Differenciálegyenletek gyak
2025.09.12.

(1.) $y' = x$

$y = \frac{x^2}{2} + c$

(2.) $y' = y \;/ :y$

$\frac{y'}{y} = 1 \rightarrow \log(y) = x + c_1 \rightarrow \boxed{y = \exp(x) \cdot c_2}$ "ez egy katasztrófa"

Van-e más megoldás? A megoldások konstansban külö. csak (?)

$(ye^{-x})' = \underbrace{y'}_{y} e^{-x} + y(-e^{-x}) = 0 \rightarrow ye^{-x} = c \rightarrow y = ce^{-x}$

(3.) $xy' + 2y = 0 \rightarrow xy' = -2y \rightarrow y' = -\frac{2y}{x} \rightarrow \frac{y'}{y} = -\frac{2}{x} \rightarrow$

$\rightarrow \log(y) = -2\log(x) + c_1 \rightarrow y = (e^{\log x})^{-2} c_2 = x^{-2} c_2$

(4.) $xy' + 2y = 3x^2$ LDE $(a(x)y' + b(x)y = c(x))$
homogén, ha $c(x) = 0$

<u>Mo/</u> először homogén rész

$xy' + 2y = 0 \rightarrow y_h = x^{-2} c_2$

az eredeti inhomogén LDE-t 3féleképp lehet megoldani

i) Ha már y' eh-ja 1, akkor osztunk a hom. mo-sal*

$xy' + 2y = 3x^3 \;/ :x$

$y' + \frac{2}{x}y = 3x^2 \;/ : \frac{1}{x^2}$

$x^2y' + 2xy = \frac{3}{5}x^4 + c \;/ \int \ldots$

$x^2y = \frac{3}{5}x^5 + c$

$y = \frac{3}{5}x^3 + \frac{c}{x^2}$ $\rightsquigarrow y = \overbrace{y_p}^{\text{partikuláris}} + y_h$

* c_2 nélkül

ii) „Állandók variálása"

Keressük a most $y(x) = c(x) \cdot y_h(x)$ alakban

Most $y = c(x) \cdot \frac{1}{x^2} \rightarrow y' = c'(x)x^{-2} + c(x)(-2)x^{-3}$

Behelyettesítve:

$$xy' + 2y = 3x^3$$

$$c'(x) \cdot x^{-1} + \underbrace{c(x) \cdot (-2)x^{-2} + 2c(x) \cdot x^{-2}}_{0} = 3x^3$$

$$c'(x) = 3x^4$$

$$c(x) = \frac{3}{5}x^5 + c_2$$

$$x^2 y = \frac{3}{5}x^5 + c_2 \rightarrow y = \frac{3}{5}x^3 + \frac{c_2}{x^2}$$

iii) próbafüggvény

$y_p = Ax^3$

próba/partikuláris

$y_p' = 3Ax^2$

$xy_p' + 2y_p = 3Ax^3 + 2Ax^3 = 5Ax^3 = 3x^3 \rightarrow$

$\rightarrow A = \frac{3}{5} \rightarrow y_p = \frac{3}{5}x^3$

és $y = y_h + y_p = \frac{c}{x^2} + \frac{3}{5}x^3$

5. $y' - \frac{1}{x\log(x)} y = x\log(x)$

$y' - \frac{y}{x\log(x)} = 0 \rightarrow y' = \frac{y}{x\log(x)} \rightarrow \frac{y'}{y} = \frac{1}{x\log(x)} \,/ \int \ldots$

$\log(y) = \log(\log(x)) + c$

$y_h = \log(x) \cdot c_2$

$$\frac{y'}{\log(x)} - \frac{y}{x\log^2(x)} = x$$

$$\Big(y \cdot \underbrace{\frac{1}{\log(x)}}_{y_h}\Big)' = x$$

$$y \cdot \frac{1}{\log(x)} = \frac{x^2}{2} + c_2 \rightarrow y = \frac{x^2\log(x)}{2} + c_2\log(x)$$

