Exercise 1

1 (b): FOCs and Asset Prizing

The Lagrangian for the consumer problem of consumer i looks as follows:

$$
\begin{aligned}
L = u(c_0^i) + \beta \sum_{s \in S} \pi(s) u(c_1^i(s)) \\
- \lambda_i (c_0^i + \sum_{s \in S} \{p_0^s q_0^{i,s}\} - y_0^i) \\
- \beta \sum_{s \in S} \{\pi(s) \mu_s (c_1^i(s) - q_0^{i,s} - y_1(s)\}
\end{aligned}
$$

From this problem we get three types of first order conditions (with respect to $c_0^i, c_1^i(s), q_0^{i,s}$):

$$
\begin{aligned}
\frac{\partial L}{\partial c_0^i} &= u'(c_0^i) - \lambda_i = 0 \\
\frac{\partial L}{\partial c_1^i(s)} &= \pi(s)\beta u'(c_1^i(s)) - \pi(s)\beta\mu_s = 0 \\
\frac{\partial L}{\partial q_0^i(s)} &= -\lambda_i p_0^s + \pi(s)\beta\mu_s = 0
\end{aligned}
$$

where the last two equations hold $\forall s \in S$.

From the first equation we get $\lambda_i = u'(c_0^i)$, thus λ_i is independent of the state s. The second equation implies $\mu_s = u'(c_1^i(s))$ again $\forall s \in S$. Combining the two with the last equation gives the following pricing equation:

$$
p_0^s = \beta\pi(s)\frac{u'(c_1^i(s))}{u'(c_0^i)}.
$$

Thus the price of an asset paying out in state s depends on three things:

- *Time preferences*: if β is higher, consumer have a higher preferences for future consumption thus asset demand is higher $\Rightarrow$ asset prices are higher; this holds irrespective of the state s for which the asset pays out.
- *Relative consumption (utilities)*: If consumption in the future state s is low relative to period 0 consumption, the ratio of marginal utilities in the asset pricing equation is high: an additional unit of consumption is more valuable in the future state s. Thus consumers want to shift consumption to that state by buying the asset: asset demand and thus prices increase.
- *Probability of the state*: If $\pi(s)$ is high, asset demand and thus prices increase because consumption in that state becomes more valuable in expectation (e.g. entering one state with near certainty consumer would not want to hold a lot of assets for other states).

(Note on binding constraint.)

1 (g)

Now we are in a setting in which (a) endowments in the future are (perfectly) negatively correlated assuming they are not constant (since $y_1^0(s) + y_1^1(s) = 2y$), (b) there is no *aggregate* risk (aggregate endowment is always $2y$, irrespective of state), and (c) both consumers have the same overall endowment (across time and states).

Step 1: Showing that consumption is state-invariant

We can first show that consumption levels are equalized across states for any agent.

- Combining the FOCs of agent $0, 1$ gives: $\frac{u'(c_1^1(s))}{u'(c_1^0(s))} = \frac{\lambda_1}{\lambda_0}$.
- Solve this for the consumption of agent 1: $c_1^1(s) = u'^{-1}(\frac{\lambda_1}{\lambda_0}u'(c_1^0(s)))$.
- Now plug this into the market clearing condition:
  - $\sum_{i\in\{0,1\}} c_1^1(s) = \sum_{i\in\{0,1\}} y_1^1(s)$, $\forall s \in S$, which follows from summing up budget constraints over agents and imposing market clearing for assets: $\sum_{i\in\{0,1\}} q_0^{1s} = 0$.
  - Thus

$$\sum_{i\in\{0,1\}} u'^{-1}(\frac{\lambda_i}{\lambda_0}u'(c_1^0(s)) = \sum_{i\in\{0,1\}} y_1^i(s)$$

    which implies that the consumption of agents is constant over states in $t = 1$ because the aggregate endowment on the RHS is constant and all other terms (e.g. the λ_k) do not vary by state.
- Thus, $c_t^i(s) = \bar{c}^i$ for $i \in \{0, 1\}$ and $\forall s \in S$.

Step 2: Asset prices and time zero consumption

Note further, that we can also show that $t = 0$ consumption needs to be the same constant.

The ratio of the two FOCs with respect to c_0^1 and c_0^0 gives:

$$\frac{u'(c_0^1)}{u'(c_0^0)} = \frac{\lambda_1}{\lambda_0}$$
$$c_0^i = u'^{-1}(\frac{\lambda_1}{\lambda_0}u'(c_0^j)).$$

Then using the $t = 0$ budget constraint, market clearing and common prices we have

$$\sum_{i\in\{0,1\}} y_0^i = \sum_{i\in\{0,1\}} c_0^i + \sum_{i\in\{0,1\}}\sum_{s\in S} p_0^s q_0^{is}$$
$$\sum_{i\in\{0,1\}} y_0^i = \sum_{i\in\{0,1\}} c_0^i + \sum_{s\in S} p_0^s \sum_{i\in\{0,1\}} q_0^{is}$$
$$\sum_{i\in\{0,1\}} y_0^i = \sum_{i\in\{0,1\}} c_0^i$$

and plugging the above expression for c_0^i into this gives

$$\sum_{i\in\{0,1\}} y_0^i = \sum_{i\in\{0,1\}} u'^{-1}(\frac{\lambda_i}{\lambda_0}u'(c_0^j))$$

thus c_0^j is also time-invariant. In particular, because $\sum_{i\in\{0,1\}} y_0^i = \sum_{i\in\{0,1\}} y_1^i(s)$ we also have $c_0^j = \bar{c}^j$ (equivalent for agent i).

