

CS 1510
Algorithm Design

Greedy Algorithms

Problems 6 and 7

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Problem 6

(a)

Input: An ordered sequence of words $S = \{w_1, w_2, \dots, w_n\}$ where the length of the i^{th} word is w_i .

Output: Total penalty P as described by the problem statement ($\sum P_i$ for each $P_i = K_i - L$) such that P is minimized.

Theorem: The given greedy algorithm G is correct.

Proof: Assume to reach a contradiction that there exists some input I on which G produces an unacceptable output. Let OPT be the optimal solution that agrees with G for the most number of steps. Let the first point of disagreement be the word w_{L_k} on line k . The only way OPT can disagree with G is if OPT moves w_{L_k} to a new line and G does not (obviously G will not fail to place a word on a line that it is able to).

Let OPT' be a modified version of OPT , such that the first word on each new line ($w_{L_k}, w_{L_{k+1}}, \dots, w_{L_n}$) is moved to the previous line if possible. The logic governing this movement from the i^{th} line to the $(i-1)^{th}$ line is as follows.

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1: for  $i \geq k$  do
2:   if  $w_{L_i} \leq (L - K'_{i-1})$  then
3:     if  $w_{L_i}$  is the ONLY word on the LAST line then
4:       remove the final line, as the word fits on the previous line
5:        $P'_i$  changed from  $L - w_i$  to 0
6:        $P'_{i-1} - = w_i$ 
7:        $\Delta P \leq 0$ 
8:     else
9:       Move  $w_i$  from  $L_i$  to  $L_{i-1}$ 
10:       $P_i + = w_i$ 
11:       $P_{i-1} - = w_i$ 
12:       $\Delta P = 0$ 
13:    end if
14:  else
15:    Cannot move  $w_{L_i}$  up
16:     $\Delta P = 0$ 
17:  end if
18: end for

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Thus, any state that OPT' can reach will result in $\Delta P = P' - P \leq 0$. This tells us that $\text{OPT}' \leq \text{OPT}$, and we have that OPT' is correct. Additionally, we know that OPT' can fit w_k on the prior line, since G made this selection. So for at least one word we are able to make OPT' more like G . Now OPT' is both correct and agrees with G for one step more than OPT .

\therefore OPT' contradicts the definition of OPT , and G is correct.

(b) **Input:** An ordered sequence of words $S = \{w_1, w_2, \dots, w_n\}$ where the length of the i^{th} word is w_i .

Output: Total penalty P as described by the problem statement (the maximum of the line penalties) such that P is minimized.

Theorem: The given greedy algorithm G correctly solves this problem.

Proof: Consider OPT as a counter-example to prove G incorrect.

Clearly $P_G > P_{OPT}$. Since we want to minimize P , OPT is correct.

\therefore Greedy algorithm G is not correct by way of this counter-example.

Problem 7

Algorithm: If a page is not in fast memory and an eviction must occur, evict the page that does not occur again or whose next use will occur farthest in the future.

Input: A sequence $I = \{P_1, P_2, \dots, P_n\}$ of page requests

Output: Minimum cardinality of a sequence of evicted pages E

Theorem: The "farthest in the future" algorithm F is correct.

Proof: Assume to reach a contradiction that there exists input I on which F produces unacceptable output.

Let $F(I) = \{P_{\alpha(1)}, P_{\alpha(2)}, \dots, P_{\alpha(n)}\}$ be the output of F on input I
 & $OPT(I) = \{P_{\beta(1)}, P_{\beta(2)}, \dots, P_{\beta(n)}\}$ be the optimal solution that agrees with $F(I)$ the most number of steps

Consider the first point of disagreement k between OPT and F

Let OPT' equal OPT with $P_{\alpha(k)}$ chosen at k instead of $P_{\beta(k)}$

Clearly OPT' is more similar to F (for one additional step)

To show that OPT' is still correct, consider the four possible cases for input I :

Case 1

In this case, neither $P_{\alpha(k)}$ nor $P_{\beta(k)}$ exist in the input after time-step k . Therefore $|E|$ and $|E'|$ do not change.

Case 2

In this case, only $P_{\beta(k)}$ exists in the input after time-step k . Since $P_{\beta(k)}$ is needed at some point in the future (and OPT has evicted it at time-step k), OPT is guaranteed to need at least one additional eviction. Therefore $|E| > |E'|$.

Case 3

In this case, both $P_{\alpha(k)}$ and $P_{\beta(k)}$ exist in the input after time-step k . Therefore $|E|$ must account for an eviction to bring $P_{\beta(k)}$ back into fast memory and $|E'|$ must account for an eviction to bring $P_{\alpha(k)}$ back into fast memory.

Case 4

In this case, only $P_{\alpha(k)}$ exists in the input after time-step k . By the definition of F , if $P_{\beta(k)}$ did not exist in the input after time-step k then F would have chosen it. Therefore this is an impossible state.

In all cases, $|E| \geq |E'|$. Therefore (since we are minimizing) $\text{OPT} \leq \text{OPT}'$

\therefore We have reached a contradiction because OPT' agrees with F for one additional step (and still produces a correct output) despite OPT being defined as agreeing with F for the most number of steps.