6\. $y' - \frac{y}{x} - 2x^2 = 0 \rightarrow y' - \frac{y}{x} = 2x^2 \qquad y(1) = -1$

$$y' - \frac{y}{x} = 0 \rightarrow \frac{y'}{y} = \frac{1}{x} \rightarrow \log(y) = \log(x) + c \rightarrow y_h = cx$$

$$\underbrace{\frac{y'}{x} - \frac{y}{x^2}}_{\underbrace{\frac{xy' - y}{x^2}}_{\left(\frac{y}{x}\right)'}} = 2x \rightarrow \frac{y}{x} = \frac{2x^2}{2} + c_2$$

$$y = x^3 + c_2 x$$

$$y(1) = 1 + c_2 \rightarrow c_2 = -2$$

$$y = x^3 - 2x$$

7\. $y' - 2y = x - 2 \qquad y(0) = 3$ HF megoldani

Szukcesszív approximáció

$\boxed{1.}$ integráljuk a kép-ből

Kicseréljük x-et vmi másra, majd int-unk 0-tól x-ig

$$\int_0^x y'(s)\,ds = \int_0^x 2y(s) + s - 2\,ds$$

kezdeti feltételből

$$\Big[y(s)\Big]_{s=0}^{s=x} = y(x) - y(0) = \underbrace{y(x) - 3}_{\text{kezdeti felt. haszn}} \longrightarrow$$

$$\rightarrow y(x) = 3 + \int_0^x 2y(s) + s - 2\, ds$$

Picard-féle integrálegyenlet

$\boxed{2.}$ $y_0(x) = y(0) = 3$

$$y_1(x) = 3 + \int_0^x 2y_0(s) + s - 2\, ds$$

$\vdots$

$$y_{n+1}(x) = 3 + \int_0^x 2y_n(s) + s - 2\, ds$$

$$y_1(x) = 3 + \int_0^x 2 \cdot 3 + s - 2\, ds = 3 + \int_0^x s + 4\, ds = \left[\frac{s^2}{2} + 4s\right]_0^x + 3$$

$$y_2(x) = 3 + \int_0^x 2\left(\frac{s^2}{2} + 4s + 3\right) + s - 2\, ds = 3 + \int_0^x s^2 + 9s + 4\, ds =$$

$$= 3 + \left[\frac{s^3}{3} + \frac{9}{2}s^2 + 4s\right]_0^x$$

$$y_3(x) = 3 + \int_0^x 2\left(\frac{s^3}{3} + \frac{9}{2}s^2 + 4s \cancel{+3}\right) + s - 2\, ds = 3 + \int_0^x \frac{2}{3}s^3 + 9s^2 + 9s$$

$$+ 4)\, ds = \left[3 + \frac{2}{12}s^4 + \frac{9}{3}s^3 + 9s\right]_0^x$$

$\vdots$

$$y_n = \left[\frac{2^{n-2}}{3 \cdot 4 \cdots (n+1)} s^{n+1} + \ldots\right.$$

2025.09.19.

1. $y' = 2y + x^2$

$y(0) = 4$

$y = 4 + \int_0^x 2y(s) + s^2\,ds$ ← Picard

$y_0(x) = 4$ és $y_{n+1}(x) = 4 + \int_0^x 2y_n(s) + s^2\,ds$

$y_1(x) = 4 + \int_0^x 2\cdot 4 + s^2\,ds = 4 + \left[8s + \frac{s^3}{3}\right]_0^x$

$y_2(x) = 4 + \int_0^x 2y_1(s) + s^2\,ds = 4 + \int_0^x 2\left(8s + \frac{s^3}{3} + 4\right) + s^2\,ds =$

$= 4 + \left[16\frac{s^2}{2} + 2\frac{s^4}{4\cdot 3} + \frac{s^3}{3} + 8s\right]_0^x$

$y_3(x) = 4 + \int_0^x 2\left(4 + 8s + 8s^2 + \frac{s^3}{3} + 2\frac{s^4}{3\cdot 4}\right) + s^2$

