Conjugacy in Hyperbolic Groups

A Short Recipe for Testing Conjugacy in Quasiconvex Subgroups

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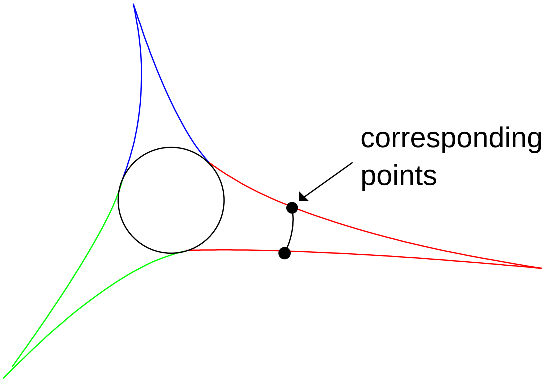

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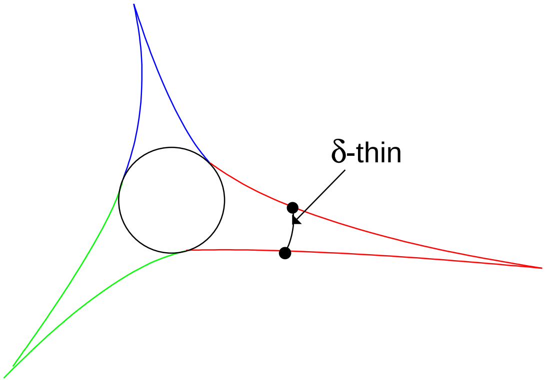

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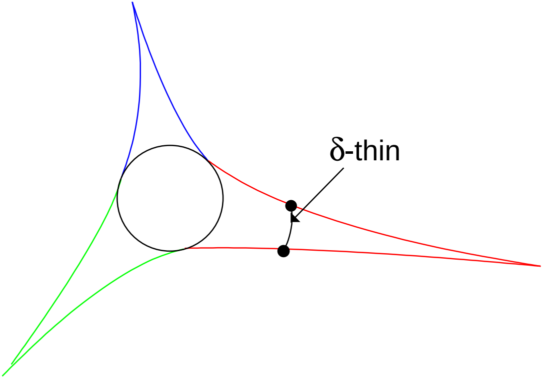

- Hyperbolic if all geodesic triangles are δ-thin.

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- Non-example: $\mathbb{Z} \times \mathbb{Z}$
- Fix $G = \langle X | R \rangle$ a finitely generated group with δ-hyperbolic Cayley graph Γ.
- Let $A = X \cup X^{-1}$.

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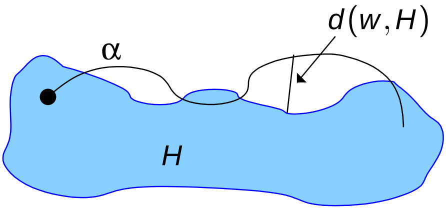

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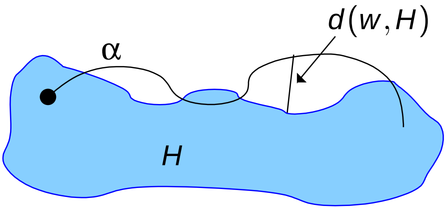

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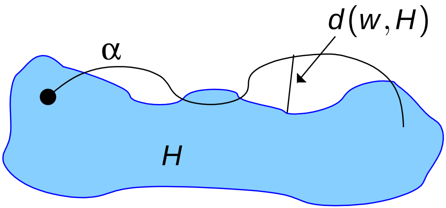

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- For instance, $<a^2>$ is a 1-quasiconvex subgroup of $F(a,b)$.

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- We assume G and H are fixed.
- Exponential running time with respect to $\delta + \varepsilon$.

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- Runs in time linear in $O(|w|)$.
- Assume that **all** words are shortest.
- If not, just run Shapiro's algorithm.

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- We therefore assume u in the problem is its own minimal conjugate.

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- Given any words $u, v \in A^*$:
    - Check if there is a $g \in A^*$ with $u =_G g^{-1}vg$.
    - Return some such g if one exists.
- Solved by Epstein and Holt (2006).
- Runs in time $O(|u| + |v|)$.

