Exercise 1

1 (b): FOCs and Asset Prizing

The Lagrangian for the consumer problem of consumer i looks as follows:

$$\begin{aligned} L = u(c_0^i) + \beta \sum_{s \in S} \pi(s) u(c_1^i(s)) \\ - \lambda_i (c_0^i + \sum_{s \in S} \{p_0^s q_0^{i,s}\} - y_0^i) \\ - \beta \sum_{s \in S} \{\pi(s) \mu_s (c_1^i(s) - q_0^{i,s} - y_1(s)\} \end{aligned}$$

From this problem we get three types of first order conditions (with respect to $c_0^i, c_1^i(s), q_0^{i,s}$):

$$\begin{aligned} \frac{\partial L}{\partial c_0^i} &= u'(c_0^i) - \lambda_i = 0 \\ \frac{\partial L}{\partial c_1^i(s)} &= \pi(s) \beta u'(c_1^i(s)) - \pi(s) \beta \mu_s = 0 \\ \frac{\partial L}{\partial q_0^i(s)} &= -\lambda_i p_0^s + \pi(s) \beta \mu_s = 0 \end{aligned}$$

where the last two equations hold $\forall s \in S$.

From the first equation we get $\lambda_i = u'(c_0^i)$, thus λ_i is independent of the state s. The second equation implies $\mu_s = u'(c_1^i(s))$ again $\forall s \in S$. Combining the two with the last equation gives the following pricing equation:

$$p_0^s = \beta \pi(s) \frac{u'(c_1^i(s))}{u'(c_0^i)}.$$

Thus the price of an asset paying out in state s depends on three things:

- *Time preferences*: if β is higher, consumer have a higher preferences for future consumption thus asset demand is higher $\Rightarrow$ asset prices are higher; this holds irrespective of the state s for which the asset pays out.
- *Relative consumption (utilities)*: If consumption in the future state s is low relative to period 0 consumption, the ratio of marginal utilities in the asset pricing equation is high: an additional unit of consumption is more valuable in the future state s. Thus consumers want to shift consumption to that state by buying the asset: asset demand and thus prices increase.
- *Probability of the state*: If $\pi(s)$ is high, asset demand and thus prices increase because consumption in that state becomes more valuable in expectation (e.g. entering one state with near certainty consumer would not want to hold a lot of assets for other states).

(Note on binding constraint.)

1 (c)

- Market clearing: Aggregate assets are net zero.
- Number of budget constraints: $|S|$.

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1 (d)

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1 (e)

In equilibrium HH do not want to change their portfolio.

- Within model HH are acting optimally.
- But: Would like to insure against aggregate risk.

1 (f)

Financial markets cannot insure against aggregate risk.

- Maybe risk-loving consumers
- Maybe differences in beliefs

1 (g)

Now we are in a setting in which (a) endowments in the future are (perfectly) negatively correlated assuming they are not constant (since $y_1^0(s) + y_1^1(s) = 2y$), (b) there is no *aggregate* risk (aggregate endowment is always $2y$, irrespective of state), and (c) both consumers have the same overall endowment (across time and states).

Step 1: Showing that consumption is state-invariant

We can first show that consumption levels are equalized across states for any agent.

- Combining the FOCs of agent $0, 1$ gives: $\frac{u'(c_1^i(s))}{u'(c_1^j(s))} = \frac{\lambda_i}{\lambda_j}$.
- Solve this for the consumption of agent i: $c_1^i(s) = u'^{-1}(\frac{\lambda_i}{\lambda_j} u'(c_1^j(s)))$.
- Now plug this into the market clearing condition:
  - $\sum_i c_1^i(s) = \sum_i y_1^i(s)$, $\forall s \in S$, which follows from summing up budget constraints over agents and imposing market clearing for assets: $\sum_i q_0^{is} = 0$.

- Thus

$$\sum_i u'^{-1}(\frac{\lambda_i}{\lambda_j} u'(c_1^j(s)) = \sum_i y_1^i(s)$$

which implies that the consumption of agents is constant over states in $t = 1$ because the aggregate endowment on the RHS is constant and all other terms (e.g. the λ_k) do not vary by state.

