(Q-1) – On Matlab Completely. 21103033

(Q-1(2)) $S = 10x^2 \frac{d^2y}{dx^2} + 5x^3 y - \sin x \; ; \; -L < x < L$.

(a) $y|_{x=-L} = 2$ and $y'|_{x=L} = -1$.

To get the weak form from the strong form –

$$\int_{\Omega} w_i \, r_\Omega(\underline{s}, x) \, d\Omega = 0 .$$

$$\Rightarrow \int_{-L}^{L} w \left(10x^2 y'' + 5x^3 y - \sin x\right) dx = 0 .$$

$$\Rightarrow \int_{-L}^{L} w\left(10x^2 y''\right) dx + \int_{-L}^{L} w\, 5x^3 y \, dx - \int_{-L}^{L} w \sin x \, dx = 0 .$$

$$\Rightarrow w\, 10x^2 y'\Big|_{-L}^{L} + \cancel{5x^3 w} \int_{-L}^{L} w\, 5x^3 y \, dx - \left[-w \cos x \Big|_{-L}^{L} + \int_{-L}^{L} \frac{dw}{dx} \cos x \, dx \right] = 0 .$$

$$-\int_{-L}^{L} \frac{d}{dx}\left(w 10x^2\right) y' \, dx$$

$$\Rightarrow \left(w\, 10x^2 y'\right)_{x=L} - \left(\cancel{w\, 10x^2 y'}\right)_{x=-L}^{0} + 5\int_{-L}^{L} w x^3 y \, dx + \left(w \cos x\right)\Big|_{x=L} - \int_{-L}^{L} \frac{dw}{dx} \cos x \, dx = 0 .$$

as $w = 0$ at Tu

$$-\int_{-L}^{L} \frac{d}{dx}\left(w 10x^2\right) y' \, dx$$

$$\Rightarrow w\, 10x^2 y'\Big|_{x=L} - \int_{-L}^{L} 10x^2 \frac{dw}{dx} y' \, dx - \int_{-L}^{L} w\, 20x y' \, dx + 5\int_{-L}^{L} w x^3 y \, dx + \left(w \cos x\right)\Big|_{x=L} - \int_{-L}^{L} \frac{dw}{dx} \cos x \, dx = 0 .$$

$$\Rightarrow -\int_{-L}^{L} \left[\left\{ \left(\frac{dw}{dx}\right) 10x^2 + 20wx \right\} y' + \left(\frac{dw}{dx}\right) \cos x \right] dx + 5\int_{-L}^{L} w x^3 y \, dx + w\left(10x^2 y' + \cos x\right)\Big|_{x=L} = 0 .$$

$$\Rightarrow \int_{-L}^{L} \left[\left\{ \left(\frac{dw}{dx}\right) 10x^2 + 20wx \right\} y' + \left(\frac{dw}{dx}\right) \cos x \right] dx - 5\int_{-L}^{L} w x^3 y \, dx = \left(-10L^2 + \cos L\right) w = 0 .$$

(as @ $x = L$; $y' = -1$)

which is the required WF-form

(c) From Vainberg's theorem if W is a function of y, w such $\frac{d}{d\xi} W(y+\xi w_2, w_2)\Big|_{\xi=0} = \frac{d}{d\xi} W(y+\xi w_2, w_1)$ @ $\xi=0$

then $\exists$ a functional $\pi(y)$ only that satisfies

$$\frac{d}{d\xi}\pi(y+\xi w)\Big|_{\xi=0} = W(y,w)$$

$$W(y,w) = \int_{-L}^{L}\left[\left\{\frac{dw}{dx}10x^2y' + 20wxy'\right\} + \left(\frac{dw}{dx}\right)\cos x\right]dx - 5\int_{-L}^{L} wx^3y\,dx - (\cos L - 10L^2)w\Big|_{x=L}$$

Now, $\frac{d}{d\xi} w(y+\xi w_1, w_2) \longrightarrow$

$$= \frac{d}{d\xi}\int_{-L}^{L}\left[\left\{\frac{dw_2}{dx}10x^2\frac{d}{dx}(y+\xi w_1) + 20w_2x\frac{d}{dx}(y+\xi w_1)\right\} + \left(\frac{dw_2}{dx}\right)\cos x\right]dx - \frac{d}{d\xi}5\int_{-L}^{L} w_2x^3(y+\xi w_1)\,dx - \frac{d}{d\xi}w_2(\cos L - 10L^2)\Big|_{x=L}$$

$$= \int_{-L}^{L}\left[\left\{\frac{dw^2}{dx}10x^2\frac{dw_1}{dx} + 20w_2x\frac{d}{dx}w_1\right\} + 0\right]dx - 5\int_{-L}^{L} 5w_2x^3w_1\,dx - 0$$

Now $\frac{d}{d\xi}(W(y+\xi w_2, w_1)] \longrightarrow$

$$\frac{d}{d\xi}\int_{-L}^{L}\left[\left\{\frac{dw_1}{dx}10x^2\frac{d}{dx}(y+\xi w_2) + 20w_1x\frac{d}{dx}(y+\xi w_2)\right\} + \left(\frac{dw_1}{dx}\cos x\right)\right]dx - \frac{d}{d\xi}\int_{-L}^{L} 5w_1x^3(y+\xi w_2)\,dx - \frac{d}{d\xi}w_2(\cos L - 10L^2)\Big|_{x=L}$$

$$= \int_{-L}^{L}\left[\left\{\frac{dw_1}{dx}10x^2\frac{dw_2}{dx} + 20w_1x\frac{dw_2}{dx}\right\} + 0\right]dx - 5\int_{-L}^{L} w_1x^3w_2\,dx - 0.$$

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(c) Due to the existence of the weight w and y', this introduces a non-symmetric term in the W-form. For symmetry to exist, we must always have either both $\bar{w}'$ and $\bar{y}'$ in non-derivative sense or the order of derivative for both $\bar{w}$ and $\bar{y}'$ must be same.

Hence a functional (Π) form of the problem does not exist.

(b) The TNT-form of the above case can be written in a shorter form as –

$$\int_{-L}^{L} (10x^2 y'')w\,dx + \underbrace{\int_{-L}^{L} 5x^3 y\, w\,dx - \int_{-L}^{L} w \sin x\,dx}_{=I} = 0.$$

By parts –

$$\left[10x^2 w^{n} y'^{t}\right] + \left[\cancel{10x^2 w^{n} y'}\right]_{L} - \int_{-L}^{L} \frac{d}{dx}(w\,10x^2)\,y'\,dx + I = 0$$

$$\Rightarrow \int_{-L}^{L} \frac{d}{dx}(w\,10x^2)\,y'\,dx - I - \left[10x^2 w y'^{n}\right]_{t} = 0.$$

By chain rule of differentiation

$$\Rightarrow \int_{-L}^{L} 20wxy'\,dx + \int_{-L}^{L} 10x^2\left(\frac{dw}{dx}\right)y'\,dx - I + \left[10L^2 w\right]_{@\,x=L} = 0.$$

$y' = -1$.

$$\Rightarrow \int_{-L}^{L}\left[20wx + 10x^2\left(\frac{dw}{dx}\right)\right]y'\,dx - \int_{-L}^{L} 5x^3 y w\,dx + \int_{-L}^{L} w \sin x\,dx + \left[10L^2 w\right]_{x=L} = 0.$$

which is a simplified W-form.

(b) Now coefficient of $w = \frac{\partial L}{\partial \bar{y}}$ @ $\xi = 0$

" " " $w' = \frac{\partial L}{\partial \bar{y}'}$ @ $\xi = 0$

at $\xi = 0$, $\bar{y}' = y'$ and $\bar{y} = y$.

$\therefore$ The differential form is nothing but summation of $\frac{\partial L}{\partial \bar{y}} - \frac{d}{dx}\left(\frac{\partial L}{\partial \bar{y}'}\right) = 0$

$$= \cancel{20xy' + 5x^3 y - \sin x + 10x^2}$$

$$\Rightarrow 20xy' - 5x^3 y + \sin x - \frac{d}{dx}(10x^2 y') = 0$$

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$$20x\cancel{y'} - 5x^3y + \sin x - \cancel{20xy'} - 10x^2y'' = 0$$

$$\Rightarrow \quad 10x^2y'' + 5x^3y - \sin x = 0$$

$$\Rightarrow \quad \boxed{10x^2y'' + 5x^3y = \sin x.}$$

which is our original ODE.

Also from boundary condition terms –

$$10x^2 w y' n\Big|_{\Gamma_u} = 0.$$

$$\Rightarrow \quad y'\Big|_{\Gamma_u} = 0 \quad \Rightarrow \quad y = \bar{y} = 2 \ @\ x = -L$$

as per given essential BC.

also $10x^2 w y' n\Big|_{\Gamma_t} \neq 0.$

i.e $y'\Big|_{\Gamma_t} \neq 0 \Rightarrow y'\Big|_{\Gamma_t} = -1$ as per given Natural BC.

(d) Since our weak form consists of a product term of w and y' and in Galerkin's approach, $w = N^e w^e$; $y = N^e d^e$ considering w is composed of the same basis shape f^n as u, $\therefore y' = B^e d^e$ and $w' = B^e w^e$, For symmetry to exist throughout the system, the matrix multiplication must be of form $(B^{eT}B^e)$ or $(N^{eT}N^e)$ and non-symmetric for product like $(B^{eT}N^e)$ or $(N^{eT}B^e)$ which occurs due to the product $(w \times y')$ in our equation.

Thus in our case the stiffness matrix will be non-symmetric.

e) $10x^2y'' + 5x^2y' + 5x^3y - \sin x = 0$, with same B.Cs.

$$\underbrace{\int_{-L}^{L} w\, 10x^2 y''\, dx}_{I_1} + \underbrace{\int_{-L}^{L} w\, 5x^2 y'\, dx}_{I_2} + \underbrace{\int_{-L}^{L} 5x^3 y\, w\, dx - \int_{-L}^{L} w \sin x\, dx}_{= I} = 0.$$

Doing separately, $I_1 = \int_{-L}^{L} w\, 10x^2 y''\, dx.$

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$\Rightarrow$ Using By parts —

$$I_1 = \int_{-L}^{L} w\,10x^2 y'' dx = \left[w\,10x^2 y' n\right]_{\Gamma_u} + \left[w\,10x^2 y' n\right]_{\Gamma_t} - \int_{-L}^{L} \frac{d}{dx}(w\,10x^2)\, y' dx$$

as $w=0$ at Γ_u

$$= \cancel{\left[w\,10x^2 y' n\right]_{\Gamma_u}}^{\,0} + \left[w\,10x^2 y' n\right]_{\Gamma_t} - \int_{-L}^{L} w'\,10x^2 y' dx - \int_{-L}^{L} 20xw\,y' dx.$$

$$I_2 = \int_{-L}^{L} 5x^2 w y' dx = \cancel{\left[5x^2 w y' n\right]_{\Gamma_u}}^{\,0} + \left[5x^2 w y n\right]_{\Gamma_t} - \int_{-L}^{L} \frac{d}{dx}(5x^2 w)\, y\, dx$$

$$= \cancel{\left[5x^2 w y n\right]_{\Gamma_t}}^{\,0} - \int_{-L}^{L} 10xwy\,dx - \int_{-L}^{L} 5x^2 w' y\, dx.$$

at y is not specified at Γ_t

$$= -\int_{-L}^{L} 10xwy\,dx - \int_{-L}^{L} 5x^2 w' y\, dx.$$

$$\therefore W = \int_{-L}^{L} w'\,10x^2 y' dx + \int_{-L}^{L} 20wxy'\,dx + \int_{-L}^{L} 10xwy\,dx + \int_{-L}^{L} 5x^2 w' y\, dx - \int_{-L}^{L} 5x^3 y w\, dx + \int_{-L}^{L} w \sin x\, dx = 0.$$

$$\Rightarrow \int_{-L}^{L} \cancel{(10x^2 y' + 5x^2 y')}\,(w'\,10x^2 + 20xw)\, y' dx + \int_{-L}^{L} (5x^2 w' + 10wx + 5x^3 w)\, y\, dx + \int_{-L}^{L} w \sin x\, dx = 0$$

Here also a $\Pi(y)$ form does not exist for the same reason as before, the existense of either $(w'y)$ or $(y'w)$ will lead to non-symmetry as shown before. Thus here also a $\Pi(u)$ form does not exist.

Lastly, K will not be symmetric as the existence of $(w'y)$ or $(y'w)$ will result in the terms $(B^{e^T} N^e)$ ~~or~~ in product form which will lead to non-symmetrical 'K' matrix.

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Q:-) (3) $\Pi = \int_0^L \left[\frac{1}{2}\left(\frac{d\phi}{dx}\right)^2 - 50\phi\right] dx$.

(a) essential boundary conditions :-

$\phi = 0$; $x = 0$, $\phi = 20$; $x = L$.

consider $\mathcal{L}(x, \phi, \phi')$ as $= \frac{1}{2}\left(\frac{d\phi}{dx}\right)^2 - 50\phi$.

$$\therefore \Pi(x, \phi, \phi') = \int_0^L \mathcal{L}(x, \phi, \phi')\, dx.$$

for some $\bar{\phi} = \phi + \xi\omega$ where both $\bar{\phi}$ and ϕ are subsets of 'Φ' the set of admissible functions satisfying the essential boundary conditions, and ω is a random variable quantity $\in \Phi_0$

Now, for Π to be minimum, we know,

$\left.\frac{d\Pi}{d\xi}\right|_{\xi=0} = 0$ so that $\bar{\phi} = \phi + \cancel{\xi}^{\,0}\omega$

$= \phi$ becomes that

subset of $\bar{\Phi}$ which minimises the P.E functional

$$\text{So } \Pi(\bar{\phi}) = \int_0^L \mathcal{L}(x, \bar{\phi}, \bar{\phi}')\, dx.$$

$$\therefore \left.\frac{d\Pi}{d\xi}\right|_{\xi=0} = \int_0^L \left[\frac{\partial \mathcal{L}}{\partial x}\frac{dx}{d\xi} + \frac{\partial \mathcal{L}}{\partial \phi}\frac{\partial \phi}{\partial \xi} + \frac{\partial \mathcal{L}}{\partial \phi'}\cdot\frac{d\phi'}{d\xi}\right] dx$$

$$\bar{\phi} = \phi + \xi\omega$$

$$\frac{d\bar{\phi}}{d\xi} = \omega \quad \text{and} \quad \bar{\phi}' = \phi' + \xi\omega'$$

$$\frac{d\bar{\phi}'}{d\xi} = 0 + \omega'$$

$$\Rightarrow \left.\frac{d\Pi}{d\xi}\right|_{\xi=0} = \int_0^L \frac{\partial \mathcal{L}}{\partial x}\cancel{\left(\frac{dx}{d\xi}\right)}^{\,0} + \left(\frac{\partial \mathcal{L}}{\partial \phi}\omega + \frac{\partial \mathcal{L}}{\partial \bar{\phi}'}\cdot\omega'\right) dx$$

$$\Rightarrow \int_0^L \left.\left(\frac{\partial \mathcal{L}}{\partial \phi}\omega + \frac{\partial \mathcal{L}}{\partial \bar{\phi}'}\cdot\omega'\right) dx\right|_{\xi=0} = 0.$$

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Now, considering the 2nd term +

$$\int_0^L \frac{\partial L}{\partial \bar{\phi}'} w' \, dx$$

Using integral by parts -

$$\frac{\partial L}{\partial \bar{\phi}'} w_n \Big|_{\Gamma_u} - \int_0^L \frac{d}{dx}\left(\frac{\partial L}{\partial \bar{\phi}'}\right) w \, dx .$$

Since no. traction boundary is present

$\therefore$ substituting in the equation, -

$$\int_0^L \underbrace{\left[\frac{\partial L}{\partial \bar{\phi}} \cdot w - \frac{d}{dx}\left(\frac{\partial L}{\partial \bar{\phi}'}\right) w\right] dx}_{\text{(I)}} + \frac{\partial L}{\partial \bar{\phi}'} w_n \Big|_{\Gamma_u} = 0 .$$

from (I) :- As w is randomand arbitrary,

$$\frac{\partial L}{\partial \bar{\phi}} - \frac{d}{dx}\left(\frac{\partial L}{\partial \bar{\phi}'}\right) = 0 .$$

at $\xi = 0$, $\phi = \bar{\phi}$ $\therefore$ replace ξ with $\bar{\phi}$.

Now, $L = \frac{1}{2}\left(\frac{d\bar{\phi}}{dx}\right)^2 - 50\bar{\phi}$

$$\frac{\partial L}{\partial \bar{\phi}} = -50 .$$

$$\frac{\partial L}{\partial \bar{\phi}'} = \cancel{2} \times \frac{1}{\cancel{2}} \bar{\phi}'$$

Thus, $-50 - \frac{d}{dx}(\phi)' = 0 .$

$$\Rightarrow \boxed{\frac{d}{dx}\left(\frac{d\phi}{dx}\right) + 50 = 0}$$

Which is our original ODE in strong form

and from (II) :-

$$\frac{\partial L}{\partial \bar{\phi}'} w' \Big|_{\Gamma_u} = 0 \Rightarrow \frac{d\phi}{dx} w' \Big|_{\Gamma_u} = 0$$

$\Rightarrow \phi = \bar{\phi}$ at Γ_u.

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Exact solution.

$$\frac{d^2\phi}{dx^2} + 50 = 0.$$

Now, $$\frac{d^2\phi}{dx^2} = -50.$$

$\Rightarrow$ Integrating once –

$$\frac{d\phi}{dx} = -50x + C_1;$$

again integrating,

$$\phi = -\frac{\cancel{50}^{25}x^2}{\cancel{2}} + C_1x + C_2$$

at $x=0$, $\phi=0$; $\Rightarrow C_2 = 0$.

$$\therefore \phi = -\cancel{50}^{25}x^2 + C_1x.$$

at $x=L$ $\phi=20$,

$$\therefore 20 = -\cancel{50}^{25}L^2 + C_1L.$$

$$\Rightarrow \therefore C_1 = \frac{20 + \cancel{50}^{25}L^2}{L}$$

$$= \cancel{50}^{25}L + \frac{20}{L}.$$

$$\therefore \phi = -\cancel{50}^{25}x^2 + \cancel{50}^{25}Lx + \frac{20}{L}x.$$

Exact solution :- $\Rightarrow$. $$\phi = -\cancel{50}^{25}x^2 + \left(\cancel{50}^{25}L + \frac{20}{L}\right)x.$$

(b) (i) Rayleigh - Ritz approach for linear approx ; $\phi = \sum_{i=1}^{2} a_i f_i(x)$.

$$= \cancel{a_1x + a_2x} \quad a_0 + a_1x$$

substituting in the 'Π' form and putting $\frac{\partial \Pi}{\partial a_i} = 0$ to obtain the coefficients.

