# Budgeted Reinforcement Learning in Continuous State Space

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# Introduction — Use-case

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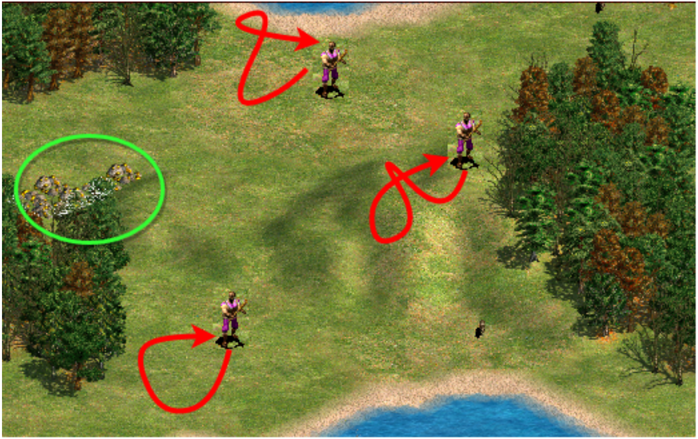

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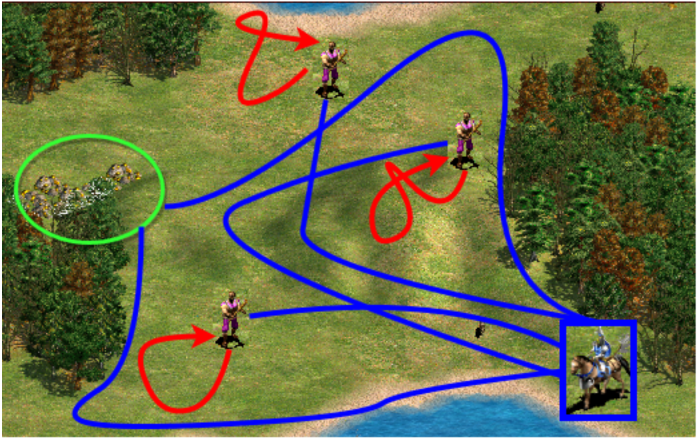

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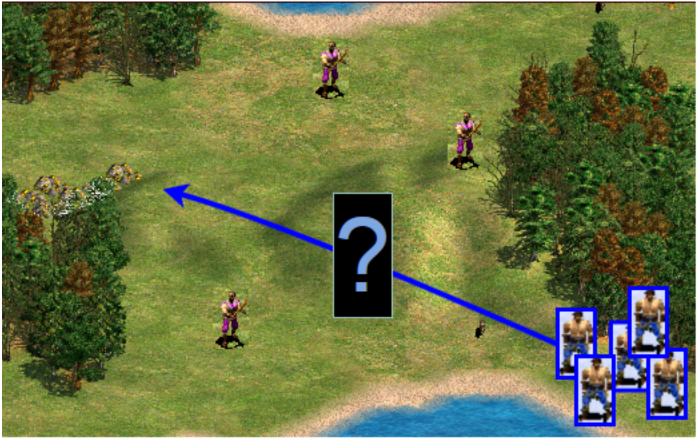

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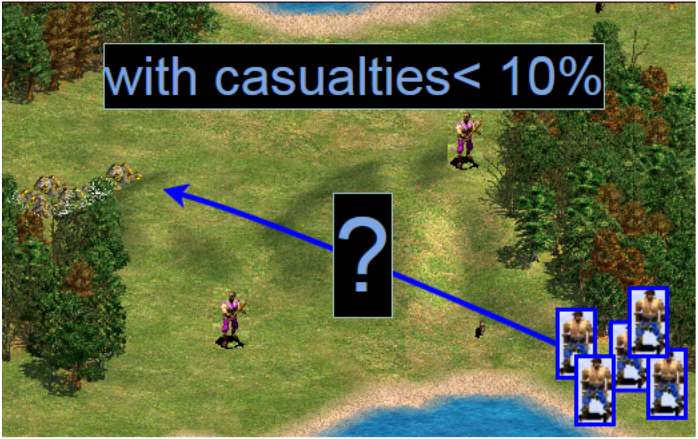


# **Introduction** — Use-case

### Problem

Given past trajectories, find a way to:

- gather gold as much as possible;
- limit the villager casualties under some budget;
- being able to change the budget in real time.

### Solution

- This problem can be cast as a Budgeted Markov Decision Process.

