Some Constructions in Coset Cayley Graphs in Hyperbolic Groups

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January 13, 2008

Abstract

Inside the coset Cayley graph of a quasiconvex subgroup with a specific property, there is a ball about the origin such that if w is a geodesic word based at some point p so that w lies entirely outside of the ball, and p' is some other point such that w also lies outside of the ball when based at p', then w is also geodesic based at p'.

1 Introduction

This note is largely based on work by Foord in his PhD thesis. In it, some interesting properties of coset Cayley graphs are discussed, relating to the properties as one moves further from the group in question. In fact, the constructions take place inside something more general than Cayley graphs; we will use the term "Cayley-like" here:

Definition 1.1. *Suppose Γ is a labelled directed graph with labels in some set X and the path metric. Then Γ is* ***Cayley-like*** *if, for each vertex $p \in \Gamma$ and each $x \in X$, there are edges e and e' labelled by x, so that e starts at p and e' terminates at p.*

In such a graph, for each vertex $p \in \Gamma$, and for each word w in X^*, there is a path based at p labelled by w. Clearly, Cayley graphs and coset Cayley graphs are Cayley-like. We can define a concept of isomorphism in these graphs:

Definition 1.2. *Suppose Γ'_1 and Γ'_2 are subgraphs of Cayley-like graphs Γ_1 and Γ_2 respectively that both have the same alphabet, and that $p \in \Gamma'_1$ and $q \in \Gamma'_2$ are vertices. We construct a partial map $\tau_{p,q} : \Gamma'_1 \to \Gamma'_2$ as follows:*

Given a word w in the alphabet of Γ_1, let $\tau'(w)$ be the vertex in Γ_2 reached by following the path starting from q and labelled by w. Given $x \in B_k(p)$, if we have $\tau'(w) = \tau'(u) \in \Gamma'_2$ for all words w and u labelling paths from p to x which lie entirely inside Γ'_1, let $\tau_{p,q}(x) = \tau'(w)$. If $\tau_{p,q}(x)$ is defined for all vertices in Γ'_1 and we can perform the same construction when exchanging Γ'_1 and Γ'_2, we say Γ'_1 and Γ'_2 are isomorphic about p and q.

For a vertex $p \in \Gamma_1$ we define the ball $B_k^{\Gamma_1}(p)$ to be the subgraph of Γ_1 containing all points v such that $d(v,p) \leq k+\frac{1}{2}$. We add $\frac{1}{2}$ here in order to avoid adding it every time we use a ball later on. We will omit the superscript if the graph is clear. If q is a vertex in Γ_2, we say the balls $B_k^{\Gamma_1}(p)$ and $B_k^{\Gamma_2}(q)$ are isomorphic if they are isomorphic about p and q.

If a word labels a geodesic path starting at some p in a Cayley-like graph Γ, we say it is a Γ-geodesic word at p. If it labels a geodesic regardless of starting point, we simply say that it is a Γ-geodesic word.

With Cayley graphs there is a natural base point of the identity element, and with coset Cayley graphs the natural base point is the subgroup in question. This allows us to make the following definitions:

Definition 1.3. *Suppose that Γ is a Cayley-like graph with base point b. Then for non-negative $k \in \mathbb{Z}$ we say:*

*Γ has **IB(k)** if there exists some K such that if $p,q \in \Gamma$ are vertices with $d(b,p) \geq K$ and $d(b,q) \geq K$ then $B_k(p)$ and $B_k(q)$ are isomorphic.*

*Suppose G is some finitely generated group with the same alphabet as Γ and Cayley graph Γ_G. Then Γ has **GIB(k)** if there exists some K such that if $p \in \Gamma$ is a vertex with $d(b,p) \geq K$, then $B_k^{\Gamma}(p)$ and $B_k^{\Gamma_G}(1)$ are isomorphic.*

These properties can be regarded to be some indication that the space made up by the coset Cayley graph is "mostly homogeneous." Where the presentation is understood, we will refer to a subgroup of a hyperbolic group as having IB(k) or GIB(k) if its associate coset Cayley graph does.

In his thesis, Foord proves that a quasiconvex subgroup H of a hyperbolic group G has GIB(k) for all non-negative $k \in \mathbb{Z}$ (Foord refers to this as GIB(∞)) if and only if the index $|C_G(h) : C_G(h) \cap H|$ is finite for any $h \in H$. In particular, this is true of any torsion free quasiconvex subgroup. However, no bounds on the constants involved are given, and the notions are only explored as far as is required to show some growth properties. It is the aim of this note to further expand this work.

We will suppose throughout this note that the ambient group and presentation $G = <X|R>$ has been picked, and its Cayley graph Γ is hyperbolic in the sense of thin triangles: There exists a δ such that given a geodesic triangle with corners A, B and C, and points x and y on the sides connecting to C, if $d(C,x) = d(C,y) \leq \frac{d(A,C)+d(B,C)-d(A,B)}{2}$ then $d(x,y) \leq \delta$. x and y are referred to as corresponding points. We will use the phrase x n-corresponds to y if there is a sequence $x = z_0, z_1, z_2, \dots z_n = y$ of points such that z_i corresponds (via a previously constructed triangle) to z_{i+1} for all i. More generally, we will say that points chain-correspond if there exists an $n \in \mathbb{N}$ such that they n-correspond.