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$$\pi = \int_0^L \left\{ \frac{1}{2}\left[\frac{d}{dx}(a_0 + a_1 x)\right]^2 - 50(a_0 + a_1 x) \right\} dx .$$

$$\cancel{\frac{\partial \pi}{\partial a_0} = \int_0^L \frac{1}{2} \cdot 2 (a_0 + a_1 x) \cdot 1 - 50}$$

imposing the BCs,

$$\phi = a_0 + a_1 x .$$

$$\phi = 0 \text{ at } x = 0 \; ; \; a_0 = 0 .$$

$$\text{at } x = L, \; \phi = 20$$

$$\therefore a_1 = \frac{20}{L} .$$

$$\therefore \phi = \frac{20}{L} x .$$

Which is a linear approx and does not come any way closer to the approx solution.

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(ii) Quadratic approximation –

$$\phi = a_0 + a_1 x + a_2 x^2 .$$

applying BCs –

at $x=0$, $\phi=0 \rightarrow a_0 = 0$.

$$\phi = a_1 x + a_2 x^2 .$$

at $x = L$; $\phi = 20 \rightarrow 20 = a_1 L + a_2 L^2$.

$$\Rightarrow a_2 L^2 = 20 - a_1 L .$$

$$\Rightarrow a_2 = \left(\frac{20}{L^2} - \frac{a_1}{L}\right) .$$

$$\therefore \phi = a_1 x + \left(\frac{20}{L^2} - \frac{a_1}{L}\right) x^2 .$$

$$\phi' = a_1 + 2\left(\frac{20}{L^2} - \frac{a_1}{L}\right) x .$$

$$\therefore \pi = \int_0^L \left\{ \frac{1}{2} (\phi')^2 - 50\phi \right\} dx$$

$$= \int_0^L \frac{1}{2} \left(a_1 + 2x\left(\frac{20}{L^2} - \frac{a_1}{L}\right)\right]^2 - 50\left(a_1 x + \left(\frac{20}{L^2} - \frac{a_1}{L}\right) x^2\right] dx$$

$$\frac{\partial \pi}{\partial a_1} = \int_0^L \left[\frac{1}{\cancel{2}} \times \cancel{2} \left(a_1 + 2x\left(\frac{20}{L^2} - \frac{a_1}{L}\right)\right] \times \left(1 + 2x\left(-\frac{1}{L}\right)\right] - 50\left(x - \frac{x^2}{L}\right)\right] dx = 0$$

$$\Rightarrow \int_0^L \left\{ a_1 + 2x\left(\frac{20}{L^2} - \frac{a_1}{L}\right)\right\} \left\{ 1 - \frac{2x}{L} \right\} dx - \int_0^L 50\left(x - \frac{x^2}{L}\right) dx = 0$$

$$\Rightarrow \int_0^L \left\{ a_1 - \frac{2x}{L} a_1 + 2x\left(\frac{20}{L^2} - \frac{a_1}{L}\right) - \frac{4}{L}\left(\frac{20}{L^2} - \frac{a_1}{L}\right) x^2 \right\} dx - \int_0^L 50x\, dx + \int_0^L \frac{50x^2}{L} dx = 0 .$$

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$$\Rightarrow \cancel{a_1 L} - \cancel{a_1 L} + L^2\left(\frac{20}{L^2} - \frac{a_1}{L}\right) - \frac{4}{3}L^2\left(\frac{20}{L^2} - \frac{a_1}{L}\right)$$

$$- 25L^2 + \frac{50}{3}L^2 = 0 .$$

$$\Rightarrow 20 - a_1 L - \frac{80}{3} + \frac{4}{3}a_1 L - \frac{25}{3}L^2 = 0 .$$

$$\Rightarrow \frac{1}{\cancel{3}}a_1 L - \frac{20}{\cancel{3}} - \frac{25}{\cancel{3}}L^2 = 0 .$$

$$\Rightarrow a_1 L = 25L^2 + 20 .$$

$$\Rightarrow a_1 = 25L + \frac{20}{L} .$$

$$\therefore a_2 = \left(\frac{20}{L^2} - \frac{1}{L}\left(25L + \frac{20}{L}\right)\right] .$$

$$= \cancel{\frac{20}{L^2}} - 25 - \cancel{\frac{20}{L^2}}$$

$$\therefore q = \left(25L + \frac{20}{L}\right)x - 25x^2 .$$

which is same as the exact soln.

(i) $\pi = \int_0^L \frac{1}{2}(\phi')^2 - 50\phi \, dx$ $\quad \phi = a_0 + a_1 x + a_2 x^2 + a_3 x^3$. (Cubic approx).

applying boundary condition –

@ $x=0$; $\phi = 0 \Rightarrow a_0 = 0$.

@ $x = L$; $\phi = 20$

$\Rightarrow 20 = a_1 L + a_2 L^2 + a_3 L^3$. $\Rightarrow a_1 L = 20 - a_2 L^2 - a_3 L^3$

$$\Rightarrow a_1 = \frac{20}{L} - \frac{a_2 L^2}{L} - \frac{a_3 L^3}{L}$$

$$= \left(\frac{20}{L} - a_2 L - a_3 L^2\right)$$

$$\therefore \phi = \left(\frac{20}{L} - a_2 L - a_3 L^2\right) x + a_2 x^2 + a_3 x^3.$$

$$\phi' = \left(\frac{20}{L} - a_2 L - a_3 L^2\right) + 2a_2 x + 3a_3 x^2.$$

$$\pi = \int_0^L \frac{1}{2}\left[\left(\frac{20}{L} - a_2 L - a_3 L^2\right) + 2a_2 x + 3a_3 x^2\right]^2 - 50\left[\left(\frac{20}{L} - a_2 L - a_3 L^2\right) x + a_2 x^2 + a_3 x^3\right] dx.$$

$$\frac{\partial \pi}{\partial a_2} = \int_0^L \left\{\left(\frac{20}{L} - a_2 L - a_3 L^2\right) + 2a_2 x + 3a_3 x^2\right\}(2x - L)\, dx$$

$$\cancel{- 50\left[\left(\frac{20}{L} - a_2 L - a_3 L^2\right) x + a_2 x^2 + a_3 x^3\right]}\left(x^2 - xL\right) dx = 0$$

$$\Rightarrow \int_0^L \left(\frac{20}{L} - a_2 L - a_3 L^2\right) 2x\, dx - \left(\frac{20}{L} - a_2 L - a_3 L^2\right) L\, dx + \int_0^L 2a_2 x \cdot 2x$$

$$- \int_0^L 2a_2 x L\, dx + \int_0^L 3a_3 x^2 (2x)\, dx - \int_0^L 3a_3 x^2 L\, dx - \int_0^L 50(x^2 - xL)\, dx = 0$$

$$\cancel{- \int_0^L 50\left(\frac{20}{L} - a_2 L - a_3 L^2\right) x^3 + 50\left(\frac{20}{L} - a_2 L - a_3 L^2\right) x^2 L}\, dx$$

$$\cancel{- \int_0^L 50 a_2 x^4 + \int_0^L 50 a_2 x^3 L\, dx - \int_0^L 50 a_3 x^5\, dx + \int_0^L 50 a_3 x^4 L\, dx} = 0.$$

$$\left[\frac{\cancel{40}^{20}}{\cancel{L}}\left(\frac{L^{\cancel{2}}}{\cancel{2}}\right) - \cancel{2} a_2 L \frac{(L)^2}{\cancel{2}} - \cancel{2} a_3 L^2 \left(\frac{L^2}{\cancel{2}}\right)\right] - \left[20(L) - a_2 (L)^3 - a_3 (L)^4\right].$$

$$+ 4a_2\left(\frac{L^3}{3}\right) - 2a_2 L\left(\frac{L^2}{2}\right) + 6a_3\left(\frac{L^4}{4}\right) - 3a_3 L \frac{(L)^3}{3}.$$

$$\cancel{- 50\left[\frac{20}{L} \cdot \frac{(L)^4}{4} - a_2 L \cdot \frac{(L)^4}{4} - a_3 L^2 \left(\frac{L^4}{4}\right)\right]} - 50\left(\frac{L^3}{3} - L \cdot \frac{L^2}{2}\right) = 0.$$

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$$\Rightarrow \cancel{20l} - a_2 l^3 - a_3 l^4 - \cancel{20l} + a_2 l^3 + a_3 l^4 + 4a_2 \frac{l^3}{3} - a_2 l^3$$

$$+ \frac{3}{2} a_3 l^4 - a_3 l^4 + 50\left(\frac{l^3}{6}\right) = 0.$$

$$\Rightarrow \frac{a_2 l^3}{3} + \frac{a_3 l^4}{2} = -\frac{\cancel{50}^{25} l^3}{\cancel{6}\,3}. \quad -(i).$$

Similarly $\frac{\partial \pi}{\partial a_3} = 0$. —

$$\int_0^l \left[\left(\frac{20}{l} - a_2 l - a_3 l^2\right) + 2a_2 x + 3a_3 x^2\right]\left(3x^2 - l^2\right) dx - \int_0^l 50\left(x^3 - x l^2\right) dx = 0.$$

$$\Rightarrow \int_0^l \left(\frac{20}{l} - a_2 l - a_3 l^2\right)\left(3x^2 - l^2\right) dx + \int_0^l 2a_2 x\left(3x^2 - l^2\right) dx$$

$$+ \int_0^l 3a_3 x^2\left(3x^2 - l^2\right) dx - 50\left[\frac{l^4}{4} - l^2 \cdot \frac{l^2}{2}\right] = 0.$$

$$\Rightarrow \cancel{20}\ \frac{\cancel{60}^{20}}{\cancel{l}} \cdot \left(\frac{l^3}{\cancel{3}}\right) - \frac{20}{\cancel{l}} \cdot l^3 - \cancel{3}a_2 l\left(\frac{l^3}{\cancel{3}}\right) + a_2 l^4 - \cancel{3}a_3 l^2\left(\frac{l^3}{\cancel{3}}\right)$$

$$+ a_3 l^5 + 6a_2\left(\frac{l^4}{4}\right) - 2a_2 l^2\left(\frac{l^2}{2}\right) + 9a_3\left(\frac{l^5}{5}\right) - 3a_3 l^2\left(\frac{l^3}{3}\right).$$

$$- 50\left(\frac{l^4}{4} - \frac{l^4}{2}\right) = 0.$$

$$\Rightarrow \cancel{20l^2} - \cancel{20l^2} - \cancel{3a_2 l^4} + \cancel{3a_2 l^4} - \cancel{a_3 l^5} + \cancel{a_3 l^5} + \frac{3}{2} a_2 l^4 - a_2 l^4$$

$$+ \frac{9}{5} a_3 l^5 - a_3 l^5 + \frac{50 l^4}{4} = 0.$$

$$\Rightarrow \frac{1}{2} a_2 l^4 + \frac{4}{5} a_3 l^5 = -\frac{25}{2} l^4 \quad -(ii).$$

$\therefore$ The two simultaneous equations are –

$$\frac{a_2}{3} + \frac{a_3 l}{2} = -\frac{25}{3} \cdot \left(\times \frac{1}{2}\right) \quad -(i)$$

$$\frac{a_2}{2} + \frac{4}{5} a_3 l = -\frac{25}{2} \cdot \left(\times \frac{1}{3}\right). \quad -(ii)$$

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$)\times\frac{1}{2} - (ii)\times\frac{1}{3}$ gives –

$$\frac{a_2 L}{6} + a_3\frac{L}{4} = -\frac{25}{6}$$

$$+\frac{a_2 L}{6} + \frac{4a_3 L}{15} = -\frac{25}{6}$$

$(-)$ $(-)$ $(+)$

$$\Rightarrow a_3 = 0.$$

$$\therefore \frac{a_2 L}{6} + 0 = -\frac{25}{6}$$

$$\Rightarrow a_2 = -25.$$

$$\therefore a_1 = \frac{20}{L} - a_2 L$$

$$= \frac{20}{L} + 25L.$$

$$\therefore \phi(x) = \left(\frac{20}{L} + 25L\right)x - 25x^2 + 0x^3$$

which is exactly same as the analytical and quadratic approximation and $a_3 = 0$ as is due to the fact that quadratic approx was able to capture the exact variation of ϕ with x hence all higher approximations will yield higher order coefficients $= 0$.

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Q(4). $\Pi = \int_0^L \left(\frac{M^2}{2EI} + \left(\frac{dM}{dx}\right)\left(\frac{dw}{dx}\right) + qw \right) dx$.

here the Lagrangian is

$$\mathcal{L} = \frac{M^2}{2EI} + \left(\frac{dM}{dx}\right)\left(\frac{dw}{dx}\right) + qw$$

$$= \frac{M^2}{2EI} + M'w' + qw$$

independent variable 'x'

dependent " → M, w

$\therefore$ 1st equation – $\frac{\partial \mathcal{L}}{\partial M} - \frac{d}{dx}\left(\frac{\partial \mathcal{L}}{\partial M'}\right) = 0$.

$$\frac{\partial \mathcal{L}}{\partial M} = \frac{2M}{2EI} = \frac{M}{EI} + 0 + 0$$

$$\frac{\partial \mathcal{L}}{\partial M'} = 0 + w' + 0.$$

$\therefore$ the equation becomes,

$$\frac{M}{EI} - \frac{d}{dx}(w') = 0.$$

$$\Rightarrow w'' = \frac{M}{EI}$$

Bernoulli's theorem

$$\frac{1}{R} = \frac{M}{EI} \Rightarrow \left(\frac{M}{I} = \frac{E}{R}\right) \Rightarrow \boxed{\frac{d^2w}{dx^2} = \frac{M}{EI}}$$

which relation between bending moment and the curvature of the beam

Similarly 2nd equation – $\frac{\partial \mathcal{L}}{\partial w} - \frac{d}{dx}\left(\frac{\partial \mathcal{L}}{\partial w'}\right) = 0$.

$$\frac{\partial \mathcal{L}}{\partial w} = 0 + 0 + q$$

$$\frac{\partial \mathcal{L}}{\partial w'} = 0 + M' + 0$$

$\therefore$ the equation becomes –

$$q - \frac{d(M)}{dx} = 0.$$

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which is also a standard result

as we know, $\frac{dM}{dx} = V$ (shear force @ a section).

thus $\frac{dV}{dx} = q$ is also a valid expression

Since the slope of the shear force diagram at a section gives the loading ~~int~~ intensity at that section.

There both the euler-Lagrange equations bear the form & exactly as obtained from classical Euler-Bernoulli beam theory.

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(5):- $$-\frac{\partial}{\partial x}\left(a_{11}\frac{\partial u}{\partial x}+a_{12}\frac{\partial u}{\partial y}\right)-\frac{\partial}{\partial y}\left(a_{21}\frac{\partial u}{\partial x}+a_{22}\frac{\partial u}{\partial y}\right)+f=0 \text{ on } \Omega.$$

$u=\bar{u}$ on Γ_1

and $$\left[a_{11}\frac{\partial u}{\partial x}+a_{12}\frac{\partial u}{\partial y}\right]n_x+\left[a_{21}\frac{\partial u}{\partial x}+a_{22}\frac{\partial u}{\partial y}\right]n_y=\bar{t} \text{ on } \Gamma_2$$

$$\Rightarrow \frac{\partial}{\partial x}\left(a_{11}\frac{\partial u}{\partial x}+a_{12}\frac{\partial u}{\partial y}\right)+\frac{\partial}{\partial y}\left(a_{21}\frac{\partial u}{\partial x}+a_{22}\frac{\partial u}{\partial y}\right)-f=0.$$

$$\Rightarrow \int_\Omega w\left[\frac{\partial}{\partial x}\left(a_{11}\frac{\partial u}{\partial x}+a_{12}\frac{\partial u}{\partial y}\right)+\frac{\partial}{\partial y}\left(a_{21}\frac{\partial u}{\partial x}+a_{22}\frac{\partial u}{\partial y}\right)-f\right]d\Omega=0.$$

$\Rightarrow$ Lets write the expression $\frac{\partial}{\partial x}\left(a_{11}\frac{\partial u}{\partial x}+a_{12}\frac{\partial u}{\partial y}\right)+\frac{\partial}{\partial y}\left(a_{21}\frac{\partial u}{\partial x}+a_{22}\frac{\partial u}{\partial y}\right)$ in matrix form -

$$\underbrace{\begin{pmatrix}\frac{\partial}{\partial x} & \frac{\partial}{\partial y}\end{pmatrix}}_{\nabla^T}\underbrace{\begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}}_{A}\underbrace{\begin{Bmatrix}\partial/\partial x\\ \partial/\partial y\end{Bmatrix}}_{\nabla}(u)$$

$$\therefore (\nabla^T A\nabla)u \quad \text{or} \quad \underline{\underline{\nabla\cdot(A\nabla u)}}$$

$$\Rightarrow \int_\Omega\left[w\,\nabla\cdot(A\nabla u)-wf\right]d\Omega=0$$

$$\Rightarrow \int_\Omega \nabla\cdot(w\,A\nabla u)\,d\Omega-\int_\Omega wf\,d\Omega=0$$

Recalling the ~~Gauss divergence~~ Green's integral theorem -

$$\int_\Omega \nabla\cdot F\,d\Omega=\int_\Gamma n\cdot F\,d\Gamma$$

$$\Rightarrow \int_\Gamma n\cdot w\,A\nabla u\,d\Gamma-\int_\Omega \nabla w\cdot A\nabla u\,d\Omega-\int_\Omega wf\,d\Omega=0$$

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We have the Natural boundary conditions as

$n.(A\nabla u) = \bar{t}$ on Γ_2 and $u = \bar{u}$ at Γ_1.

