Prerequisites:

Proposition 0.1. *Define $L := 34\delta + 1$. Let w be some short-lex least word, and let $w = w_L w_R$ with $|w_L| \le |w_R| \le |w_L| + 1$. Let w_C be the short-lex least representative of $w_R w_L$. If w_C has length strictly greater than $2L$, then all positive powers of w_C label L-local $(1, 2\delta)$-quasigeodesics, and there exists some integer $0 < k \le Q^2$ and some word a whose length is less than 4δ such that $((w_C)^k)^a$ is short-lex straight.*

Moreover, k and a can be computed in time linear in $|w|$.

In particular, if w is of finite order, then $|w_C|_G \le 2L$.

Proposition 0.2. *Let $M := 26000\delta^5 L^3 V^4$, where V is the volume of the 2δ-ball in Γ and let w be any infinite order Γ-geodesic word with $|w| \le 2L$. Let w' be the short-lex least representative of w^M. Then $|(w')_C| > 2L$.*

New stuff:

Lemma 0.3. *Suppose that $w =_G w_1 w_2$ is some shortest word representing an element of G. Let $k := (w_1^{-1}, w_2)_1$, and $K := d(H, Hw_1) - k - 3\delta - \epsilon$. If $K > 0$ then $d(H, Hw) \ge K + |w_2| - k$.*

Proof. Let $w_1 =_G h_1 w_1'$ and $w =_G h_2 w_2'$ where $h_1, h_2 \in H$ and w_1' and w_2' are of minimal length words for expressions of this form. Then $|w_1'| = d(H, Hw_1)$ and $|w_2'| = d(H, Hw)$.

Let $d_1 = (1, w_1)_{h_1}$, $d_2 = (1, w)_{h_2}$, $s_1 = (h_1, w_1)_1$ and $s_2 = (h_2, w)_1$. Note that $d_i \le \delta + \epsilon$, or w_i' would not be minimal as defined above. We aim to show that s_2 is not much larger than s_1.

Suppose $K' := s_2 - s_1 + d_1 - 3\delta - \epsilon > 0$. Let $K'' = min\{3\delta + \epsilon + K', |w_1'| - k\} > 3\delta + \epsilon$, and let p be the point on $[h_1, w_1]$ with $d(p, h_1) = K''$. As $d(p, h_1) > 3\delta + \epsilon > d_1$, p corresponds to a point q on $[1, w_1]$. As $d(q, w_1) = d(p, w_1) = |w_1'| - K'' \ge k$, q must correspond to a point q' on $[1, w]$.

Note that

$$\begin{aligned} d(1, q') &= d(1, q) \\ &= d(h_1, p) - d_1 + s_1 \\ &\le K' - 3\delta - \epsilon - d_1 + s_1 \\ &= s_2 \end{aligned}$$

so q' corresponds to a point on $[1, h_2]$ and we have $d(p, H) \le 3\delta + 3\epsilon$ since $h_2 \in H$. But w_1' was picked to be minimal so $d(p, H) = d(p, h_1) = K'' > 3\delta + \epsilon$, a contradiction. Thus we must have $K' \le 0$, ie. $s_2 \le s_1 - d_1 + 3\delta + \epsilon$.

Note by observing corresponding lengths that $s_1 = |h_1| - d_1 = (|w_1| - |w_1'| + 2d_1) - d_1$. Similarly, we find $|w_2'| = |w| - s_2 + d_2|$ and $|w| = |w_1| +$

$|w_2| - 2k$, so

$$\begin{aligned}
d(H, Hw) &= |w_2'| \\
&= |w| - s_2 + d_2 \\
&\geq |w| - s_1 + d_1 - 3\delta - \epsilon + d_2 \\
&= (|w_1| + |w_2| - 2k) - (|w_1| - |w_1'| + d_1) + d_1 - 3\delta - \epsilon + d_2 \\
&= |w_2| - 2k + |w_1'| - 3\delta - \epsilon + d_2 \\
&= |w_2| - k + K + d_2 \\
&\geq |w_2| - k + K
\end{aligned}$$

as required. □

Theorem 0.4. *Suppose that $g \in G$ with $g^n \in H$ for some $n \in \mathbb{N}$. Then $g^{n'} \in H$ for some $n' \leq MQ^2R$ where Q is as in Proposition 0.1, M is as in Proposition 0.2 and R is the number of vertices in the $B_{5\delta+\epsilon}(H)$ in the coset Cayley graph of H.*

Proof. Note that for some $0 < m \leq MQ^2$, g^m has a short-lex straight conjugate by Propositions 0.2 and 0.1. Let $x = pg^mp^{-1}$ be a short-lex straight conjugate word with minimal length p. We first aim to show that there is little cancellation between p and x.

Suppose $j = min\{(x,p)_1, (x^{-1}p,p)_1\} > \frac{3}{2}\delta$. Then letting $q = p(j)$, we find $|q^{-1}xq|_G < |x|_G$ which is impossible since x is straight.

Therefore, at least one of $(x,p)_1 \leq \frac{3}{2}\delta$ and $(x^{-1}p,p)_1 \leq \frac{3}{2}\delta$ must hold. Suppose $(x,p)_1 \leq \frac{3}{2}\delta$ and $(x^{-1}p,p)_1 > 2\delta$. Let $w = (x^{-1}(j))^{-1}p(j)$, and $q = wp(j)^{-1}p$. Noting that since $(x^{-1}, x^{-1}p)_1 = |x| - (x,p)_1 \geq j$, we have $|w|_G \leq 2\delta < j$. Now $|p(j)^{-1}p|_G = |p| - j$, so we see $|q|_G < |p|$. But the short-lex reduction of qg^mq^{-1} is just a cyclic conjugate of x, and hence is short-lex straight which contradicts minimality of p.

Similarly if $(x,p)_1 > \delta$ and $(x^{-1}p,p)_1 \leq 2\delta$ we will get a contradiction. Therefore, $(x,p)_1 \leq \delta$ and $(x^{-1}p,p)_1 \leq 2\delta$. We actually want to show that $(x^{-n}p, x^lp)_1 \leq 2\delta$.

Now suppose for some n that $d(H, Hp^{-1}x^n) > 5\delta + \epsilon$. Then by 0.3 we have $d(H, Hp^{-1}x^{n+l}p) > 0$ for all $l > 0$, and so no power of $g^m = p^{-1}xp$ can be an element of H. Thus $d(H, Hp^{-1}x^np) \leq 5\delta + \epsilon$ for all n.

If $Hp^{-1}x^ip = Hp^{-1}x^jp$ for some i and j, then $p^{-1}x^{i-j}p$ is in H, so if $g^{mk} \in H$ with k minimal, we must have $Hp^{-1}x^ip \neq Hp^{-1}x^jp$ for all $0 \leq i < j < mk$. Since $d(H, Hp^{-1}x^ip) \leq 5\delta + \epsilon$ for all $i \in \mathbb{N}$, we must have $k \leq R$. Let $n' = mk \leq MQ^2R$ and we are done. □

Theorem 0.5. *Suppose H is an ϵ-quasiconvex subgroup of a δ-hyperbolic group G and that $g \in G$ with $w^{-1}gw \in H$ for some $w \in G$. Then either $|h| \leq ...$ or $|w| \leq ...$.*

Theorem 0.6. *Suppose H and K are ϵ_H and ϵ_K-quasiconvex subgroups, respectively, of a δ-hyperbolic group G. Then and that $g \in G$ with $w^{-1}gw \in H$ for some $w \in G$. Then either $|h| \leq \ldots$ or $|w| \leq \ldots$.*