CS 1510
Algorithm Design

Greedy Algorithms

Problems 4 and 5

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Problem 4

(a)
Input: Locations x_1, x_2, ..., x_n of gas stations along a highway.

Output: Stoppage time at each station which minimizes the total time which you are stopped for gas.

Theorem: The given "next-station" algorithm (NS) is correct / optimal.

Proof: Assume to reach a contradiction that there exists an input on which NS produces an unacceptable output.

Let S_i be the stoppage time at each station
& NS(I)=$\{S_{\alpha(1)}, S_{\alpha(2)}, ..., S_{\alpha(n)}\}$ be the output of NS on input I
& OPT(I)=$\{S_{\beta(1)}, S_{\beta(2)}, ..., S_{\beta(n)}\}$ be the optimal solution that agrees with NS(I) the most number of steps

Let x_k be the first stop of the trip upon which NS(I) differs from OPT(I)

By the definition of NS, filling up for any time less than $S_\alpha(k)$ would not get you to the next station and since $S_{\alpha(k)} \neq S_{\beta(k)}$, then we know that $S_{\beta(k)} > S_{\alpha(k)}$

Let time t be defined as the difference between those stoppage times, t = $S_{\beta(k)}$ - $S_{\alpha(k)}$

Therefore $S_{\beta(k)} = S_{\alpha(k)}$ + t:

Let OPT'(I) = OPT(I) with time t shifted to the next stop:

$$S'_k = S_k \text{ - t}$$
$$S'_{k+1} = S_{k+1} + t$$

Clearly, we haven't changed the total stoppage time and we know that we can reach the next stop by the definition of NS. Furthermore we add the difference to the stop at x_{k+1}, so we will undoubtably be able to reach the station at x_{k+2} – and all future stations.

∴ We have reached a contradiction because OPT' agrees with NS for one additional step despite OPT being defined as agreeing with NS for the most number of steps.

(b)
Input: Locations x_1, ..., x_n of gas stations along a highway.

Output: Stoppage time at each station which minimizes the total time which you are stopped for gas.

Theorem: The given "fill-up" algorithm (FU) is not correct or optimal.

Proof:

Consider the input I = $\{x_1, x_2\}$ where $x_2 - x_1 = \frac{1}{2}\frac{C}{F}$

Note that to reach x_2 from x_1 it would take $F(x_2 - x_1) = F(\frac{1}{2}\frac{C}{F}) = \frac{1}{2}C$ liters of gas

At x_1 the tank is filled to capacity according to FU, taking $\frac{C}{r}$ minutes. But it is possible and correct to reach x_2 using only $\frac{1}{2}C$ liters, making the optimal time $\frac{1}{2}\frac{C}{r}$ minutes.

$\therefore$ FU is not correct or optimal.

Problem 5

(a)
Input: A set of n points on the real line A where $a_i \in \mathbb{R}$.

Output: Minimum cardinality collection S of unit intervals that cover every point in A.

Theorem: The given "most points" algorithm (MP) is incorrect.

Proof:

Consider the input P = {0.5, 1.0, 1.49, 1.51, 2.0, 2.5}

Clearly the interval I_1 = (1, 2) covers the most points in P, so MP adds I_1 to S. Now MP has no choice but to add separate intervals I_2 = (0.5, 1.5) and I_3 = (1.5, 2.5) to cover the remaining two points (0.5 and 2.5, respectively).

Thus MP has arrived at S = $\{I_1, I_2, I_3\}$

Now consider S = {(0.5, 1.5), (1.5, 2.5)}

Clearly OPT(P) = S, $|OPT(P)| < |MP(P)|$, and MP is not optimal for P.

$\therefore$ MP is not correct.

(b)
Input: A set of n points on the real line A where $a_i \in \mathbb{R}$.

Output: Minimum cardinality collection S of unit intervals that cover every point in A.

Theorem: The given "left-most" algorithm (LM) is correct / optimal.

Proof: Assume to reach a contradiction that there exists an input on which LM produces an unacceptable output.

Let I_i be unit-intervals
& LM(A) = $\{I_{\alpha(1)}, ..., I_{\alpha(n)}\}$ be the output of LM on input I
& OPT(A) = $\{I_{\beta(1)}, ..., I_{\beta(n)}\}$ be the optimal solution that agrees with LM(I) for the most number of steps

Let a_k be the first point where LM(I) differs from OPT(I)

By the definition of LM, we know that the interval which covers a_k is $I_{\alpha(k)} = (a_k, a_k + 1)$

In order to define $I_{\beta(k)}$, we should make a few observations:

• All points to the left of a_k are covered by some interval in OPT (by the definition of LM) and a_k is not covered

• The interval $I_{\beta(k)}$ can not start to the right of a_k (because a_k would not be covered) and $I_{\alpha(k)} \neq I_{\beta(k)}$

• Therefore, $I_{\beta(k)}$ must start to the **left** of a_k in OPT

So for some distance Δ, the interval in OPT which covers point a_k is $I_{\beta(k)} = (a_k - \Delta, a_k + 1 - \Delta)$

Let OPT' = OPT with $I_{\beta(k)}$ shifted to the right by Δ:

$$I'_{\beta(k)} = I_{\beta(k)} + (\Delta,\, \Delta) = (a_k, a_k + 1) = I_{\alpha(k)}$$

As stated earlier, every point a_i where i < k has been covered by some previous interval, therefore we can safely move the interval $I_{\beta(k)}$ to the right by a distance Δ. This move may cause a potential overlap to the right, however this overlapping is inconsequential

Further note that OPT' is still optimal because it does not create any additional intervals and covers all the points

$\therefore$ We have reached a contradiction because OPT' agrees with LM for one additional step despite OPT being defined as agreeing with LM for the most number of steps.