We will also suppose that some subgroup H has been picked, and that H is ε-quasiconvex, so that for each Γ-geodesic w representing a word in H if we pick an integer $1 \leq i \leq |w|$ there exists $h \in H$ and a word u with $|u| \leq \varepsilon$ such that $w(i) =_G hw$. Unless stated, no other properties of H are assumed, so that constants in $O(\varepsilon)$ the same for any ε-quasiconvex subgroup of G.

2 A Tighter Bound on the Thinness of Triangles

Lemma 2.1. *Suppose that u is a Γ-geodesic word, that w is a Γ'-geodesic word at H and that u labels a loop at Hw. Then there exists a Γ-geodesic word $v \in H$ such that $wvw^{-1} =_G u$ and either*

- $|v| \leq |u| - 2|w| + 6\delta + 4\varepsilon \leq |u| + 3\delta + 2\varepsilon$ *and* $2|w| \leq |u| + 3\delta + 2\varepsilon$ *or*
- $|v| \leq 3\delta + 2\varepsilon$ *and* $|u| \leq 2\delta$ *or*
- $|v| \leq 3\delta + 2\varepsilon$ *and letting* $i = \left\lfloor \frac{|u|}{2} \right\rfloor - \delta$, *there is a word* u' *with* $|u'| \leq 5\delta + 1$ *such that* $u =_G w(-i)u'(w(-i))^{-1}$

Moreover, if $2|w| > |u| + 3\delta + 2\varepsilon$ then the quadrilateral in Γ constructed using the above words and split into any two triangles will have the meeting point on the side labelled u within δ of the midpoint of that side.

Proof. It is clear that u conjugates into H, let us pick any Γ-geodesic representing wuw^{-1} for v. Form a geodesic quadrilateral in Γ with sides v, w_1 (labelled w and connecting the points v and vw), u and w_2 (labelled w and connecting the points 1 and w) and corners $A := 1$, $B := v$, $C := vw$ and $D := w$. Pick some geodesic t connecting A to C, and we have divided the quadrilateral into 2 geodesic triangles.

Suppose that y is on w_1 and corresponds to some point y' on v. There exists some $h \in H$ such that $d(y', h) \leq \varepsilon$, so we must have $d(v, y) = d(y, H) \leq \delta + \varepsilon$ since w is a Γ'-geodesic at H. Therefore the meeting point on w_1 must lie at most $\delta + \varepsilon$ from B. Similarly, if y is on w_2 and correpsonds to some point on v, the distance from y to A must be less than or equal to $2\delta + \varepsilon$.

For the first case, suppose some point on u 2-corresponds to a point on v. Then all points on w_1 must chain-correspond to points not on w_2 and vice-versa, so if n and m are the distances from A and B to the meeting points on w_2 and w_1 respectively we find

$$\begin{aligned} |v| &= (|u| - (|w| - n) - (|w| - m)) + n + m \\ &= |u| - 2|w| + 2n + 2m. \end{aligned}$$

We know that $2|w| \geq n + m \leq 3\delta + 2\varepsilon$, so we can derive $|v| \leq |u| - 2|w| + 6\delta + 4\varepsilon$ and $|v| \leq |u| + 3\delta + 2\varepsilon$. Similarly, $|v| \geq n + m$ so we obtain

$$\begin{aligned} 2|w| &= |u| - |v| + 2n + 2m \\ &\leq |u| + 3\delta + \varepsilon. \end{aligned}$$

This is everything we need to be in the first case.

Now suppose that at most one point on u chain-corresponds to a point on v. Then clearly all points on u chain-correspond to points on w_1 or w_2, and $|u| \leq$

$2\varepsilon + 3\delta$ as required by the second two cases. Since if $|u| \leq 2\delta$, all points on u are within δ of its midpoint, it remains to show that if $|u| > 2\delta$, we satisfy the conditions of the final case.

Note that if a point p on w_1 corresponds to a point p' on t which corresponds to a point q on w_2 then $d(B,p) - 2\delta \leq d(A,q) \leq d(B,p) + 2\delta$, otherwise we can find a path from A to p or from h to q which is shorter than w, contradicting the fact that w labels a geodesic in Γ'. We can use the correspondances to show that

$$\begin{aligned} |t| - |w| &= d(A,q) - |w| + |t| - d(A,q) \\ &= d(A,q) - |w| + d(C,p') \\ &= d(A,q) - |w| + d(C,p) \\ &= d(A,q) - d(B,p) \\ &\leq 2\delta \end{aligned}$$

so $|w| - 2\delta \leq |t| \leq |w| + 2\delta$, and we find the meeting point m on u must have

$$\begin{aligned} d(D,m) &= \frac{|u| + |t| - |w|}{2} \\ &\leq \frac{|u|}{2} + \delta \end{aligned}$$

and similarly $d(C,m) \leq \frac{|u|}{2} + \delta$. Now it is clear that the points a on w_1 and b on w_2 at distance $i = \left\lfloor \frac{|u|}{2} \right\rfloor - \delta$ from C and D respectively correspond to points on u at most $2\delta + 1$ apart. So we have $d(a,b) \leq 5\delta + 1$ and letting u' be a word labelling a path between a and b, we have $u_G = w(-i)u'w^{-1}(i)$ as required. $\square$

Foord proves that the coset Cayley graph of a ε-quasiconvex subgroup of a hyperbolic group is hyperbolic, and that the slim triangles constant is at worst exponential in ε. We demonstrate here that one can reduce this bound to linear. The result is stated in terms of having slim triangles (which implies having thin triangles), but of course it's possible to improve the bound on the thin triangles constant by direct, but more involved, proof.