# Setting

## Markov Decision Process

We define a MDP as a tuple $(\mathcal{S}, \mathcal{A}, P, R_r, \gamma)$ where:

- $\mathcal{S}$ is the state space, $\mathcal{A}$ the action space,
- $R_r \in \mathbb{R}^{\mathcal{S}\times\mathcal{A}}$ the rewards,
- $P \in \mathcal{M}(\mathcal{S})^{\mathcal{S}\times\mathcal{A}}$ the dynamics,
- and $\gamma$ the discounted factor.

## Objective

- $G_r^\pi = \sum_{t=0}^{\infty} \gamma^t R_r(s_t, a_t)$ the $\gamma$-discounted return of rewards.
- Find $\pi^*$ s.t $\forall s \in \mathcal{S}$:

$$\pi^* \in \underset{\pi\in\mathcal{M}(\mathcal{A})^{\mathcal{S}}}{\arg\max}\, \mathbb{E}[G_r^\pi | s_0 = s] \tag{1}$$

# Setting

## Constrained Markov Decision Process

We define a CMDP as a tuple $(\mathcal{S}, \mathcal{A}, P, R_r, R_c, \gamma, \beta)$ where:

- $\mathcal{S}$ is the state space, $\mathcal{A}$ the action space,
- $R_r \in \mathbb{R}^{\mathcal{S}\times\mathcal{A}}$ the rewards, and $R_c \in \mathbb{R}^{\mathcal{S}\times\mathcal{A}}$ the costs
- $P \in \mathcal{M}(\mathcal{S})^{\mathcal{S}\times\mathcal{A}}$ the dynamics,
- $\gamma$ the discounted factor, and $\beta$ the budget.

## Objective

- $G_r^\pi = \sum_{t=0}^{\infty} \gamma^t R_r(s_t, a_t)$ the $\gamma$-discounted return of rewards.
- $G_c^\pi = \sum_{t=0}^{\infty} \gamma^t R_c(s_t, a_t)$ the $\gamma$-discounted return of costs.
- Find $\pi^*$ s.t $\forall s \in \mathcal{S}$:

$$\begin{aligned} \pi^* \in \underset{\pi \in \mathcal{M}(\mathcal{A})^{\mathcal{S}}}{\arg\max}\ & \mathbb{E}[G_r^\pi | s_0 = s] \\ \text{s.t. } & \mathbb{E}[G_c^\pi | s_0 = s] \leq \beta \end{aligned} \tag{2}$$

# Setting

## Budgeted Markov Decision Process

We define a BMDP as a tuple $(\mathcal{S}, \mathcal{A}, P, R_r, R_c, \gamma, \mathcal{B})$ where:

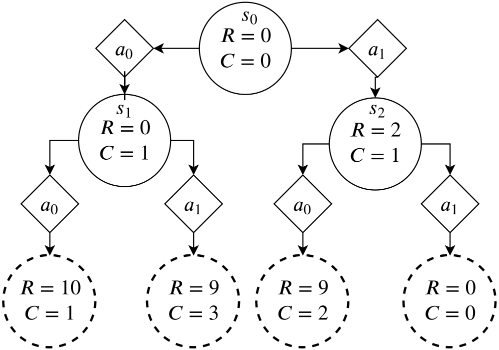
- $\mathcal{S}$ is the state space, $\mathcal{A}$ the action space,
- $R_r \in \mathbb{R}^{\mathcal{S}\times\mathcal{A}}$ the rewards, and $R_c \in \mathbb{R}^{\mathcal{S}\times\mathcal{A}}$ the costs
- $P \in \mathcal{M}(\mathcal{S})^{\mathcal{S}\times\mathcal{A}}$ the dynamics,
- $\gamma$ the discounted factor, and $\mathcal{B}$ the budget space.

## Objective

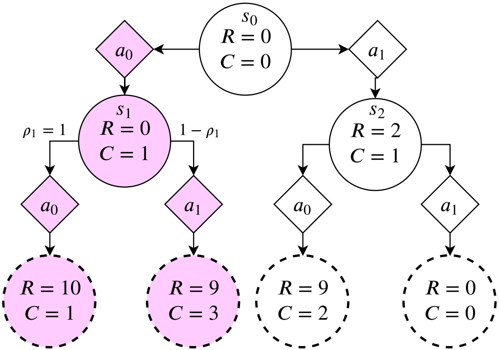
- $G_r^\pi = \sum_{t=0}^{\infty} \gamma^t R_r(s_t, a_t)$ the $\gamma$-discounted return of rewards.
- $G_c^\pi = \sum_{t=0}^{\infty} \gamma^t R_c(s_t, a_t)$ the $\gamma$-discounted return of costs.
- Find $\pi^*$ s.t $\forall (s, \beta) \in \mathcal{S} \times \mathcal{B}$:

$$\begin{aligned} \pi^* \in \; & \underset{\pi \in \mathcal{M}(\mathcal{A}\times\mathcal{B})^{\mathcal{S}\times\mathcal{B}}}{\arg\max} \; \mathbb{E}[G_r^\pi | s_0 = s, \beta_0 = \beta] \\ \text{s.t.} \; & \mathbb{E}[G_c^\pi | s_0 = s, \beta_0 = \beta] \leq \beta \end{aligned} \tag{3}$$

# The Dynamic Programming solution: intuition



# The Dynamic Programming solution: intuition

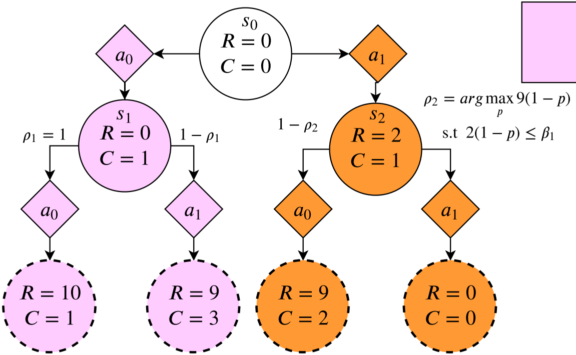



$r = 10$
$c = 2$

# The Dynamic Programming solution: intuition

$$r = 2 + 9\beta_1/2$$
$$c = 1 + \beta_1/2$$

$$r = 10$$
$$c = 2$$



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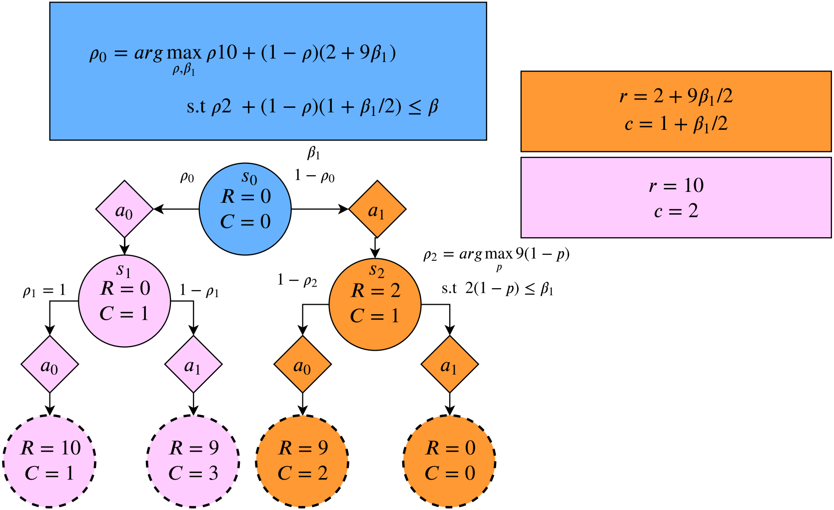


# The Dynamic Programming solution: intuition

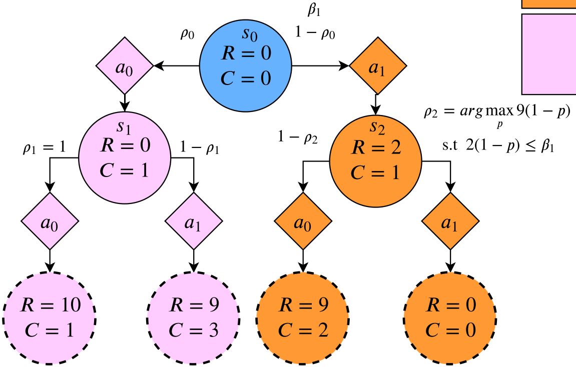
$$\rho_0 = arg\max_{\rho,\beta_1} \rho 10 + (1-\rho)(2+9\beta_1)$$

$$\text{s.t } \rho 2 + (1-\rho)(1+\beta_1/2) \leq \beta$$

$$r = \rho_0 10 + (1-\rho_0)(2+9\beta_1)$$
$$c = \rho_0 2 + (1-\rho_0)(1+\beta_1/2)$$

$$r = 2 + 9\beta_1/2$$
$$c = 1 + \beta_1/2$$

$$r = 10$$
$$c = 2$$



# Augmented Settings

**Budgeted policies** $\pi$

- Take a budget $\beta$ as an additional input
- Output a next budget $\beta'$
- $\pi : \underbrace{(s, \beta)}_{\overline{s}} \to \underbrace{(a, \beta')}_{\overline{a}}$

