

**CS 1510**  
**Algorithm Design**

Greedy Algorithms

Problems 6 and 7

Due September 3, 2014

**Buck Young and Rob Brown**

**Problem 4****(a)****Input:** Locations  $x_1, x_2, \dots, x_n$  of gas stations along a highway.**Output:** Stoppage time at each station which minimizes the total time which you are stopped for gas.**Theorem:** The given "next-station" algorithm (NS) is correct / optimal.**Proof:** Assume to reach a contradiction that there exists an input on which NS produces an unacceptable output.Let  $S_i$  be the stoppage time at each station&  $NS(I) = \{S_{\alpha(1)}, S_{\alpha(2)}, \dots, S_{\alpha(n)}\}$  be the output of NS on input  $I$ &  $OPT(I) = \{S_{\beta(1)}, S_{\beta(2)}, \dots, S_{\beta(n)}\}$  be the optimal solution that agrees with  $NS(I)$  the most number of stepsLet  $x_k$  be the first stop of the trip upon which  $NS(I)$  differs from  $OPT(I)$ By the definition of NS, filling up for any time less than  $S_{\alpha(k)}$  would not get you to the next station and since  $S_{\alpha(k)} \neq S_{\beta(k)}$ , then we know that  $S_{\beta(k)} > S_{\alpha(k)}$ Let time  $t$  be defined as the difference between those stoppage times,  $t = S_{\beta(k)} - S_{\alpha(k)}$ Therefore  $S_{\beta(k)} = S_{\alpha(k)} + t$ :Let  $OPT'(I) = OPT(I)$  with time  $t$  shifted to the next stop:

$$\begin{aligned} S'_k &= S_k - t \\ S'_{k+1} &= S_{k+1} + t \end{aligned}$$

Clearly, we haven't changed the total stoppage time and we know that we can reach the next stop by the definition of NS. Furthermore we add the difference to the stop at  $x_{k+1}$ , so we will undoubtedly be able to reach the station at  $x_{k+2}$  – and all future stations. $\therefore$  We have reached a contradiction because  $OPT'$  agrees with NS for one additional step despite  $OPT$  being defined as agreeing with NS for the most number of steps.

(b)

**Input:** Locations  $x_1, \dots, x_n$  of gas stations along a highway.

**Output:** Stoppage time at each station which minimizes the total time which you are stopped for gas.

**Theorem:** The given "fill-up" algorithm (FU) is not correct or optimal.

**Proof:**

Consider the input  $I = \{x_1, x_2\}$  where  $x_2 - x_1 = \frac{1}{2} \frac{C}{F}$

Note that to reach  $x_2$  from  $x_1$  it would take  $F(x_2 - x_1) = F(\frac{1}{2} \frac{C}{F}) = \frac{1}{2}C$  liters of gas

At  $x_1$  the tank is filled to capacity according to FU, taking  $\frac{C}{r}$  minutes. But it is possible and correct to reach  $x_2$  using only  $\frac{1}{2}C$  liters, making the optimal time  $\frac{1}{2} \frac{C}{r}$  minutes.

$\therefore$  FU is not correct or optimal.

**Problem 5**

(a)

**Input:** A set of  $n$  points on the real line  $A$  where  $a_i \in \mathbb{R}$ .

**Output:** Minimum cardinality collection  $S$  of unit intervals that cover every point in  $A$ .

**Theorem:** The given "most points" algorithm (MP) is incorrect.

**Proof:**

Consider the input  $P = \{0.5, 1.0, 1.49, 1.51, 2.0, 2.5\}$

Clearly the interval  $I_1 = (1, 2)$  covers the most points in  $P$ , so MP adds  $I_1$  to  $S$ . Now MP has no choice but to add separate intervals  $I_2 = (0.5, 1.5)$  and  $I_3 = (1.5, 2.5)$  to cover the remaining two points (0.5 and 2.5, respectively).

**Thus MP has arrived at  $S = \{I_1, I_2, I_3\}$**

Now consider  $S = \{(0.5, 1.5), (1.5, 2.5)\}$

Clearly  $\text{OPT}(P) = S$ ,  $|\text{OPT}(P)| < |\text{MP}(P)|$ , and MP is not optimal for  $P$ .

$\therefore$  MP is not correct.

(b)

**Input:** A set of  $n$  points on the real line  $A$  where  $a_i \in \mathbb{R}$ .**Output:** Minimum cardinality collection  $S$  of unit intervals that cover every point in  $A$ .**Theorem:** The given "left-most" algorithm (LM) is correct / optimal.**Proof:** Assume to reach a contradiction that there exists an input on which LM produces an unacceptable output.Let  $I_i$  be unit-intervals&  $LM(A) = \{I_{\alpha(1)}, \dots, I_{\alpha(n)}\}$  be the output of LM on input  $A$ &  $OPT(A) = \{I_{\beta(1)}, \dots, I_{\beta(n)}\}$  be the optimal solution that agrees with LM( $A$ ) for the most number of stepsLet  $a_k$  be the first point where  $LM(A)$  differs from  $OPT(A)$ By the definition of LM, we know that the interval which covers  $a_k$  is  $I_{\alpha(k)} = (a_k, a_k + 1)$ In order to define  $I_{\beta(k)}$ , we should make a few observations:

- All points to the left of  $a_k$  are covered by some interval in  $OPT$  (by the definition of LM) and  $a_k$  is not covered
- The interval  $I_{\beta(k)}$  can not start to the right of  $a_k$  (because  $a_k$  would not be covered) and  $I_{\alpha(k)} \neq I_{\beta(k)}$
- Therefore,  $I_{\beta(k)}$  must start to the **left** of  $a_k$  in  $OPT$

So for some distance  $\Delta$ , the interval in  $OPT$  which covers point  $a_k$  is  $I_{\beta(k)} = (a_k - \Delta, a_k + 1 - \Delta)$ Let  $OPT' = OPT$  with  $I_{\beta(k)}$  shifted to the right by  $\Delta$ :

$$I'_{\beta(k)} = I_{\beta(k)} + (\Delta, \Delta) = (a_k, a_k + 1) = I_{\alpha(k)}$$

As stated earlier, every point  $a_i$  where  $i < k$  has been covered by some previous interval, therefore we can safely move the interval  $I_{\beta(k)}$  to the right by a distance  $\Delta$ . This move may cause a potential overlap to the right, however this overlapping is inconsequential

Further note that  $OPT'$  is still optimal because it does not create any additional intervals and covers all the points

$\therefore$  We have reached a contradiction because  $OPT'$  agrees with LM for one additional step despite  $OPT$  being defined as agreeing with LM for the most number of steps.