$$\therefore \int_{\Gamma_1} w\, n.\nabla A u\, d\Gamma_1 + \int_{\Gamma_2} w\, n.A(A\nabla u)\, d\Gamma_2 - \int_{\Omega} \nabla w . A\nabla u\, d\Omega - \int_{\Omega} w f\, d\Omega = 0$$

$$\Rightarrow 0 + \cancel{\left[w\bar{t} \right]_{\Gamma_2}} - \cancel{\int_{\Omega}} \int_{\Gamma_2} w\bar{t}\, d\Gamma_2 - \int_{\Omega} \nabla w . A\nabla u\, d\Omega - \int_{\Omega} w f\, d\Omega = 0$$

as $w = 0$ on Γ_1 with is ~~ya~~ essential B.C

$$\therefore \int_{\Omega} \nabla w . (A\nabla u)\, d\Omega + \int_{\Omega} w f\, d\Omega - \int_{\Gamma_2} w\bar{t}\, d\Gamma_2 = 0$$

Now, as $u^e(x,y) = \sum_{i=1}^{n} N_i^e(x,y) d_i^e$ which is the weak form.

the w's are ~~also~~ also comprised of the same basis functions -

$w = \sum_{i=1}^{n} N_i^e(x,y) \underline{w}_i^e \rightarrow$ arbitrary.

for one w_i

$$\therefore \int_{\Omega} \underbrace{\left(\nabla N_i^{eT} A \nabla \sum_{j=1}^{n} N_j^e\, d\Omega \right)}_{\underset{\sim}{K}^e} \underbrace{d_j^e}_{\underset{=}{d}^e} + \int_{\Omega} N_i^{eT} f\, d\Omega - \underbrace{\int_{\Gamma_2} N_i^{eT} \bar{t}\, d\Gamma_2}_{f^e} = 0$$

$$\Rightarrow \boxed{\underset{\sim}{K}^e \underset{=}{d}^e = \underset{=}{f}^e}$$

Now $K_G =$

$$\begin{bmatrix} \nabla N_1^{eT} A \nabla N_1^e & \nabla N_1^{eT} A \nabla N_2^e & \cdots & \nabla N_1^{eT} A \nabla N_n^e \\ \nabla N_2^{eT} A \nabla N_1^e & \nabla N_2^{eT} A \nabla N_2^e & \cdots & \\ \vdots & \vdots & & \\ \nabla N_n^{eT} A \nabla N_1^e & \nabla N_n^{eT} A \nabla N_2^e & \cdots & \end{bmatrix}$$

$\Rightarrow$ Comparing bothsides

$A^T = A$

$\Rightarrow \boxed{a_{ij} = a_{ji}}$

A matrix must be symmetric as well (Ans).

for a symmetric K_G matrix, for any i, j, $\underline{\underline{k_{ij} = k_{ji}}}$

$$\therefore \left(\nabla N_i^{eT} A \nabla N_j^e \right)^T = \nabla N_j^{eT} A \nabla N_i^e$$

$$\Rightarrow \nabla N_j^{eT} A^T \nabla N_i^e = \nabla N_j^{eT} A \nabla N_i^e \quad (*)$$

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Q(6) (a) $\frac{d^2u}{dx^2} + \cos(\pi x) = 0$.

$$\Rightarrow r_\Omega(\underline{a}, x) = \frac{d^2u}{dx^2} + \cos(\pi x).$$

$$\int_\Omega w_i \, r_\Omega(\underline{a}, x) \, d\Omega = 0.$$

$$\Rightarrow \int_0^1 w\left[u'' + \cos(\pi x)\right] dx = 0.$$

$$\Rightarrow \int_0^1 \left(wu'' + w\cos \pi x\right) dx = 0.$$

$$\Rightarrow \int_0^1 wu'' \, dx + \int_0^1 w(\cos \pi x) \, dx = 0.$$

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$$\Rightarrow \cancel{w u' \Big|_0^L - \int_0^L \frac{dw}{dx} u' \, dx + \left[w \frac{\sin(\pi x)}{\pi} \right]_0^L - \int_0^L \frac{dw}{dx} \left(\frac{\sin \pi x}{\pi} \right) dx} = 0 .$$

$$\Rightarrow \int_0^L \left(\frac{dw}{dx} \right) u' \, dx + \int_0^L \frac{dw}{dx} \left(\frac{\sin \pi x}{\pi} \right) dx - \cancel{\left[w u' + \frac{w \sin \pi x}{\pi} \right]_{\Gamma_t}^{L}} = 0 .$$

$$\Rightarrow \left[w u' \right]_{\Gamma_u \cup \Gamma_t} - \int_0^L \left(\frac{dw}{dx} \right) \left(\frac{du}{dx} \right) dx + \int_0^L w \overset{\cos}{\cancel{\sin}}(\pi x) \, dx = 0 .$$

as $w = 0$ at Γ_u,

$\therefore$ the weak form becomes -

$$W = \int_0^1 \left(\frac{dw}{dx} \right) \left(\frac{du}{dx} \right) dx - \int_0^1 w \overset{\cos}{\cancel{\sin}}(\pi x) \, dx - (w u')\Big|_{\Gamma_t} = 0 .$$

(a) Now, since no traction BC is given, we ~~choose~~ ~~w such that it is = 0 at Γ_t~~ can omit the $(w u')_{\Gamma_t}$ term from the weak form.

Now $u^e = N^e d^e$
$w^e = N^e w^e$
$= w^{eT} N^{eT}$

$\left.\right\}$ 'w' is composed of same basis shape functions as 'u' as assumed by Galerkin.

$u^{e\prime} = B^e d^e$

$w^{e\prime} = \cancel{N} B^e w^e$.

$= w^{eT} B^{eT}$ (as it is a scalar).

$$\therefore W = \int_0^{l^e} w^{eT} B^{eT} B^e d^e \, dx^e - \int_0^{l^e} w^{eT} N^{eT} \overset{\cos}{\cancel{\sin}} \cancel{p} \cdot \pi x \, dx^e = 0 .$$

as w^{eT} is arbitrary; cancel from the equation

$\therefore$ fem equation thus becomes -

$$\Big[\underbrace{\int_0^{l^e} B^{eT} B^e \, dx^e}_{k^e d^e} \Big] d^e - \Big[\underbrace{\int_0^{l^e} N^{eT} \overset{\cos}{\cancel{\sin}}(\pi x^e) \, dx^e}_{F^e} \Big] = 0 .$$

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$$\Rightarrow \underline{k^e d^e = F^e}.$$

$$k^e = \int_0^{l^e} B^{eT} B^e \, dx^e \quad ; \quad F^e = \int_0^{l^e} N^{eT} \overset{\cos}{\cancel{\sin}}(\pi x) \, dx$$

Now for two noded element,

$$N^e = \begin{bmatrix} 1 - \frac{x^e}{l^e} & \frac{x^e}{l^e} \end{bmatrix}$$

$$N^{e'} = B^e = \begin{bmatrix} -\frac{1}{l^e} & \frac{1}{l^e} \end{bmatrix} = \frac{1}{l^e}\begin{pmatrix} -1 & 1 \end{pmatrix}.$$

$$\therefore k^e = \int_0^{l^e} \frac{1}{l^e} \underset{2\times 1}{\begin{pmatrix} -1 \\ 1 \end{pmatrix}} \frac{1}{l^e} \underset{1\times 2}{\begin{pmatrix} -1 & 1 \end{pmatrix}} dx^e.$$

$$= \int_0^{l^e} \frac{1}{(l^e)^2} \times \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} dx^e.$$

Since we have to consider 3, 2-nodel elements, here $l^e = \frac{1}{3}$.

$$k^e = \frac{l^e}{l^{e^2}} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = \frac{1}{l^e}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

$$= 3\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

$$F^e = \int_{x_1^e}^{x_2^e} \begin{bmatrix} 1 - \frac{x^e}{l^e} \\ \frac{x^e}{l^e} \end{bmatrix} \overset{\cos}{\cancel{\sin}}(\pi x^e) \, dx^e$$

$$F^e = \int_{x_1^e}^{x_2^e} \begin{bmatrix} \overset{\cos}{\cancel{\sin}} \pi x^e \, dx - \frac{x^e \overset{\cos}{\cancel{\sin}}(\pi x^e)}{l^e} dx^e \\ \frac{x^e \overset{\cos}{\cancel{\sin}}(\pi x^e)}{l^e} dx^e \end{bmatrix}.$$

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$$F^1 = 3\left[\frac{1}{3}\frac{\sin \pi x}{\pi}\Big|_0^{\frac{1}{3}} - \left[\frac{\cos \pi x + \pi x \sin \pi x}{\pi^2}\right]_0^{\frac{1}{3}}\right\}.$$

$$\frac{\cos \pi x + \pi x \sin \pi x}{\pi^2}\Big|_0^{\frac{1}{3}}$$

$$\int_0^{\frac{1}{3}} x \cos \pi x \, dx = x\frac{\sin \pi x}{\pi}\Big|_0^{\frac{1}{3}} - \int_0^{\frac{1}{3}} \frac{\sin \pi x}{\pi} dx.$$

$$= \frac{x \sin \pi x}{\pi}\Big|_0^{\frac{1}{3}} + \frac{\cos \pi x}{\pi^2}\Big|_0^{\frac{1}{3}}.$$

$$= \left(\frac{\pi x \sin \pi x + \cos \pi x}{\pi^2}\right)\Big|_0^{\frac{1}{3}}.$$

Substituting all values -

$$\Rightarrow F^1 = 3\begin{Bmatrix} 0.0507 \\ 0.0412 \end{Bmatrix}.$$

Now for F^2; $x_2^e = \frac{2}{3}$; $x_1^e = \frac{1}{3}$; $l^e = \frac{1}{3}$.

$$F^2 = 3\int_{\frac{1}{3}}^{\frac{2}{3}} \begin{bmatrix} \left(\frac{2}{3} - x\right) \\ \left(x - \frac{1}{3}\right) \end{bmatrix} \cos(\pi x)\, dx.$$

$$= 3\left[\begin{matrix} \frac{2}{3}\left(\frac{\sin \pi x}{\pi}\right)\Big|_{\frac{1}{3}}^{\frac{2}{3}} - \left[\frac{\cos \pi x + \pi x \sin \pi x}{\pi^2}\right]_{\frac{1}{3}}^{\frac{2}{3}} \\ \left(\frac{\cos \pi x + \pi x \sin \pi x}{\pi^2}\right)_{\frac{1}{3}}^{\frac{2}{3}} - \frac{1}{3}\left(\frac{\sin \pi x}{\pi}\right)_{\frac{1}{3}}^{\frac{2}{3}} \end{matrix}\right\}$$

$$= 3\begin{bmatrix} 0.0094 \\ -0.0094 \end{bmatrix}.$$

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and $F^3 = 3 \int_{x_1^3}^{x_2^3} \begin{bmatrix} (1-x) \\ \left(x - \frac{2}{3}\right) \end{bmatrix} \cos(\pi x)\, dx$

As for $e=3$, $x_2^3 = 1$; $x_1^3 = \frac{2}{3}$

$$F^3 = 3 \begin{bmatrix} \frac{\sin \pi x}{\pi} \Big|_{2/3}^{1} - \left(\frac{\cos \pi x + \pi x \sin \pi x}{\pi^2}\right)_{2/3}^{1} \\ \left(\frac{\cos \pi x + \pi x \sin \pi x}{\pi^2}\right)_{2/3}^{1} - \frac{2}{3}\left(\frac{\sin \pi x}{\pi}\right)_{2/3}^{1} \end{bmatrix}$$

$$F^3 = 3 \begin{bmatrix} -0.0412 \\ -0.0507 \end{bmatrix}$$

$$\therefore \quad K_G = 3 \begin{array}{c} \begin{array}{cccc} u_1 & u_2 & u_3 & u_4 \end{array} \\ \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \end{array} \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \end{array}$$

$$F_G = 3 \begin{bmatrix} 0.0507 \\ 0.0412 + 0.0094 \\ -0.0094 - 0.0412 \\ -0.0507 \end{bmatrix}$$

$\therefore \quad \underset{\sim}{K_G}\, \underset{\sim}{d} = \underset{\sim}{F_G}$ form —

$$\cancel{3} \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{Bmatrix} = \cancel{3} \begin{Bmatrix} \cancel{0.0507} \\ 0.0507 \\ -0.0507 \\ \cancel{-0.0507} \end{Bmatrix}$$

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As $d_1 = d_4 = 0$,

The reduced system –

$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \begin{Bmatrix} d_2 \\ d_3 \end{Bmatrix} = \begin{Bmatrix} 0.0507 \\ -0.0507 \end{Bmatrix}.$$

$$\Rightarrow \begin{Bmatrix} d_2 \\ d_3 \end{Bmatrix} = \frac{1}{3} \begin{bmatrix} 2 & +1 \\ +1 & 2 \end{bmatrix} \begin{Bmatrix} 0.0507 \\ -0.0507 \end{Bmatrix}.$$

$$\Rightarrow \begin{pmatrix} d_2 \\ d_3 \end{pmatrix} = \begin{pmatrix} 0.0169 \\ -0.0169 \end{pmatrix}.$$

Exact solution; $u(x) = \dfrac{\cos \pi x + 2x - 1}{\pi^2}$

$$d_2 = u\left(\tfrac{1}{3}\right) = \frac{\cos \pi/3 + 2/3 - 1}{\pi^2} = 0.01688 \approx 0.0169.$$

$$d_3 = u\left(\tfrac{2}{3}\right) = \frac{\cos \frac{2\pi}{3} + \frac{4}{3} - 1}{\pi^2} = -0.01688 \approx -0.0169.$$

Hence we see that the FEM solution is almost same as the exact analytical solution.

(b) $u(0) = 0$ and $u'\big|_{x=1} = 0$.

here in the weak form a $[w\,u']$ term will exist for the last element though it will have no effect on the force vector.

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the reduced global stiffness matrix will be 3x3

now -

$$\underset{\sim}{k_G}\,\underset{\sim}{u} = \underset{\sim}{F}$$

$$\Rightarrow \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} 0.0507 \\ -0.0507 \\ -0.0507 \end{Bmatrix}$$

Doing the inverse of $\underset{\sim}{k_G}$ matrix using '\' command in matlab with the global $\underset{\sim}{F}$ vector, we get -

$$\begin{Bmatrix} u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{pmatrix} -0.0507 \\ -0.01521 \\ -0.2028 \end{pmatrix}$$

Analytical -

$$u_2 = u\left(\tfrac{1}{3}\right) = \frac{\cos \pi/3 - 1}{\pi^2} = -0.05066$$
$$\simeq -0.0507$$

$$u_3 = u\left(2/3\right) = \frac{\cos \frac{2\pi}{3} - 1}{\pi^2} = -0.1520$$
$$\simeq -0.1521$$

$$u_4 = u(1) = \frac{\cos \pi - 1}{\pi^2} = -0.2026$$
$$\simeq -0.2028$$

Again all solutions in both cases are almost identical.

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(c) Since No disp. boundary condition is given our stiffness matrix will be singular, therefore considering an arbitrary constant $u(0) = \bar{u}$ as u_1 to remove the rigid body body mode —

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} \bar{u}_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} 0.0507 \\ 0.0507 \\ -0.0507 \\ -0.0507 \end{Bmatrix}.$$

[The end traction conditions donot contribute to F_g as they are all = 0]

$$\Rightarrow \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} \bar{u}_1 + \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} 0.0507 \\ -0.0507 \\ -0.0507 \end{Bmatrix}.$$

$$\Rightarrow \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} \begin{Bmatrix} u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{bmatrix} 0.0507 + \bar{u}_1 \\ -0.0507 \\ -0.0507 \end{bmatrix}.$$

Inverting and obtaining the solution vector —

$$\begin{Bmatrix} u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{bmatrix} -0.0587 + \bar{u}_1 \\ -0.1521 + \bar{u}_1 \\ -0.2028 + \bar{u}_1 \end{bmatrix}.$$

the values are again same as the previous case with only difference being the addition of $\bar{u}_1 = u(0)$ term which a non-unique arbitrary constant to remove the rigid body mode.

(d) $u(0) = u_1 = 0 \rightarrow$ displacement B.C.

$$u'\Big|_{x=1} = 2u(1) + \frac{1}{\pi^2} = 2u_4 + \frac{1}{\pi^2}$$

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Here the element F^e vector for last element will have a non-zero traction term.

$x_2^3 = 1$ $\quad \therefore F^{e^3} = \int_{x_1^3}^{x_2^3} \frac{1}{L^e}\begin{bmatrix}(1-x)\\(x-2/3)\end{bmatrix}\cos \pi x\, dx + \frac{L}{L^e}\begin{bmatrix}(1-x)\\(x-\frac{2}{3})\end{bmatrix} u'$

$x_1^3 = 2/3$ $\qquad\qquad\qquad

7(a) Given ODE –

$$\frac{d^2u}{dx^2} + 5u = \sin \pi x \quad . \; 0 < x < 1$$

$$u(0) = 0 \quad ; \quad u'\Big|_{x=1} = -1 \; .$$

Analytical solution –

Let $\frac{du}{dx} = Du$. $\therefore$ $u'' = D^2u$.

(i) homogeneous solution –

$$u'' + 5u = 0 \; .$$

$$\Rightarrow \quad D^2u + 5u = 0 \; .$$

$$\Rightarrow \quad (D^2+5)u = 0 \quad , \Rightarrow f(D).u = 0 \; .$$

$$\Rightarrow \quad D^2 = -5 \; . \Rightarrow D = \pm\sqrt{5}i$$

$$\therefore \quad u = a_1 \cos\sqrt{5}x + a_2 \sin\sqrt{5}x \; .$$

(ii) Particular solution –

$$P.I = \frac{1}{f(D)} \cdot \sin \pi x \; .$$

$$= \frac{1}{(D^2+5)} \sin \pi x \; .$$

$$= \frac{1}{-(\pi)^2+5} \sin \pi x \; . = \frac{1}{5-\pi^2} \sin \pi x \; .$$

When RHS is of form $\sin(ax)$ or $\cos(ax)$ we replace the D^2 terms only of $f(D)$ function but by $-(a^2)$.

$$\therefore \quad u = a_1 \cos\sqrt{5}x + a_2 \sin\sqrt{5}x + \frac{\sin(\pi x)}{5-\pi^2} \; .$$

Now satisfying the boundary conditions –

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$u(0) = 0$ –

$$0 = a_1(1) + 0 + 0$$

$$\Rightarrow a_1 = 0.$$

$$\therefore u = a_2 \sin\sqrt{5}x + \frac{\sin \pi x}{5 - \pi^2}$$

Now, $u'|_{x=1} = -1$

$$u' = a_2\sqrt{5}\cos\sqrt{5}x + \frac{\pi}{(5-\pi^2)}\cos \pi x$$

$$u'|_{x=1} = a_2\sqrt{5}\cos\sqrt{5} + \frac{\pi}{(5-\pi^2)}\cos(\pi).$$

$$\Rightarrow a_2\sqrt{5}\cos\sqrt{5} - \frac{\pi}{(5-\pi^2)} = -1.$$

$$\Rightarrow -a_2(1.3803) = -0.6451 - 1.$$

$$\Rightarrow a_2 = 1.192.$$

$$\therefore u(x) = \underline{1.192\sin(\sqrt{5}x) - 0.2054\sin(\pi x)}$$

(b) Quadratic approximation of u.

$$u(x) = a_0 + a_1 x + a_2 x^2$$

satisfying the essential BC –

$$u(0) = 0;$$

$$\therefore a_0 = 0.$$

$$\therefore u(x) = a_1 x + a_2 x^2.$$

for finding the two unknown coefficients we choose two points in total, one in the domain i.e within the body

(i) i.e at $x = 0.5$ and other at the natural BC i $x = 1$.