**Domain**

- States $\overline{\mathcal{S}} = \mathcal{S} \times \mathcal{B}$.
- Actions $\overline{\mathcal{A}} = \mathcal{A} \times \mathcal{B}$.
- Dynamics $\overline{P}\left((s', \beta') \mid (s, \beta), (a, \beta_a)\right) \stackrel{\text{def}}{=} P(s'|s, a)\delta(\beta' - \beta_a)$.

**2D signals**

1. Rewards $R = (R_r, R_c)$
2. Returns $G^\pi = (G_r^\pi, G_c^\pi)$
3. $V^\pi(\overline{s}) = (V_r^\pi, V_c^\pi) \stackrel{\text{def}}{=} \mathbb{E}\left[G^\pi \mid \overline{s_0} = \overline{s}\right]$
4. $Q^\pi(\overline{s}, \overline{a}) = (Q_r^\pi, Q_c^\pi) \stackrel{\text{def}}{=} \mathbb{E}\left[G^\pi \mid \overline{s_0} = \overline{s}, \overline{a_0} = \overline{a}\right]$

**Policy Evaluation**

The Bellman Expectation equationsare preserved, and the Bellman Expectation Operator $\mathcal{T}^\pi$ is a $\gamma$-contraction.

# Augmented Optimality

### Definition

In that order, we want to:

(i) Respect the budget $\beta$:

$$\Pi_a(\overline{s}) \stackrel{\text{def}}{=} \{\pi \in \Pi : V_c^\pi(s, \beta) \leq \beta\}$$

(ii) Maximise the rewards:

$$V_r^*(\overline{s}) \stackrel{\text{def}}{=} \max_{\pi \in \Pi_a(\overline{s})} V_r^\pi(\overline{s}) \qquad \Pi_r(\overline{s}) \stackrel{\text{def}}{=} \arg\max_{\pi \in \Pi_a(\overline{s})} V_r^\pi(\overline{s})$$

(iii) Minimise the costs:

$$V_c^*(\overline{s}) \stackrel{\text{def}}{=} \min_{\pi \in \Pi_r(\overline{s})} V_c^\pi(\overline{s}), \qquad \Pi^*(\overline{s}) \stackrel{\text{def}}{=} \arg\min_{\pi \in \Pi_r(\overline{s})} V_c^\pi(\overline{s})$$

We define the budgeted action-value function $Q^*$ similarly

# Budgeted Bellman Optimality Equation

### Theorem (Budgeted Bellman Optimality Equation)

*$Q^*$ verifies the following equation:*

$$Q^*(\overline{s}, \overline{a}) = \mathcal{T} Q^*(\overline{s}, \overline{a})$$

$$\stackrel{\text{def}}{=} R(\overline{s}, \overline{a}) + \gamma \sum_{\overline{s}' \in \overline{\mathcal{S}}} \overline{P}(\overline{s'} | \overline{s}, \overline{a}) \sum_{\overline{a'} \in \overline{\mathcal{A}}} \pi_{greedy}(\overline{a'} | \overline{s'}; Q^*) Q^*(\overline{s'}, \overline{a'}),$$

*where the greedy policy $\pi_{greedy}$ is defined by:*

$$\pi_{greedy}(\overline{a} | \overline{s}; Q) \in \operatorname*{arg\,min}_{\rho \in \Pi_r^Q} \mathop{\mathbb{E}}_{\overline{a} \sim \rho} Q_c(\overline{s}, \overline{a}),$$

$$\text{where} \quad \Pi_r^Q \stackrel{\text{def}}{=} \operatorname*{arg\,max}_{\rho \in \mathcal{M}(\overline{\mathcal{A}})} \mathop{\mathbb{E}}_{\overline{a} \sim \rho} Q_r(\overline{s}, \overline{a})$$

$$\text{s.t.} \quad \mathop{\mathbb{E}}_{\overline{a} \sim \rho} Q_c(\overline{s}, \overline{a}) \leq \beta$$