- Thus, $c_t^j(s) = \bar{c}^j$ for $j \in \{0, 1\}$ and $\forall s \in S$.

Step 2: Asset prices and time zero consumption

Note further, that we can also show that $t = 0$ consumption needs to be the same constant.

The ratio of the two FOCs with respect to c_0^i and c_0^j gives:

$$\frac{u'(c_0^i)}{u'(c_0^j)} = \frac{\lambda_i}{\lambda_j}$$
$$c_0^i = u'^{-1}(\frac{\lambda_i}{\lambda_j} u'(c_0^j)).$$

Then using the $t = 0$ budget constraint, market clearing and common prices we have

$$\sum_i y_0^i = \sum_i c_0^i + \sum_i \sum_{s \in S} p_0^s q_0^{is}$$
$$\sum_i y_0^i = \sum_i c_0^i + \sum_{s \in S} p_0^s \sum_i q_0^{is}$$
$$\sum_i y_0^i = \sum_i c_0^i$$

and plugging the above expression for c_0^i into this gives

$$\sum_i y_0^i = \sum_i u'^{-1}(\frac{\lambda_i}{\lambda_j} u'(c_0^j))$$

thus c_0^j is also time-invariant. In particular, because $\sum_i y_0^i = \sum_i y_1^i(s)$ we also have $c_0^j = \bar{c}^j$ (equivalent for agent i).

Lastly, we derive the asset pricing equation under constant consumption:

$$p_0^s = \beta\pi \frac{u'(\bar{c}^i)}{\lambda_i}, \forall s \in S$$

and since $\lambda_i = u'(c_0^i) = u'(\bar{c}^i)$ we have in our potential equilibrium that

$$p_0^s = \beta\pi.$$

Step 3: Consumption levels

We can then use the budget constraints to derive the constant consumption levels:

$$
\begin{gathered}
y = y_0^i = \bar{c}^i + \sum_s p_0^s q_0^{is} \\
y = \bar{c}^i + \sum_s \beta\pi(c_1^i(s) - y_1^i(s)) \\
y = \bar{c}^i + \sum_s \beta\pi(\bar{c}^i - y_1^i(s)) \\
0 = (\bar{c}^i - y) + \beta\pi(|S|\bar{c}^i - \sum_s y_1^i(s)) \\
0 = (\bar{c}^i - y) + \beta\pi(|S|\bar{c}^i - |S|y) \\
0 = (1 + \beta\pi|S|)(\bar{c}^i - y),
\end{gathered}
$$

where we use the budget constraints from $t = 0$ and $t = 1$, $p_0^s = \pi\beta$, and the feature of the endowment process that $y_0^i = y$ and $\sum_s y_1^i(s) = \sum_s y = |S|y$. The last step then implies $\bar{c}^i = y$. Note the problem for j is the same as total endowments (t = 0 and summed over states) are identical. Thus, also $\bar{c}^j = y$.

While we never explicitly use the covariance/variance assumption note this is an implication of $\sum_i y_1^i(s) = 2y \Rightarrow Corr(y_1^0(s), y_1^1(s))) = -1$. This holds unless one of the agents receives the full endowment for all periods (so there is no randomness).

Complete Solution

- Consumption allocation: $c_t^i = y$ which holds $\forall i \in \{0,1\}, \forall t \in \{0,1\}, \forall s \in S$
- Asset demands: $q_0^{is} = y - y_1^i(s)$ holding $\forall i \in \{0,1\}, \forall s \in S$
- Prices: $\{1, \{\pi\beta\}_{s\in S}\}$, where the first entry is the price of period $t = 0$ consumption (the numeraire).

1 (i)

- same weights --> same consumption
- to rationalize (h) --> choose weights $

1(j)

- Assumption: two or more HH, but don't know before t = 0 which HH type
- in extreme case might not buy assets at all although would like ex-post
- social planner knows what type HH will be --> buys out insurance for them
- or: so many HH that social planer can pool the risk

1(k)

- achieve: insurance against idio-syncratic risk of HHs
- not achieve: insurance against aggregate risks

1(l)

With complete markets and same total endowments --> same consumptions in all periods and states --> all HH identical