Proposition 2.2. *All triangles in Γ' are $10\delta + \varepsilon$-slim.*

If a triangle in Γ' has side lengths adding up to $n \in \mathbb{N}$, and at least two corners are at distance greater than $\frac{n}{2} + 3\delta + \varepsilon$ from H, then the triangle is 6δ-slim.

Proof. Suppose we are given a geodesic triangle in Γ' with corners A', B' and C', and side labels a from A' to B', b from B' to C' and c from C' to A'. Suppose that we are given some point Hp' on the side labelled by a (for points elsewhere on the triangle, we can just relabel appropriately). Our aim is to find a bound on the minimum distance between p' and and some point on the other two sides.

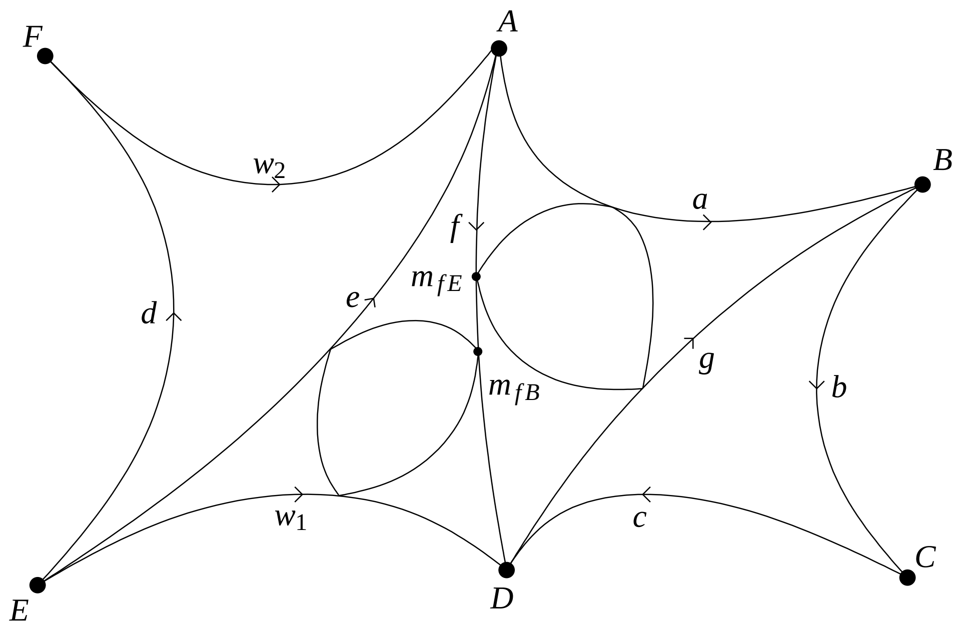

Figure 1: Dividing up the hexagon

Pick a geodesic path from H to A', and let w be its label. Clear that $Hwabcw^{-1} = H$, hence we can pick some $h \in H$ so that $wabcw^{-1} =_G h$. Let d be a geodesic path from 1 to h in Γ. Then, since any geodesic word based at a point in Γ' is a geodesic in Γ based at any point, we have a geodesic hexagon in Γ.

Label the corners of this hexagon by $A := hw$, $B := hwa$, $C := hwab$, $D := hwabc =_G w$, $E := 1$, and $F := h$. We will refer to the sides by their labels, with w_1 referring to the side connecting D and E and w_2 referring to the remaining side labelled w. Divide the hexagon into 4 triangles by picking geodesic paths with labels e from E to A, f from A to D and g from D to B.

Having constructed this triangle, we let p be the point on the side labelled a such that $Hp = Hp'$ (that is, it lies at the same distance along its respective side in Γ as x' did in Γ').

We now branch into a number of cases, based on which side of the hexagon p chain-corresponds to. We can first eliminate the side w_1 as follows:

Let m_{fE} be the meeting point on f resulting from the triangle connecting A, D and E, and let m_{fB} be the other meeting point. Note that $|e| = |hw|_\Gamma \geq |hw|_{\Gamma'} = |w|_{\Gamma'} = |w|$, and similarly $|g| \geq |a|$. Then

$$\begin{aligned} d(A, m_{fE}) &= \frac{d(A,D) + d(A,E) - d(D,E)}{2} \\ &\geq \frac{d(A,D)}{2} \end{aligned}$$

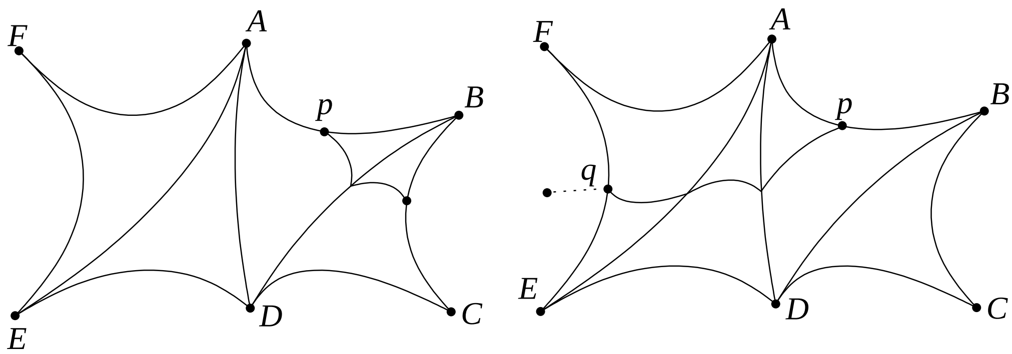

Figure 2: Cases 1 and 2

and

$$\begin{aligned} d(A, m_{fB}) &= \frac{d(A,D)+d(A,B)-d(B,D)}{2} \\ &\leq \frac{d(A,D)}{2} \end{aligned}$$

so if x 2-corresponds to a point on w_1 it also 3-corresponds to a point on either d or w_2.

