sqrt{3}} \begin{bmatrix} 0 & -1 & -1 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix}$

$+ \left(\frac{3}{2\pi} - \frac{1}{2}\frac{1}{\sqrt{3}}\right) \frac{1}{3} \begin{bmatrix} 0 & -1 & -1 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 & -1 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix}$

re-arrange

$= \frac{3}{2\pi}\left(I + \frac{1}{3}[\;]\right) + \frac{1}{2\sqrt{3}}\left(-[\;] - \frac{1}{3}\begin{bmatrix} -2 & -1 & 1 \\ -1 & -2 & -1 \\ 1 & -1 & -2 \end{bmatrix}\right)$

$= \frac{3}{2\pi}\left(\begin{bmatrix} 3 & -1 & -1 \\ 1 & 3 & -1 \\ 1 & 1 & 3 \end{bmatrix} \frac{1}{3}\right) + \frac{1}{6\sqrt{3}}\left(\begin{bmatrix} 2 & 4 & 2 \\ -2 & 2 & 4 \\ -4 & -2 & 2 \end{bmatrix}\right)$

using a calculator:

$= \begin{bmatrix} 0.7 & 0.2 & 0.03 \\ -0.03 & 0.86 & 0.22 \\ -0.22 & -0.03 & 0.67 \end{bmatrix} \to v = G^{-1} p = \begin{bmatrix} 2 \\ -.1 \\ -.67 \end{bmatrix}$

$\to [S]\theta = \begin{bmatrix} [\omega]\theta & v\theta \\ 0 & 0 \end{bmatrix} =$ I got my math wrong, but python tells me it's $\begin{bmatrix} 0 & -1.2 & -1.2 & 2.2 \\ 1.2 & 0 & -1.2 & -2.2 \\ 1.2 & 1.2 & 0 & -1.4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$[S] = \frac{\quad}{\theta}$

$\Rightarrow v = 1.05, -1.05, -0.67$

$\Rightarrow \boxed{S = \begin{bmatrix} 0.57 \\ -0.57 \\ 0.57 \\ 1.05 \\ -1.05 \\ -0.67 \end{bmatrix}}$ $\boxed{\theta = \frac{2\pi}{3} \sim 2.1}$

$\mathcal{V} = S\dot{\theta} = \begin{bmatrix} \omega \\ v \end{bmatrix} = \begin{bmatrix} \hat{s}\dot{\theta} \\ -\hat{s}\dot{\theta} \times q + h\hat{s}\dot{\theta} \end{bmatrix}$ pg 101

$\hat{s} = S$ from above

$\dot{\theta} = 2.1$ from above

$h = \hat{\omega}^T v / \dot{\theta} = \sim 0.81$

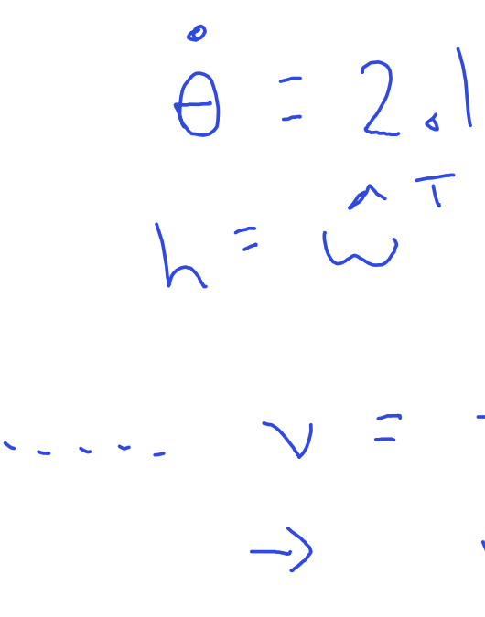
….. $v = -[\hat{s}]\dot{\theta} q + h\hat{s}\dot{\theta}$

