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CS 1510
Algorithm Design

Greedy Algorithms

Problems 6 and 7

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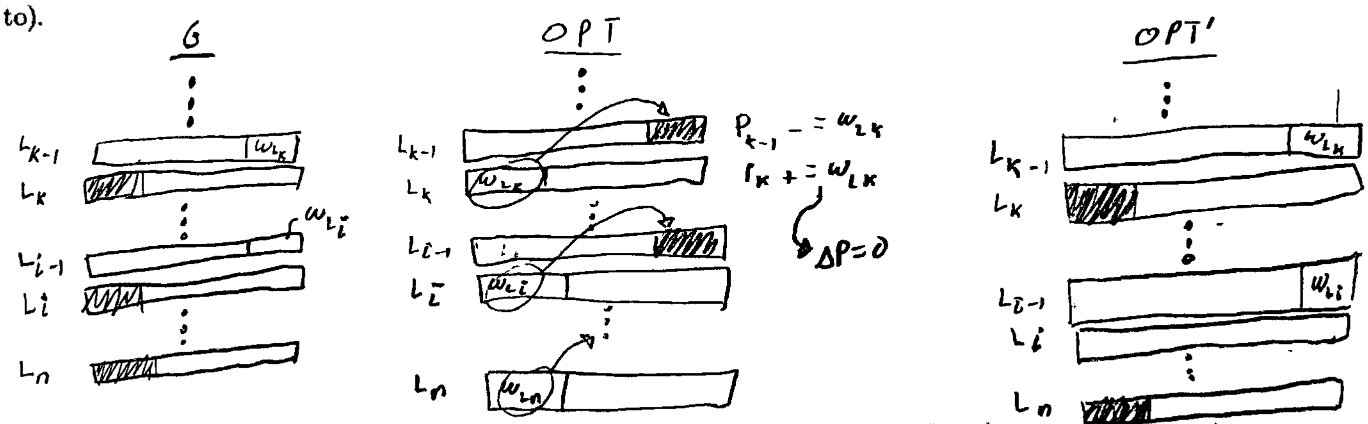
11/14

Problem 6

(a)
Input: An ordered sequence of words $S = \{w_1, w_2, ..., w_n\}$ where the length of the i^{th} word is w_i.
Output: Total penalty P as described by the problem statement (ΣP_i for each $P_i = K_i - L$) such that P is minimized.
Theorem: The given greedy algorithm G is correct.

Proof: Assume to reach a contradiction that there exists some input I on which G produces an unacceptable output. Let OPT be the optimal solution that agrees with G for the most number of steps. Let the first point of disagreement be the word w_{L_k} on line k. The only way OPT can disagree with G is if OPT moves w_{L_k} to a new line and G does not (obviously G will not fail to place a word on a line that it is able to).



Let OPT' be a modified version of OPT, such that the first word on each new line ($w_{L_k}, w_{L_{k+1}}, ..., w_{L_n}$) is moved to the previous line if possible. The logic governing this movement from the i^{th} line to the $(i-1)^{th}$ line is as follows.

for $i \geq k$ **do**
  if $w_{L_i} \leq (L - K'_{i-1})$ **then**
    if w_{L_i} is the ONLY word on the LAST line **then**
      remove the final line, as the word fits on the previous line
      P'_i changed from $L - w_i$ to 0
      $P'_{i-1} -= w_i$
      $\Delta P \leq 0$
    else
      Move w_i from L_i to L_{i-1}
      $P_i += w_i$
      $P_{i-1} -= w_i$
      $\Delta P = 0$
    end if
  else
    Cannot move w_{L_i} up
    $\Delta P = 0$
  end if
end for

English is ~~free~~ Preferred:
Do not use code

Thus, any state that OPT' can reach will result in $\Delta P = P' - P \leq 0$. This tells us that $OPT' \leq OPT$, and we have that OPT' is correct. Additionally, we know that OPT' can fit w_k on the prior line, since G made this selection. So for at least one word we are able to make OPT' more like G. Now OPT' is both correct and agrees with G for one step more than OPT.

$\therefore$ OPT' contradicts the definition of OPT, and G is correct.

