

Differenciáleqgyenletek gyakorlata
2025.09.12.

① $y' = x$

$$y = \frac{x^2}{2} + C$$

② $y' = y \quad / :y$

$$\frac{y'}{y} = 1 \rightarrow \log(y) = x + C_1 \rightarrow y = e^{x+C_1} = e^x \cdot e^{C_1}$$

"ez egy hatasztrófa"
Van-e más megoldás? A megoldások konstansban kívánhatóak?

$$(ye^{-x})' = \underbrace{y'e^{-x}}_{y} + y(-e^{-x}) = 0 \rightarrow ye^{-x} = C \rightarrow y = Ce^{-x}$$

③ $xy' + 2y = 0 \rightarrow x \cdot y' = -2y \rightarrow y' = -\frac{2y}{x} \rightarrow \frac{y'}{y} = -\frac{2}{x} \rightarrow$
 $\rightarrow \log(y) = -2 \log(x) + C_1 \rightarrow y = (e^{\log x})^{C_1} = x^{-2} C_1$

④ $xy' + 2y = 3x^2 \quad \text{LDE } (a(x)y' + b(x)y = c(x))$
 homogén, ha $c(x) = 0$

Mielőtt homogén rész

$$xy' + 2y = 0 \rightarrow y_h = x^{-2} C_2$$

az eredeti inhomogen LDE-t 3 féléppelhető megoldása

i) Ha már y' elh. ja 1, akkor osztunk a hom. m. rész

$$xy' + 2y = 3x^3 / :x$$

$$y' + \frac{2}{x}y = 3x^2 / : \frac{1}{x^2}$$

$$x^2 y' + 2x y = \frac{3}{5} x^4 + C / \int \dots \quad \text{partikularis}$$

$$x^2 y = \frac{3}{5} x^5 + C$$

$$y = \frac{3}{5} x^3 + \frac{C}{x^2}$$

$$y = \underbrace{y_p}_{y_p + y_h} + y_h$$

* C_2 nélküli

ii) "Ellenzéki variálás"

Rétezésük a módt $y(x) = c(x) \cdot y_h(x)$ aláírban

$$\text{Mivel } y = c(x) \cdot \frac{1}{x^2} \rightarrow y' = c'(x)x^{-2} + c(x)(-2)x^{-3}$$

Bemutatott:

$$xy' + 2y = 3x^3$$

$$c'(x) \cdot x^{-1} + c(x) \cdot (-2)x^{-2} + 2c(x) \cdot x^{-3} = 3x^3$$

0

$$c'(x) = 3x^4$$

$$c(x) = \frac{3}{5}x^5 + C_2$$

$$x^2y = \frac{3}{5}x^5 + C_2 \rightarrow y = \frac{3}{5}x^3 + \frac{C_2}{x^2}$$

iii) részfüggvény

$$y_p = A x^3$$

\nwarrow résza/particuláris

$$y'_p = 3Ax^2$$

$$xy'_p + 2y_p = 3Ax^3 + 2Ax^3 = 5Ax^3 = 3x^3 \rightarrow$$

$$\rightarrow A = \frac{3}{5} \rightarrow y_p = \frac{3}{5}x^3$$

$$\leftarrow y = y_h + y_p = \frac{C}{x^2} + \frac{3}{5}x^3$$

5. $y' - \frac{1}{x \log(x)} y = x \log(x)$

$$y' - \frac{y}{x \log(x)} = 0 \rightarrow y' = \frac{y}{x \log(x)} \rightarrow \frac{y'}{y} = \frac{1}{x \log(x)} / \int \dots$$

$$\log(y) = \log(\log(x)) + C$$

$$y_h = \log(x) \cdot C_2$$

$$3. \quad \frac{y}{\log(x)} \cdot \frac{y}{\log^2(x)} = x$$

$$\left(y \cdot \frac{1}{\log(x)} \right)^2 = x$$

$\uparrow y_n$

$$y \cdot \frac{1}{\log(x)} = \frac{x^2}{2} + c_2 \rightarrow y = \frac{x^2 \log(x)}{2} + c_2 \log(x)$$

