Algebra 1 gyakorlat
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1.1.

1. asszoc. $b \pmod n$ inverze $n-b \pmod n$, neutr. $0 \pmod n$
2. asszoc., de nincs mindig inverz pl. $2 \pmod 6$-nak nincs inverze
3. Pl. mod 7, akkor $1, 2, 3, 4, 5, 6$ 8 15 22 29 36 43 50 57 . 64
   0
4. asszoc ? igen, inverz és neutr. van
5. $\{\mathbb{R} \to \mathbb{R},\ x \mapsto ax+b : 0 \neq a \in \mathbb{R},\ b \in \mathbb{R}\}$

   f_1, f_2, f_3

   $$f_1(f_2 f_3) = a_1(a_2a_3x + a_2b_3 + b_2) + b_1 = a_1a_2a_3x + a_1a_2b_3 + a_1b_2 + b_1$$

   $$a_2(a_3x + b_3) + b_2 = a_2a_3x + a_2b_3 + b_2$$

   $$(f_1 f_2) f_3 = a_1a_2(a_3x + b_3) + a_1b_2 + b_1 = a_1a_2a_3x + a_1a_2b_3 + a_1b_2 + b_1$$
6. $\mathbb{R}^3$, $a \times (b \times c) \neq (a \times b) \times c$

   nincs egység (neutr.
7.
8.

1.2

$$Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$$

$\forall q \in Q_8$ $1q = q1 = q$

$(-1)q = q(-1) = -q$

$ij = k,\ jk = i,\ ki = j$

$ji = -k,\ kj = -i,\ ik = -j$

$i^2 = j^2 = k^2 = -1$

	1	i	j	k
1	1	i	j	k
i	i	-1	k	-j
j	j	-k	-1	i
k	k	j	-i	-1

$r(1) = 1$ $r(i) = 4$ $r(-i) = 4$

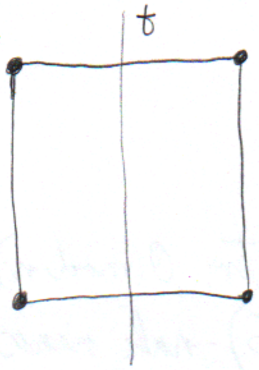
$r(-1) = 2$ $r(j) = 4$ $r(-j) = 4$

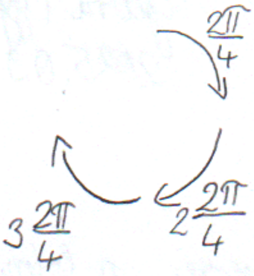
$r(k) = 4$ $r(-k) = 4$

$i^2 = (-1)$ $(-1)^2$

$(-i)^4$ $\underbrace{\overbrace{-i \cdot -i \cdot -i}^{i}}_{-1} \cdot -i = 1$

1,3. $D_{2\cdot 4}$





$$D_{2\cdot 4}=\{t, f, f^2, f^3, tf, tf^2, tf^3, id\}$$

rend $\quad 2, 4, 2, 4, 2, 2, 2, 1$

$$\begin{array}{cc}1&2\\4&3\end{array} \xrightarrow{t} \begin{array}{cc}2&1\\3&4\end{array} \xrightarrow{f} \begin{array}{cc}3&2\\4&1\end{array} \xrightarrow{t} \begin{array}{cc}2&3\\1&4\end{array} \xrightarrow{f} \begin{array}{cc}1&2\\4&3\end{array}.$$

$$\begin{array}{cc}1&2\\4&3\end{array} \xrightarrow{t} \begin{array}{cc}2&1\\3&4\end{array} \xrightarrow{f^2} \begin{array}{cc}4&3\\1&2\end{array} \xrightarrow{t} \begin{array}{cc}3&4\\2&1\end{array} \xrightarrow{f^2} \begin{array}{cc}1&2\\4&3\end{array}$$

$$\begin{array}{cc}1&2\\4&3\end{array} \xrightarrow{t} \begin{array}{cc}2&1\\3&4\end{array} \xrightarrow{f^3} \begin{array}{cc}1&4\\2&3\end{array} \xrightarrow{t} \begin{array}{cc}4&1\\3&2\end{array} \xrightarrow{f^3} \begin{array}{cc}1&2\\4&3\end{array}$$

$$D_{2n}=\{id, t, f, \ldots, f^{n-1}, tf, \ldots, tf^{n-1}\}$$

rend: $\quad 1, 2, n, \quad n,$

f^k esetén ha $k|n$, akkor $o(f^k)=\frac{n}{k}$

ha $k\nmid n$, akkor $o(f^k)=n$

$tf^k tf^k \qquad o(tf^k)=2$

megforgatja a forgatás irányát

(1.4) $|G|=2n$

$G=\{g_1,\dots,g_{2n}\}$, $g_1 = e$ $o(e)=1$ ha $i\in 2n$ $g_i=:g$

$g\neq e \;\; \exists g'\in G \quad gg'=e=gg'$

ha $g'=e \quad gg'=g \rightarrow g=e$ ↯ $\rightarrow g'\neq e$

olyan elemet keresünk, ami nem e és inverze önmaga, ~~az inverz~~ így inverz-párok lesznek, így $2n-1$ ptlan számú elem, így

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(2.1) /1. $g^m=e \rightarrow o(g)\mid m$

ha $o(g)\nmid m \Rightarrow m=q\,o(g)+r \quad r<o(g)$

$g^m=g^{q\,o(g)}\cdot g^r \longrightarrow g^r=e$ ↯ ($o(g)$ ∄-ja miatt)