# Optimality of the policy

**Proposition (Optimality of the policy)**

*$\pi_{greedy}(\cdot\ ; Q^*)$ is simultaneously optimal in all states $\bar{s} \in \overline{\mathcal{S}}$:*

$$\pi_{greedy}(\cdot\ ; Q^*) \in \Pi^*(\bar{s})$$

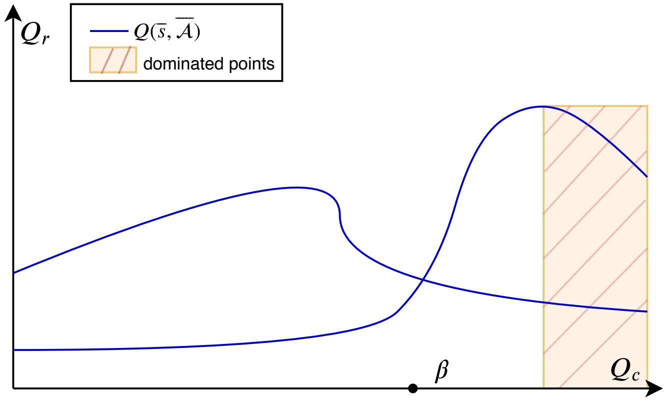
*In particular, $V^{\pi_{greedy}(\cdot\,;Q^*)} = V^*$ and $Q^{\pi_{greedy}(\cdot\,;Q^*)} = Q^*$.*

# Solving the untractable program

Proposition ($\pi_{\mathrm{greedy}} = \pi_{\mathrm{hull}}$)

*$\pi_{greedy}$ can be computed explicitly, as a mixture of two points that lie on the convex hull of $Q$.*

# Solving the non-linear programming problem: intuition



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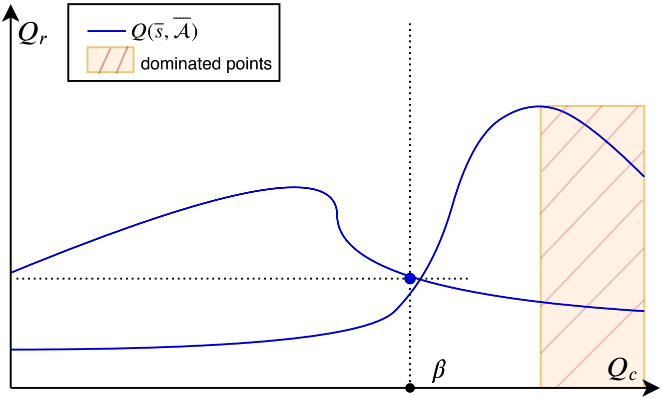

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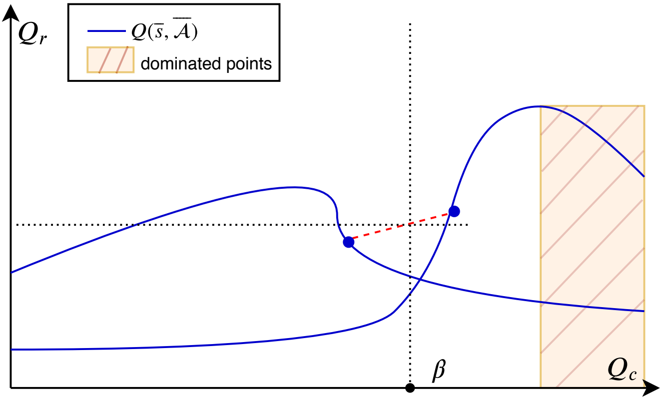

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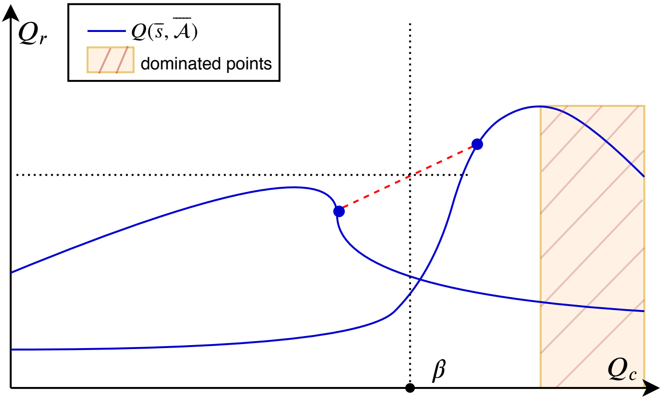