Using point collocation, we know -

$$\int_{\Omega} w_i \, r_{\Omega}(\underline{a}, x) \, d\Omega = 0 .$$

where $w_i = \delta(x - x_i)$

thus, $\int_{\Omega} \delta(x - x_i) \, r_{\Omega}(\underline{a}, x) = \underline{\underline{r_{\Omega}(\underline{a}, x_i)}} = 0$

from the body

and $\cancel{\Omega} \int_{\Gamma_t} \delta(x - x_i) \, r_{\Gamma_t}(\underline{a}, x) \, d\Gamma_t = 0 .$

$$\Rightarrow r_{\Gamma_t}(\underline{a}, x_i) = 0 .$$

[As $\int_{-\infty}^{\infty} f(x) \, \delta(x - x_a) \, dx = f(x_a)$ by the property of the Dirac-delta func.].

$\therefore r_{\Omega}(\underline{a}, x) = u'' + 5u - \sin \pi x = 0$

$r_{\Gamma_t}(\underline{a}, x) = u' + 1 = 0 .$

Now, $r_{\Omega}(\underline{a}, x)\Big|_{x=0.5} = \left(u'' + 5u - \sin \pi x\right)\Big|_{x=0.5} = 0$

and $r_{\Gamma_t}(\underline{a}, x)\Big|_{x=1} = (u' + 1)\Big|_{x=1} = 0 .$

$u = a_1 x + a_2 x^2 \; ; \; u' = a_1 + 2a_2 x .$

$u'' = 2a_2 .$

$\therefore r_{\Omega}(\underline{a}, x) = 2a_2 + 5(a_1 x + a_2 x^2) - \sin \pi x$

$\therefore r_{\Omega}(\underline{a}, x)\Big|_{x=0.5} = 2a_2 + 5\left(\frac{a_1}{2} + \frac{a_2}{4}\right) - 1 = 0$

$\Rightarrow 2a_2 + \frac{5}{2} a_1 + \frac{5}{4} a_2 = 1 .$

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$$\Rightarrow \frac{5}{2}a_1 + \frac{13a_2}{4} = 1 \quad \text{--(i)}$$

Similarly –

$$r_{rt}(\underline{a}, x) = a_1 + 2a_2 x + 1$$

$$r_{rt}(\underline{a}, x)\Big|_{x=1} = a_1 + 2a_2 + 1 = 0$$

$$\Rightarrow a_1 + 2a_2 = -1 \quad \text{(ii)}$$

solving (i) and (ii) simultaneously,

$$\frac{5}{2}a_1 + \frac{13}{4}a_2 = 1$$

$$\frac{5}{2}a_1 + 5a_2 = -\frac{5}{2}$$

$$(-) \qquad (-) \qquad (+)$$

$$\left(\frac{13}{4} - 5\right)a_2 = \left(1 + \frac{5}{2}\right)$$

$$\Rightarrow -\frac{\cancel{7}}{\cancel{4}_2}a_2 = \frac{\cancel{7}}{\cancel{2}}$$

$$\Rightarrow a_2 = -2$$

$$\therefore a_1 = -1 - 2a_2$$

$$= -1 + 4 = 3$$

$\therefore u(x) = 3x - 2x^2$ is the approximation.

(ii) choosing $x = 0.25$ instead of 0.5,

$$r_2(\underline{a}, x)\Big|_{x=0.25} = 2a_2 + 5\left(\frac{a_1}{4} + \frac{a_2}{16}\right) = 1/\sqrt{2}$$

$$\Rightarrow 2a_2 + \frac{5a_2}{16} + \frac{5a_1}{4} = 1/\sqrt{2}$$

$$\Rightarrow \frac{5a_1}{4} + \frac{37a_2}{16} = 1/\sqrt{2}$$

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The second equation remains the same from the previous part at $x=1$.

i.e $a_1 + 2a_2 = -1$.

Solving both the equations simultaneously, we get,

$$a_1 = 19.8756$$

$$a_2 = -10.438.$$

$$\therefore u = 19.8756x - 10.438x^2.$$

(iii) <u>For subdomain collocation</u> - $0<x<1$

For subdomain collocation, the w_i's become the unit step function.

i.e $w_i = \begin{cases} 1 & \text{for } x \in \Omega_i \text{ (subdomain)} \\ 0 & \text{otherwise.} \end{cases}$

for $0<x<1$; the subdomain is the entire domain itself.

$$\therefore \left.\begin{aligned} &\int_0^1 r_\Omega(\underline{a}, x)\,dx = 0 \\ \text{and } &r_\Gamma(\underline{a}, x) = 0 \end{aligned}\right\} \text{ 2 equations in 2 unknowns.}$$

$$\therefore \int_0^1 \left[2a_2 + 5(a_1 x + a_2 x^2) - \sin \pi x\right] dx = 0.$$

$$\Rightarrow \left\{2a_2 + 5\left(a_1\left(\frac{1}{2}\right) + a_2\frac{(1)^3}{3}\right)\right\} + \frac{\cos \pi x}{\pi}\Big|_0^1 = 0.$$

$$\Rightarrow 2a_2 + 5\left(\frac{a_1}{2} + \frac{a_2}{3}\right) + \frac{1}{\pi}(\cos \pi - 1) = 0.$$

$$\Rightarrow 2a_2 + \frac{5a_1}{2} + \frac{5a_2}{3} - \frac{2}{\pi} = 0.$$

$$\Rightarrow \frac{5a_1}{2} + \frac{11a_2}{3} = \frac{2}{\pi} \quad \text{--(i)}.$$

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and $r_{\Gamma_t}(\underline{a}, x) = 0$

$$\Rightarrow \left(a_1 + 2a_2 x\right)\Big|_{x=1} = -1$$

$$\Rightarrow a_1 + 2a_2 = -1 \quad \text{--(ii)}.$$

Solving the two equations simultaneously -

$$a_1 = 3.705 \; ; \; a_2 = -2.353.$$

$$\therefore \; u(x) = \underline{3.705x - 2.353x^2}.$$

(iv) <u>Continuous Least squares</u> :- $(\alpha = 1)$.

$$I = \int_{\Omega} \left[r_{\Omega}(\underline{a}, x)\right]^2 d\Omega + \int_{\Gamma_t} \left[r_{\Gamma_t}(\underline{a}, x)\right]^2 d\Gamma_t.$$

then do, $\dfrac{\partial I}{\partial a_i} = 0$

$$r_{\Omega}(\underline{a}, x) = 2a_2 + 5\left(a_1 x + a_2 x^2\right) - \sin \pi x.$$

$$r_{\Gamma_t}(\underline{a}, x) = a_1 + 2a_2 x + 1$$

$$I = \int_{\Omega} \left\{2a_2 + 5\left(a_1 x + a_2 x^2\right) - \sin \pi x\right\}^2 d\Omega + \left(a_1 + 2a_2 x + 1\right)^2 \text{ at } x = 1.$$

$$\frac{\partial I}{\partial a_1} = 0 = \int_{\Omega} \left\{2a_2 + 5\left(a_1 x + a_2 x^2\right) - \sin \pi x\right\}(5x)\, dx + \left(a_1 + 2a_2 + 1\right)(1) = 0.$$

$$\Rightarrow \int_0^1 \cancel{2a_2 5} \; 10a_2 x + 25\left(a_1 x^2 + a_2 x^3\right) - 5x \sin \pi x \, dx + a_1 + 2a_2 + 1 = 0.$$

$$\Rightarrow \int_0^1 10a_2 x \, dx + \int_0^1 25 a_1 x^2 dx + \int_0^1 25 a_2 x^3 dx - 5\int_0^1 x \sin \pi x \, dx + \left(a_1 + 2a_2 + 1\right) = 0.$$

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doing $\int_0^1 x \sin \pi x \, dx$ separately —

$$= -x \frac{\cos \pi x}{\pi}\Big|_0^1 + \int_0^1 1 \cdot \frac{\cos \pi x}{\pi} dx$$

$$= -\left(\frac{x \cos \pi x}{\pi}\right)\Big|_0^1 + \cancel{\frac{\sin \pi x}{\pi^2}}^{\,0}\Big|_0^1$$

$$= -\left(\frac{1 \cdot \cos \pi - 0}{\pi}\right) = \frac{1}{\pi}.$$

$\therefore$ the integral becomes —

$$\Rightarrow \int_0^1 10 a_2 x \, dx + \int_0^1 25 a_1 x^2 dx + 25 \int_0^1 a_2 x^3 dx - \frac{5}{\pi} + a_1 + 2a_2 + 1 = 0$$

$$\Rightarrow 5a_2 + 25a_1 \frac{(1)^3}{3} + 25 a_2 \frac{(1)^4}{4} - \frac{5}{\pi} + a_1 + 2a_2 + 1 = 0.$$

$$\Rightarrow \left(a_1 + \frac{25}{3} a_1\right) + \left(7a_2 + \frac{25}{4} a_2\right) = \frac{5}{\pi} - 1$$

$$\Rightarrow \frac{28 a_1}{3} + \frac{53 a_2}{4} = \frac{5}{\pi} - 1. \quad \text{—(i)}.$$

Now $\frac{\partial I}{\partial a_2} = 0$.

$$\Rightarrow \int_0^1 \left\{2a_2 + 5(a_1 x + a_2 x^2) - \sin \pi x\right\}(2 + 5x^2) dx + (a_1 + 2a_2 + 1)(2) = 0.$$

$$\Rightarrow \int_0^1 2a_2 (2 + 5x^2) dx + \int_0^1 (10 + 25x^2)(a_1 x + a_2 x^2) dx - \int_0^1 (2 + 5x^2) \sin \pi x \, dx + 2a_1 + 4a_2 + 2 = 0.$$

$$\Rightarrow \int_0^1 2(2 + 5x^2) a_2 dx + \int_0^1 (10 + 25x^2) x a_1 dx + \int_0^1 (10 + 25x^2) a_2 x^2 dx - \int_0^1 (2 + 5x^2) \sin \pi x \, dx + 2a_1 + 4a_2 + 2 = 0.$$

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Doing this integrals online from Wolfram -

$$\Rightarrow \frac{22}{3}a_2 + \frac{45}{4}a_1 + \frac{25}{3}a_2 - (2.22) + 2a_1 + 4a_2 + 2 = 0.$$

$$\Rightarrow \left(\frac{45}{4}a_1 + 2a_1\right) + \left(4a_2 + \frac{47}{3}a_2\right) = 0.22.$$

$$\Rightarrow \frac{53}{4}a_1 + \frac{59}{3}a_2 = 0.22 \quad -(ii).$$

Solving equations (i) and (ii) :-

$$a_1 = 1.0908 \; ; \; a_2 = -0.7237$$

$$\therefore U(x) = 1.0908x - 0.7237x^2$$

(v) Point Least squares ($\alpha=1$) at $x = 0.5, 1$. i.e ($m=n$)

$$I = \sum_{i=1}^{p} [r_\Omega(\underline{a}, x_i)]^2 + \sum_{i=p+1}^{m} [r_\Gamma(\underline{a}, x_i)]^2$$

then take $\frac{\partial I}{\partial a_i}$

$$I = \left(2a_2 + 5(a_1 x + a_2 x^2) - \sin \pi x\right)^2\Big|_{x=0.5} + \left(a_1 + 2a_2 x + 1\right)^2\Big|_{x=1}$$

$$\Rightarrow I = \left(2a_2 + 5\left(\frac{a_1}{2} + \frac{a_2}{4}\right) - \sin\frac{\pi}{2}\right)^2 + \left(a_1 + 2a_2 + 1\right)^2$$

$$= \left(\frac{5}{2}a_1 + \frac{13a_2}{4} - 1\right)^2 + \left(a_1 + 2a_2 + 1\right)^2.$$

$$\frac{\partial I}{\partial a_1} = \left(\frac{5}{2}a_1 + \frac{13a_2}{4} - 1\right)\frac{5}{2} + \left(a_1 + 2a_2 + 1\right)\cdot 1 = 0.$$

$$\Rightarrow \frac{25}{4}a_1 + \frac{65}{8}a_2 - \frac{5}{2} + a_1 + 2a_2 + 1 = 0.$$

$$\Rightarrow \left(a_1 + \frac{25}{4}a_1\right) + \left(2a_2 + \frac{65}{8}a_2\right) = \frac{3}{2}.$$

$$\Rightarrow \frac{29}{4}a_1 + \frac{81}{8}a_2 = \frac{3}{2} \quad -(i)$$

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$$\frac{\partial I}{\partial a_2} = \left(\frac{5}{2}a_1 + \frac{13}{4}a_2 - 1\right)\left(\frac{13}{4}\right) + \left(a_1 + 2a_2 + 1\right)(2) = 0.$$

$$\Rightarrow \frac{65}{8}a_1 + \frac{169}{16}a_2 - \frac{13}{4} + 2a_1 + 4a_2 + 2 = 0.$$

$$\Rightarrow \left(2a_1 + \frac{65}{8}a_1\right) + \left(4a_2 + \frac{169}{16}a_2\right) = \frac{13}{4} - 2$$

$$\Rightarrow \frac{81}{8}a_1 + \frac{233}{16}a_2 = \frac{5}{4} \quad -(ii).$$

Solving simultaneously eqns (i) and (ii) -

$$a_1 = 3 \; ; \; a_2 = -2.$$

$$\therefore u(x) = 3x - 2x^2$$ (Exact as the analytical soln.)

(vi) <u>Point least squares</u> ($\alpha = 1$) $x = 0.25, 0.75, 1$ i.e $m > n$.

$$I = \left[2a_2 + 5(a_1 x + a_2 x^2) - \sin \pi x\right]^2_{@x=0.25} + \left[2a_2 + 5(a_1 x + a_2 x^2) - \sin \pi x\right]^2_{@x=0.75}$$

$$+ \left(a_1 + 2a_2 x + 1\right)^2_{@x=1}$$

$$\Rightarrow I = \left[2a_2 + 5\left(\frac{a_1}{4} + \frac{a_2}{16}\right) - \frac{1}{\sqrt{2}}\right]^2 + \left[2a_2 + 5\left(\frac{3a_1}{4} + \frac{9a_2}{16}\right) - \frac{1}{\sqrt{2}}\right]^2$$

$$+ \left(a_1 + 2a_2 + 1\right)^2.$$

$$\Rightarrow I = \left(\frac{5a_1}{4} + \frac{37a_2}{16} - \frac{1}{\sqrt{2}}\right)^2 + \left(\frac{15}{4}a_1 + \frac{77}{16}a_2 - \frac{1}{\sqrt{2}}\right)^2$$

$$+ \left(a_1 + 2a_2 + 1\right)^2$$

$$\frac{\partial I}{\partial a_1} = \left(\frac{5}{4}a_1 + \frac{37a_2}{16} - \frac{1}{\sqrt{2}}\right)\left(\frac{5}{4}\right) + \left(\frac{15}{4}a_1 + \frac{77}{16}a_2 - \frac{1}{\sqrt{2}}\right)\left(\frac{15}{4}\right)$$

$$+ \left(a_1 + 2a_2 + 1\right) = 0.$$

$$\Rightarrow \frac{25}{16}a_1 + \frac{185}{64}a_2 - \frac{5}{4\sqrt{2}} + \frac{225a_1}{16} + \frac{1155}{64}a_2 - \frac{15}{4\sqrt{2}}$$

$$+ a_1 + 2a_2 + 1 = 0.$$

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$$\Rightarrow \quad 16.625a_1 + 2\cancel{0}^{2}.9375a_2 = 2.5355. \quad -(i).$$

$$\frac{\partial I}{\partial a_2} = \left(\frac{5}{4}a_1 + \frac{37}{16}a_2 - \frac{1}{\sqrt{2}}\right)\left(\frac{37}{16}\right) + \left(\frac{15}{4}a_1 + \frac{77}{16}a_2 - \frac{1}{\sqrt{2}}\right)\left(\frac{77}{16}\right)$$

$$+ 2a_1 + 4a_2 + 2 = 0.$$

$$\Rightarrow \quad \frac{185}{64}a_1 + \cancel{3}\,\frac{1369}{256}a_2 - \frac{37}{16\sqrt{2}} + \frac{1155}{64}a_1$$

$$+ \frac{5929}{256}a_2 - \frac{77}{16\sqrt{2}} + 2a_1 + 4a_2 + 2 = 0.$$

$$\Rightarrow \quad 22.9375a_1 + 32.508a_2 = 3.038 \quad -(ii).$$

Solving (i) and (ii) simultaneously

$$a_1 = 0.8898 \; ; \; a_2 = -0.5344.$$

$$U(x) = \underline{0.8898x - 0.5344x^2}.$$

(vii) <u>Galerkin's approach</u> :-

Lets convert the differential equation into its weak form -

$$\int_{\Omega} w_i \, r_{\Omega}(q, x)\, d\Omega = 0.$$

$$\int_{\Omega} w\left(u'' + 5u - \sin \pi x\right) dx = 0.$$

$$\Rightarrow \int_0^1 w u'' dx + 5\int_0^1 w u\, dx - \int_0^1 w \sin \pi x\, dx = 0.$$

$$\Rightarrow \quad \cancel{wu'\Big|_0^1 - \int_0^1 \frac{dw}{dx} w u' dx + 5\int_0^1 w u\, dx + \frac{w \cos \pi x}{\pi}\Big|_0^1 = 0.}$$

Doing by parts on 'u''' term $\rightarrow$

$$I_1 = \int_0^1 \cancel{ww}\, w u'' dx = \left[wu'\right]_0^{1} - \int_0^1 \frac{dw}{dx} u' dx + \int_0^1 5wu\, dx$$

$$- \int_0^1 w \sin \pi x\, dx = 0.$$

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$v = a_1 x + a_2 x^2$. $\quad w_1 = \frac{\partial v}{\partial a_1} = x$. $\quad v' = a_1 + 2a_2 x$.