6.) $y' - \frac{y}{x} - 2x^2 = 0 \rightarrow y' - \frac{y}{x} = 2x^2 \quad y(1) = -1$

$$y' - \frac{y}{x} = 0 \rightarrow \frac{y'}{y} = \frac{1}{x} \rightarrow \log(y) = \log(x) + c \rightarrow y_n = cx$$

$$y' + \frac{y}{x^2} = 2x \rightarrow \frac{y}{x} = \frac{2x^2}{2} + c_2$$

$$y = x^3 + c_2 x$$

$$\frac{xy'}{x^2} = y$$

$$\underbrace{\qquad}_{\frac{y}{x}}$$

$$\left(\frac{y}{x}\right)'$$

$$y(1) = 1 + c_2 \rightarrow c_2 = -2$$

$$y = x^3 - 2x$$

7.) $y' - 2y = x - 2 \quad y(0) = 3 \quad \text{HF megoldani}$

Szukcesszív approximáció

1. integráleget a hep-ből

Rieseljük x -et von műsorba, majd int. műk 0-tól x -ig

$$\int y'(s) ds = \int 2y(s) + s - 2 ds$$

azdeti
feltételeből

$$\left[y(s) \right]_{s=0}^x = y(x) - y(0) = y(x) - 3 \rightarrow$$

azdeti felt. hozzá

/3

$$\rightarrow y(x) = 3 + \int_0^x 2y(s) + s - 2 \, ds$$

Picard-féle integrálegyenlet

2.

$$y_0(x) = y(0) = 3$$

$$y_1(x) = 3 + \int_0^x 2y_0(s) + s - 2 \, ds$$

$$y_{n+1}(x) = 3 + \int_0^x 2y_n(s) + s - 2 \, ds$$

$$y_1(x) = 3 + \int_0^x 2 \cdot 3 + s - 2 \, ds = 3 + \int_0^x s + 4 \, ds = \left[\frac{s^2}{2} + 4s \right]_0^x + 3$$

$$y_2(x) = 3 + \int_0^x 2 \left(\frac{s^2}{2} + 4s + 3 \right) + s - 2 \, ds = 3 + \int_0^x s^2 + 9s + 4 \, ds =$$

$$= 3 + \left[\frac{s^3}{3} + \frac{9}{2}s^2 + 4s \right]_0^x$$

$$y_3(x) = 3 + \int_0^x 2 \left(\frac{s^3}{3} + \frac{9}{2}s^2 + 4s \right) + s - 2 \, ds = 3 + \int_0^x \frac{2}{3}s^3 + 9s^2 + 9s + 4 \, ds =$$

$$+ 4 \, ds = \left[3 + \frac{2}{12}s^4 + \frac{9}{3}s^3 + 9s^2 + 9s \right]_0^x$$

$$y_n = \left[\frac{2^{n-2}}{3 \cdot 4 \cdots (n+1)} s^{n+1} + \dots \right]$$

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① $y' = 2y + x^2$

$y(0) = 4$

$y = 4 + \int_0^x 2y(s) + s^2 ds \quad \curvearrowleft \text{Picard}$

$y_0(x) = 4 \quad \text{is} \quad y_{n+1}(x) = 4 + \int_0^x 2y_n(s) + s^2 ds$

$y_1(x) = 4 + \int_0^x 2 \cdot 4 + s^2 ds = 4 + \left[8s + \frac{s^3}{3} \right]_0^x$

$y_2(x) = 4 + \int_0^x 2y_1(s) + s^2 ds = 4 + \int_0^x 2\left(8s + \frac{s^3}{3}\right) + s^2 ds =$
 $= 4 + \left[16 \frac{s^2}{2} + 2 \cdot \frac{8s^4}{4 \cdot 3} + \frac{s^3}{3} + 8s \right]$

$y_3(x) = 4 + \int_0^x 2\left(16s^2 + 8s^4 + \frac{s^3}{3} + 2s^4\right) + s^2 ds$

$y_n(x) = 4 + \frac{s^3}{3} + \int_0^x 2y_{n-1}(s) ds = 4 + \frac{s^3}{3} + 2 \int_0^x y_{n-1}(s) ds$