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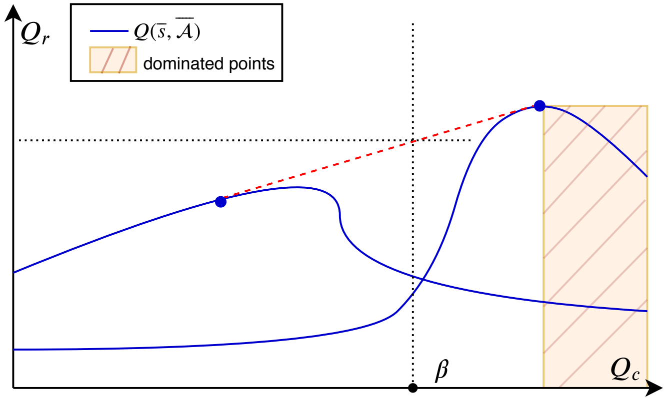


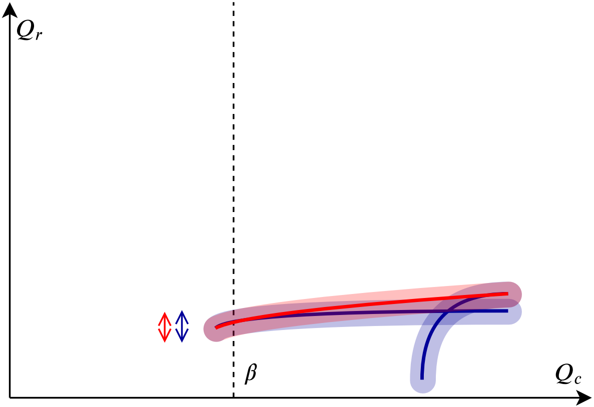
# Not a contraction

Theorem (Non-Contractivity)

*For any BMDP* $(\mathcal{S}, \mathcal{A}, P, R_r, R_c, \gamma)$ *with* $|\mathcal{A}| \geq 2$*,* $\mathcal{T}$ *is not a contraction.*

$$\forall \varepsilon > 0, \exists Q^1, Q^2 \in (\mathbb{R}^2)^{\overline{\mathcal{S}\mathcal{A}}} : \|\mathcal{T} Q^1 - \mathcal{T} Q^2\|_\infty \geq \frac{1}{\varepsilon} \|Q^1 - Q^2\|_\infty$$

# Not a contraction: intuition



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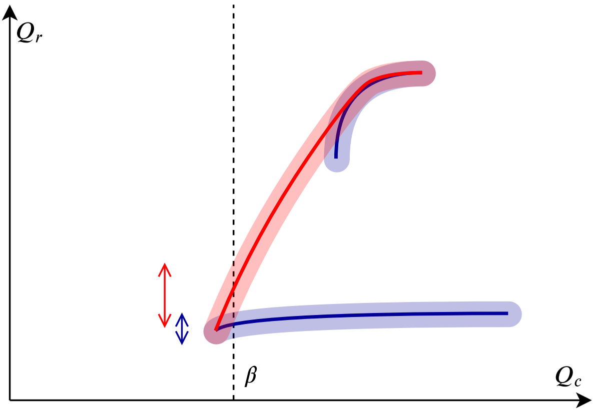
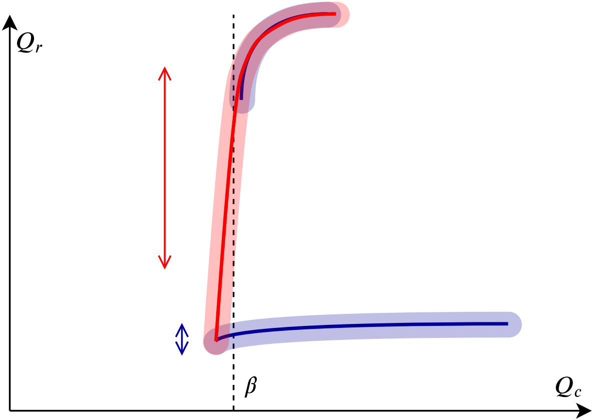


# Not a contraction: intuition



# Contractivity on smooth $Q$-functions

**Conjecture (Contractivity $\mathcal{L}_\gamma$)**

*$\mathcal{T}$ is a contraction when restricted to the subset $\mathcal{L}_\gamma$ of $Q$-functions such that "$Q_r$ is $L$-Lipschitz with respect to $Q_c$", with $L < \frac{1}{\gamma} - 1$.*

