1 Probability Theory

Theorem 1.3: $A_1, ..., A_2$ partition of S and $B \subset S$, then $P(B) = \sum i = 1^{\infty} P(B|A_i)P(A_i)$.

2 Asymptotic Theory

2.1 Inequalities

Thm 5.1 (Markov's Inequality): X r.v., $g : \mathbb{R} \to [0, \infty)$, then $\forall \epsilon > 0$, $P(g(X) > \epsilon) \leq \frac{E(g(X))}{\epsilon}$.

Cor 5.1 (Chebyshev's Inequality): X r.v., then $\forall \epsilon > 0$, $P(|X - E(X)| \geq \epsilon) \leq \frac{Var(X)}{\epsilon^2}$.

2.2 Modes of Convergence

Def 5.2: $plim_{n\to\infty} X_n = X \leftrightarrow \lim_{n\to\infty} P(|X_n - X| < \epsilon) = 1$. **Def 5.3:** $\hat{\theta}_n consistent for \theta \leftrightarrow plim \hat{\theta}_n = \theta$.

Def 5.4: $\{X_n\}_{n=1}^{\infty}$ converges in *distribution* to $X \leftrightarrow \lim_{n\to\infty} F_{X_n}(x) = F_X(x)$ for every continuity point of x of $F_X(\cdot)$.

Def 5.5: $\{X_n\}_{n=1}^{\infty}$ converges in *mean square* to $X \leftrightarrow \lim_{n\to\infty} E[(X_n - X)^2] = 0$.

Thm 5.2: $X_n \xrightarrow{m.s.}\Rightarrow X_n \xrightarrow{p} X$. Proof by Chebyshev's inequality. The reverse is not true, consider $X_n \in 0, \sqrt{n}$ with probabilities $1 - 1/n, 1/n$.

Thm 5.3: $X_n \xrightarrow{p}\Rightarrow X_n \xrightarrow{d} X$. Proof uses definition of $\xrightarrow{p}$ and continuity. The reverse is generally *not true*, consider $X_n = Z \sim N(0,1)$ and $X, Z \sim N(0,1)$, have $F_{X_n}(x) = F_X(x)$ but $P(|Z - X| \geq \epsilon) > 0$.

Exception: $X_n \xrightarrow{d} c \in \mathbb{R} \Rightarrow X_n \xrightarrow{p} c$.

2.3 Law of Large Numbers

Thm 5.4 (LLN): $X_{i}{}_{i=1}^{\infty}$ seq. of uncorrelated rvs from F_X with $\mu = E(X)$, $Var(X)$ existing and finite. Then $\overline{X}_n \xrightarrow{p} \mu$. Proof: Chebyshev's inequality.

Thm 5.5 (WLLN): $X_{i}{}_{i=1^{\infty}}$ seq. of uncorrelated rvs. Suppose $\mu_i = E(X_i)$ and $\sigma_i^2 = Var(X_i)$ exist and finite. If $\lim_{n\to\infty} \frac{1}{n^2}\sum_{i=1}^{n} \sigma_i^2 = 0$ then $\overline{X}_n - \frac{1}{n}\sum_{i=1}^{n} \mu_i \xrightarrow{p} 0$.

Thm 5.6 (LLN i.i.d): $\{X_i\}_{i=1}^{\infty}$ seq. of iid rvs from F_X with $\mu = E(X)$ exist and finite. Then $\overline{X}_n \xrightarrow{p} \mu$.

Convergence Criteria: Need a combination of three assumptions: (1) finite mean and/or variance (no LLN for Cauchy), (2) bounds on asymptotic variance (e.g. not growing too fast with i), (3) restricted dependence.

2.4 Central Limit Theorem

Thm 5.7 (Lindeberg-Levy CLT): $\{X_i\}_{i=1}^{\infty}$ seq. of iid rvs from F_X, μ and σ^2 finite. Then $\sqrt{n}(\overline{X}_n - \mu) \xrightarrow{d} N(0, \sigma^2)$.