$y_n(x) = 4 + \frac{s^3}{3} + \int_0^x 2\cdot y_{n-1}(s)\,ds = 4 + \frac{s^3}{3} + 2\int_0^x y_{n-1}(s)\,ds$

$= 4 + \frac{s^3}{3} + 2\int_0^x \left(4 + \int_0^x 2y_{n-1}(s) + s^2\,ds\right) =$

$= 4 + \frac{s^3}{3} + 8s + \int_0^x\int_0^x 4 + 2y_{n-1}(s) + \int_0^x\int_0^x s^2\,ds = 4 + \frac{s^3}{3} + 8s + 2\iiint_0^x \left(\int_0^x 2y_n(s) + s^2\,ds\right) +$

$+ \frac{x^4}{2\cdot 3}$

$$y(x)=4+8x+8x^2+\frac{17}{2^2}\sum^{\infty}\frac{2^k x^k}{k}=4+8x+8x^2+\frac{17}{2}\left(e^{2x}-1-2x-2x^2\right)$$

sima mo. $y'=2y+x^2$

homogén: $y'=2y \to \frac{y'}{y}=2 \to \log(y)=2x+C \to y=Ce^{2x}$

$$\underbrace{y'e^{-2x}-2ye^{-2x}}_{(ye^{-2x})'}=x^2e^{-2x}$$

$$ye^{-2x}=\int \underset{f}{x^2}\,\underset{g'}{e^{-2x}}=-\frac{1}{2}x^2e^{-2x}+\int \underset{f}{2x}\,\underset{g'}{\left(\frac{1}{2}e^{-2x}\right)}dx=$$

$$=-\frac{1}{2}x^2e^{-2x}-\frac{1}{2}xe^{-2x}+\int \frac{1}{2}e^{-2x}dx=-\frac{1}{2}e^{-2x}\left(x^2-x\right)+$$

$$+\frac{1}{2}\cdot\left(-\frac{1}{2}\right)e^{-2x}+c$$

$$y=-\frac{x^2}{2}+\frac{x}{2}+\frac{1}{4}+c$$

próbafüggvény

$\boxed{\text{Taylor-sor-módszer}}$

$y'=2y+x^2$, ha $x=0$ $y'(0)=2y(0)+0=8$

deriváljuk DE-t és behelyettesítek 0-t

$$y''=2y'+2x \xrightarrow[x=0]{} y''(0)=16$$

$$y^{(3)}=2y''+2 \xrightarrow{x=0} y^{(3)}(0)=34$$

$$y^{(4)}=2y^{(3)} \xrightarrow{x=0} y^{(4)}(0)=2\cdot 34=2^2\cdot 17$$

$$\vdots$$

$$y^{(n)}=2y^{(n-1)} \longrightarrow y^{(n)}(0)=2y^{(n-1)}(0)=\ldots=2^{n-3}y'''(0)=$$

$$=2^{n-2}\cdot 17$$

Taylor-sor képlete

$$y(x)=\sum_{k=0}^{\infty}\frac{y^{(k)}}{k!}(x-x_0)^k=y(0)+y'(0)x+\frac{y''(0)}{2}\cdot x^2+$$

$$+\sum_{k=3}^{\infty}\frac{y^{(k)}(0)}{k!}x^k=4+8x+8x^2+\sum_{k=3}^{\infty}\frac{2^{k-2}\cdot 17}{k!}x^k=\ldots=$$

$$=-\frac{1}{2}x^2-\frac{1}{2}x-\frac{1}{4}+\frac{17}{4}e^{2x}$$

Határozatlan együtthatók módszere

$$y=c_0+c_1(x-x_0)+c_2(x-x_0)^2+\ldots+c_k(x-x_0)^k+\ldots$$ alakú megoldást keresünk

$$\to y'=c_1+2c_2(x-x_0)+\ldots+kc_k(x-x_0)^{k-1}+\ldots$$

|| konvergenciasugáron belül lehet deriválni

$y(x_0)=c_0$, ha $x_0=0$, akkor $c_0=y(0)=4$ és a $\overbrace{\text{DE}}^{\text{diff. egyenl.}}$