$\Rightarrow w' = 1$

$\therefore$ 1st equation –

$$\left[x v'\right]_{x=1} - \int_0^1 x \cdot (a_1 + 2a_2 x)\, dx + \int_0^1 5x (a_1 x + a_2 x^2)\, dx$$

$$- \int_0^1 x \sin \pi x \, dx = 0.$$

$$\Rightarrow 1 \cdot (-1) - \left[\frac{a_1 (1)^2}{1} + \frac{\cancel{2} a_2 (1)^2}{\cancel{2}}\right] + \left[\frac{5a_1 (1)^3}{3} + \frac{5a_2 (1)^4}{4}\right]$$

$$- \frac{1}{\pi} = 0.$$

$$\Rightarrow -\frac{a_1}{1} - \frac{1 a_2}{1} + \frac{5}{3} a_1 + \frac{5}{4} a_2 = 1 + \frac{1}{\pi}$$

$$\Rightarrow \left(\frac{5-3}{3}\right) a_1 + \left(\frac{-4+5}{4}\right) a_2 = \left(1 + \frac{1}{\pi}\right).$$

$$\Rightarrow \frac{2}{3} a_1 + \frac{1}{4} a_2 = \left(1 + \frac{1}{\pi}\right) \quad -(i).$$

$w_2 = \frac{\partial v}{\partial a_2} = x^2$. $\quad w' = 2x$

$\therefore$ 2nd equation –

$$\left[x^2 v'\right]_{x=1} - \int_0^1 2x (a_1 + 2a_2 x)\, dx + \int_0^1 5x^2 (a_1 x + a_2 x^2)\, dx$$

$$- \int_0^1 x^2 \sin \pi x \, dx = 0.$$

$$\Rightarrow 1 \cdot (-1) - \left[\frac{\cancel{2} a_1 (1)^2}{\cancel{2}} + \frac{4 a_2 (1)^3}{3}\right] + \left[\frac{5 a_1 (1)^4}{4} + \frac{\cancel{5} a_2 (1)^5}{\cancel{5}}\right]$$

$$- \left(\frac{1}{\pi} - \frac{4}{\pi^3}\right) = 0.$$

$$\Rightarrow -\frac{a_1}{1} - \frac{4a_2}{3} + \frac{5}{4} a_1 + a_2 = \left(1 + \frac{1}{\pi} - \frac{4}{\pi^3}\right) \quad -(ii)$$

$$\Rightarrow \frac{-4a_1 + 5a_1}{4} - \frac{a_2}{3} = \left(1 + \frac{1}{\pi} - \frac{4}{\pi^3}\right). \quad -(ii).$$

$$\Rightarrow \frac{1}{4} a_1 - \frac{a_2}{3} = \left(1 + \frac{1}{\pi} - \frac{4}{\pi^3}\right) \quad -(ii)$$

Solving simultaneously –

$$a_1 = \underline{2.588 x - 1.627 x^2}.$$

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7(c) Cubic approximation of $u = a_0 + a_1 x + a_2 x^2 + a_3 x^3$ satisfying the essential BC, $u(0) = 0$

$\therefore a_0 = 0$ as before

$\therefore u = a_1 x + a_2 x^2 + a_3 x^3$.

$u' = a_1 + 2a_2 x + 3a_3 x^2$; $u'' = 2a_2 + 6a_3 x$.

$\therefore r_\Omega(\underline{a}, x) = u'' + 5u - \sin \pi x$.

$= (2a_2 + 6a_3 x) + 5(a_1 x + a_2 x^2 + a_3 x^3) - \sin \pi x$

$r_\Gamma(\underline{a}, x) = u' + 1$ @ $x = 1$

$= a_1 + 2a_2 x + 3a_3 x^2 + 1$

(i) <u>Point collocation</u> @ $x = 0.25, 0.75$ and $x = 1$.

$r_\Omega(\underline{a}, x)\Big|_{x=0.25} = \left(2a_2 + \cancel{6}^{3} a_3 \times \frac{1}{\cancel{4}_2}\right) + 5\left(\frac{a_1}{4} + \frac{a_2}{16} + \frac{a_3}{64}\right) - \sin\left(\frac{\pi}{4}\right) = 0$.

$\Rightarrow \left(2a_2 + \frac{3a_3}{2}\right) + \frac{5a_1}{4} + \frac{5a_2}{16} + \frac{5a_3}{64} = \frac{1}{\sqrt{2}}$.

$\Rightarrow \frac{5}{4}a_1 + \frac{37a_2}{16} + \frac{101 a_3}{64} = \frac{1}{\sqrt{2}}$ —(i).

$r_\Omega(\underline{a}, x)\Big|_{x=0.75} = \left(2a_2 + \cancel{6}^{3} a_3 \times \frac{3}{\cancel{4}_2}\right) + 5\left(a_1 \times \frac{3}{4} + a_2\left(\frac{9}{16}\right) + a_3 \times \frac{27}{64}\right) - \sin\frac{3\pi}{4} = 0$.

$\Rightarrow \left(2a_2 + \frac{9a_3}{2}\right) + \frac{15a_1}{4} + \frac{45a_2}{16} + \frac{135a_3}{64} = \frac{1}{\sqrt{2}}$.

$\Rightarrow \frac{15a_1}{4} + \frac{77a_2}{16} + \frac{423}{64}a_3 = \frac{1}{\sqrt{2}}$ —(ii).

$r_\Gamma(\underline{a}, x)\Big|_{x=1} = a_1 + 2a_2 + 3a_3 = -1$ —(iii).

Solving both equations simultaneously —

$a_1 = 1.913$; $a_2 = -0.1204$; $a_3 = -0.8907$

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$\therefore u(x) = 1.913x - 0.1204x^2 - 0.8907x^3$

(ii) <u>Subdomain collocation</u> with $\Omega_1 = 0 < x < 0.5$

$\Omega_2 = 0.5 < x < 1$.

$\therefore$ <u>Equation 1</u> —

$$\int_0^{0.5} (2a_2 + 6a_3 x) + 5(a_1 x + a_2 x^2 + a_3 x^3) - \sin \pi x \, dx = 0.$$

$$\Rightarrow 2a_2 x \Big|_0^{0.5} + \cancel{6}^{3} a_3 \left(\frac{x^2}{\cancel{2}}\right)\Big|_0^{0.5} + 5a_1\left(\frac{x^2}{2}\right)\Big|_0^{0.5} + 5a_2\left(\frac{x^3}{3}\right)\Big|_0^{0.5} + 5a_3\left(\frac{x^4}{4}\right)\Big|_0^{0.5}$$

$$+ \frac{\cos \pi x}{\pi}\Big|_0^{0.5} = 0.$$

$$\Rightarrow a_2 + 3a_3 \times \frac{1}{4} + \frac{5}{2}a_1 \times \frac{1}{4} + \frac{5a_2}{3}\left(\frac{1}{8}\right) + \frac{5a_3}{4} \times \frac{1}{16}$$

$$+ \frac{1}{\pi}(0 - 1) = 0.$$

$$\Rightarrow a_2 + \frac{3}{4}a_3 + \frac{5a_1}{8} + \frac{5a_2}{24} + \frac{5a_3}{64} = \frac{1}{\pi}$$

$$\Rightarrow \frac{5}{8}a_1 + \frac{29a_2}{24} + \frac{53a_3}{64} = \frac{1}{\pi} \quad -(i).$$

<u>Equation 2</u> :-

$$\int_{0.5}^{1} (2a_2 + 6a_3 x) + 5(a_1 x + a_2 x^2 + a_3 x^3) - \sin \pi x \, dx = 0.$$

$$\Rightarrow 2a_2 x \Big|_{0.5}^{1} + 3a_3 x^2 \Big|_{0.5}^{1} + \frac{5a_1}{2} x^2 \Big|_{0.5}^{1} + \frac{5a_2}{3} x^3 \Big|_{0.5}^{1}$$

$$+ \frac{5}{4}a_3 x^4 \Big|_{0.5}^{1} + \frac{\cos \pi x}{\pi}\Big|_{0.5}^{1} = 0.$$

$$\Rightarrow a_2 + 3a_3\left(1 - \frac{1}{4}\right) + \frac{5a_1}{2}\left(1 - \frac{1}{4}\right) + \frac{5a_2}{3}\left(1 - \frac{1}{8}\right)$$

$$+ \frac{5}{4}a_3\left(1 - \frac{1}{16}\right) + \frac{1}{\pi}(\cos \pi - 0) = 0.$$

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$$\Rightarrow a_2 + 3a_3 \times \left(\frac{3}{4}\right) + \frac{5}{2}a_1\left(\frac{3}{4}\right) + \frac{5a_2}{3}\left(\frac{7}{8}\right) + \frac{5}{4}a_3\left(\frac{15}{16}\right) - \frac{1}{\pi} = 0.$$

$$\Rightarrow a_2 + \frac{9a_3}{4} + \frac{15a_1}{8} + \frac{35a_2}{24} + \frac{75a_3}{64} = \frac{1}{\pi}$$

$$\Rightarrow \frac{15a_1}{8} + \frac{59a_2}{24} + \frac{219a_3}{64} = \frac{1}{\pi} \quad -(ii).$$

Equation 3 —

$$r_{\Gamma t}(\underline{a}, x)\Big|_{x=1} = a_1 + 2a_2 + 3a_3 = -1 \quad -(iii).$$

Solving the 3 equations simultaneously —

$$a_1 = 2.028 \; ; \; a_2 = -0.1729 \; ; \; a_3 = -0.8942$$

$$\therefore u(x) = 2.028x - 0.1729x^2 - 0.8942x^3$$

(iii) Continuous Least squares — $(\alpha = 1)$.

$$I = \int_{\Omega} [r_{\Omega}(\underline{a}, x)]^2 d\Omega + \int_{\Gamma t} [r_{\Gamma t}(\underline{a}, x)]^2 d\Gamma t$$

$$I = \int_0^1 \left[(2a_2 + 6a_3 x) + 5(a_1 x + a_2 x^2 + a_3 x^3) - \sin \pi x\right]^2 dx + (a_1 + 2a_2 x + 3a_3 x^2)\Big|_{x=1}^2$$

$$\frac{\partial I}{\partial a_1} = \int_0^1 \left[(2a_2 + 6a_3 x) + 5(a_1 x + a_2 x^2 + a_3 x^3) - \sin \pi x\right] 5x\,dx + (a_1 + 2a_2 + 3a_3)(1) = 0.$$

$$\Rightarrow \int_0^1 10xa_2 + 30a_3 x^2 + 25a_1 x^2 + 25a_2 x^3 + 25a_3 x^4 - 5x \sin \pi x \, dx + (a_1 + 2a_2 + 3a_3) = 0.$$

$$\Rightarrow 5a_2 + \cancel{30}^{10} a_3 \left(\frac{1}{\cancel{3}}\right) + \frac{25}{3}a_1 (1)^3 + \frac{25}{4}a_2 (1)^4 + \frac{\cancel{25}^{5}}{\cancel{5}} a_3 (1)^5$$

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$$-\frac{5}{\pi} + a_1 + 2a_2 + 3a_3 + 1 = 0.$$

$$\Rightarrow \left(a_1 + \frac{25}{3}a_1\right) + \left(7a_2 + \frac{25}{4}a_2\right) + \left(5a_3 + 10a_3 + 3a_3\right) = \frac{5}{\pi} - 1$$

$$\Rightarrow \boxed{\frac{28}{3}a_1 + \frac{53a_2}{4} + \cancel{15}\,18\,a_3 = \frac{5}{\pi} - 1} \quad -(i).$$

$$\frac{\partial I}{\partial a_2} = \int_0^1 \left[(2a_2 + 6a_3x) + 5(a_1x + a_2x^2 + a_3x^3) - \sin \pi x\right] \times (2 + 5x^2)\,dx + (a_1 + 2a_2x + 3a_3x + 1) \times 2x = 0.$$

$$\Rightarrow \int_0^1 2a_2(2+5x^2)\,dx + \int_0^1 6a_3x(2+5x^2)\,dx + \int_0^1 5a_1x(2+5x^2)\,dx + \int_0^1 5a_2x^2(2+5x^2)\,dx + \int_0^1 5a_3x^3(2+5x^2)\,dx - \int_0^1 \sin \pi x\,(2+5x^2)\,dx + (2a_1 + 4a_2 + 6a_3 + 2) = 0.$$

Doing all the integrals online →

$$\Rightarrow \frac{22}{3}a_2 + \frac{27}{2}a_3 + \frac{45}{4}a_1 + \frac{25}{3}a_2 + \frac{20}{3}a_3 - 2.22 + 2a_1 + 4a_2 + 6a_3 + 2 = 0.$$

$$\Rightarrow \frac{53}{4}a_1 + \left(\frac{47}{3} + 4\right)a_2 + \left(\frac{27}{2} + \frac{20}{3} + 6\right)a_3 = 0.22$$

$$\Rightarrow \boxed{\frac{53}{4}a_1 + \frac{59}{\cancel{4}\,3}a_2 + \frac{157}{6}a_3 = 0.22} \quad -(ii).$$

Now, $$\frac{\partial I}{\partial a_3} = \int_0^1 \left[(2a_2 + 6a_3x) + 5(a_1x + a_2x^2 + a_3x^3) - \sin \pi x\right] \times (6x + 5x^3)\,dx + (a_1 + 2a_2 + 3a_3 + 1) \times 3 = 0$$

$$\Rightarrow \int_0^1 2a_2(6x+5x^3)\,dx + \int_0^1 6a_3x(6x+5x^3)\,dx + \int_0^1 5a_1x(6x+5x^3)\,dx + \int_0^1 5a_2x^2(6x+5x^3)\,dx + \int_0^1 5a_3x^3(6x+5x^3)\,dx - \int_0^1 \sin \pi x\,(6x+5x^3)\,dx + (3a_1 + 6a_2 + 9a_3 + 3) = 0$$

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Again doing the integrals online –

$$\frac{17}{2}a_2 + 18a_3 + 15a_1 + \frac{35}{3}a_2 + \frac{67}{7}a_3$$

$$- 2.538 + 3a_1 + 6a_2 + 9a_3 + 3 = 0.$$

$$\Rightarrow 18a_1 + \left(\frac{17}{2} + \frac{35}{3} + 6\right)a_2 + \left(\frac{67}{7} + 18 + 9\right)a_3 = 3 - 2.534$$

$$\Rightarrow \boxed{18a_1 + \frac{157}{6}a_2 + \frac{256}{7}a_3 = 0.466} \quad \text{--(iii)}.$$

So, it is a symmetric (3x3) system.

and solving simultaneously –

$$a_1 = 1.239 \; ; a_2 = -0.607 \; ; a_3 = -0.163$$

$$u(x) = 1.239x - 0.607x^2 - 0.163x^3$$

(iv) Point least squares ($\alpha = 1$) $x = 0.25, 0.5, 0.75, 1$

$$I = \sum_{i=1}^{p}\left[r_\Omega(\underline{a}, x)\right]^2 + \sum_{i=p+1}^{m}\left[r_\Gamma(\underline{a}, x)\right]^2 \Rightarrow \underline{m > n}$$

$$I = \left[2a_2 + 6a_3\left(\tfrac{1}{4}\right) + 5\left(a_1 \times \tfrac{1}{4}\right) + a_2\left(\tfrac{1}{16}\right) + a_3\left(\tfrac{1}{64}\right)\right\} - \sin\frac{\pi}{4}\right]^2 + \left[2a_2 + 6a_3\left(\tfrac{1}{2}\right) + 5\left\{a_1\left(\tfrac{1}{2}\right) + a_2\left(\tfrac{1}{4}\right) + a_3 \times \left(\tfrac{1}{8}\right)\right\} - \sin\frac{\pi}{2}\right]^2 + \left[2a_2 + 6a_3\left(\tfrac{3}{4}\right) + 5\left\{a_1\left(\tfrac{3}{4}\right) + a_2\left(\tfrac{9}{16}\right) + a_3 \times \left(\tfrac{27}{64}\right)\right\} - \sin\frac{3\pi}{4}\right]^2 + \left[a_1 + 2a_2(1) + 3a_3(1) + 1\right]^2$$

simplifying –

$$\cancel{\frac{\partial I}{\partial a_1} = \left[2a_2 + \frac{3}{2}a\right.}$$

$$I = \left[2a_2 + \frac{3}{2}a_3 + \frac{5}{4}a_1 + \frac{5a_2}{16} + \frac{5a_3}{64} - \frac{1}{\sqrt{2}}\right]^2 + \left[2a_2 + 3a_3 + \frac{5a_1}{2} + \frac{5a_2}{4} + \frac{5a_3}{8} - 1\right]^2 + \left[2a_2 + \frac{9a_3}{2} + \frac{15a_1}{4} + \frac{45a_2}{16} + \frac{135}{64}a_3 - \frac{1}{\sqrt{2}}\right]^2 + \left(a_1 + 2a_2 + 3a_3 + 1\right)^2.$$

