Subgroups of Hyperbolic Groups

A Default Beamer Presentation

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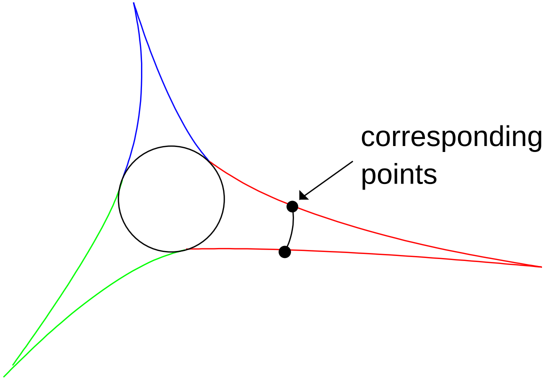

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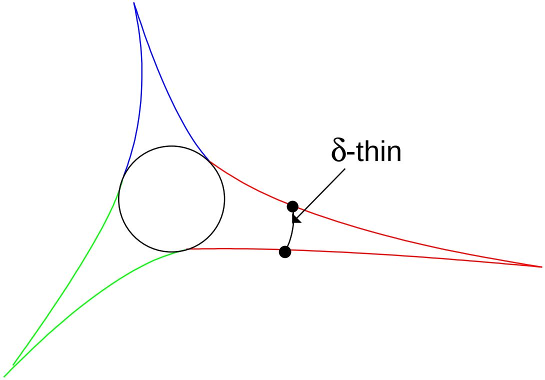

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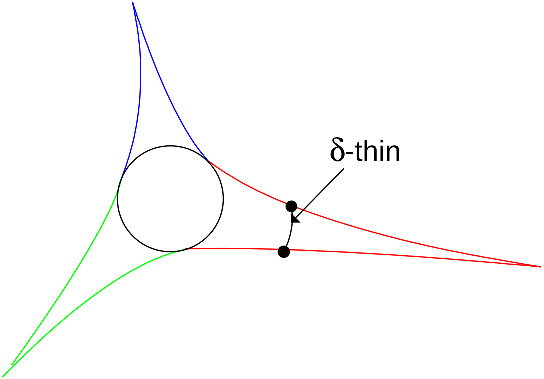

- Hyperbolic if all geodesic triangles are δ-thin.

Divergence

- Metric space X.
- A map $e : [0,\infty] \to [0,\infty]$ is a divergence function if:
  - When $r_1, r_2 \subset X$ two rays which start at the same point
  - and $d(r_1(a), r_2(a)) > e(0)$ for some $a > 0$
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- Will assume divergence is strict:
- Then $e(b) < d_{a+b}(r_1(a+b), r_2(a+b)) < ke(b)$ for all $b > 0$.

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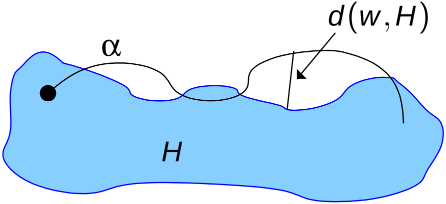
- Non-example: $\mathbb{Z} \times \mathbb{Z}$
- Fix $G = \langle X | R \rangle$ a finitely generated group with δ-hyperbolic Cayley graph Γ.
- Let $A = X \cup X^{-1}$.

Quasiconvex Subgroups

- Let w be a word in A^*.
- Let α be the path starting at 1 and labelled by w.
- Let $H < G$ any subgroup.
- $d(w,H)$ is the max distance from a point on α to closest point in H.



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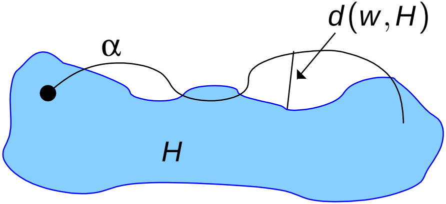

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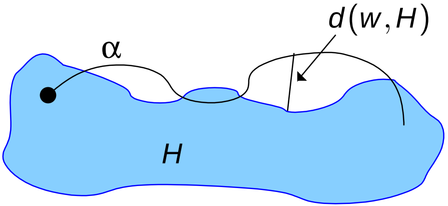

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- For instance, $<a^2>$ is a 1-quasiconvex subgroup of $F(a, b)$.

Cut points

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- Example: Every closed 1-ball in $\mathbb{R} \times [0,1]$ is a cut set.
- Y cuts set A from set B if $A \setminus Y$ and $B \setminus Y$ lie in different components of $X \setminus Y$.

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- Geodesic rays r_1, r_2 (path isometric to $[0, \infty]$)
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- Space of ends is space of equivalence classes of rays r.
- Compare to boundary - equivalence classes of geodesic rays under $r \equiv s$ if there exists M such that $d(x, s) \leq M$ for all $x \in r$ and vice versa.

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- Example: $\mathbb{Z}$ has a 2-ended Cayley graph (there's precisely 2 directions).
- Example: Non-cyclic free groups have ∞-ended Cayley graphs (any string of positive powers like *abb* can be repeated indefinitely to give a unique end).

Problem

- G is a hyperbolic group with Cayley graph Γ.
- G has 1 end.
- H is a quasiconvex subgroup.
- Γ has a strict divergence function e.
- What can we say about ends of coset Cayley graph?