$$c_1+2c_2x+\ldots+kc_kx^{k-1}+\ldots=(2c_0+2c_1x+\ldots+2c_kx^k+\ldots)+x^2$$

$$c_1=2c_0$$

$$2c_2=2c_1=16$$

$$3c_3=2c_2+1=17$$

$$4c_4=2c_3 \to c_4=\frac{2\cdot 17}{3\cdot 4}$$

$$\vdots$$

$$nc_n=2c_n \to c_n=2^{n-3}\cdot 2\cdot\frac{17}{n!}$$

$y(x) = 4 + 8x + 8x^2 + \sum_{k=3}^{\infty} \frac{2^{k-2}\cdot 17}{k!} x^k$ ~~same story as befor~~

oh shit, here we go again

(2.) $y' = y^4\cos(x) + y\,tg(x) \quad / y^4$ (Bernoulli-féle DE)

$\frac{y'}{y^4} = \cos(x) + \underbrace{y^{-3}}_{u} tg(x) \qquad \Big\| \quad u(x) = [y(x)]^{-3}, \quad u' = -3y^{-4}\cdot y'$

$-3y^{-4}y' = {}^{-3}\cos(x) + {}^{-3}y^{-3}tg(x)$

$u' = -3\cos(x) - 3u\,tg(x)$

homogén

$\frac{u'}{u} = -3tg(x) \to \log(x) = -3\int\frac{\sin x}{\cos x} = 3\log(\cos x) + c$

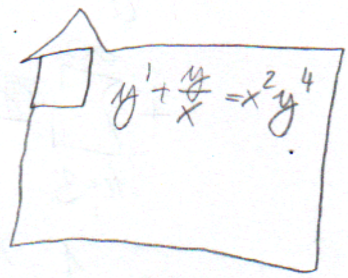
$x = e^{\cos^3(x)}\cdot c_2$

$\underbrace{u'\cos^{-3}(x) + 3u\,tg(x)\cdot\cos^{-3}x}_{u\cos^3 x} = -3\cos^{-2}(x)$

$u\cos^{-3} = \int -3\cos^{-2}(x) = -3tg(x) + c$

$u = -3\sin(x)\cos^2(x) + c\cos^3(x)$

$y = u^{-\frac{1}{3}} = (-3\sin(x)\cos^2(x) + c\cos^3(x))^{-\frac{1}{3}}$ és $\overbrace{y = 0}^{\text{függvény}}$



2025. 09. 26.

1. $y' = \frac{y^2}{\cos(x)} - y\,tg(x) - \cos(x)$ emiatt nem Bernoulli

$y' = a(x)y^2 + b(x)y + c(x)$ Riccati-egyenlet

y_0 megoldás, hh

$y_0' = a(x)y_0^2 + b(x)y_0 + c(x)$

$(y-y_0)' = a(x)(y^2-y_0^2) + b(x)(y-y_0)$

$\underbrace{(y-y_0)}_{z}(y+y_0) =$

$y' = a(x)z(z+2y_0) + b(x)z$ Bernoulli-féle DE

$\boxed{z = y - y_0 \text{ -ra} \updownarrow y = y_0 + z}$

$\cos = \frac{\cos^2}{\cos} - \cos tg - \cos = -\sin$

$y = z + \cos(x)$

$z' - \sin(x) = \frac{(z+\cos(x))^2}{\cos(x)} - (z+\cos(x))tg(x) - \cos(x)$

$z' - \cancel{\sin(x)} = \frac{z^2 + 2\cos z + \cancel{\cos^2(x)}}{\cos(x)} - z\,tg(x) - \cancel{\sin(x)} - \cancel{\cos(x)}$ $/ : z^2$