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$$\Rightarrow I = \left[\frac{5a_1}{4} + \frac{37}{16}a_2 + \frac{101}{64}a_3 - \frac{1}{\sqrt{2}}\right]^2 + \left[\frac{5}{2}a_1 + \frac{13}{4}a_2 + \frac{29}{8}a_3 - 1\right]^2$$

$$+ \left[\frac{15a_1}{4} + \frac{77a_2}{16} + \frac{423}{64}a_3 - \frac{1}{\sqrt{2}}\right]^2 + \left[a_1 + 2a_2 + 3a_3 + 1\right]^2$$

$$\frac{\partial I}{\partial a_1} = \left(\frac{5}{4}a_1 + \frac{37a_2}{16} + \frac{101a_3}{64} - \frac{1}{\sqrt{2}}\right)\left(\frac{5}{4}\right) + \left(\frac{5}{2}a_1 + \frac{13a_2}{4}\right.$$

$$\left. + \frac{29}{8}a_3 - 1\right)\left(\frac{5}{2}\right) + \left(\frac{15}{4}a_1 + \frac{77}{16}a_2 + \frac{423}{64}a_3 - \frac{1}{\sqrt{2}}\right)\left(\frac{15}{4}\right)$$

$$+ \left(a_1 + 2a_2 + 3a_3 + 1\right) = 0.$$

$$\Rightarrow \left(\frac{25}{16}a_1 + \frac{185}{64}a_2 + \frac{505}{256}a_3 - \frac{5}{4\sqrt{2}}\right) + \left(\frac{25}{4}a_1 + \frac{65}{8}a_2\right.$$

$$\left. + \frac{145}{16}a_3 - \frac{5}{2}\right) + \left(\frac{225}{16}a_1 + \frac{1155}{64}a_2 + \frac{6345}{256}a_3\right.$$

$$\left. - \frac{15}{4\sqrt{2}}\right) + \left(a_1 + 2a_2 + 3a_3 + 1\right) = 0.$$

$$\Rightarrow \left(a_1 + \frac{25}{16}a_1 + \frac{25}{4}a_1 + \frac{225}{16}a_1\right) + \left(2a_2 + \frac{185}{64}a_2 + \frac{65}{8}a_2\right.$$

$$\left. + \frac{1155}{64}a_2\right) + \left(3a_3 + \frac{505}{256}a_3 + \frac{145}{16}a_3 + \frac{6345}{256}a_3\right)$$

$$= \left(\frac{5}{4\sqrt{2}} + \frac{15}{4\sqrt{2}} + \frac{5}{2} - 1\right)$$

$$\Rightarrow 22.875\,a_1 + 31.063\,a_2 + 38.820\,a_3 = 5.036 \quad \text{— (i)}$$

$$\frac{\partial I}{\partial a_2} = \left(\frac{5}{4}a_1 + \frac{37}{16}a_2 + \frac{101a_3}{64} - \frac{1}{\sqrt{2}}\right)\left(\frac{37}{16}\right) + \left(\frac{5}{2}a_1 + \frac{13}{4}a_2\right.$$

$$\left. + \frac{29}{8}a_3 - 1\right)\left(\frac{13}{4}\right) + \left(\frac{15}{4}a_1 + \frac{77}{16}a_2 + \frac{423}{64}a_3 - \frac{1}{\sqrt{2}}\right)\left(\frac{77}{16}\right)$$

$$+ \left(a_1 + 2a_2 + 3a_3 + 1\right)(2) = 0.$$

$$\Rightarrow \left(\frac{185}{64}a_1 + \frac{1369}{256}a_2 + \frac{3737}{1024}a_3 - \frac{37}{16\sqrt{2}}\right) + \left(\frac{65}{8}a_1\right.$$

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$$+\frac{169}{16}a_2+\frac{377}{32}a_3-\frac{13}{4}\Big)+\Big(\frac{1155}{64}a_1+\frac{(77)^2}{256}a_2$$

$$+\frac{32571}{1024}a_3-\frac{77}{\sqrt{2}(16)}\Big)+2a_1+4a_2+6a_3+2=0.$$

$$\Rightarrow \quad 31.063a_1+43.07a_2+53.238a_3=6.288 \quad \text{— (ii)}$$

$$\frac{\partial I}{\partial a_3}=\left(\frac{5}{4}a_1+\frac{37}{16}a_2+\frac{101}{64}a_3-\frac{1}{\sqrt{2}}\right)\left(\frac{101}{64}\right)+\left(\frac{5}{2}a_1+\frac{13}{4}a_2\right.$$

$$\left.+\frac{29}{8}a_3-1\right)\left(\frac{29}{8}\right)+\left(\frac{15}{4}a_1+\frac{77}{16}a_2+\frac{423}{64}a_3-\frac{1}{\sqrt{2}}\right)\left(\frac{423}{64}\right)$$

$$+\left(a_1+2a_2+3a_3+1\right)(3)=0.$$

$$\Rightarrow \left(\frac{505}{256}a_1+\frac{3737}{1024}a_2+\frac{(101)^2}{64^2}a_3-\frac{101}{64\sqrt{2}}\right)+\left(\frac{145}{16}a_1\right.$$

$$\left.+\frac{377}{32}a_2+\frac{841}{64}a_3-\frac{29}{8}\right)+\left(\frac{6345}{256}a_1+\frac{32571}{1024}a_2\right.$$

$$\left.+\left(\frac{423}{64}\right)^2a_3-\frac{423}{\cancel{16}\,64\sqrt{2}}\right)+3a_1+6a_2+9a_3+3=0.$$

$$\Rightarrow \quad 38.820a_1+53.238a_2+68.315a_3=\cancel{23.783}\ 6.414. \quad \text{— (iii)}.$$

Which gives us a symmetric 3x3 system.

solving simultaneously —

$a_1=1.825$; $a_2=-0.12$; $a_3=-0.85$.

$$\therefore v(x)=1.825x-0.12x^2-0.85x^3$$

(v) Galerkin's approach :-

The W-form remains the same as before

$$\left[-wv'\right]_{x=\pm 1}-\int_0^1 w'v'\,dx+\int_0^1 5xwv\,dx-\int_0^1 w\sin x\,dx=0.$$

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$$+\frac{169}{16}a_2+\frac{377}{32}a_3-\frac{13}{4}\Big)\cdot+\Big(\frac{1155}{64}a_1+\frac{(77)^2}{256}a_2$$

$$+\frac{32571}{1024}a_3-\frac{77}{\sqrt{2}(16)}\Big)+2a_1+4a_2+6a_3+2=0.$$

$$\Rightarrow\quad 31.063a_1+43.07a_2+53.238a_3=6.288 \quad \text{— (ii)}$$

$$\frac{\partial I}{\partial a_3}=\Big(\frac{5}{4}a_1+\frac{37}{16}a_2+\frac{101}{64}a_3-\frac{1}{\sqrt{2}}\Big)\Big(\frac{101}{64}\Big)+\Big(\frac{5}{2}a_1+\frac{13}{4}a_2$$

$$+\frac{29}{8}a_3-1\Big)\Big(\frac{29}{8}\Big)+\Big(\frac{15}{4}a_1+\frac{77}{16}a_2+\frac{423}{64}a_3-\frac{1}{\sqrt{2}}\Big)\Big(\frac{423}{64}\Big)$$

$$+\Big(a_1+2a_2+3a_3+1\Big)(3)=0.$$

$$\Rightarrow\Big(\frac{505}{256}a_1+\frac{3737}{1024}a_2+\frac{(101)^2}{64^2}a_3-\frac{101}{64\sqrt{2}}\Big)+\Big(\frac{145}{16}a_1$$

$$+\frac{377}{32}a_2+\frac{841}{64}a_3-\frac{29}{8}\Big)+\Big(\frac{6345}{256}a_1+\frac{32571}{1024}a_2$$

$$+\Big(\frac{423}{64}\Big)^2a_3-\frac{423}{\cancel{16}\,64\sqrt{2}}\Big)+3a_1+6a_2+9a_3+3=0.$$

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Which gives us a symmetric 3x3 system.

solving simultaneously —

$a_1=1.825$; $a_2=-0.12$; $a_3=-0.85$.

$$\therefore\ v(x)=1.825x-0.12x^2-0.85x^3$$

(v) Galerkin's approach :-

The W-form remains the same as before

$$\Big[-wv'\Big]_{x=\cancel{-}1}-\int_0^1 w'v'\,dx+\int_0^1 5xwv\,dx-\int_0^1 w\sin x\,dx=0.$$

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$v = a_1 x + a_2 x^2 + a_3 x^3$.

$v' = a_1 + 2a_2 x + 3a_3 x^2$.

$w_1 = \frac{\partial v}{\partial a_1} = x$. $\quad w_2 = \frac{\partial v}{\partial a_2} = x^2$

$w_3 = \frac{\partial v}{\partial a_3} = x^3$.

1st equation:-

$w_1' = 1$ $\quad [xv']_{x=1} - \int_0^1 \cancel{\frac{dx}{dx}} \cdot 1 \cdot (a_1 + 2a_2 x + 3a_3 x^2)\,dx + \int_0^1 5x(a_1 x + a_2 x^2 + a_3 x^3)\,dx$

$- \int_0^1 x \sin^7 x\,dx = 0$.

$\Rightarrow (-1) - \left[a_1(1) + a_2(1)^2 + a_3(1)^3\right] + \left[5a_1\frac{(1)^3}{3} + \frac{5a_2(1)^4}{4} + \frac{\cancel{5}a_3(1)^5}{\cancel{5}}\right]$

$- \left(\frac{1}{12}\right) = 0$.

$\Rightarrow -a_1 - a_2 - \cancel{a_3} + \frac{5}{3}a_1 + \frac{5}{4}a_2 + \cancel{a_3} = \left(1 + \frac{1}{12}\right)$

$\Rightarrow \left(\frac{5-3}{3}\right)a_1 + \left(\frac{5-4}{4}\right)a_2 + 0\,a_3 = \left(1 + \frac{1}{12}\right)$.

$\Rightarrow \frac{2}{3}a_1 + \frac{1}{4}a_2 + 0\,a_3 = \left(1 + \frac{1}{12}\right)$ $\quad -(i)$.

2nd equation:- $(w_2' = 2x)$.

$(-1) - \int_0^1 2x(a_1 + 2a_2 x + 3a_3 x^2)\,dx + \int_0^1 5x^2(a_1 x + a_2 x^2 + a_3 x^3)\,dx$

$- \int_0^1 x^2 \sin^7 x\,dx = 0$.

$\Rightarrow (-1) - \left[\frac{\cancel{2}a_1(1)^2}{\cancel{2}} + \frac{4a_2(1)^3}{3} + \frac{\cancel{6}^{3}a_3\cancel{x}^{(1)^4}}{\cancel{4}2}\right] + \left[\frac{5}{4}a_1(1)^4 + \frac{\cancel{5}}{\cancel{5}}a_2(1)^5 + \frac{5}{6}a_3(1)^6\right] - \left(\frac{1}{\pi} - \frac{4}{\pi^3}\right) = 0$.

$\Rightarrow -a_1 - \frac{4}{3}a_2 - \frac{3}{2}a_3 + \frac{5}{4}a_1 + a_2 + \frac{5}{6}a_3 = \left(1 + \frac{1}{\pi} - \frac{4}{\pi^3}\right)$

$\Rightarrow \frac{1}{4}a_1 - \frac{1}{3}a_2 \frac{-9+5}{6}a_3 = \left(1 + \frac{1}{\pi} - \frac{4}{\pi^3}\right)$.

$\Rightarrow \frac{1}{4}a_1 - \frac{1}{3}a_2 - \frac{4}{6}a_3 = \left(1 + \frac{1}{\pi} - \frac{4}{\pi^3}\right)$. $-(ii)$.

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3rd equation — $w_3' = 3x^2$.

$$(-1)\left[-\int_0^1 3x^2(a_1 + 2a_2 x + 3a_3 x^2)\,dx + \int_0^1 5x^3(a_1 x + a_2 x^2 + a_3 x^3)\,dx - \int_0^1 x^3 \sin \pi x\right] = 0.$$

$$\Rightarrow (-1)\left[-\left[a_1(1)^3 + \frac{6a_2(1)^4}{4} + \frac{9a_3(1)^5}{5}\right] + \left[\frac{5a_1(1)^5}{5} + \frac{5a_2(1)^6}{6} + \frac{5}{7}a_3(1)^7\right] - \left(\frac{1}{\pi} - \frac{6}{\pi^3}\right)\right] = 0.$$

$$\Rightarrow -\cancel{a_1} - \frac{3}{2}a_2 - \frac{9}{5}a_3 + \cancel{a_1} + \frac{5}{6}a_2 + \frac{5}{7}a_3 = 1 + \frac{1}{\pi} - \frac{6}{\pi^3}$$

$$\Rightarrow 0a_1 - \frac{4}{6}a_2 \frac{-63+25}{35}a_3 = \left(1 + \frac{1}{\pi} - \frac{6}{\pi^3}\right) \quad \cancel{(iii)}.$$

$$\Rightarrow 0a_1 - \frac{4}{6}a_2 - \frac{38}{35}a_3 = \left(1 + \frac{1}{\pi} - \frac{6}{\pi^3}\right).$$

Solving the 3x3 symmetric system of linear equations —

$a_1 = 2.067$ $\cancel{a_2 = 0.24}$ $a_2 = -0.241$

$a_3 = -0.889$.

$\therefore u(x) = 2.067x - 0.241x^2 - 0.889x^3$.

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8.7 Given constraint condition :

$$d_1 = d_3$$

$$\Rightarrow d_1 - d_3 = 0$$

$$\Rightarrow d_3 - d_1 = 0$$

In matrix form -

$$\therefore \underset{\sim}{A}\,\underset{\sim}{d} = \underset{\sim}{b}$$

$$\underbrace{\begin{pmatrix} 1 & \vdots & 0 & -1 \end{pmatrix}}_{A_c \;\cdot\; A_r} \begin{Bmatrix} d_3 \\ d_2 \\ d_1 \end{Bmatrix} = \begin{Bmatrix} 0 \end{Bmatrix}.$$

d_c ; d_r

$$\Rightarrow A_c d_c + A_r d_r = 0$$

$$\Rightarrow \boxed{d_c = -A_c^{-1} A_r d_r}$$

$$\underset{\sim}{d} = \begin{Bmatrix} d_3 \\ d_2 \\ d_1 \end{Bmatrix} = \begin{Bmatrix} d_c \\ d_r \end{Bmatrix} \quad \text{--(i)}$$

Now $d_c = -A_c^{-1} A_r d_r$

$A_c = (1) \rightarrow$ scalar

$A_r = \begin{pmatrix} 0 & -1 \end{pmatrix}$

$d_r = \begin{Bmatrix} d_2 \\ d_1 \end{Bmatrix}$.

$$d_c = -1 \begin{pmatrix} 0 & -1 \end{pmatrix} \begin{Bmatrix} d_2 \\ d_1 \end{Bmatrix}.$$

Putting this in (i) $\rightarrow$

$$\underset{\sim}{d} = \begin{Bmatrix} -\begin{pmatrix} 0 & -1 \end{pmatrix} \begin{Bmatrix} d_2 \\ d_1 \end{Bmatrix} \\ \begin{Bmatrix} d_2 \\ d_1 \end{Bmatrix} \end{Bmatrix} = \underbrace{\begin{Bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{Bmatrix}}_{T} \underbrace{\begin{Bmatrix} d_2 \\ d_1 \end{Bmatrix}}_{d_r}.$$

Given stiffness matrix relation -

$$\underset{\sim}{K}\,\underset{\sim}{d} = \underset{\sim}{F} \quad \underset{\sim}{K}\,\underset{\sim}{d} \Rightarrow \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0 \\ 2 \end{Bmatrix}.$$

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Rearranging the 1st and 3rd rows and columns to maintain congruency with the α considered sequences of 'd' vector in the constraint equations -

Row interchange →

$$\underset{\sim}{K} = \begin{bmatrix} 0 & -1 & 2 \\ -1 & 2 & -1 \\ 2 & -1 & 0 \end{bmatrix} \text{ and } \underset{\sim}{F} = \begin{Bmatrix} 2 \\ 0 \\ 1 \end{Bmatrix}$$

Column interchange →

$$\underset{\sim}{K} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}.$$

Therefore new system -

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{Bmatrix} d_3 \\ d_2 \\ d_1 \end{Bmatrix} = \begin{Bmatrix} 2 \\ 0 \\ 1 \end{Bmatrix}.$$

$$\Rightarrow \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ \cancel{2}\,0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} d_2 \\ d_1 \end{Bmatrix}$$

$$= \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}.$$

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Pre multiply with $T' \rightarrow$

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} d_r \end{Bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}.$$

$$\begin{bmatrix} 2 & -2 \\ -2 & 4 \end{bmatrix} \begin{Bmatrix} d_2 \\ d_1 \end{Bmatrix} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}.$$

$\Rightarrow \quad 2d_2 - 2d_1 = 0 \Rightarrow d_1 = d_2.$

and $-2d_2 + 4d_1 = 3.$

$\Rightarrow \quad -2d_1 + 4d_1 = 3.$

$\Rightarrow \quad 2d_1 = 3$

$d_1 = \frac{3}{2}.$

$\therefore \quad d_2 = d_1 = \frac{3}{2}$ and $d_3 - d_1 = 0.$

$\Rightarrow \quad d_3 = d_1 = \frac{3}{2}.$

$$\therefore d = \begin{Bmatrix} \frac{3}{2} \\ \frac{3}{2} \\ \frac{3}{2} \end{Bmatrix}. = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \frac{3}{2} \\ \frac{3}{2} \end{Bmatrix} = \begin{Bmatrix} \frac{3}{2} \\ \frac{3}{2} \\ \frac{3}{2} \end{Bmatrix}.$$

(B) Penalty function approach :-

Equation — $(\underset{\sim}{K} + \underset{\sim}{A}^T \underset{\sim}{\beta} \underset{\sim}{A}) \underset{\sim}{d} = \underset{\sim}{F} + \underset{\sim}{A}^T \underset{\sim}{\beta} \underset{\sim}{b}.$

Same as above example

$\underset{\sim}{A} = (1 \quad 0 \quad -1) \quad \underset{\sim}{b} = 0.$

$\underset{\sim}{\beta} = \beta$ (only one scalar value since we have $nc = 1$).

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$$A^T \beta A = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \beta \begin{pmatrix} 1 & 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} \beta \\ 0 \\ -\beta \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} \beta & 0 & -\beta \\ 0 & 0 & 0 \\ -\beta & 0 & \beta \end{pmatrix}.$$

$$\underset{\sim}{k} + \underset{\sim}{A}^T \beta \underset{\sim}{A} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} + \begin{bmatrix} \beta & 0 & -\beta \\ 0 & 0 & 0 \\ -\beta & 0 & \beta \end{bmatrix}.$$

$$\hat{\underset{\sim}{k}} = \begin{bmatrix} (2+\beta) & -1 & -\beta \\ -1 & 2 & -1 \\ -\beta & -1 & (2+\beta) \end{bmatrix}$$

Since $\underset{\sim}{b} = 0$; $\underset{\sim}{F} + \underset{\sim}{A}^T \beta \underset{\sim}{b} = \underset{\sim}{F} + 0 = \underset{\sim}{F}$.

$$\begin{bmatrix} (2+\beta) & -1 & -\beta \\ -1 & 2 & -1 \\ -\beta & -1 & (2+\beta) \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0 \\ 2 \end{Bmatrix}.$$

Now to solve by Cramer's Rule –

take the determinant of $\hat{\underset{\sim}{k}}$ first –

$$|\hat{\underset{\sim}{k}}| = (2+\beta)\left[2(2+\beta) - 1\right] + 1\left(-(2+\beta) - \beta\right)$$
$$- \beta(1 + 2\beta)$$

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$= (2+\beta)(4+2\beta-1) + 1(-2-\beta-\beta) - \beta - 2\beta^2$

$\Rightarrow (2+\beta)(3+2\beta) - 2 - 2\beta - \beta - 2\beta^2$

$= 6 + 4\beta + \cancel{3\beta} + \cancel{2\beta^2} - 2 - \cancel{3\beta} - \cancel{2\beta^2}$

$= \underline{4(1+\beta)}$

Now, $d_1 = \dfrac{|\hat{k}_1|}{|\hat{k}|}$; $d_2 = \dfrac{|\hat{k}_2|}{|\hat{k}|}$; $d_3 = \dfrac{|\hat{k}_3|}{|\hat{k}|}$

Now, $|\hat{k}_1| \rightarrow$ replace the 1st column with RHS values -

$$|\hat{k}_1| = \begin{vmatrix} 1 & -1 & -\beta \\ 0 & 2 & -1 \\ 2 & -1 & 2+\beta \end{vmatrix}$$

$= \cancel{2(2+\beta)+2}\ [2(2+\beta)-1] + 2(1+2\beta)$.