$\to v - h\hat{s}\dot{\theta} = -[\hat{s}]\dot{\theta} q \to 3\times1$

$[s] \cdot q = -\frac{(v - h\hat{s}\dot{\theta})}{\dot{\theta}}$

using np.linalg.lstsq

$\hookrightarrow \boxed{q \approx (0.21, 0.75, 0.53)^T}$



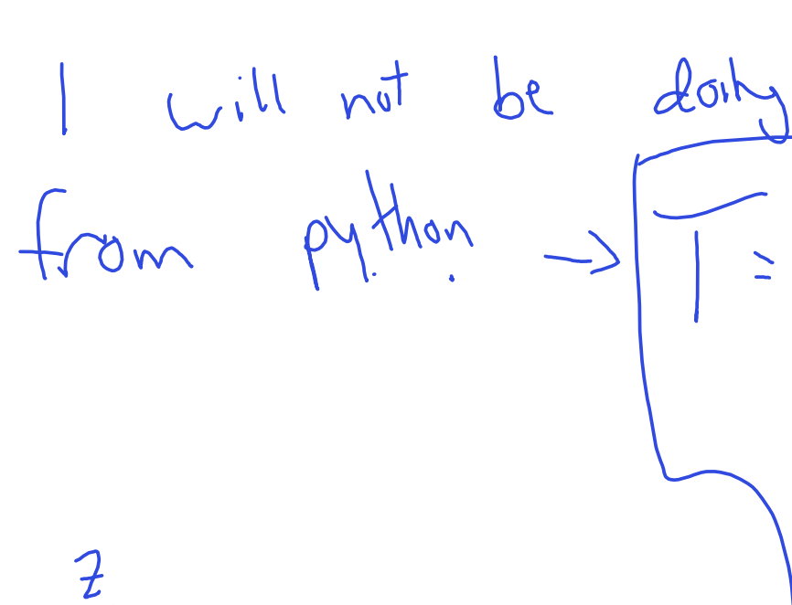
j) $S\theta = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \\ 0 \\ 0 \end{bmatrix}$ calc $\exp([S]\theta)$

eq 3.86 $e^{[S]\theta} = \begin{bmatrix} e^{[\omega]\theta} & G(\theta)v \\ 0 & 1 \end{bmatrix}$

eq. 3.87 $G(\theta) = I\theta + (1 - \cos\theta)[\omega] + (\theta - \sin\theta)[\omega]^2$

I will not be doing this by hand.

from python $\to$ $\boxed{T = \begin{bmatrix} -0.61 & -0.7 & 0.35 & 1.05 \\ 0.7 & -0.29 & 0.64 & 1.94 \\ -0.35 & 0.64 & 0.67 & -.97 \\ 0 & 0 & 0 & 1 \end{bmatrix}}$