$= 4 + \frac{s^3}{3} + 2 \int_0^x \left(4 + \int_0^s 2y_{n-1}(t) dt \right) ds =$

$= 4 + \frac{s^3}{3} + 8s + \iint_0^x 4 + 2y_{n-1}(s) + s^2 ds = 4 + \frac{s^3}{3} + 8s + 2 \iint_0^x \left(\int_0^s 2y_{n-1}(t) dt \right) +$

$+ \frac{x^4}{2 \cdot 3}$

✓ 5

$$y(x) = 4 + 8x + 8x^2 + \frac{17}{2^2} \sum_{k=2}^{\infty} \frac{2^k x^k}{k} = 4 + 8x + 8x^2 + \frac{17}{2} (e^{2x} - 1 - 2x - 2x^2)$$

simultan: $y' = 2y + x^2$

homogen: $y' = 2y \rightarrow \frac{y'}{2} = 2 \rightarrow \log(y) = 2x + C \rightarrow y = Ce^{2x}$

$$\underbrace{y' e^{-2x} - 2y e^{-2x}}_{(y e^{-2x})'} = x^2 e^{-2x}$$

$$y e^{-2x} = \int x^2 e^{-2x} = -\frac{1}{2} x^2 e^{-2x} + \int \frac{1}{2} x^2 e^{-2x} dx =$$

$$= -\frac{1}{2} x^2 e^{-2x} - \frac{1}{2} x e^{-2x} + \int \frac{1}{2} x e^{-2x} dx = -\frac{1}{2} e^{-2x} (x^2 - x) +$$

$$+ \frac{1}{2} \left(-\frac{1}{2}\right) e^{-2x} + C$$

$$y = -\frac{x^2}{2} + \frac{x}{2} + \frac{1}{4} + C$$

 probafüggvény

Taylor-sor-módszer

$$y' = 2y + x^2, \text{ ha } x=0 \quad y'(0) = 2y(0) + 0 = 8$$

deriváljuk DE-t és behelyettesítik 0-t

$$y'' = 2y' + 2x \xrightarrow{x=0} y''(0) = 16$$

$$y^{(3)} = 2y'' + 2 \xrightarrow{x=0} y^{(3)}(0) = 34$$

$$y^{(4)} = 2y^{(3)} \xrightarrow{x=0} y^{(4)}(0) = 2 \cdot 34 = 2^2 \cdot 17$$

$$y^{(n)} = 2y^{(n-1)} \rightarrow y^{(n)}(0) = 2y^{(n-1)}(0) = \dots = 2^{n-3} y'''(0) = 2^{n-2} \cdot 17$$

Taylor-sor hiplete

$$\begin{aligned} y(x) &= \sum_{k=0}^{\infty} \frac{y^{(k)}}{k!} (x-x_0)^k = y(0) + y'(0)x + \frac{y''(0)}{2} x^2 + \\ &+ \sum_{k=3}^{\infty} \frac{y^{(k)}(0)}{k!} x^k = 4 + 8x + 8x^2 + \sum_{k=3}^{\infty} \frac{2^{k-2} \cdot 17}{k!} x^k = \dots = \\ &= -\frac{1}{2}x^2 - \frac{1}{2}x - \frac{1}{4} + \frac{17}{4} e^{2x} \end{aligned}$$

Határozatlanszegíthető módszer

$$y = c_0 + c_1(x-x_0) + c_2(x-x_0)^2 + \dots + c_k(x-x_0)^k + \dots$$

alakin megoldását keresünk

$$\rightarrow y' = c_1 + 2c_2(x-x_0) + \dots + kc_k(x-x_0)^{k-1} + \dots$$

|| konvergenciáságára belül lehet deriválni

$$y(x_0) = c_0, \text{ha } x_0 = 0, \text{ akkor } c_0 = y(0) = 4 \text{ és a } \widetilde{DE}$$

$$c_1 + 2c_2x + \dots + kc_kx^{k-1} + \dots = (c_0 + 2c_1x + \dots + 2c_nx^n + \dots) + x^2$$

$$c_1 = 2c_0$$

$$2c_2 = 2c_1 = 16$$

$$3c_3 = 2c_2 + 1 = 17$$

$$4c_4 = 2c_3 \rightarrow c_4 = \frac{2 \cdot 17}{3 \cdot 4}$$

$$nc_n = 2c_n \rightarrow c_n = 2^{n-3} \cdot \frac{17}{n!}$$

diff. egyenl.