$= 4 + 2\beta - 1 + 2 + 4\beta$.

$= 6\beta + 5$

$$|\hat{k}_2| = \begin{vmatrix} (2+\beta) & 1 & -\beta \\ -1 & 0 & -1 \\ -\beta & 2 & (2+\beta) \end{vmatrix} = (2+\beta)[0+2] - 1(-(2+\beta)-\beta) - \beta(-2)$$

$= 2(2+\beta) + 2 + \beta + \beta + 2\beta$.

$= 4 + 2\beta + 2\beta + 2 + 2\beta = \underline{6\beta + 6}$.

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$$|\hat{K}_3| = \begin{vmatrix} (2+\beta) & -1 & 1 \\ -1 & 2 & 0 \\ -\beta & -1 & 2 \end{vmatrix}$$

$$= 1(1+2\beta) + 2(4+2\beta-1).$$

$$= 1+2\beta+8+4\beta-2.$$

$$= 7+6\beta.$$

$$\therefore d_1 = \frac{5+4\beta}{4+4\beta}; \quad d_2 = \frac{6+6\beta}{4+4\beta}; \quad d_3 = \frac{7+6\beta}{4+4\beta}.$$

Now as $\beta \to \infty$; $\frac{x}{\beta} = 0$ ($x \to$ some value)

$$d_1 = \frac{5/\beta + 6}{4/\beta + 4} = \frac{0+6}{0+4} = \frac{3}{2}$$

$$d_2 = \frac{6/\beta + 6}{4/\beta + 4} = 3/2.$$

$$d_3 = \frac{7/\beta + 6}{4/\beta + 4} = \frac{3}{2}.$$

Thus the values of d_1, d_2 and d_3 are exactly same as in part (a) when $\beta \to \infty$ which is theoretical.

Values of d_1, d_2 and d_3 for different values β are tabulated below—

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β	d_1	d_2	d_3	
0	Infinitely many	"	"	(as K matrix will be singular and hence we cannot find a unique solution as BC's are not taken into account.
1	$11/8 = 1.375$	$12/8 = 1.5$	$13/8 = 1.625$	
10	1.477	1.5	1.523	
100	1.498	1.5	1.502	

Thus we observe that as β increases, the values of $\beta_1 \to \beta$ d_1, d_2 and d_3 start converging towards a single value $(=1.5)$, as is evident from the case when $\beta \to \infty$ all were 1.5. We also observe that d_2 never changes and is $(=1.5)$ for any non-zero value of β.

(c). <u>Lagrange Multiplier method</u>:-

$$\underset{\sim}{A} = \begin{pmatrix} 1 & 0 & -1 \end{pmatrix} ; \quad \underset{\sim}{b} = 0 ; \quad \hat{\underset{\sim}{F}} = \begin{Bmatrix} 1 \\ 0 \\ 2 \end{Bmatrix}.$$

$$\underset{\sim}{K} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}.$$

$\underset{\sim}{\lambda} = \lambda_1$ (scalar) as only 1 constraint equation is available data in the

Thus arranging the given form —

$$\begin{pmatrix} \underset{\sim}{K} & \underset{\sim}{A}^T \\ \underset{\sim}{A} & 0 \end{pmatrix} \begin{Bmatrix} \underset{\sim}{d} \\ \underset{\sim}{\lambda} \end{Bmatrix} = \begin{Bmatrix} \underset{\sim}{F} \\ \underset{\sim}{b} \end{Bmatrix}.$$

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$$|\hat{K}_3| = \begin{vmatrix} (2+\beta) & +\cancel{\beta} & 1 \\ -1 & 2 & 0 \\ -\beta & -1 & 2 \end{vmatrix} = (1+2\beta) + 2(4+2\beta-1)$$

$$= 1+2\beta+8+4\beta-2.$$

$$=$$

$$\begin{bmatrix} 2 & -1 & 0 & 1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 1 & 0 & -1 & 0 \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \\ \lambda_1 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{Bmatrix}.$$

Initialising the above system as —

$\hat{\underset{\sim}{K}}\ \hat{\underset{\sim}{d}} = \hat{\underset{\sim}{F}}$ in MATLAB

and inverting in $(\hat{\underset{\sim}{K}}) = \begin{bmatrix} 2 & -1 & 0 & 1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 1 & 0 & -1 & 0 \end{bmatrix}$.

$$\hat{\underset{\sim}{F}} = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix}$$

we get the resulting solution vector

$$\hat{\underset{\sim}{d}} = \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \\ \lambda_1 \end{Bmatrix} = \begin{Bmatrix} 1.5 \\ 1.5 \\ 1.5 \\ -0.5 \end{Bmatrix}$$

thus the nodal displacements

$$\begin{Bmatrix} d_1 \\ d_2 \\ d_3 \end{Bmatrix} = \begin{Bmatrix} 1.5 \\ 1.5 \\ 1.5 \end{Bmatrix}$$ same as previous cases.

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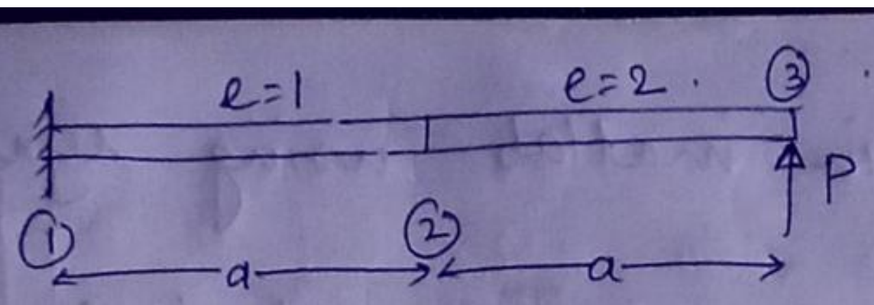
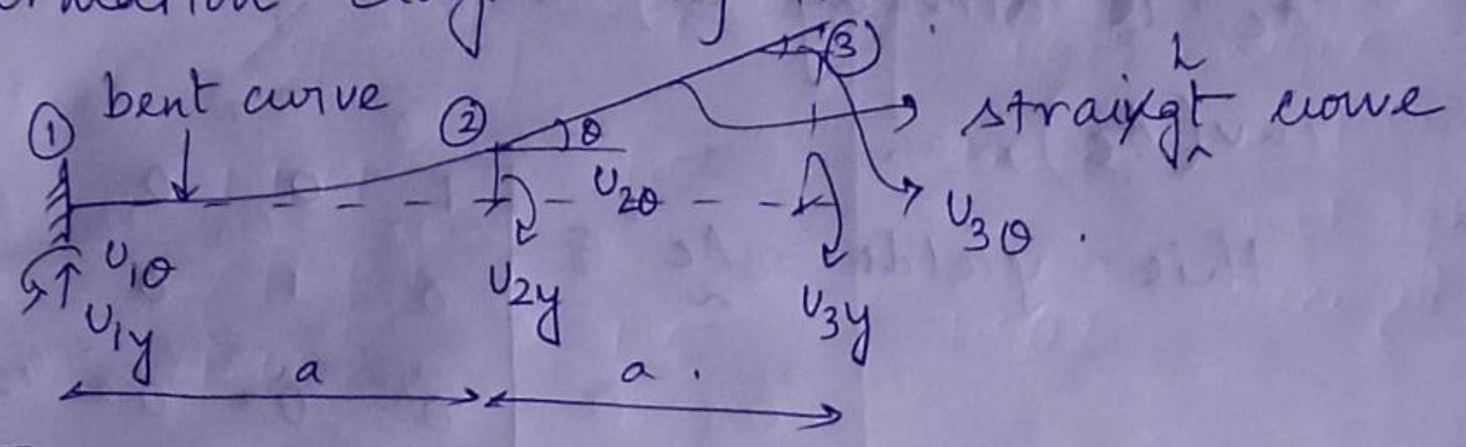
3:-(a)



Given element '2' is perfectly rigid, then the deformation diagram of the beam -



Thus the constraint relations will be -

(i) $\boxed{U_{1y} = 0}$

(ii) $\boxed{U_{2\theta} = 0}$

(iii) $U_{2\theta} = U_{3\theta} \Rightarrow U_{2\theta} - U_{3\theta} = 0 \Rightarrow \boxed{U_{3\theta} - U_{2\theta} = 0}$

(iv) $U_{3y} = U_{2y} + \theta_{2\theta} \times a$

$\Rightarrow \boxed{U_{3y} - U_{2y} - \theta_{2\theta} \times a = 0}$

∴ Writing in matrix form -

$$\underbrace{\left[\begin{array}{cccc|cc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & +1 \\ 0 & 0 & -1 & -a & 1 & 0 \end{array}\right]}_{a_c\,(4\times4)\quad a_r\,(4\times2)} \begin{Bmatrix} \overbrace{\begin{matrix} U_{1y} \\ U_{1\theta} \\ U_{2y} \\ U_{2\theta} \end{matrix}}^{d_c} \\ \underbrace{\begin{matrix} U_{3y} \\ U_{3\theta} \end{matrix}}_{d_r} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

$\therefore A_c d_c + A_r d_r = 0$

$\Rightarrow d_c = -A_c^{-1} A_r d_r$

$$d = \begin{Bmatrix} d_c \\ d_r \end{Bmatrix} = \underset{\sim}{T}\, \underset{\sim}{d_r} = \begin{bmatrix} -\underset{\sim}{A_c}^{-1} \underset{\sim}{A_r} \\ \underset{\sim}{I} \end{bmatrix} \underset{\sim}{d_r}$$

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Doing the A_c^{-1} in matlab using symbolic commands -

syms a; → as 'a' is a symbol here.

A_c is already defined,

inv(A_c);

this yields $\underset{\sim}{A_c}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$.

Now doing $-A_c^{-1}A_r$:- (on matlab).

$$-\underset{\sim}{A_c}^{-1}\underset{\sim}{A_r} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & (-a) \\ 0 & 1 \end{bmatrix}.$$

$$\therefore \underset{\sim}{T} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & (-a) \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}. \quad \text{and } \cancel{d_c} \; \underset{\sim}{d_c} = \underset{\sim}{T} \cdot \underset{\sim}{d_r}$$

Since the beam is uniform in every way i.e it has same EI's every where

So the elemental stiffness matrices for both elements will be same -

$$k_1 = \frac{EI}{a^3} \begin{array}{c} \begin{array}{cccc} u_{1y} & u_{1\theta} & u_{2y} & u_{2\theta} \end{array} \\ \begin{bmatrix} 12 & 6a & -12 & 6a \\ 6a & 4a^2 & -6a & 2a^2 \\ -12 & -6a & 12 & -6a \\ 6a & 2a^2 & -6a & 4a^2 \end{bmatrix} \end{array} \begin{array}{c} \\ u_{1y} \\ u_{1\theta} \\ u_{2y} \\ u_{2\theta} \end{array}$$

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$$K_2 = \frac{EI}{a^3}\begin{array}{c} \begin{array}{cccc} U_{2y} & U_{2\theta} & U_{3y} & U_{3\theta} \end{array} \\ \begin{bmatrix} 12 & 6a & -12 & 6a \\ 6a & 4a^2 & -6a & 2a^2 \\ -12 & -6a & 12 & -6a \\ 6a & 2a^2 & -6a & 4a^2 \end{bmatrix} \end{array} \begin{array}{c} U_{2y} \\ U_{2\theta} \\ U_{3y} \\ U_{3\theta} \end{array}$$

$$\therefore K_G = \frac{EI}{a^3}\begin{array}{c} \begin{array}{cccccc} U_{1y} & U_{1\theta} & U_{2y} & U_{2\theta} & U_{3y} & U_{3\theta} \end{array} \\ \begin{bmatrix} 12 & 6a & -12 & 6a & 0 & 0 \\ 6a & 4a^2 & -6a & 2a^2 & 0 & 0 \\ -12 & -6a & (12+12) & (-6a+6a) & -12 & 6a \\ 6a & 2a^2 & (-6a+6a) & (4a^2+4a^2) & -6a & 2a^2 \\ 0 & 0 & -12 & -6a & 12 & -6a \\ 0 & 0 & 6a & 2a^2 & -6a & 4a^2 \end{bmatrix} \end{array} \begin{array}{c} U_{1y} \\ U_{1\theta} \\ U_{2y} \\ U_{2\theta} \\ U_{3y} \\ U_{3\theta} \end{array}$$

$$\Rightarrow K_G = \frac{EI}{a^3}\begin{array}{c} \begin{array}{cccccc} U_{1y} & U_{1\theta} & U_{2y} & U_{2\theta} & U_{3y} & U_{3\theta} \end{array} \\ \begin{bmatrix} 12 & 6a & -12 & 6a & 0 & 0 \\ 6a & 4a^2 & -6a & 2a^2 & 0 & 0 \\ -12 & -6a & 24 & 0 & -12 & 6a \\ 6a & 2a^2 & 0 & 8a^2 & -6a & 2a^2 \\ 0 & 0 & -12 & -6a & 12 & -6a \\ 0 & 0 & 6a & 2a^2 & -6a & 4a^2 \end{bmatrix} \end{array} \begin{array}{c} U_{1y} \\ U_{2\theta} \\ U_{2y} \\ U_{2\theta} \\ U_{3y} \\ U_{3\theta} \end{array}$$

Now, $\underset{\sim}{K_G}\,\underset{\sim}{d} = \underset{\sim}{F}$ where $\underset{\sim}{F} = \begin{array}{c} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ +P \\ 0 \end{Bmatrix} \end{array} \begin{array}{c} U_{1y} \\ U_{1\theta} \\ U_{2y} \\ U_{2\theta} \\ U_{3y} \\ U_{3\theta} \end{array}$

Replacing $\underset{\sim}{d} = \underset{\sim}{T}\,\underset{\sim}{d_r}$ in above eqn.

$$\underset{\sim}{K_G}\,\underset{\sim}{T}\,\underset{\sim}{d_r} = \underset{\sim}{F}.$$

pre multiply with T'

$$\underbrace{(\underset{\sim}{T'}\,\underset{\sim}{K_G}\,\underset{\sim}{T})}_{\hat{\underset{\sim}{K}}}\,\underset{\sim}{d_r} = \underbrace{(\underset{\sim}{T'}\,\underset{\sim}{F})}_{\hat{\underset{\sim}{F}}}.$$

$$\hat{\underset{\sim}{K}}\,\underset{\sim}{d_r} = \hat{\underset{\sim}{F}}.$$

Again finding $T' K_G T$ using Matlab —

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$\underset{\sim}{T}' \underset{\sim}{K_G} \underset{\sim}{T}$:-

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & (-a) & 1 & 0 & 1 \end{bmatrix}_{2\times 6} \frac{EI}{a^3} \begin{bmatrix} 12 & 6a & -12 & 6a & 0 & 0 \\ 6a & 4a^2 & -6a & 2a^2 & 0 & 0 \\ -12 & -6a & 24 & 0 & -12 & 6a \\ 6a & 2a^2 & 0 & 8a^2 & -6a & 2a^2 \\ 0 & 0 & -12 & -6a & 12 & -6a \\ 0 & 0 & 6a & 2a^2 & -6a & 4a^2 \end{bmatrix}_{6\times 6} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & -a \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}_{6\times 2}$$

$$= \begin{bmatrix} -12 & -6a & 12 & -6a & 0 & 0 \\ 18a & 8a^2 & -18a & 10a^2 & 0 & 0 \end{bmatrix}_{2\times 6} * \frac{EI}{a^3} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & (-a) \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}_{6\times 2}$$

$$= \begin{pmatrix} 12 & -18a \\ -18a & 28a^2 \end{pmatrix} * \frac{EI}{a^3}$$

Now doing $\underset{\sim}{T}' * \underset{\sim}{F}$

$$\underset{\sim}{\hat{F}} = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & (-a) & 1 & 0 & 1 \end{bmatrix}_{2\times 6} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ P \\ 0 \end{bmatrix}_{6\times 1}$$

$$= \begin{pmatrix} P \\ 0 \end{pmatrix}$$

Therefore $\underset{\sim}{\hat{K}}\, \underset{\sim}{dr} = \underset{\sim}{\hat{F}}$ becomes -

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~~8(b)~~

$$\frac{EI}{a^3}\begin{pmatrix} 12 & -18a \\ -18a & 28a^2 \end{pmatrix}\begin{Bmatrix} U_{3y} \\ U_{3\theta} \end{Bmatrix} = \begin{Bmatrix} P \\ 0 \end{Bmatrix}.$$