$\frac{z'}{z^2} = \frac{1}{\cos(x)} + \frac{1}{z}(2 - tg(x))$ $/ (-1)$

$-\frac{z'}{z^2} = -\frac{1}{\cos(x)} + \frac{1}{z}(tg(x) - 2)$

$u = \frac{1}{z}$ $\quad u' = \frac{z'}{z^2}$

$$u' = \frac{-1}{\cos^2(x)} + u(tg(x)-2)$$

homogén: $u'(tg(x)-2)$

$$\frac{u'}{u} = tg(x)-2$$

$$\log(u) = -\log(\cos(x)) - 2x + c \to u_h = Ce^{-2x} \cdot \cos^{-1}(x) =$$

$$= C\frac{e^{-2x}}{\cos(x)}$$

$$u' + u(2-tg(x)) = -\frac{1}{\cos(x)} \quad / e^{2x}\cos(x)$$

$$e^{2x}\cos(x)(u' + u(2-tg(x))) = -e^{2x}$$

$$e^{2x}\cos(x)(u' + u(2-tg(x))) = -e^{2x}$$

$$(e^{2x}\cos(x)u)' = -e^{2x} \quad /\int dx$$

$$e^{2x}\cos(x)\cdot u = -\frac{1}{2}e^{2x} + c$$

Azt az orosz könyvet, amit ajánlott el tudod küldeni?

$$u = -\frac{1}{2\cos(x)} + \frac{c}{e^{2x}\cos(x)} = \frac{-e^{2x} + 2c}{2e^{2x}\cos(x)}$$

$$z = \frac{1}{u} = \frac{2e^{2x}\cos(x)}{-e^{2x}+2c} \to y = z + \cos(x) =$$

$$= \frac{2e^{2x}\cos(x)}{-e^{2x}+2c} + \cos(x)$$

lineáris Bernoulli SZÉP Riccati

→ megyünk tovább

(2.) a) $y' = y^2 \to -\frac{y'}{y^2} = -1 \xrightarrow{\int} \frac{1}{y} = -x + c \to y = \frac{1}{c-x}$

b) $y' = \frac{1}{y^2} \to 3y^2y' = 3 \xrightarrow{\int} y^3 = 3x + c \to y = \sqrt[3]{3x+c}$

(3.) $xy' = y^2 + 4y$ szétválasztható

nézzük meg a konstans megoldásokat

$$\frac{y'}{y^2+4y} = \frac{1}{x}$$

$$\frac{\frac{dy}{dx}}{y^2+4y} = \frac{1}{x} \rightsquigarrow \int \frac{dy}{y^2+4y} = \int \frac{dx}{x} \rightsquigarrow \int \frac{1}{y(y+4)} dy =$$

$$= \frac{1}{4}\int \frac{1}{y} - \frac{1}{y+4} dy = \frac{1}{4}(\log(y) - \log(y+4)) + c = \frac{1}{4}\log\left(\frac{y}{y+4}\right) + c$$

$$\log\left(\frac{y}{y+4}\right) = 4\log(x) + 4c$$

$$\frac{y}{y+4} = cx^4 \to y = \frac{4cx^4}{1-cx^4} \text{ és } y = -4$$

(4.) Egyöntetű fokszámú DE

$y' = f\left(\frac{y}{x}\right) \to u(x) = \frac{y(x)}{x}$ -et kell helyettesíteni

$$y' = (ux)' = u' \cdot x + u = f(u)$$

$u' \cdot x = f(u) - u$ szétválasztható

$$y' = \frac{2x+y}{x-y} \qquad y = ux$$

$$u'x + u = \frac{2x+ux}{x-ux} = \frac{2+u}{1-u} \to u'x = \frac{2+u}{1-u} - \frac{u-u^2}{1-u} = \frac{2+u^2}{1-u}$$

$$\frac{1-u}{2+u^2}\,u' = \frac{1}{x}$$

$$\int \frac{1-u}{2+u^2}\,du = \int \frac{dx}{x}$$

$$\int \frac{1}{2+u^2}\,du = \int \frac{\frac{1}{2}}{\frac{1}{2}u^2+1}\,du = \frac{\sqrt{2}}{2}\,\mathrm{arctg}\left(\frac{u}{\sqrt{2}}\right)+c$$

$$\int \frac{u}{2+u^2}\,du = \frac{1}{2}\int \frac{2u}{2+u^2}\,du = \frac{1}{2}\log(u^2+2)+c$$