We then need only consider the cases where p chain-corresponds to a point on the remaining 4 sides. When treating these cases, we note that each case occurs only for one continuous section of points on a. As we run from the corner B to A, we get to cases 1, 2, 3, 4 and 5 (assuming any points exist in each) in order.

Case 1: Suppose p 2-corresponds to a point on b or c as in the left of Figure **??**. Then clearly p' is within 2δ of a point on b' or c' as required.

Case 2: Suppose p 3-corresponds to some point q on d as in the right of **??**. Then $d(Hp',H) = d(Hp,H) \leq 3\delta+\varepsilon$, since d is a Γ'-geodesic word at H, which is ε-quasiconvex. We will treat this case further later.

It remains to cover those p which 3-correspond to some point q on w_2 as in the top left of Figure 3. We find that since $Hhw = Hw$ there is some point r on w_1 such that, when q and r are viewed as group elements, we have $Hq = Hr$. Since w labels a geodesic in Γ', we have $d(F,q) = d(E,r)$. We branch into further cases depending on which side r chain-corresponds to (note that it cannot chain-correspond to a point on a, since by previous arguments points on a cannot chain-correspond to points on w_1).

Case 3: Suppose r 2-corresponds to a point on d as in the top right of Figure 3. Then similarly to Case 2, we find $d(Hp',H) \leq 5\delta+\varepsilon$.

Case 4: Suppose r 2-corresponds to a point s on w_2 as in the bottom left of Figure 3. Then because w is a Γ'-geodesic word at H, $d(q,s) = d(Hq,Hs) \leq 2\delta$, and noting the paths taken by the correspondances we see that if m_{w_2} is the meeting

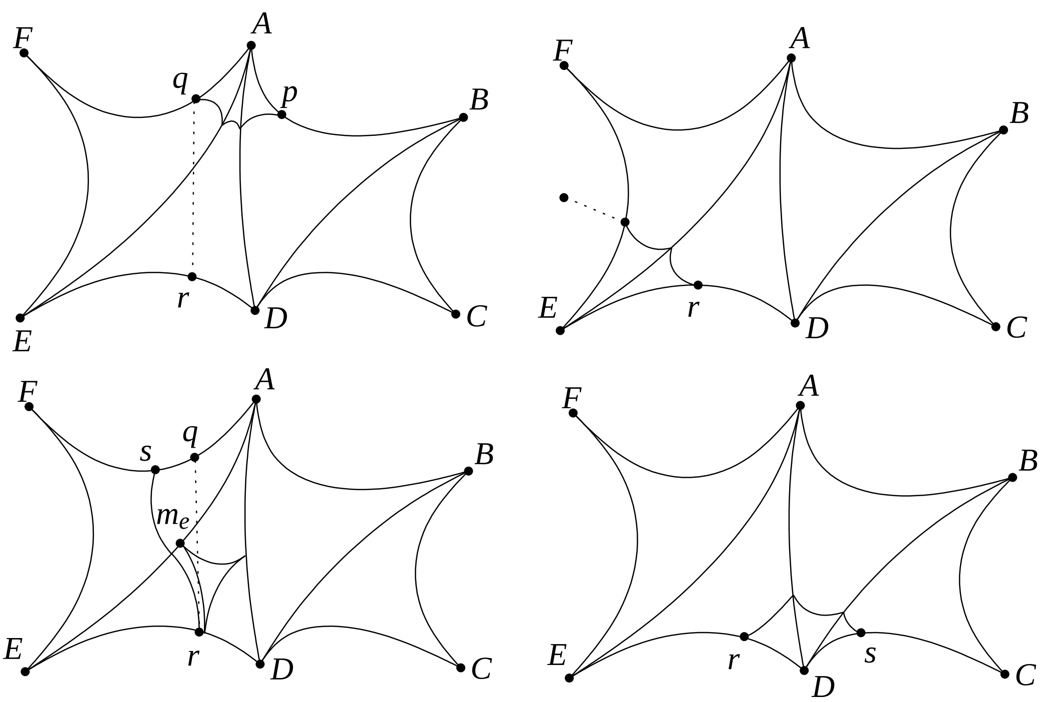

Figure 3: Cases 3, 4 and 5

point on w_2, we have $d(F,q) - 2\delta \leq d(F,s) \leq d(F,m_{w_2}) \leq d(F,q)$. There is then a 2δ long segment of w_2 in which q can reside, so the distance between any two points p following this case is at most 2δ. We treat this case, along with Cases 2 and 3 after Case 5.

Case 5: Suppose r 3-corresponds to a point s on b or c as in the bottom right of Figure 3. Then Hs lies on a different side of the triangle to Hp', and $d(Hp', Hs) \leq 6\delta$ which is within the state bound.