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$$y(x) = 4 + 8x + 8x^2 + \sum_{k=3}^{\infty} \frac{2^{k-2} \cdot 17}{k!} x^k$$

same story as
before

oh shit, here we
go again

$$\textcircled{2} \quad y' = y^4 \cos(x) + y^3 \operatorname{tg}(x) \quad | \quad y^4 \quad (\text{Bernoulli-Jede DE})$$

$$\frac{y'}{y^4} = \cos(x) + y^3 \operatorname{tg}(x)$$

$\underbrace{}_u$

$$\parallel u(x) = [y(x)]^{-3}$$

$$\parallel u' = -3y^{-4} \cdot y'$$

$$-3y^{-4}y' = \cos(x) + y^{-3} \operatorname{tg}(x)$$

$$u' = -3 \cos(x) - 3u \operatorname{tg}(x)$$

homogen

$$\frac{u'}{u} = -3 \operatorname{tg}(x) \rightarrow \log(u) = -3 \int \frac{\sin x}{\cos^2 x} = 3 \log(\cos x) + C$$

$$x = e^{\cos^3(x) \cdot C_2}$$

$$\underbrace{u' \cos^3(x) + 3u \operatorname{tg}(x) \cdot \cos^{-3} x}_{u \cos^3 x} = -3 \cos^2(x)$$

$$u \cos^{-3} x = \int -3 \cos^2(x) = -3 \operatorname{tg}(x) + C$$

$$u = -3 \sin(x) \cos^2(x) + C \cos^3(x)$$

$$y = u^{\frac{1}{3}} = \left(-3 \sin(x) \cos^2(x) + C \cos^3(x) \right)^{-\frac{1}{3}}$$

es $y=0$ függőleg

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$$\textcircled{1} \quad y' = \frac{y^2}{\cos(x)} - y \operatorname{tg}(x) - \cos(x) \quad \text{emitt nem Bernoulli}$$

$$y' = a(x)y^2 + b(x)y + c(x) \quad \text{Riccati-egyenlet}$$

y_0 megoldás, hh

$$y_0' = a(x)y_0^2 + b(x)y_0 + c(x)$$

$$(y - y_0)' = a(x)(y^2 - y_0^2) + b(x)(y - y_0)$$

$$\underbrace{(y - y_0)}_z(y + y_0) =$$

$$\underline{y' = a(x)\frac{z}{z+2y_0} (z+2y_0) + b(x)\frac{z}{z+2y_0}} \quad \text{Bernoulli-féle DE} \quad z = y - y_0$$

$$\cos(x) = \frac{\cos^2}{\cos} - \cos \operatorname{tg} - \cos = -\sin$$

$$z = y - y_0$$

$$y = z + \cos(x)$$

$$z' - \sin(x) = \frac{(z + \cos(x))^2}{\cos(x)} - (z + \cos(x))\operatorname{tg}(x) - \cos(x)$$

$$z' - \sin(x) = \frac{z^2 + 2z\cos(x) + \cos^2(x)}{\cos(x)} - z\operatorname{tg}(x) - \sin(x) - \cos(x)$$

$$/z^2$$

$$\frac{z'}{z^2} = \frac{1}{\cos(x)} + \frac{1}{z}(2 - \operatorname{tg}(x)) / (-1)$$

$$\frac{z'}{z^2} = -\frac{1}{\cos(x)} + \frac{1}{z}(\operatorname{tg}(x) - 2)$$

$$u = \frac{1}{z} \quad u' = \frac{z'}{z^2}$$

g

$$u' = \frac{-1}{\cos^2(x)} + u(\operatorname{tg}(x) - 2)$$

homogen: $u'(\operatorname{tg}(x) - 2)$

$$\frac{u'}{u} = \operatorname{tg}(x) - 2$$

$$\log(u) = -\log(\cos(x)) - 2x + C \rightarrow u_h = C e^{-2x} \cdot \cos^{-1}(x) = \\ = C \frac{e^{-2x}}{\cos(x)}$$