(2.3) /1. $\begin{pmatrix}1&2&3&4\\2&4&1&3\end{pmatrix}=(1243)$

/2. $\begin{pmatrix}1&2&3&4&5&6\\2&4&6&1&3&5\end{pmatrix}=(124)(365)$

/3. $(12345)^{-1}(123)(45)(12345)=\begin{pmatrix}1&2&3&4&5\\5&1&2&3&4\end{pmatrix}^{-1}(123)(45)(12345)=$

$=\underbrace{\begin{pmatrix}1&2&3&4&5\\2&3&4&5&1\end{pmatrix}}(123)(45)(12345)=(15)(234)$

/4. $(12)(13)=(123)$

$(12)(13)(14)=(1234)$ stb teljes indukció

(2.5) /1 $o((123)(4567)(89))=\text{lkkt}(3,4,2)$

/2. $o((123)(234)=(13)(24))=2$

/3. $o((123)(4567)(567)=$

$=(1246573))=7$

2.4 /1.

	$(\ )^2$	$(\ )^3$	$(\ )^{-1}$
(12345678)	$(1357)(2468)$	(14725836)	(87654321)
$(1234)(567)$	$(13)(24)(576)$	(1432) id	$(4321)(765)$

$(1234)^2=(13)(24) \quad (567)^2=(576)$

$(1234)^3=(1432) \quad (567)^3=\cancel{(567)}$ id

2.5 /1.

$\sigma((123)(4567)(89))=12$

/2. π_2 $(123)(234)=(13)(24) \quad \sigma(\pi_2)=2$

/3. π_3 $(123)(34567)(567)=(1246573) \quad \sigma(\pi_3)=7$

2.7 S_n összes permutációja (MO)

2.9. /1. Legyen $H\subset\mathbb{Z}$

* $H\cap\mathbb{Z}^+=\emptyset \rightarrow H=\{0\}$
* $H\cap\mathbb{Z}^+\neq\emptyset$ ~~leghisebb~~ $n\in H\cap\mathbb{Z}^+$ a leghisebb ilyen $\xrightarrow{H\text{ csoport}} n$ $\forall$ többszöröse is benne van

$\exists k\in H-n\mathbb{Z} \rightarrow k=q\cdot n+r \quad r<n$

$k-qn=r\rightarrow r\in H$ $\lightning$ emiatt

Tehát csak $n\mathbb{Z}$-k részcsoportjai $(\mathbb{Z},+)\rightarrow \forall k\in n\mathbb{Z}$

vtlg1G

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$|G| = |H| \cdot |H:G|$

(3.1) Lagrange alapján

$|H| \in \{1,2,4,8\}$

$$\begin{tikzcd}[column sep=small]
& Q_8 \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
\langle i \rangle \arrow[dr, dash] & \langle j \rangle \arrow[d, dash] & \langle k \rangle \arrow[dl, dash] \\
& \{\pm 1\} \arrow[d, dash] & \\
& \{1\} &
\end{tikzcd}$$

részben rendezett halmazrendszer

(3.2) /1 $G = \{\mathbb{Z}, +\}$, $H = n\mathbb{Z}$

H rés. $\Longleftrightarrow n\mathbb{Z}$

Bal mellékosztályok : $3+H = \{3+x\cdot n : x \in \mathbb{Z}\} =$

$= (2n+3)+H = \{2n+3+h : h \in H = n\cdot\mathbb{Z}\}$

/2 $G = Q_8$, $H = \langle -1 \rangle = \{1,-1\}$

$1H = \{1,-1\}$

$iH = \{i,-i\} = -iH$

$jH = \{j,-i\}$

$kH = \{k,-k\}$

/3 $G = S_3$, $H = \{(),(12)\}$ $\qquad |G| = 3! = 6 \quad H = 2$

$|G:H| = \frac{|G|}{|H|} = 3$

$()H = \{(),(12)\}$

$(13)H = \{(13),(13)(12)\} = \{(13),(132)\}$

$(23)H = \{(23),(123)\}$

$H() = \{(), (12)\}$

$H(13) = \{(13), (123)\}$

$H(23) = \{(23), (132)\}$

(3.4) $n = |G| \in \mathbb{N} \rightarrow \forall g \in G : g^n = e$

$\exists g \in G : g^n \neq e \rightarrow |G| \neq n \in \mathbb{N}$

másik $|G| = n$ $\quad o(g) \mid |G|$ $\quad \exists k \in \mathbb{N} : k\,o(g) = |G| = n$

$$g^n = g^{k o(g)} = \left(g^{o(g)}\right)^k = e^k = e$$

(3.5) Ciklikus csoportban $\{\text{osztók}\} \leftrightarrow \{\text{részcsoportok}\}$

(3.7) első rész trivi

$a = 0 \rightarrow b = 0$

utlen eset $\langle g^a \rangle = \langle g^b \rangle$ $\quad \langle g^a \rangle = \{g^0, g^{-a}, g^{a}, g^{-2a}, g^{2a}, \ldots\}$ $\quad a \neq 0$

$\langle g^a \rangle = \langle g^b \rangle \leftrightarrow \forall n \in \mathbb{Z}, \exists k \in \mathbb{Z}\ nb = ka \rightarrow$

$\rightarrow k = n\frac{b}{a} \rightarrow$