The maximum distance between two points p lying in Cases 2, 3 or 4 is $2(5\delta + \varepsilon) + 2\delta = 12\delta + 2\varepsilon$, so if p lies in one of these cases, it is within $8\delta + \varepsilon$ of a point t in Case 1 (so that Ht lies within 2δ of some point on another side of the triangle), or within $4\delta + \varepsilon$ of a point t in Case 5 (so that Ht is within 6δ of some point on another side of the triangle). In either case, Hp' lies within $10\delta + \varepsilon$ of a point on b or c, and we are done.

For the second part of the statement, note that we can always relabel our corners to ensure that $|w| > \frac{n}{2} + 3\delta + \varepsilon$ since the side containing x' is connected to two out of the three corners. In particular, the final part of Lemma 2.1 will apply, so points on w_i chain-corresponding to points on f must lie within $\frac{|f|}{2} + \delta \leq \frac{n}{2} + \delta$ of their respective ends of f.

In fact, since we saw in the lemma that the meeting points on w_1 and w_2 must be within $2\delta + \varepsilon$ and $\delta + \varepsilon$ of E and F respectively, we find all points on f must be intermediate points from points on a in Case 5 (or an analogue with p on b or c). Since Cases 2, 3 and 4 all require intermediate points on f, there can be no point on a which is not in either Case 1 or Case 5. Thus the triangle is 6δ-slim. $\square$

It is known (for instance, by [**?**]) that if triangles in a metric space are Δ-slim, they are 6Δ-thin, so triangles in Γ' are $60\delta + 6\varepsilon$-thin, and triangles which lie a relatively large distance from H are simply 36δ-thin.

This is the first hint that there is some ball about H where much of the behaviour occurs.

3 A Linear Bound on the GIB(k) Constant for Torsion Free Subgroups

For torsion free subgroups, Foord demonstrates the GIB(k) property for any k for a specific class of subgroups, however he gives no bound on the value of the constant associated to it. It is the aim of this section to demonstrate that for torsion-free subgroups, such a bound exists and is in $O(k\varepsilon)$. The argument specific to torsion free subgroups is again largely based on the argument given in Foord's thesis, with some observations on the specific class of groups giving us the eventual bound.

We first note the following result from Foord's thesis for which only a sketch proof is provided here:

Proposition 3.1. *Suppose x and y are vertices in some Cayley-like graph Γ. If for some non-negative integer k the balls $B_k(x)$ and $B_k(y)$ are not isomorphic, then* *there is some word w of length at most $2k+1$ which labels a loop based at the centre of one ball but a path which is not a loop based at the centre of the other.*

Proof. If the balls are not isomorphic, then the construction in Definition 1.2 must not be possible. One can break down the possibilities into cases and construct a word as required in each case. For example, suppose the construction given does not give an injective function. Find two points in $B_k(x)$ which are mapped to the same points in $B_k(y)$. Pick shortest words u and v labelling paths to these points, and uv^{-1} is a word of length at most $2k$ which labels a loop at x but not at y. □

Proposition 3.2. *If H has IB($\frac{5}{2}\delta$) with constant K then it has IB(k) for $k \geq \frac{5}{2}\delta$ with constant* $\max\{K+k-\delta+1, \varepsilon+\frac{3\delta+2k+1}{2}\}$.

If H has GIB($\frac{5}{2}\delta$) with constant K' then it has GIB(k) for $k \geq \frac{5}{2}\delta$ with constant $\max\{K'+k-\delta+1, \varepsilon+\frac{3\delta+2k+1}{2}\}$.

Proof. Suppose H has IB($\frac{5}{2}\delta$). Suppose that w and w' are Γ'-geodesics words at H with both words being longer than $\varepsilon+\frac{3\delta+2k+1}{2}$, and suppose that the k-balls around these two points are not isomorphic.

By Proposition 3.1, there is a loop of length at most $2k+1$ based at the centre of one of the balls whose label does not label a loop about the centre of the other. Suppose that this loop is present at Hw, and has label u. We can suppose that u is a Γ-geodesic word, since if it isn't, we can replace it with a word which labels a geodesic between 1 and u, and find that the new word has the same properties.

Using Lemma 2.1 we must have either $|u| \leq 2\delta$ or for $i = \left\lfloor \frac{|u|}{2} \right\rfloor - \delta$, $u =_G w(-i)u'w^{-1}(i)$ with $|u'| \leq 5\delta$. In the former case, u must lie inside $B_{2\delta}(Hw)$ which now is clearly not isomorphic to $B_{2\delta}(Hw')$, so we must in particular have $|w| \leq K$ or $|w'| \leq K$.

In the second case, note that since all group relators label loops in Γ', u' labels a loop based at $Hw(|w|-i)$ inside $B_{\frac{5}{2}\delta}(Hw(|w|-i))$ but does not label a loop based at $Hw'w(-i)$. Thus either $|w| = d(H, Hw(|w|-i)) + i \leq K+i$ or $|w'| \leq d(H, Hw'w(-i)) + i \leq K+i$. Since $i \leq k+1-\delta$, we have shown that Γ' has IB(k) with the required constant.

For the GIB case, the same method applies, but we can use 1 in Γ in place of Hw'. □

In order to find a bound on GIB(k), we now only need to exhibit a constant for GIB($\frac{5}{2}\delta$). We do this below, after stating some results which help us get to that goal; some notation and constants are introduced in the statements which will be used from this point on. First, the following is rephrased from Proposition 2.3 in [**?**]:

Proposition 3.3. *Suppose a and b are words which are conjugate in G. Let Q be the volume of the 4δ ball in Γ. Then there exists a word x such that $x^{-1}ax =_G b$, and*

$$|x| \leq |a| + |b| + Q + 4\delta,$$

where δ is the slimness constant of Γ.