(b) Input: An ordered sequence of words $S = \{w_1, w_2, ..., w_n\}$ where the length of the i^{th} word is w_i.

Output: Total penalty P as described by the problem statement (the maximum of the line penalties) such that P is minimized.

Theorem: The given greedy algorithm G correctly solves this problem.

Proof: Consider OPT as a counter-example to prove G incorrect.

L=6

G

1) b y e b y e $P_1 = \emptyset$

2) a m (P$_2$ = 4)

→ $P_G = 4$

OPT

1) b y e $P_1 = 3$

2) b y e a m $P_2 = 1$

→ $P_{OPT} = 3$

Clearly $P_G > P_{OPT}$. Since we want to minimize P, OPT is correct.

$\therefore$ Greedy algorithm G is not correct by way of this counter-example.

Problem 7

Algorithm: If a page is not in fast memory and an eviction must occur, evict the page that does not occur again or whose next use will occur farthest in the future.

Input: A sequence I = $\{P_1, P_2, ..., P_n\}$ of page requests

Output: Minimum cardinality of a sequence of evicted pages E

Theorem: The "farthest in the future" algorithm F is correct.

Proof: Assume to reach a contraction that there exists input I on which F produces unacceptable output.

Let F(I)=$\{P_{\alpha(1)}, P_{\alpha(2)}, ..., P_{\alpha(n)}\}$ be the output of F on input I
& OPT(I)=$\{P_{\beta(1)}, P_{\beta(2)}, ..., P_{\beta(n)}\}$ be the optimal solution that agrees with F(I) the most number of steps

Consider the first point of disagreement k between OPT and F

$$F(I) = \{P_1, P_2, \ldots, \overset{k}{\downarrow}{P_{\alpha k}}, \ldots\}$$
$$OPT(I) = \{P_1, P_2, \ldots, P_{\beta k}, \ldots\}$$
$$OPT'(I) = \{P_1, P_2, \ldots, P_{\alpha k}, \ldots\}$$

Let OPT' equal OPT with $P_{\alpha(k)}$ chosen at k instead of $P_{\beta(k)}$

Clearly OPT' is more similar to F (for one additional step)

To show that OPT' is still correct, consider the four possible cases for input I:

Case 1

$$I = \{P_1, P_2, \overset{k}{\downarrow}{\ldots}, P_n\}$$

In this case, neither $P_{\alpha(k)}$ nor $P_{\beta(k)}$ exist in the input after time-step k. Therefore $|E|$ and $|E'|$ do not change.

Case 2

$$I = \{P_1, P_2, \overset{k}{\downarrow}{\ldots}, P_{\beta k}, \ldots P_n\}$$

In this case, only $P_{\beta(k)}$ exists in the input after time-step k. Since $P_{\beta(k)}$ is needed at some point in the future (and OPT has evicted it at time-step k), OPT is guaranteed to need at least one additional eviction. Therefore $|E| > |E'|$.

Case 3

$$I = \{P_1, P_2, \overset{k}{\downarrow}\dots, P_{\beta k}, \dots, P_{\alpha k}, \dots, P_n\}$$

it is not clear in this case what OPT' is or why it's better.

In this case, both $P_{\alpha(k)}$ and $P_{\beta(k)}$ exist in the input after time-step k. Therefore $|E|$ must account for an eviction to bring $P_{\beta(k)}$ back into fast memory and $|E'|$ must account for an eviction to bring $P_{\alpha(k)}$ back into fast memory.

Case 4

$$I = \{P_1, P_2, \overset{k}{\downarrow}\dots, P_{\alpha k}, \dots, P_n\}$$

In this case, only $P_{\alpha(k)}$ exists in the input after time-step k. By the definition of F, if $P_{\beta(k)}$ did not exist in the input after time-step k then F would have chosen it. Therefore this is an impossible state.

In all cases, $|E| \geq |E'|$. Therefore (since we are minimizing) OPT $\leq$ OPT'

$\therefore$ We have reached a contradiction because OPT' agrees with F for one additional step (and still produces a correct output) despite OPT being defined as agreeing with F for the most number of steps.