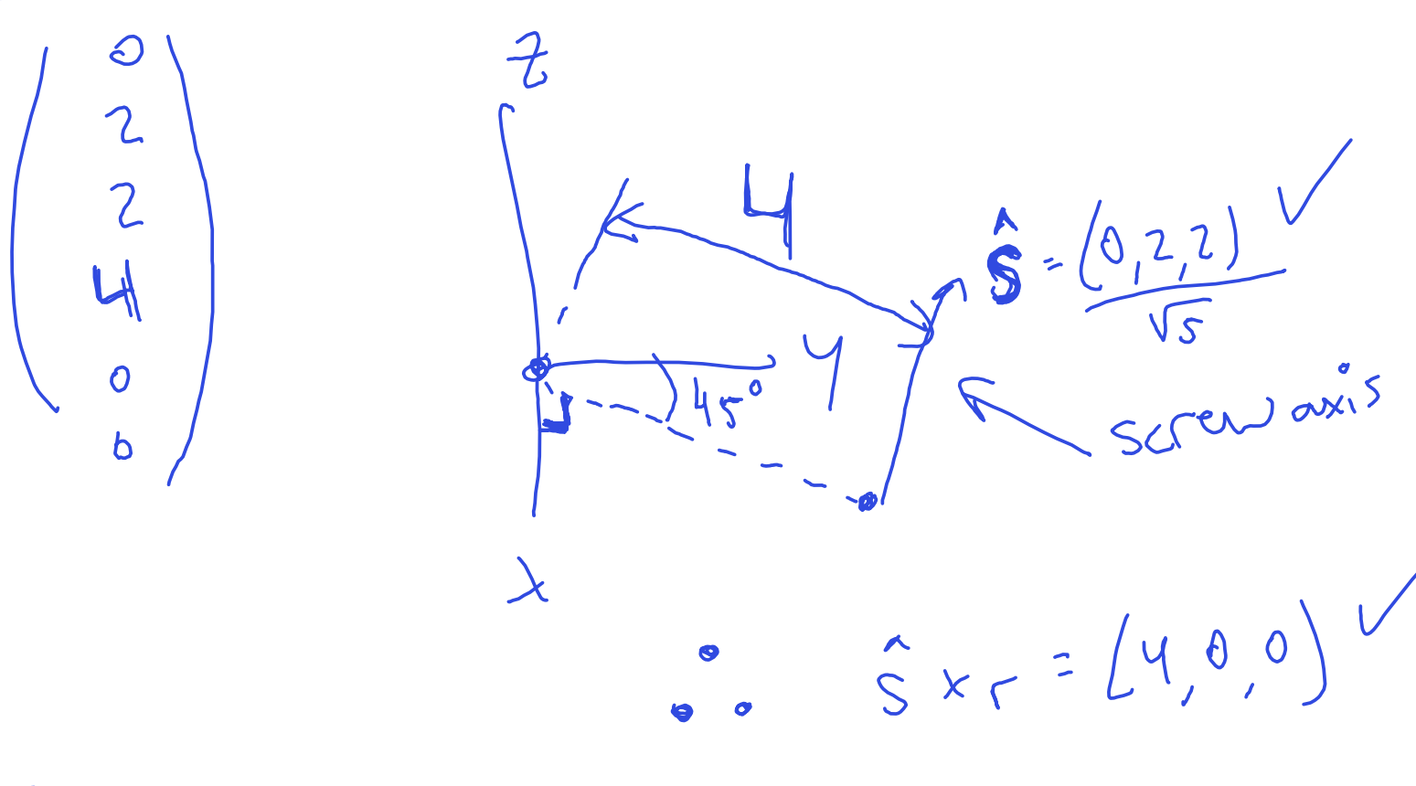
3.27

$\mathcal{V} = \begin{bmatrix} 0 \\ 2 \\ 2 \\ 4 \\ 0 \\ 0 \end{bmatrix}$

$\hat{S} = \frac{(0,2,2)}{\sqrt{5}}$ ✓

$\therefore \hat{s} \times r = (4, 0, 0)$ ✓

4.2



$M = \begin{bmatrix} I & \begin{matrix} 0 \\ l_1 + l_2 \\ l_0 \end{matrix} \\ 0 \; 0 \; 0 & 1 \end{bmatrix}$ $\# J = 4 \sim \theta \in \mathbb{R}^{4\times1}$

$\Rightarrow J(\theta) \in {}^{6\times4}$

by inspection,

$S_s = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & +l_1 & +(l_1+l_2) & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