This result is paraphrased also from [**?**]:

Proposition 3.4. *Define $L := 34\delta + 1$. Let w be some short-lex least word, and let $w = w_L w_R$ with $|w_L| \leq |w_R| \leq |w_L| + 1$. Let w_C be the short-lex least representative of $w_R w_L$. If w_C has length strictly greater than $2L$, there exists some integer $0 < k \leq Q^2$ and some word a whose length is less than 4δ such that $((w_C)^k)^a$ is short-lex straight.*

Moreover, k and a can be computed in time linear in $|w|$.

In particular, if w is of finite order, then $|w_C|_G \leq 2L$.

The following result is from [**?**]:

Proposition 3.5. *Let $M := 26000\delta^5 L^3 V^4$, where V is the volume of the 2δ-ball in Γ and let w be any infinite order Γ-geodesic word with $|w| \leq 2L$. Let w' be the short-lex least representative of w^M. Then $|(w')_C| > 2L$.*

Finally, also paraphrased from [**?**]:

Proposition 3.6. *Suppose that z is a short-lex straight word, and c is in its centraliser in G. Then there exists $n \in \mathbb{Z}$, a prefix z_1 of z and a word y with $|y| \leq 2\delta$ such that $c =_G z^n z_1 y$.*

Now we can move onto our result.

Proposition 3.7. *If H is any ε-quasiconvex torsion-free subgroup of G, then H has GIB($\frac{5}{2}\delta$) with constant $(M + MQ^2)(5\delta + 1) + 22\delta + Q + 2\varepsilon$.*

Proof. Suppose that w is the label of a geodesic path from starting at H, that w is strictly longer than $\varepsilon + 4\delta$, and that $B_{\frac{5}{2}\delta}(Hw)$ is not isomorphic to $B_{\frac{5}{2}\delta}(1)$ (in Γ).

As in the above proof, using Proposition 3.1 we find a word u of length at most $5\delta + 1$ which labels a geodesic in Γ but labels a loop at Hg in Γ'. Application of Lemma 2.1 gives us a Γ-geodesic word v which represents an element of H, so that $wuw^{-1} =_G v$ and $|v| \leq 3\delta + 2\varepsilon$.

By Proposition 3.3, there is a word w' such that $w'uw'^{-1} = v$ and

$$
\begin{aligned}
|w'| &\leq |u| + |v| + Q + 4\delta \\
&\leq 3\delta + 2\varepsilon + 5\delta + 1 + Q + 4\delta \\
&= Q + 12\delta + 2\varepsilon + 1.
\end{aligned}
$$

Then $wuw^{-1} =_G wuw'^{-1}$, so we can let $c =_G w'^{-1}w \in C_G(u)$.

Let u'' be the short-lex least representative of u. If $|(u'')_C| \leq 2L$, let $m = M$, else let $m = 1$. Let u' be the short-lex least representative of u^m. By Proposition 3.5 we have $|(u')_C| > 2L$, so by Proposition 3.4 we have some z, a and k with $|a| \leq 4\delta$ and $0 < k \leq Q^2$ such that $z =_G ((u')_C^k)^a$ and z is short-lex straight.

Clearly since $c \in C_G(u)$ we have $c \in C_G(u^{mk})$, so there exists a $c' \in C_G(z)$ such that $c' =_G c^{(u')_L a}$. By Proposition 3.6, $c' =_G z^n z_1 b$ for some $n \in \mathbb{N}$, z_1 a prefix of z and b a word of length at most 2δ, so we see

$$\begin{aligned} c &=_G (z^n z_1 b)^{a^{-1}(u')_L^{-1}} \\ &=_G u^{mkn}(u')_L a z_1 b a^{-1}(u')_L^{-1}. \end{aligned}$$

Note that $|z_1| \leq |z|$, and since z is straight it must be a shortest conjugate of u^{mk}. In particular, $|z_1| \leq mk|u|$.

Since the leading term of this form is some power of $u = w^{-1}u$ and we know $Hw'u = Hw'$, we have

$$\begin{aligned} d(Hw'c, Hw') &= d(Hw', Hw'u^{mkn}(u')_L a z_1 b a^{-1}(u')_L^{-1}) \\ &= d(Hw', Hw'(u')_L a z_1 b a^{-1}(u')_L^{-1}) \\ &\leq 2|(u')_L| + 2|a| + |z_1| + |b| \\ &\leq M|u| + 8\delta + MQ^2|u| + 2\delta \\ &= (M + MQ^2)(5\delta + 1) + 10\delta. \end{aligned}$$

Now since $w =_G w'c$, it is just a case of applying the triangle equality to show that

$$\begin{aligned} d(H, Hw) &= d(H, Hw'c) \\ &\leq d(H, Hw') + d(Hw', Hw'c) \\ &\leq (M + MQ^2)(5\delta + 1) + 22\delta + Q + 2\varepsilon. \end{aligned}$$

□

It is interesting to note that the factor of ε above does not depend on δ (although the leading constant does rather heavily).

4 A Stronger Sense of Local Homogeneity

In this section, we will give some results which show that not just balls but geodesics in Γ' behave in a homogeneous way when they are relatively distant from H, assuming IB($\frac{5}{2}\delta$).