Example

- Let G be the hyperbolic triangle group $\langle x, y | x^2, y^3, (xy)^7 \rangle$.
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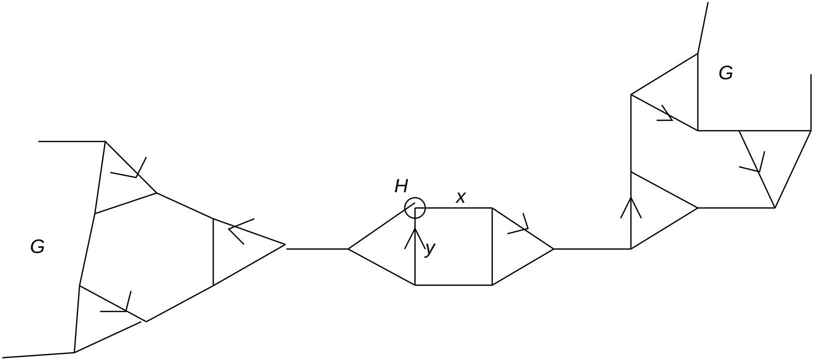

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- Also, that w is long enough (about $\varepsilon + 2\delta$).
- Suppose wa does not have much cancellation
- (Roughly, $|wa| > |w| + |a| - k$).
- Then Hwa does not have much cancellation.
- (Roughly, $d(H, Hwa) > d(H, Hw) + |a| - k - \varepsilon - 2\delta$).

Divergence Made Useful

- Divergence is a bit of a pain to use, but strict divergence is (obviously) easier!
- Suppose $d(r_1(a), r_2(a)) > e(0)$ with a minimal.
- If α is a path connecting $r_1(a+b)$ to $r_2(a+b)$, then there exists a word w of length nearly a and near-suffixes $r_1' =_G w^{-1} r_1$ and $r_2' =_G w^{-1} r_2$ such that:

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- wr_1' and wr_2' don't have much cancellation
- (Roughly $e^{-1}(k)$ where k is the k you got along with e).
- for all paths x to points in α, there is $x' =_G w^{-1}x$ such that wx' don't have much cancellation.

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- Take a huge ball $B_K(H)$.

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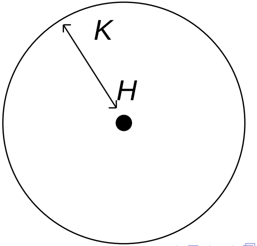
- Proof? You want proof?
- Let Hx, Hy be points in not in the ball, and in infinite components.



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- So there's only a finite number of ends of G relative to H.
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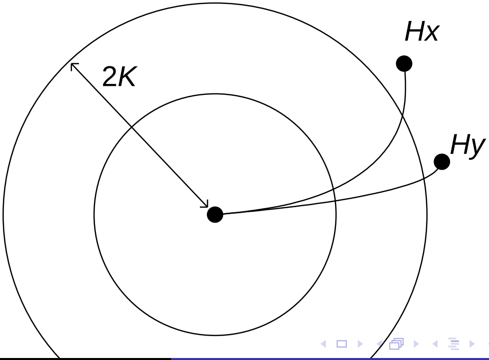
- Since they're in infinite components, may as well assume $d(Hx, H) \geq 2K$ and $d(Hy, H) \geq 2K$.



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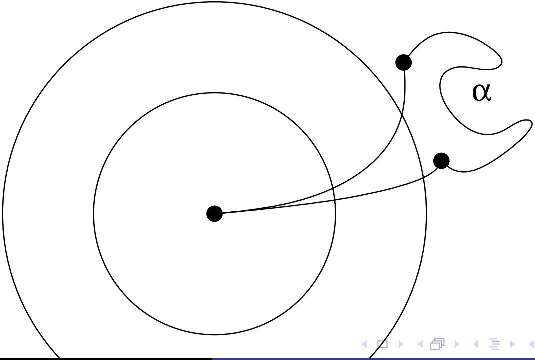
- Proof? You want proof?
- There's a path α connecting x to y in the group Cayley graph



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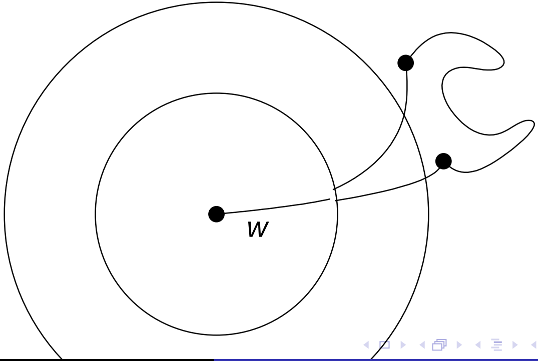
- There's a magical path w which doesn't cancel too much with x, y, or paths from 1 to points on α.



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- So there's only a finite number of ends of G relative to H.
- Proof? You want proof?
- If w is big enough, all points Hx on $H\alpha$ have $d(H, Hx)$ a bit lower than K. Thus $H\alpha$ connects Hx and Hy and lies outside the ball.
- Basically, this means the component a point Hx lies in is determined completely by its behaviour inside a small ball around H.
- There's only a finite number of short prefix words, so only a finite number of ends.