# Budgeted Dynamic Programming

**Algorithm 1:** Budgeted Value-Iteration

**Data:** $P, R_r, R_c$

**Result:** $Q^*$

```
Q_0 ← 0
repeat
    Q_{k+1} ← 𝒯Q_k
until convergence
```

# Budgeted Reinforcement Learning

We address several limitations of Budgeted Value-Iteration

1. If the $P$, $R_r$ and $R_c$ are unknown:
   - Work with a batch of samples $\mathcal{D} = \{(\overline{s}_i, \overline{a}_i, r_i, \overline{s}'_i\}_{i\in[0,N]}$
   - Replace $\mathcal{T}$ with a sampling operator $\hat{\mathcal{T}}$:

$$\hat{\mathcal{T}} Q(\overline{s}_i, \overline{a}_i, r_i, \overline{s}'_i) \stackrel{\text{def}}{=} r_i + \gamma \sum_{\overline{a'_i} \in \mathcal{A}_i} \pi_{\text{greedy}}(\overline{a'_i} | \overline{s'_i}; Q) Q(\overline{s'_i}, \overline{a'_i}).$$

2. If $\mathcal{S}$ is continuous:
   - Employ function approximation $Q_\theta$, and minimise a regression loss

$$\mathcal{L}(Q_\theta, Q_{\text{target}}; \mathcal{D}) = \sum_{\mathcal{D}} ||Q_\theta(\overline{s}, \overline{a}) - Q_{\text{target}}(\overline{s}, \overline{a}, r, \overline{s}')||_2^2$$

# Budgeted Fitted-Q

**Algorithm 2:** Budgeted Fitted-Q Iteration

**Data:** $\mathcal{D}$

**Result:** $Q^*$

$Q_{\theta_0} \leftarrow 0$

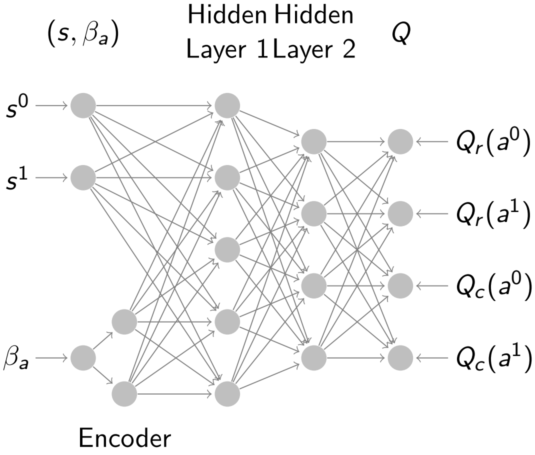
**repeat**

  $\theta_{k+1} \leftarrow \arg\min_\theta \mathcal{L}(Q_\theta, \hat{\mathcal{T}} Q_{\theta_k}; \mathcal{D})$

**until** *convergence*

# More scaling

- CPU parallel computing of the targets $\sum_{\overline{a_i'} \in \mathcal{A}_i} \pi_{\text{greedy}}(\overline{a_i'} | \overline{s_i'}; Q) Q(\overline{s_i'}, \overline{a_i'}) \ \forall i$
- Same for samples generation.
- Neural Network as function approximator:



# Experiments: performances of BFTQ

- Baseline: $\lambda$-FTQ, Lagrangian relaxation
  - $R_r(s, a) \leftarrow R_r(s, a) - \lambda R_c(s, a)$ where $\lambda \geq 0$

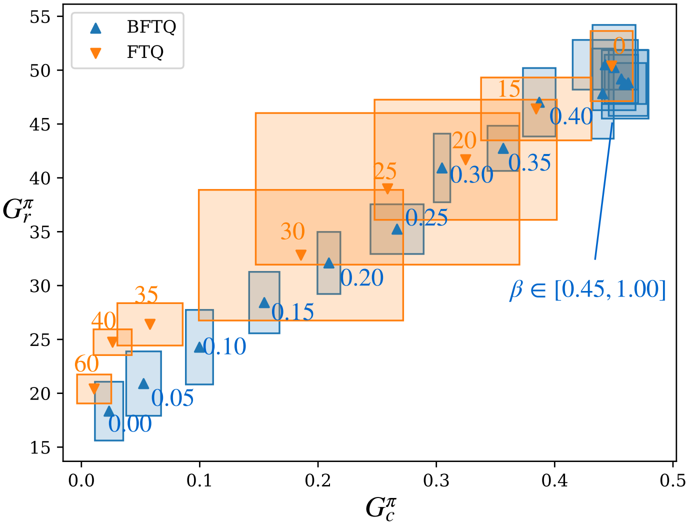
.