Thm 5.9 (Berry-Esseen): $\{X_i\}_{i=1}^{\infty}$ seq. of iid rvs from F_X, μ and σ^2 finite and $\lambda = E(|X - E(X)|^3)$ exist and finite. Let $Z \sim N(0,1)$. Then $\left|P\left(\frac{\sqrt{n}(\overline{X}_n - \mu)}{\sigma} \leq x\right) - P(Z \leq x)\right| \leq \frac{C\lambda}{\sigma^3\sqrt{n}}$.

2.5 Convergence of Random Vectors

Def 5.7: $\mathbf{X}_n \xrightarrow{p} \mathbf{X} \leftrightarrow \lim_{n\infty} P(\|\mathbf{X}_n - \mathbf{X}\| < \epsilon) = 1$.

Def 5.8: $\mathbf{X}_n \xrightarrow{ms} \mathbf{X} \leftrightarrow \lim_{n\infty} E(\|\mathbf{X}_n - \mathbf{X}\|^2) = 0$.

Def 5.9: $\mathbf{X}_n \xrightarrow{d} \mathbf{X} \leftrightarrow \lim_{n\infty} F_{\mathbf{X}_n}(x) = F_{\mathbf{X}}(x)$ for every continuity point x of $F_{\mathbf{X}}(\cdot)$.

Thm 5.10 (Cramér-Wold): $\{\mathbf{X}_n\}_{n=1}^{\infty}$ seq. of K-dimensional random vectors. Then, $\forall \lambda \in \mathbb{R}^{\mathbb{K}}$ we have $\lambda'\mathbf{X}_n \xrightarrow{d} \lambda'\mathbf{X} \leftrightarrow \mathbf{X}_n \xrightarrow{d} \mathbf{X}$.

2.6 CMT and Slutzky's

Thm 5.11 (CMT): Let $\{\mathbf{X}_n\}_{n=1}^{\infty}$ be a sequence of K-dim. rvecs $\mathbf{X}$ K-dim rvec, and $g : \mathbb{R}^{\mathbb{K}} \to \mathbb{R}$ with discontinuity points D such that $P(\mathbf{X} \in D) = 0$.

(a) $\mathbf{X}_n \xrightarrow{p} \mathbf{X} \Rightarrow g(\mathbf{X}_n) \xrightarrow{p} g(\mathbf{X})$.

(b) $\mathbf{X}_n \xrightarrow{d} \mathbf{X} \Rightarrow g(\mathbf{X}_n) \xrightarrow{d} g(\mathbf{X})$.

Implication: Sums and products of convergent sequences converge. Does *not* hold for *mean square* convergence.

Thm 5.12 (Slutzky's): X_n, Y_n seq of rvs with $X_n \xrightarrow{d} X$ and $Y_n \xrightarrow{p} c \in \mathbb{R}$, then $X_n + Y_n \xrightarrow{d} X + c$ and $X_n Y_n \xrightarrow{d} cX$, and if $c \neq 0$, $X_n/Y_n \xrightarrow{d} X/c$.

Extension to rvecs: $\mathbf{X}_n \xrightarrow{d} \mathbf{X}$ and $\mathbf{Y}_n \xrightarrow{p} \mathbf{C} \in \mathbb{R}^{K\times K}$, $\mathbf{C}$ invertible, then $\mathbf{Y}_n^{-1}\mathbf{X}_n \xrightarrow{d} \mathbf{C}^{-1}\mathbf{X}$.

Example CMT: $\left(\frac{\sqrt{n}(\overline{X}_n - \mu)}{\sigma}\right)^2 \xrightarrow{d} N(0,1)^2 = \xi_1^2$.

Thm 5.13 (Delta-Method): X_n seq of rvs with LL-CLT applying. $g : \mathbb{R} \to \mathbb{R}$ *continuously diff.* at μ with $g'(\mu) \neq 0$. Then $\sqrt{n}(g(X_n) - g(\mu)) \xrightarrow{d} N(0, g'(\mu)^2\sigma^2)$. Proof: CMT and Slutzky's applied to Taylor's/intermediate value theorem.