Proposition 4.1. *Suppose Γ is a Cayley-like graph with base point b such that all geodesic triangles in Γ are Δ-thin. Suppose that Γ has IB($\Delta + 1$) with constant K, that w is a labels a geodesic that lies entirely outside of $B_K(b)$, and that γ is any other path labelled by w and lying entirely outside of $B_K(b)$. Then γ is a geodesic.*

Proof. Suppose the conclusion is false, and suppose the geodesic that w labels starts from p and γ starts from q. Let $w = w_1 a w_2$, where w_1 is the longest subword which does label a geodesic starting at q, and a is a word of length 1. Let w_1' be a the label of a geodesic such that $qw_1' = qw_1 a$, so that we must have $|w_1'| \leq |w_1|$.

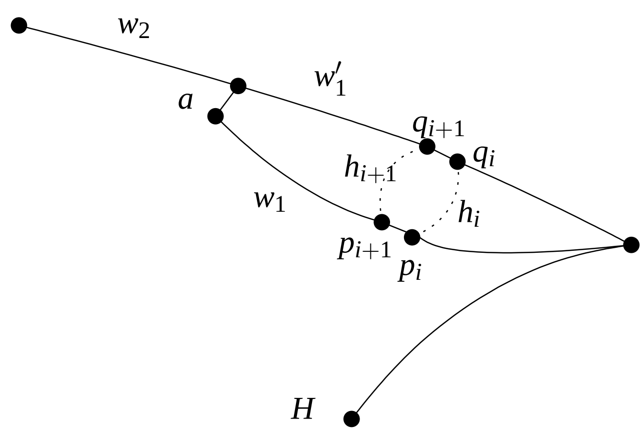

Figure 4: Geodesic triangle constructed outside of $B_{K-1}(b)$

Then we have a geodesic triangle with corners q, qw_1 and qw_1' and the obvious sides connecting them. Let $n := |w_1|$, and for $0 \leq i < n$, let $p_i := qw(i)$ and $q_i := qw_1'(i)$. Let $p_n := qw_1$ and $q_n := qw_1'$. This is illustrated in Figure 4.

Now, since the triangle above is Δ-thin, we can pick, for each i, a word h_i joining p_i and q_i so that $|h_i| \leq \Delta$. Now we find that for $0 \leq i < n$, each quadrilateral with corners p_i, p_{i+1}, q_i, q_{i+1} lies within $\Delta + 1$ of p_i, hence it is contained inside the $\Delta + 1$-ball around $qw(i)$, which is isomorphic to the $\Delta + 1$-ball around $pw(i)$ (since this point is at a distance of at least K from H).

Using a simple induction, $d(p, pw_1'(i)) = d(q, qw_1'(i)) = i$ for $0 \leq i \leq n$ and $pw_1a = pw_1'$. But this is a clear contradiction, since $|w_1a| > |w_1'|$, and w_1a labels a geodesic path starting at p. Hence no such w_1' existed, and w labels a geodesic starting at q. $\square$

By substituting the point 1 in the group Cayley graph for q in the above argument, we derive the following similar result:

Proposition 4.2. *Suppose all triangles in Γ' are Δ-thin, and H has GIB($\Delta+1$) with constant K and that w is a shortest word representing some group element. Then any path in Γ' labelled by w which lies outside of $B_{K-1}(H)$ is a geodesic.*

We see the emergence of one "bad" ball, centred at H in the coset Cayley graph. By the previous two sections, note that for torsion-free subgroups, $K \leq K'\varepsilon$, where K' depends only on the group.

It should be clear that if shortest words in the group label geodesics in the coset Cayley graph outside a certain radius, the same is true of quasigeodesics. We show this explicitly for L-local $(1,k)$-quasigeodesics, using Proposition 2.3 in [**?**]:

Proposition 4.3. *Let w be an L-local $(1,2\delta)$-quasigeodesic in Γ. Let u be a geodesic connecting the endpoints of w. Then each point on w is within 4δ of some* *vertex on u and each point on u is within* 4δ *of some vertex on w. Furthermore, if* $|w| > L$*, then* $|u| \geq \frac{7}{17}|w|$.

Lemma 4.4. *If w is a word which labels a* $(1,k)$*-quasigeodesic path in* Γ *and labels a path* γ *in* Γ' *which lies outside of* $B_{K+\frac{k}{2}+\delta-1}(H)$*, then* γ *is a* $(1,k)$*-quasigeodesic.*

Proof. Let γ be a $(1,k)$-quasigeodesic in Γ labelled by w. By [**?**] (constant hyperbolicity), α lies within $\frac{k}{2}+\delta$ of a geodesic α' with label w' say. Since any loop in Γ is present at all points in Γ', we can copy the construction over to Γ' and find that this property translates exactly. Therefore a path labelled by w based at any point in Γ' lies within $\frac{k}{2}+\delta$ of a path labelled by w' based at the same point.

If γ lies outside of $B_{K+\frac{k}{2}+\delta-1}(H)$ then the path γ' labelled by w' based at the same point (say Hg) must lie outside of $B_{K-1}(H)$, so is a geodesic by Propositon 4.2. But then $d_\gamma(Hg,Hgw) = |w| \leq |w'|+k = d(Hg,Hgw')+k$, so γ is a $(1,k)$-quasigeodesic as required. □

Lemma 4.5. *If w is a word with* $|w| \geq L$ *which labels an L-local* $(1,2\delta)$*-quasigeodesic path in* Γ *and labels a path* γ *in* Γ' *which lies outside of* $B_{K+4\delta-1}(H)$*, then* γ *is an L-local* $(1,2\delta)$*-quasigeodesic and a* $(\frac{17}{7},0)$*-quasigeodesic.*

Proof. Firstly, each subpath of γ of length at most L is clearly a $(1,2\delta)$-quasigeodesic, by Lemma 4.4, so it's clear that γ is an L-local $(1,2\delta)$-quasigeodesic.