$$u' + u(2 - \operatorname{tg}(x)) = -\frac{1}{\cos(x)} \quad | \cdot e^{2x} \cos(x)$$

$$e^{2x} \cos(x) (u' + u(2 - \operatorname{tg}(x))) = -e^{2x}$$

$$e^{2x} \cos(x) (u' + u(2 - \operatorname{tg}(x))) = -e^{2x}$$

$$(e^{2x} \cos(x) u)' = -e^{2x} \quad | \int dx$$

$$e^{2x} \cos(x) \cdot u = -\frac{1}{2} e^{2x} + C$$

$$u = -\frac{1}{2 \cos(x)} + \frac{C}{e^{2x} \cos(x)} = \frac{-e^{2x} + 2C}{2e^{2x} \cos(x)}$$

$$\frac{y}{u} = \frac{1}{u} = \frac{2e^{2x} \cos(x)}{-e^{2x} + 2C} \rightarrow y = \frac{2e^{2x} \cos(x)}{-e^{2x} + 2C} + \cos(x) =$$

$$= \frac{2e^{2x} \cos(x)}{-e^{2x} + 2C} + \cos(x)$$

lineáris

Bernoulli SZEPEP

Picard

\rightarrow megjűnik tovább

Azaz az
arra leg könny-
vet, amit
ajánlott
el tudod
növelni?

$$\textcircled{2} \text{ a) } y' = y^2 \rightarrow -\frac{y'}{y^2} = -1 \xrightarrow{\int} \frac{1}{y} = -x + C \rightarrow y = \frac{1}{C-x}$$

$$\text{b) } y' = \frac{1}{y^2} \rightarrow 3y^2 y' = 3 \xrightarrow{\int} y^3 = 3x + C \rightarrow y = \sqrt[3]{3x + C}$$

$$\textcircled{3} \text{ x } y' = y^2 + 4y \text{ sz\'etv\'alaszthat\'o}$$

$$\frac{y'}{y^2 + 4y} = \frac{1}{x}$$

n\'egyz\'ek meg a
hosszus megold\'as
sor\'at

$$\frac{\frac{dy}{dx}}{y^2 + 4y} = \frac{1}{x} \rightsquigarrow \int \frac{dy}{y^2 + 4y} = \int \frac{dx}{x} \rightsquigarrow \int \frac{1}{y(y+4)} dy =$$

$$= \frac{1}{4} \int \frac{1}{y} - \frac{1}{y+4} dy = \frac{1}{4} (\log(y) - \log(y+4)) + C = \frac{1}{4} \log\left(\frac{y}{y+4}\right) + C$$

$$\log\left(\frac{y}{y+4}\right) = 4 \log(x) + 4C$$

$$\frac{y}{y+4} = Cx^4 \rightarrow y = \frac{4Cx^4}{1-Cx^4} \text{ \'es } y = -4$$

\textcircled{4} Egy\'ontet\'is fokozam\'as DE

$$y' = f\left(\frac{y}{x}\right) \rightarrow u(x) = \frac{y(x)}{x} \text{ -et helyettesiteni}$$

$$y' - (ux) = y' \cdot x + u = f(u)$$

$$u' \cdot x = f(u) - u \text{ sz\'etv\'alaszthat\'o h\'ato!}$$

$$y' = \frac{2x+y}{x-y} \quad y = ux$$

$$u'x + u = \frac{2x+ux}{x-ux} = \frac{2+u}{1-u} \rightarrow u'x = \frac{2+u}{1-u} - \frac{u-u^2}{1-u} = \frac{2+u^2}{1-u}$$