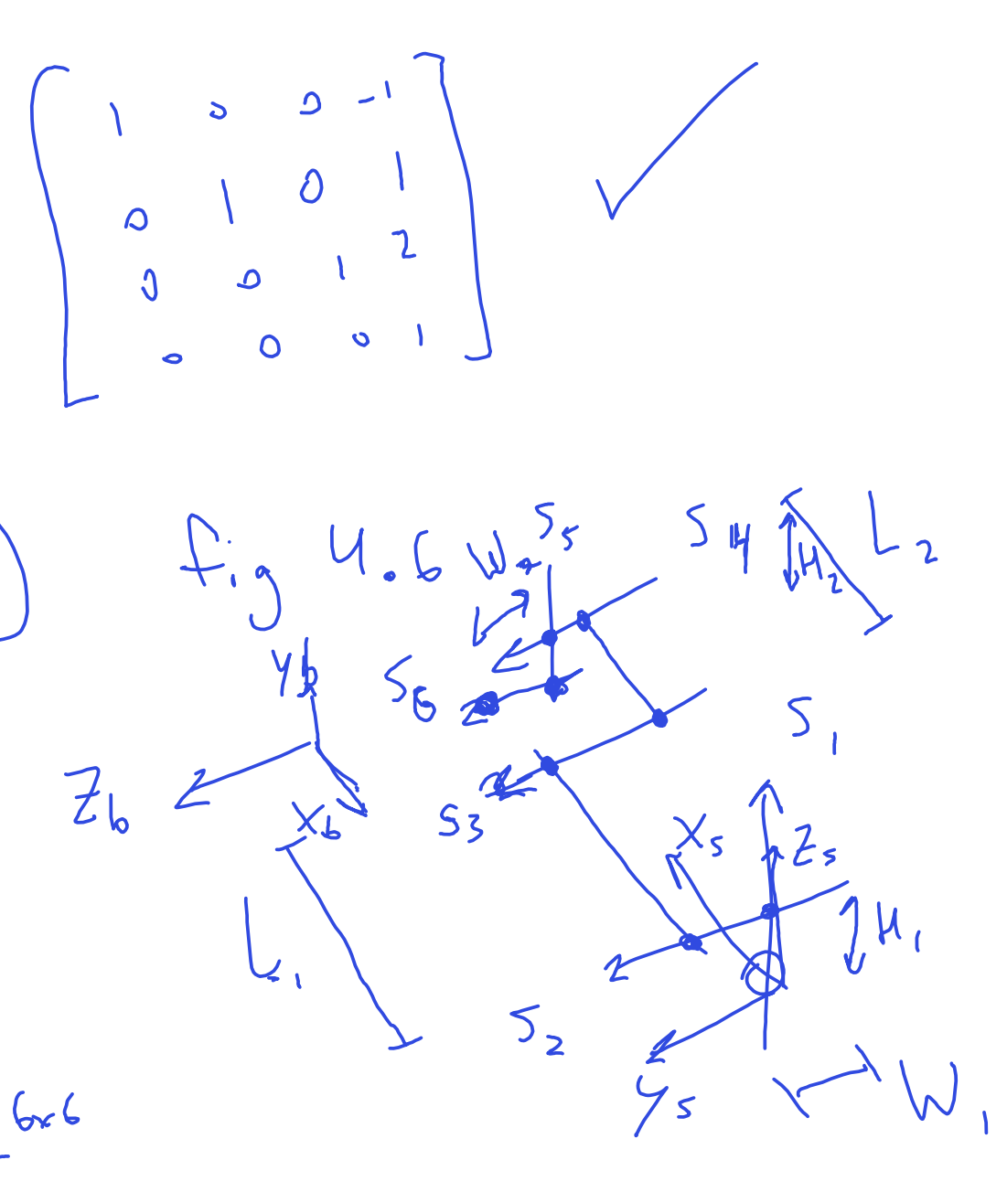
by inspection

$B'_s = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ -(l_1+l_2) & -l_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$T_{ee} = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ ✓

4.5

fig 4.6



$B's \in \mathbb{R}^{6\times6}$

$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ W_1 + W_2 & H_2 & H_2 & H_2 & -W_2 & 0 \\ 0 & -(L_1+L_2) & -L_2 & 0 & 0 & 0 \\ L_1 + L_2 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

4.10

by inspection,

$M = \begin{bmatrix} I \\ 0 \; 0 \; 0 \; 1 \end{bmatrix}$