Let α be an L-local $(1,2\delta)$-quasigeodesic in Γ labelled by w. Similarly to the previous proof, by Proposition 4.3, α is a $(\frac{17}{7},0)$-quasigeodesic, and lies within 4δ of any geodesic α' (with label w', say) between its endpoints and vice versa. Suppose γ is based at Hg, and copy the whole construction to Γ' at Hg, noting that the above distance proporties are preseved on the paths.

This implies that γ' lies entirely outside of $B_{K-1}(H)$, so is a geodesic, and thus γ must be a $(\frac{17}{7},0)$-quasigeodesic. □

We can now prove the following statement:

Corollary 4.6. *If H has GIB(*$\frac{5}{2}\delta$*), there is an algorithm which, given* $g \in G$*, can decide if there exists an* $a \in G$ *such that* $a^{-1}ga \in H$ *in time linear in* $|g|_G$ *(assuming a fixed subgroup and presentation).*

Proof. Suppose K is the constant associated to the GIB$(\Delta+1)$ property. Let w be the short-lex least representative of g, so that by 3.4, either $|w_C| \leq 2L$ or positive powers of w_C (in particular w_C^3) label L-local $(1,2\delta)$-quasigeodesics in Γ.

We first treat the case where $|w_C| > 2L$. If w_C labels a loop γ at Hb, then by Lemma 4.5 if all of γ lies outside of $B_{K+4\delta-1}(H)$ we have $d(Hb,Hbw_C^2) \geq \frac{7}{17}|w_C| > 0$. This clearly contradicts our assumption that w_C labels a loop at Hb, so at some point, γ must pass through $B_{k+4\delta-1}(H)$. Extend γ using initial and terminal subwords of w_C to a path γ' whose endpoints lie inside this ball. Clearly the label of this path is a subword of w_C^3, so it still labels an L-local $(1,2\delta)$-quasigeodesic in Γ.

Now let α be some segment of γ which lies outside of $B_{k+4\delta-1}(H)$ except for its endpoints (so α is of locally maximal length.) If the length of α is greater than L, it labels a $(\frac{17}{7},0)$-quasigeodesic and so, since its endpoints lie at distance at most $2K+8\delta-2$ apart, must be of length at most $\frac{17}{7}(2K+8\delta-2)$. In particular, letting $N=(1+\frac{17}{7})(K+4\delta-1)$, α lies inside of $B_N(H)$, so the same is true of all of γ.

Now, if w_C conjugates into H, it labels a path inside $B_N(H)$. We can check for this by, for each point in this ball, simply following the path w_C. If w_C labels a loop at Hb then $Hb = Hbw_Rw_L = Hbw_L^{-1}ww_L$ and $h = bw_L^{-1}gw_Lb^{-1}$ for some $h \in H$, so we can return w_Lb^{-1} as a conjugating element. If w_C does not label a loop at any point, it does not conjugate into H, so neither does g. Each check can be done in time linear in $|g|_G$ and the number of checks is dependent only on the subgroup and presentation, so we have proved the statement for this case.

If $|w_C| \leq 2L$ then if w_C labels a loop based at Hb in the coset Cayley graph, we must have $d(H,Hb) < K'$ (where K' is the constant associated to GIB($2L$)). We need simply check, for every word b such that $|b| < K'$ if w_C labels a loop at Hb and deal with the results as before. Clearly we are checking a number of points dependent only on the subgroup and presentation, and each check takes time dependent only on the subgroup and presentation, so we can complete this check in constant time, and we are done. ☐

We can also easily show:

Corollary 4.7. *If H has GIB($\Delta+1$) with constant K, $g,a \in G$ and $g^n \in a^{-1}Ha$, then $g^m \in a^{-1}Ha$ for some $m \leq M|B_{(1+\frac{17}{7})(K+4\delta-1)}(H)| + |B_{2\delta}(1)|$.*

Proof. Let w be the short-lex least word representing g. If $|w_C| > 2L$ then w_C^i labels an L-local $(1,2\delta)$-quasigeodesic in Γ for any i. Then since w^n labels a loop based at Ha, we find w_C^n must form a loop around Haw_L. This loop must pass into the $K+4\delta-1$-ball as in the previous proof, and so we again find it must lie entirely within $B_{(1+\frac{17}{7})(K+4\delta-1)}(H)$.

If $Haw_L(w_C)^i = Haw_L(w_C)^j$ for some integers such that $j > i$, we find $Hag^i = Hag^j$ and so $Ha = Hag^{j-i}$. But there can be at most $|B_{(1+\frac{17}{7})(K+4\delta-1)}(H)|$ distinct values for $Haw_L(w_C)^j$, hence we are done.

Suppose then that $|w_C| \leq 2L$. Then either g was of finite order (in which case there is a bound of $|B_{2\delta}(1)|$ on its order), or it is of infinite order in which case by Proposition 3.5, if u is the short-lex least representative of w_C^M, $|(u)_C| > 2L$ and we can use the above on u. ☐

Here it is interesting to note that the constant above is independent of a, thus depends only on the minimum value of the GIB constant over all subgroups conjugate to H.