Lastly, we derive the asset pricing equation under constant consumption:

$$p_0^s = \beta\pi\frac{u'(\bar{c}^i)}{\lambda_i}, \forall s \in S$$

and since $\lambda_i = u'(c_0^i) = u'(\bar{c}^i)$ we have in our potential equilibrium that

$$p_0^s = \beta\pi.$$

Step 3: Consumption levels

We can then use the budget constraints to derive the constant consumption levels: For any $i \in \{0,1\}$ we have:

$$y = y_0^i = \bar{c}^i + \sum_s p_0^s q_0^{is}$$

Now plugging in the price derived above and the second period budget constraints gives

$$y = \bar{c}^i + \sum_s \beta\pi(c_1^i(s) - y_1^i(s))$$

Using constant consumption across all periods as well as total endowments in period 1 of agent i being $|S|y$ we can derive $\bar{c}^i$:

$$\begin{aligned}
y &= \bar{c}^i + \sum_s \beta\pi(\bar{c}^i - y_1^i(s)) \\
0 &= (\bar{c}^i - y) + \beta\pi(|S|\bar{c}^i - \sum_s y_1^i(s)) \\
0 &= (\bar{c}^i - y) + \beta\pi(|S|\bar{c}^i - |S|y) \\
0 &= (1 + \beta\pi|S|)(\bar{c}^i - y).
\end{aligned}$$

The last step then implies $\bar{c}^i = y$ for $i \in \{0,1\}$.

While we never explicitly use the covariance/variance assumption note this is an implication of $\sum_i y_1^i(s) = 2y \Rightarrow Corr(y_1^0(s), y_1^1(s))) = -1$. This holds unless one of the agents receives the full endowment for all periods (so there is no randomness).

Complete Solution

- Consumption allocation: $c_t^i = y$ which holds $\forall i \in \{0,1\}, \forall t \in \{0,1\}, \forall s \in S$
- Asset demands: $q_0^{is} = y - y_1^i(s)$ holding $\forall i \in \{0,1\}, \forall s \in S$
- Prices: $\{1, \{\pi\beta\}_{s\in S}\}$, where the first entry is the price of period $t = 0$ consumption (the numeraire).

1 (h)

Now assume $y_0^0 > y_0^1$ (wlog), but assume that still $y_0^0 + y_0^1 = 2y$. Denote $y_0^0 = y + \delta$ with $\delta > 0$.

Heuristically, we can then argue that now the agent 0, the one with the higher total endowment, will have a higher level of consumption, while agent 1 will have a lower level: From the first order conditions we have

$$\frac{u'(c_1^1(s))}{u'(c_1^0(s))} = \frac{u'(\bar{c}^1)}{u'(\bar{c}^0)} = \frac{\lambda_1}{\lambda_0}$$

Now interpreting λ_j as the shadow price of resources available in period 1 (or of total resources if we write only one budget constraint in the problem), $y_0^0 > y_0^1$ implies $\lambda_0 < \lambda_1$, keeping the rest of the endowment sequence fixed. That is, the value of slackening the budget constraint is lower for agent 0 because they have a higher level of resources compared to the setting in (g). Thus the right-hand ratio becomes larger, so in the new equilibrium the LHS ratio also has to be larger which implies $\bar{c}^0 > \bar{c}^1$, so agent 0 has a relatively higher level of consumption thus lower marginal utility, making the LHS bigger.

We can also derive this consumption level using Step 3 from above (at least under the assumption that $y_0^0 + y_0^1 = 2y$). Note that all the arguments from Steps 1 and 2 from the (g) remain unchanged. In particular, while the ratio $\frac{\lambda_1}{\lambda_0}$ will change, consumption for both agents will still be constant across states and periods. For agent 0 we now have

$$\begin{gathered}
y + \delta = \bar{c}^0 + \sum_s \beta\pi(\bar{c}^0 - y_1^0(s)) \\
\delta = (\bar{c}^0 - y) + \beta\pi(|S|\bar{c}^0 - \sum_s y_1^0(s)) \\
\delta = (\bar{c}^0 - y) + \beta\pi(|S|\bar{c}^0 - |S|y) \\
\delta = (1 + \beta\pi|S|)(\bar{c}^0 - y). \\
\bar{c}^0 = y + \frac{\delta}{1 + \beta\pi|S|}
\end{gathered}$$

and equivalently

$$\bar{c}^1 = y - \frac{\delta}{1 + \beta\pi|S|} < \bar{c}^0$$