- Applications:
  - dialogue systems
  - autonomous driving

# Experiments: dialogue systems

- A slot-filling problem: the agent (the dialogue system) fills a form by asking the user each slot.
- Two ways to deal with recognition errors:
  - ask to repeat with voice (safe/slow),
  - Ask to repeat with numeric pad (unsafe/fast).

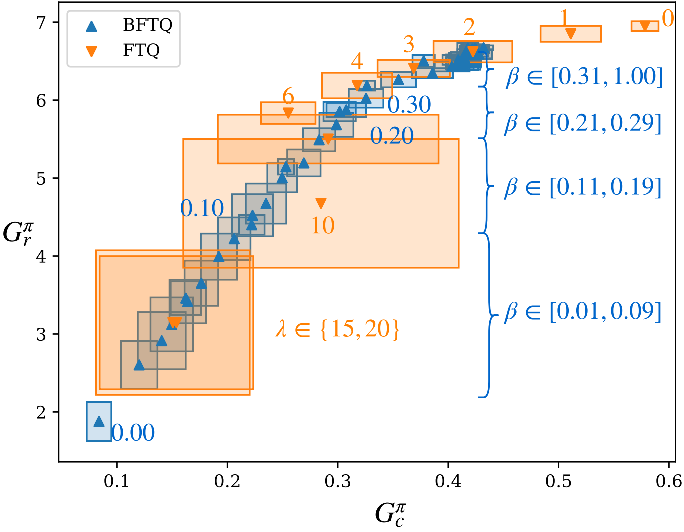
# Experiments: dialogue systems



# Experiments: autonomous driving

- the agent (the car) is on a two-way road with a car in front of it:
  - it can stay behind (safe/slow),
  - it can overtake (unsafe/fast).

# Experiments: autonomous driving



# Experiments: autonomous driving

▸ BFTQ on the highway environment

# Risk-sensitive exploration

How to collect the batch $\mathcal{D}$?

- We propose an $\varepsilon$-greedy exploration procedure
  - Sample an initial budget $\beta_0$
  - At each step, where $\overline{s} = (s, \beta)$ only explore feasible budgets:

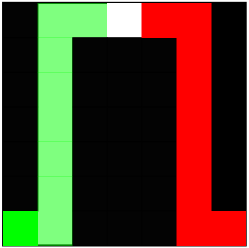
$$\overline{a} = (a, \beta_a) \sim \mathcal{U}(\Delta_{\mathcal{AB}})$$
where $\Delta$is s.t. $\mathbb{P}\,(a, \beta_a | s, \beta)$ verifies $\mathbb{E}[\beta_a] \leq \beta$

# Experiments: risk-sensitive exploration

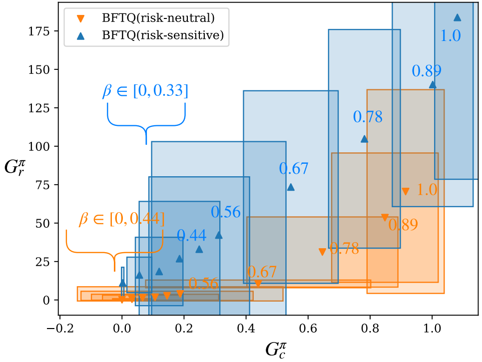
- Validate the risk-sensitive exploration procedure on the corridor environment
- Learn 2 BFTQ policies with respectively:
  - A batch generated by a risk-neutral $\varepsilon$-greedy procedure
  - A batch generated by a risk-sensitive $\varepsilon$-greedy procedure

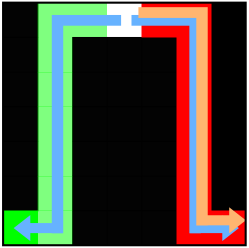
# Experiments: corridors

- 2 corridors:
  - high costs/high rewards around the starting state
  - no costs/low rewards around the starting state
- The outermost cell is the one yielding the most reward



# Experiments: corridors



# Experiments: corridors

▸ Risk-sensitive vs Risk-Neutral on the corridors environment

# Summary

+ Budgeted Bellman Optimality Operator.
  - Fixed point.
  - Not a contraction but converging in practice.
+ Scalable for RL in continuous state space.
  - Function approximation with Neural Network (dedicated architecture).
  - Solving of the untractable program using convex hull.
  - CPU parallel computing of the target.
  - Risk-sensitive exploration procedure.
+ Experiments on two applications.
  - BFTQ reaches similar performances as Lagrangian relaxation,
  - with no need for calibration,
  - and less variance.