$$\frac{\sqrt{2}}{2}\,\mathrm{arctg}\left(\frac{u}{\sqrt{2}}\right) - \frac{1}{2}\log(u^2+2) = \log(x)+c$$

$$e^{\frac{\sqrt{2}}{2}\mathrm{arctg}\left(\frac{u}{\sqrt{2}}\right)}\, e^{-\frac{1}{2}\log(u^2+2)} = \frac{\sqrt{e^{\sqrt{2}\,\mathrm{arctg}\left(\frac{u}{\sqrt{2}}\right)}}}{\sqrt{e^{u^2+2}}} = cx \to$$ delete stroke

$$\to e^{\sqrt{2}\,\mathrm{arctg}\left(\frac{y}{\sqrt{2}x}\right)} = c\,(y^2+2x^2)$$

5. $y' = f(ax+b)$

$$u' = a + by' \to y' = \frac{u'-a}{b}$$

$$\frac{u'-a}{b} = f(u)$$ szétválasztható

$$(1-2x-2y)\,y' = x+y+1 \qquad \begin{matrix} a=1 \\ b=1 \end{matrix} \quad f = x+1$$

$$y' = \frac{x+y+1}{1-2x-2y} \qquad u = x+y$$

$$(1-2u)(u'-1) = u+1$$

$$(1-2u)\,u' = 2-u \qquad (u=2 \text{ mo})$$

$$\frac{(1-2u)\,u'}{2-u} = 1 \to \int \frac{1-2u}{2-u}\,du = \int dx$$

$$\int \frac{1-2u}{2-u} = \int \frac{4-2u}{2-u} - \frac{3}{2-u} = \int 2 + \int \frac{3}{2+u} = 2u + 3\log(u-2) = x + c$$

$$e^{2u} \cdot (u-2)^3 = c e^x$$

$$e^{2(x+y)}(x+y-2)^3 = c e^x$$

6.) $x^3(y'-x) = y^2$ $\quad$ $y = z^m$-nel lehet-e homogén fokszámúvá tenni?

$x^3 y' - x^4 = y^2$ $\quad$ $y' = m z^{m-1}$

$m = 2$

$$x^3(\overset{y}{(z^2)'} - x) = z^4$$

$$x^3 2zz' - x^4 = z^4$$

$$x^3 2zz' = z^4 + x^4$$

$$z' = \frac{z^4 + x^4}{x^3 2z} = \frac{\left(\frac{z}{x}\right)^4 + 1}{2\left(\frac{z}{x}\right)} \qquad z = ux \quad z' = u'x + u$$

$$u'x + u = \frac{u^4+1}{2u}$$

$$u'x = \frac{u^4+1}{2u} - u = \frac{u^4+1-2u^2}{2u} = \frac{(u^2-1)^2}{2u}$$

$$\int \underbrace{\frac{2u}{(u^2-1)^2}}_{\frac{f'}{f^2}} du = \int \frac{1}{x} dx$$

$$-\frac{1}{(u^2-1)} = \log(x) + c \rightarrow e^{\frac{x^2}{x^2-y}} = cx$$

+ ellenőrizni a szabálytalan lépésekből szárm. megoldást

HF a) $y' + \frac{y}{x} = x^2 y^4$ Bernoulli

b) $y' = \frac{y^2}{x^2} - 2\frac{y}{x}$ Egyöntetű ~~együtthatójú~~ fokszámú

c) $4x - 3y + y'(2y - 3x) = 0$ Egyöntetű, fokszám

d) $e^{4x+y} - \frac{1}{2} = y'$

e) $e^{2x+\frac{y}{2}} - \frac{1}{2} = y'$

$\Big\}$ $ax + by$

f) $y' = \frac{5xy + 2y^2}{x^2}$ (lehet, hogy nem jó)

← Riccatti / ~~homo~~ egyöntetű fokszámú

g)

lineáris

↓

Bernoulli

↓

Riccatti → inkább máshogy, mert gusztustalan

↓

szétválasztható változójú