2.7 Interval Estimation

Suppose $\{X_i\}_{i=1}^{n}$ is a seq of iid random variables with μ, σ^2 finite. Then an asymptotically valid CI for μ is given by

$$CI = \left[\overline{X}_n \pm \frac{z_{1-\alpha/2}}{\sqrt{n}} S_n\right]$$

where S_n is a consistent estimator of σ and $P(\mu \in CI) \to 1 - \alpha$. Proof: CLT, CMT, Slutzky.

2.8 Moment-Based Estimation

Parameter of interest: $\theta = h(E(g(X)))$ (simple case: X, θ scalars and $g : \mathbb{R} \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$ cont. diff.).

Moment-based estimator: $\hat{\theta}_n = h\left(\frac{1}{n}\sum_{i=1}^{n} g(X_i)\right)$. *Consistency* follows from LLN and CMT.

Large-sample distribution: If $Var(X)) < \infty$ CLT applies so $\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n} g(X_i) - E(g(X))\right) \xrightarrow{d} N(0, Var(g(X)))$. By the *delta-method* if $h'(g(E(X))) \neq 0$ we have

$$\sqrt{n}(\hat{\theta}_n - \theta) = \sqrt{n}\left(h\left(\frac{1}{n}\sum_{i=1}^{n} g(X_i)\right) - h(E(g(X)\right.$$

$$\xrightarrow{d} N(0, h'(E(g(X))^2 Var(g(X))$$

3 Hypothesis Testing

3.1 Basics

Def 7.1: A *hypothesis* is a statement about the population distribution.

Def 7.2: H_0 (null hypothesis) and H_1 (alternative hypothesis) are the complementary hypothesis. We write $H_0 : \theta \in \Theta_0$ and $H_1 : \theta \in \Theta_1$ with Θ_k mutually exclusive and exhaustive.

Simple hypothesis: Θ_0 is singleton. *Composite* hypothesis: Θ_1 more than one value.

Def 7.3: A *hypothesis test* is a rule when to reject H_0 (in favor of H_1) given the data. (*Accepting* H_0 is weird, e.g. what about $\theta_0 + \epsilon$?.)

4 Size and Power

Type-I error: Reject H_0 although in fact true.

Type-II error: *Not* reject H_0 although in fact false.

Error rates: *Probabilities* of making these errors (errors are random because they depend on the sample).

I-II-trade-off: We want to minimize $P_\theta(rejectH_0)$ $\forall\theta \in \Theta_0$ and maximize $P_\theta(rejectH_0)$ $\forall\theta \in \Theta_1$ (P_θ denotes probabilities assuming θ is the true parameter).

Def 7.4 (Power function): $\beta(\theta) = P_\theta(rejectH_0)$.

Type-I error: $\beta(\theta)$ for any $\theta \in \Theta_0$.

Type-II error: $\beta(\theta)$ for any $\theta \in \Theta_1$.

Def 7.5/7.6: For $\alpha \in [0,1]$, a test is *level* α if $\sum_{\theta\in\Theta_0} \beta(\theta) \leq \alpha$ (*size*: equality).

Test choice: One approach: fix α, take the one with the best power over all $\theta \in \Theta_1$ (might not exist).

4.1 Test statistics and critical values

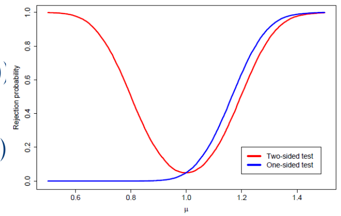
Goal: Derive statistic T and reject iff $T > c_\alpha$ controlling $sup_{\theta\in\Theta_0} P_\theta(T > c_\alpha) \Rightarrow$ need $F_T(t)$.