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$$\frac{1-u}{2+u^2} u' = \frac{1}{x}$$

$$\int \frac{1-u}{2+u^2} du = \int \frac{dx}{x}$$

$$\int \frac{1}{2+u^2} du = \int \frac{\frac{1}{2}}{\frac{1}{2}u^2 + 1} du = \frac{\sqrt{2}}{2} \operatorname{arctg}\left(\frac{u}{\sqrt{2}}\right) + C$$

$$\int \frac{u}{2+u^2} du = \frac{1}{2} \int \frac{2u}{2+u^2} du = \frac{1}{2} \log(u^2+2) + C$$

$$\frac{\sqrt{2}}{2} \operatorname{arctg}\left(\frac{u}{\sqrt{2}}\right) - \frac{1}{2} \log(u^2+2) = \log(x) + C$$

$$e^{\frac{\sqrt{2}}{2} \operatorname{arctg}\left(\frac{u}{\sqrt{2}}\right)} e^{-\frac{1}{2} \log(u^2+2)} = \frac{e^{\frac{\sqrt{2}}{2} \operatorname{arctg}\left(\frac{u}{\sqrt{2}}\right)}}{\sqrt{e^{u^2+2}}} = C x \rightarrow \text{delete stroke}$$

$$\rightarrow e^{\sqrt{2} \operatorname{arctg}\left(\frac{u}{\sqrt{2}x}\right)} = C(y^4 + 2x^2)$$

5.

$$y' = f(ax+b)$$

$$u' = a + b y' \rightarrow y' = \frac{u' - a}{b}$$

$$\frac{u' - a}{b} = f(u) \text{ szétválasztás}$$

$$(1-2x-2y)y' = x+y+1 \quad \begin{array}{l} a=1 \\ b=1 \end{array} \quad f=x+1$$

$$y' = \frac{x+y+1}{1-2x-2y} \quad u = x+y$$

$$(1-2u)(u'-1) = u+1$$

$$(1-2u)u' = 2-u \quad (u=2 \text{ mo})$$

$$\frac{(1-2u)u'}{2-u} = 1 \rightarrow \int \frac{1-2u}{2-u} du = \int dx$$

$$\int \frac{1-2u}{2-u} = \int \frac{4-2u}{2-u} - \frac{3}{2-u} = \int 2 + \int \frac{3}{2-u} = 2u + 3\log(u-2) = x+c$$

$$e^{2u} \cdot (u-2)^3 = ce^x$$

$$e^{2(x+y)}(x+y-2)^3 = ce^x$$

$$(6.) x^3(y'-x) = y^2$$

$$x^3y' - x^4 = y^2$$

$y = zg^m$ -nel lehet-e homogen fokozániúvá tenni?

$$y' = m zg^{m-1}$$

$$m=2$$

$$y$$

$$x^3((z^2)' - x) = z^4$$

$$x^3 2z^1 - x^4 = z^4$$

$$x^3 2z^1 = z^4 + x^4$$

$$z^1 = \frac{z^4 + x^4}{x^3 2z} = \frac{(z/x)^4 + 1}{2(z/x)}$$

$$z = ux$$

$$z^1 = u'x + u$$

$$u'x + u = \frac{u^4 + 1}{2u}$$

$$u'x = \frac{u^4 + 1}{2u} - u = \frac{u^4 + 1 - 2u^2}{2u} - \frac{(u^2 - 1)^2}{2u}$$

$$\underbrace{\int \frac{2u}{(u^2 - 1)^2} du}_{f'} = \int \frac{1}{x} dx$$

$$\frac{f'}{f^2} - \frac{1}{(u^2 - 1)} = \log(x) + c \rightarrow e^{\frac{x^2}{x^2 - 1}} = cx$$

+ ellenőrizni a szabálytalan leírásból járm.
megoldást

(MF) a) $y' + \frac{y}{x} = x^2 y^4$ Bernoulli

b) $y' = \frac{y^2}{x^2} - 2 \frac{y}{x}$ Egyenlőtű egynelűhatója
fokozániú

c) $4x - 3y + y'(2y - 3x) = 0$ Egyenlőtű,
fokozániú

d) $e^{4x+y} - \frac{1}{2} = y'$ } $ax+by$

e) $e^{2x+\frac{y}{2}} - \frac{1}{2} = y'$ }

f) $y' = \frac{5xy + 2y^2}{x^2}$ (lehet, hogy nem jó)
Riccati / homo egyenlőtű fokozániú

g)

lineáris

↓
Bernoulli

↓
Riccati → inkább más hogy,
mert gyorsítva van

↓
szétválasztható
változójú