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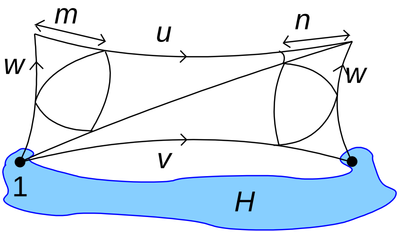
- Aim is to either:
  - find an $O(1)$ bound on the length of w, or
  - find an $O(1)$ bound on the length of v
- Bounded number of checks taking time $O(|u|)$ means $O(|u|)$ time.
- Upper bounds will all depend on constants δ and ε.

Bounding $|w|$

- Suppose a point on u corresponds to a point on v.
- m and n distances to the meeting points on w from ends of u.



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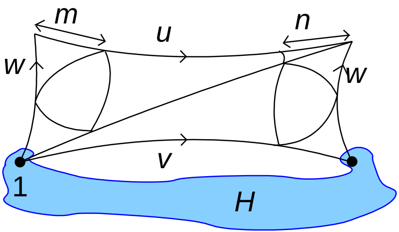

- If $\min\{m,n\} > \delta$ there is a suffix w' of w such that $|w'uw'^{-1}| < |u|$ (contradiction!).

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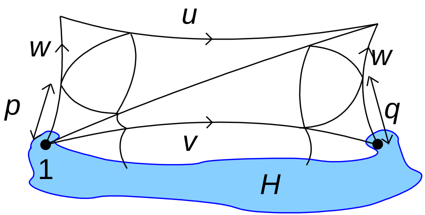

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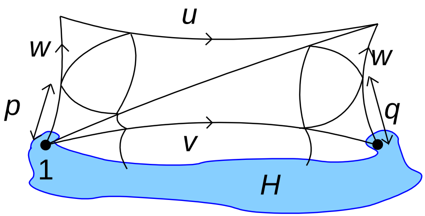

- If $\max\{p,q\} > 2\delta+\varepsilon$ there is a word $w' \in A^*$ such that $|w'| < |w|$ and $hw' =_G w$ for some $h \in H$ (contradiction!).

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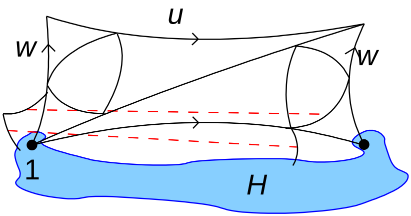

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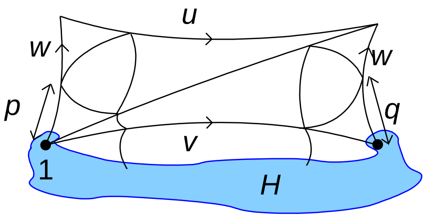

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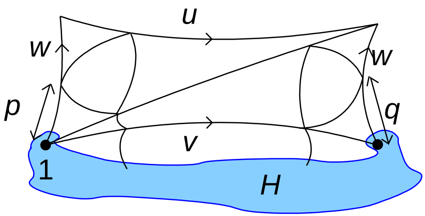

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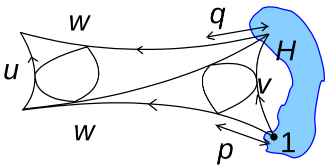
- $|w| = m+p = n+q$, so $|w| \leq 3\delta+\varepsilon$.
- $|w| \in O(1)$ and there $O(1)$ possibilities for w.
- Check if $wuw^{-1} \in H$ for all $|w| \leq 3\delta+\varepsilon$.

Bounding $|v|$

- Suppose no point on u corresponds to a point on v.



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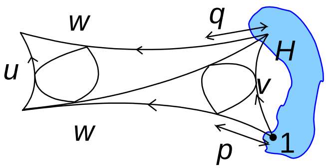

- By the same argument as before, $\max\{p,q\} \leq 2\delta + \varepsilon$.
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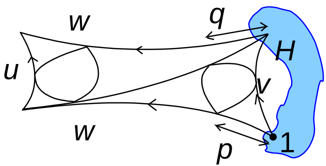

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- $|v| = p + q \le 4\delta + 2\varepsilon$.
- So $|v| \in O(1)$ and there are $O(1)$ possibilities for v.
- For each $|v| \le 4\delta + 2\varepsilon$, check if there is w such that $wuw^{-1} = v$.

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- Can with some more work extend this to checking if, given a list $a_1, \ldots, a_n$, there is one w such that $wa_iw^{-1} \in H$ for all i.

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- Useful because if there is, $<a_1, \ldots, a_n>$ conjugates into H.