Ex 7.2 (Two-sided T): $X \sim N(\mu, \sigma^2)$ so $\theta = (\mu, \sigma^2)$. Test $H_0 : \mu = \mu_0$ against $H_1 : \mu \neq \mu_0$ use $T = \frac{\sqrt{n}|\overline{X}_n - \mu_0|}{S_n}$ $|t_{n-1}|$ and reject for $T > c_\alpha = t_{n-1,1-\alpha/2}$. By construction $sup_{\theta\in\Theta_0} P_\theta(T > c_\alpha) = \alpha$. Note this holds for all $\sigma^2 \in \Gamma$ thus a test of level and size α.

Ex (One-sided T): $T = \frac{\sqrt{n}(\overline{X}_n - \mu_0)}{S_n}$ with $c_\alpha = t_{n-1,1-\alpha}$ or $Z = -T$ and c_α unchanged (symmetry). Intuition: want to reject for large $\mu > \mu_0$ (right-sided).

Deriving $\beta(\theta)$: (1) add and subtract (true) μ, (2) look at behavior as μ changes.

The one-sided test is also a test for $H_0 : \mu \leq \mu_0$ against $H_1 : \mu > \mu_0$ with size α, because $sup_{\theta\in\Theta_0, \sigma\in\Gamma} \beta^{1sided}(\theta) \leq \alpha$.



Def (p-value): For any realization T^*, $p^* = inf\{p \in [0,1] : T^* > c_p\}$. Intuition: smallest α for which we would still reject.

Under H_0, $p \sim Unif[0,1]$ (require $P(p^* < \alpha) = \alpha$, i.e. want $Pr_\theta(rejectH_0) < \alpha$), but holds $\forall\alpha$.

p-value with simple H_0: If F_0 is strictly increasing, $p* = 1 - F_0(T*)$ (again: $p\ H_0 Unif[0,1]$).

With parametric distributions with multiple parameters (e.g. $N(\mu, \sigma^2)$) usually fix one parameter (e.g. σ^2) resulting in simple test but technically *composite* H_0.

4.2 Hypothesis Testing and CIs

Test-inversion: Assume test $H_0 : \theta = \theta_0$ and have test s.t. $P_{\theta_0}(rejectH_0) = \alpha$ (size α). Assume can perform for any $\theta_0 in \Theta$. Then we have $CS = \{\theta_0 \in \Theta : notrejectH_0 : \theta = \theta_0\}$ with $P_\theta(\theta \in CS) = 1 - \alpha$.

We can also do the reverse: From any CS with coverage rate $1 - \alpha$ can construct size α test as $reject \Leftrightarrow \theta_0 \notin CS$.

Ex. one-sided CI: Testing $H_0 : \mu = \mu_0$ against $H_1 : \mu > \mu_0$ (or $H_0 : \mu \leq \mu_0$) for normal case we have $CS = \{\mu_0 \in [\overline{X}_n - \frac{t_{1-\alpha,n-1}}{\sqrt{n}} S_n, \infty)\}$.

4.3 Asymptotic Approximations

: **Asymptotic argument**: No parametric model $f(x;\theta)$, but, e.g., moments: $H_0 : E(X) = \mu$. $T \xrightarrow{d} |N(0,1)|$ and we can use $\Phi^{-1}(x)$ to control α asymptotically. In particular, $P(T > z_{1-\alpha/2}) \to \alpha$ under H_0.

Hypotheses: Set of distributions $\mathbb{P}$ with $\mathbb{P}_0 \subset \mathbb{P}$ set of distributions consitent with H_0. **Def 7.7 (Asymptotic power function)**: $\beta^\alpha(P) = lim_{n\to\infty} beta_n(P)$.

Def 7.8/7.9: test with $\beta^a(P)$ is *asymptotic level* α if $sup_{P\in P_0} \beta^a(P) \leq \alpha$ (*size*: equality).

Def 7.10 (consistency): Test *consistent* against alternative $P \in P_1$ if $\beta^a(P) = 1$.