Expanding —

$$\frac{EI}{a^3}\left(12\,U_{3y} - 18a\,U_{3\theta}\right) = P \quad (i) \quad (\times 1.5a).$$

$$\frac{EI}{a^3}\left(-18a\,U_{3y} + 28a^2\,U_{3\theta}\right) = 0 \quad -(ii)$$

$(i) \times 1.5 + (ii) \rightarrow$

$$\frac{EI}{a^3}\left(+28a^2\,U_{3\theta} - 27a^2\,U_{3\theta}\right) = \frac{3}{2}aP.$$

$$\Rightarrow \quad a^2 U_{3\theta} \times \frac{EI}{a^3} = \frac{3}{2}aP$$

$$\Rightarrow \quad \boxed{U_{3\theta} = \frac{3}{2}\left(\frac{Pa^2}{EI}\right)}.$$

and $\quad U_{3y} \times 12 - \cancel{18}^{9}a \times \frac{3}{\cancel{2}}\frac{Pa^2}{EI} = \frac{Pa^3}{EI}.$

$$\Rightarrow \quad U_{3y}(12) = \frac{28\,Pa^3}{EI}$$

$$\Rightarrow \quad U_{3y} = \frac{28}{12}\left(\frac{Pa^3}{EI}\right)$$

$$\Rightarrow \quad \boxed{U_{3y} = \frac{7}{3}\left(\frac{Pa^3}{EI}\right)}$$

And from our given Constraints relation —

$$U_{2\theta} = U_{3\theta} \Rightarrow U_{2\theta} = \frac{3}{2}\left(\frac{Pa^2}{EI}\right).$$

and $\quad U_{3y} = U_{2y} + U_{2\theta} \times a.$

$$\Rightarrow \quad U_{2y} = \frac{7}{3}\frac{Pa^3}{EI} - \frac{3}{2}\frac{Pa^3}{EI}.$$

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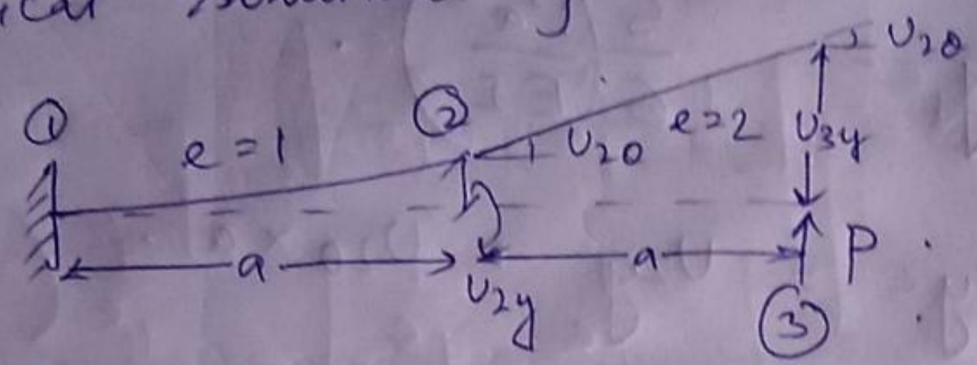
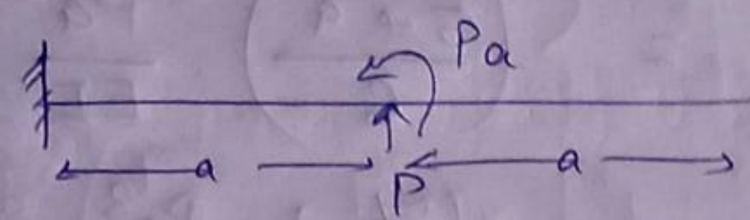
$$\Rightarrow \quad U_{2y} = \frac{(14-9)\,Pa^3}{6\,EI}$$

$$\Rightarrow \quad \boxed{U_{2y} = \frac{5\,Pa^3}{6EI}}$$

Analytical solution of the beam –



The given loading configuration can be done in the way shown below –



As shifting a force will be accompanied by the moment between the current and previous point.

Now, $U_{20} = \frac{Pa^2}{2EI} + \frac{(Pa)\,a}{EI}$

(Using standard beam results from constant deformations, i.e $\theta = \frac{Pa^2}{2EI}$ for load 'P' at free end of a cantilever and $\theta = \frac{Ma}{EI}$; for moment 'M' at the free end

Here $P = P$ and $M = (Pa)$

$$= \theta = \frac{Pa}{2EI} + \frac{(Pa)\,a}{EI}$$

$$\Rightarrow U_{20} \cancel{\theta_2} = \frac{3}{2}\left(\frac{Pa^2}{EI}\right)$$

Now since the element (2) of the beam is perfectly rigid, $U_{20} = U_{30}$.

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and $U_{2y} = \frac{Pa^3}{3EI} + \frac{(Pa)a^2}{2EI}$

(Same as 'δ', disp. of free end formula for load $P = \frac{Pa^3}{3EI}$ and for Moment 'M' $= \frac{Ma^2}{2EI}$)

$$\therefore U_{2y} = \frac{5}{6}\left(\frac{Pa^3}{EI}\right).$$

$$\therefore U_{3y} = U_{2y} + U_{2\theta} \times a.$$

$$= \left(\frac{5}{6} + \frac{3}{2}\right)\frac{Pa^3}{EI}$$

$$= \left(\frac{5+9}{6}\right)\frac{Pa^3}{EI}$$

$$\Rightarrow U_{3y} = \left(\frac{7}{3}\right)\frac{Pa^3}{EI}$$

Hence we see that our analytical and FEM results are exactly same

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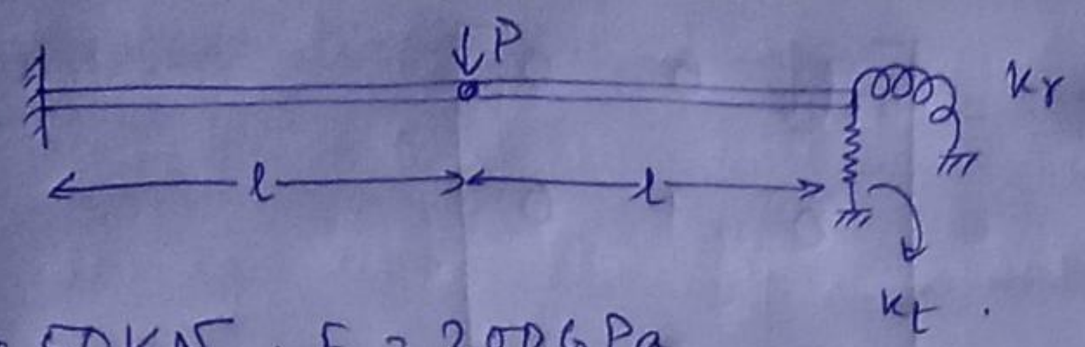
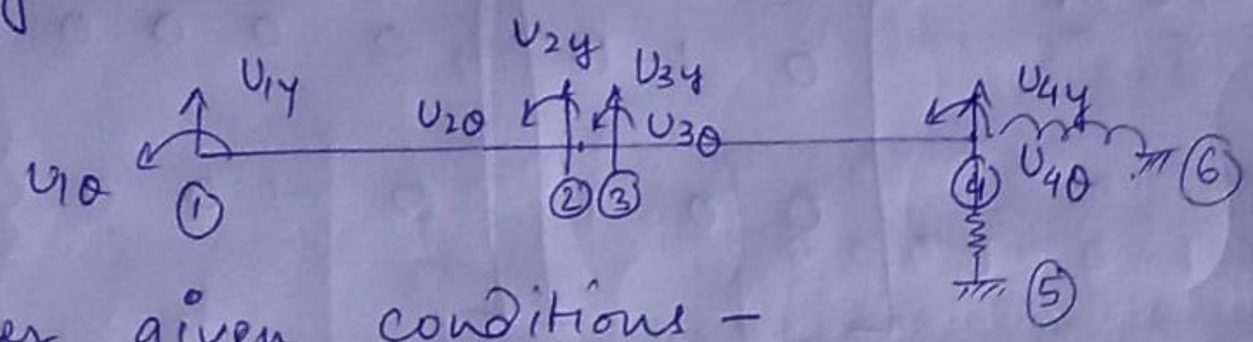
3(10)



$P = 50\,KN$; $E = 200\,GPa$ $\qquad$ $\underline{l = 3m}$.

$I = 2\times10^{-4}\,m^4$; $200\,KN/m = k_t$ and $k_r = 200\,KNm/rad$.

Modelling the internal hinge with double nodes -



As per given conditions -

$$\left.\begin{aligned} U_{1y} &= 0 \quad -(i) \\ U_{1\theta} &= 0 \quad -(ii) \\ U_{2y} - U_{3y} &= 0 \quad -(iii) \end{aligned}\right\} \text{Constraints.}$$

$U_{2\theta} \neq U_{3\theta}$ as at internal hinge left and right end rotations can be different on its account of its inability to resist bending moment.

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \end{bmatrix}}_{A_c \quad A_r \quad 3\times8} \begin{Bmatrix} U_{1y} \\ U_{1\theta} \\ U_{2y} \\ U_{2\theta} \\ U_{3y} \\ U_{3\theta} \\ U_{4y} \\ U_{4\theta} \end{Bmatrix}_{8\times1} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}.$$

d_c $\qquad$ d_r

$$\therefore \; d_c = -A_c^{-1} A_r d_r.$$

and $$d = \begin{Bmatrix} d_c \\ d_r \end{Bmatrix} = \begin{Bmatrix} \overbrace{-A_c A_r}^{T} \\ I \end{Bmatrix} \{d_r\}.$$

$$A_c = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} ; \quad A_r = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \end{bmatrix}.$$

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$$\underset{\sim}{A_c}^{-1} = \underset{\sim}{A_c} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$-\underset{\sim}{A_c}^{-1}\underset{\sim}{A_r} = \underset{3\times 3}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \underset{3\times 5}{\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \end{bmatrix}}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

$$\therefore \underset{\sim}{T} = \begin{bmatrix} -\underset{\sim}{A_c}^{-1}\underset{\sim}{A_r} \\ \underset{\sim}{I} \end{bmatrix} = \underset{8\times 5}{\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}}$$

Since we have separated the hinge into two nodes, the stiffness matrices for both elements will now be separate.

$$k^1 = \frac{EI}{l^3} \begin{array}{c} \begin{array}{cccc} v_{1y} & v_{1\theta} & v_{2y} & v_{2\theta} \end{array} \\ \begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & \cancel{4}\,2l^2 & -6l & 4l^2 \end{bmatrix} \end{array}.$$

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$$\frac{EI}{l^3} = \frac{200\times10^6 \frac{kN}{m^2} \times 2\times10^{-4} m^4}{(3)^3}$$

$$= \frac{40,000}{(3)^3} \frac{kN\,m^2}{m^3}$$

$$= 1481.48 \ kN/m.$$

$$k^1 = \frac{EI}{l^3}\begin{array}{c} \begin{bmatrix} +12 & -18 & -12 & -18 \\ -18 & -36 & -18 & 18 \\ -12 & -18 & 12 & -18 \\ 18 & 18 & -18 & 36 \end{bmatrix} \\ \begin{array}{cccc} U_{14} & U_{10} & U_{24} & U_{20} \end{array} \end{array} \begin{array}{c} U_{14} \\ U_{10} \\ U_{24} \\ U_{20} \end{array}$$

let $\frac{EI}{l^3} = a$.

as $U_{14} = U_{10} = 0$.

as $U_{14} = U_{10} = 0$

$$k^2 = \begin{array}{c} \begin{array}{cccccc} U_{34} & U_{30} & U_{44} & U_{40} & U_{54} & U_{60} \end{array} \\ \begin{bmatrix} 12a & 18a & -12a & 18a & 0 & 0 \\ 18a & 36a & -18a & 18a & 0 & 0 \\ -12a & -18a & (12a+200) & (-18a) & (-200) & \\ 18a & 18a & -18a & (36a+200) & 0 & -200 \\ 0 & 0 & -200 & 0 & 200 & 0 \\ 0 & 0 & 0 & -200 & 0 & 200 \end{bmatrix} \end{array} \begin{array}{c} U_{34} \\ U_{30} \\ U_{44} \\ U_{40} \\ U_{54} \\ U_{60} \end{array}$$

Since U_{54}, U_{60} are anyway zero, dropping them from the k^2 matrix will result in -

$$k^2 = \begin{array}{c} \begin{array}{cccc} U_{2y} & U_{30} & U_{44} & U_{40} \end{array} \\ \begin{bmatrix} 12a & 18a & -12a & 18a \\ 18a & 36a & -18a & 18a \\ -12a & -18a & (12a+200) & -18a \\ 18a & 18a & -18a & (36a+200) \end{bmatrix} \end{array}$$

where $a = \frac{EI}{l^3}$.

We can also reduce the T matrix by only considering the free dofs after applying the BCs. Thus after applying BCs, the free dofs which remain are -

$$d_f = \{U_{2y}, U_{20}, U_{3y}, U_{30}, U_{44}, U_{40}\}^T$$

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of these; there is one constraint equation i.e $U_{2y} = U_{3y}$

$$\Rightarrow U_{2y} - U_{3y} = 0.$$

$\therefore$ In matrix form:

$$\underbrace{\begin{bmatrix} 1 & \underbrace{0 \; -1 \; 0 \; 0 \; 0}_{A_r} \end{bmatrix}}_{A_c} \begin{Bmatrix} U_{2y} \\ U_{2\theta} \\ U_{3y} \\ U_{3\theta} \\ U_{4y} \\ U_{4\theta} \end{Bmatrix} = \begin{Bmatrix} 0 \end{Bmatrix}.$$

1×6 $\quad$ 6×1 $\quad$ d_c $\quad$ d_r

$\therefore A_c = (1)$

$A_r = (0 \; -1 \; 0 \; 0 \; 0)$.

$$d_c A_c + A_r d_r = 0.$$

$$\Rightarrow d_c = -A_c^{-1} A_r d_r.$$

$$d_f = \begin{pmatrix} d_c \\ d_r \end{pmatrix} = \begin{pmatrix} -A_c^{-1} A_r \\ I \end{pmatrix} d_r = T \, d_r$$

Now, $-A_c^{-1} A_r = -1 \times (0 \; -1 \; 0 \; 0 \; 0)$

$$= (0 \; 1 \; 0 \; 0 \; 0)_{1\times 5}.$$

$$\therefore T = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}_{6\times 5}.$$

Our Assembled stiffness matrix after applying the boundary conditions —

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$$K_g = \begin{array}{c} \begin{array}{cccccc} U_{2y} & U_{2\theta} & U_{3y} & U_{3\theta} & U_{4y} & U_{4\theta} \end{array} \\ \begin{bmatrix} 12a & -18a & 0 & 0 & 0 & 0 \\ -18a & 36a & 0 & 0 & 0 & 0 \\ 0 & 0 & 12a & 18a & -12a & 18a \\ 0 & 0 & 18a & 36a & -18a & 18a \\ 0 & 0 & -12a & -18a & (12a+200) & -18a \\ 0 & 0 & 18a & 18a & -18a & (36a+200) \end{bmatrix} \end{array} \begin{array}{c} U_{2y} \\ U_{2\theta} \\ U_{3y} \\ U_{3\theta} \\ U_{4y} \\ U_{4\theta} \end{array}$$

6×6

where $a = \frac{EI}{l^3} = 1481.48 \ \frac{KN}{m}$.

Now, $K_g \, d_f = F_g$ where $F_g = \begin{Bmatrix} -P/2 \\ 0 \\ -\frac{P}{2} \\ 0 \\ 0 \\ 0 \end{Bmatrix} \begin{array}{c} U_{2y} \\ U_{2\theta} \\ U_{3y} \\ U_{3\theta} \\ U_{4y} \\ U_{4\theta} \end{array}$

where $P = 50 \, KN$.

As - $\underset{\sim}{d_f} = \underset{\sim}{T} \, \underset{\sim}{d_r}$

substitute in above equation -

$$\underset{\sim}{K_g} \, \underset{\sim}{T} \, \underset{\sim}{d_r} = F_g.$$

$$\Rightarrow \underbrace{\left(\underset{\sim}{T}' \, \underset{\sim}{K_g} \, \underset{\sim}{T}\right)}_{\hat{\underset{\sim}{K}}} \underset{\sim}{d_r} = \underbrace{\left(\underset{\sim}{T}' \, \underset{\sim}{F_g}\right)}_{\hat{\underset{\sim}{F}}}.$$

Doing the matrix multiplication on Matlab - (Using symbolic command 'syms' and the inversion command inv()).

$$d_r = \begin{Bmatrix} U_{2\theta} \\ U_{3y} \\ U_{3\theta} \\ U_{4y} \\ U_{4\theta} \end{Bmatrix} = \begin{Bmatrix} -0.0056 \\ -0.0112 \\ 0.0034 \\ -0.0011 \\ 0.0033 \end{Bmatrix}.$$

disp. in 'm' and rotations in 'rads'.

$$d_c = -A_c^{-1} A_r \, d_r = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \end{pmatrix} \times d_r$$

$$\Rightarrow d_c = U_{2y} = -0.0012 \, m.$$

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(b) Lagrange multiplier method.

K_b = same as in previous case

$A = (1 \;\; 0 \;\; -1 \;\; 0 \;\; 0 \;\; 0)$.

$\lambda = \lambda_1$ only as only 1 constraint eqn is present.

Arranging the matrices in the form -

$$\begin{pmatrix} \underset{\sim}{K_b} & \underset{\sim}{A}^T \\ \underset{\sim}{A} & 0 \end{pmatrix} \begin{Bmatrix} \underset{\sim}{d} \\ \lambda \end{Bmatrix} = \begin{Bmatrix} F_b \\ b \end{Bmatrix}$$

$$\left[\begin{array}{cccccc|c} 12a & -18a & 0 & 0 & 0 & 0 & 1 \\ -18a & 36a & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 12a & 18a & -12a & 18a & -1 \\ 0 & 0 & 18a & 36a & -18a & 18a & 0 \\ 0 & 0 & -12a & -18a & (12a+200) & -12a & 0 \\ 0 & 0 & 18a & 18a & -12a & (36a+200) & 0 \\ \hline 1 & 0 & -1 & 0 & 0 & 0 & 0 \end{array}\right] \begin{Bmatrix} U_{24} \\ U_{20} \\ U_{34} \\ U_{30} \\ U_{44} \\ U_{40} \\ \hline \lambda_1 \end{Bmatrix}$$

Arranging the matrices in the above manner and inverting to find $\begin{Bmatrix} d \\ \lambda \end{Bmatrix}$ using inv() on MATLAB vector in Matlab -

$$\begin{Bmatrix} -25 \\ 0 \\ -25 \\ 0 \\ 0 \\ 0 \\ \hline 0 \end{Bmatrix}$$

as b=0 in this case.

$$\begin{Bmatrix} d \\ \lambda \end{Bmatrix} = \begin{bmatrix} -0.0112 \\ -0.0056 \\ -0.0112 \\ 0.0034 \\ -0.0011 \\ 0.0033 \\ 24.773 \end{bmatrix} \begin{matrix} U_{24} \\ U_{20} \\ U_{34} \\ U_{30} \\ U_{44} \\ U_{40} \\ \lambda_1 \end{matrix}$$

Reaction coming on to the hinge = 24.773 KN

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