CE723N: Finite Element Methods for Civil Engineering Applications
Homework 3 (Assigned: November 03, 2021; Due: November 17, 2021)

Name:
BUDHADITYA DE

Roll No.:
21103033

Problem	Points	Points awarded
1	10	
2	25	
3	60	
4	25	
Total:	**120**	

Instructions:

[1] Please submit the necessary codes (MATLAB files), along with a scanned copy (in pdf format) of your report. Also attach the question paper to the front of your report. Submit all the files as a .zip folder via email to: ***iitk.ce723a@gmail.com***. In the subject line of the email, please write: *CE723A HW3 submission*.

[2] Name the files and folder using your roll number, the HW number, and the problem number. For example, if your roll no. is 12345: your submission file name should be *HW3_12345.zip*, the report should be named *HW3_12345.pdf*, the code for problem no. 3 should be named *HW3_P3_12345.m* etc.

[3] Please show all numeric answers up to 4 decimal places.

[4] Please write neatly. Please maintain clarity in your solutions. If we do not understand what you have written, we will not grade your submission.

[5] You are encouraged to discuss amongst yourselves. But the submissions should be done individually, and you should be able to explain every single step that you have taken in solving any problem. Please do not use any unfair means.

[6] We may randomly call you in a viva and ask you to explain how you solved one/more of the problems. If we detect that you have used unfair means (e.g. you cannot explain a solution which you have included in your submission), even for a single problem, we will award a grade of zero to your entire submission.

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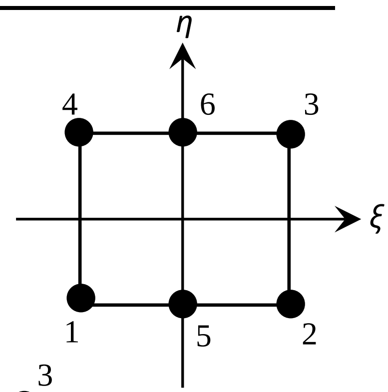
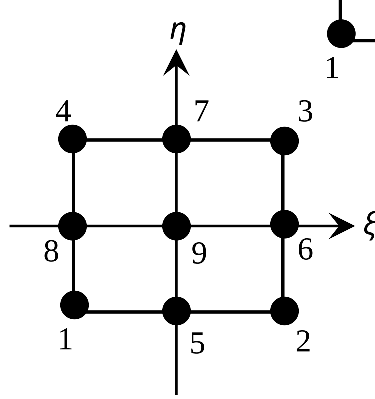
1. Write the shape functions for the following isoparametric elements, as products of appropriate 1D shape functions.

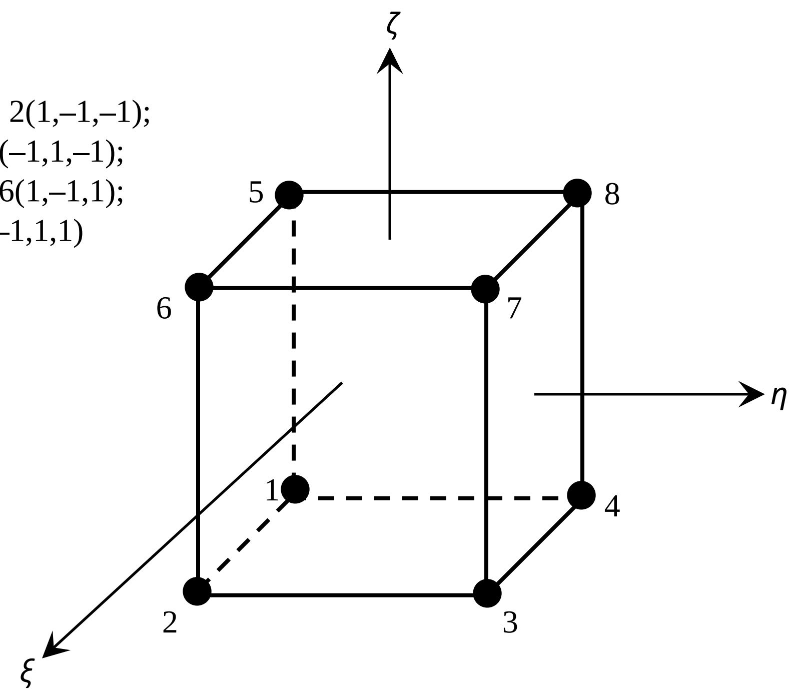
   a. 6-noded quadrilateral element
   Nodal coordinates: 1(–1,–1); 2(1,–1); 3(1,1);
   4(–1,1); 5(0,–1); 6(0,1)



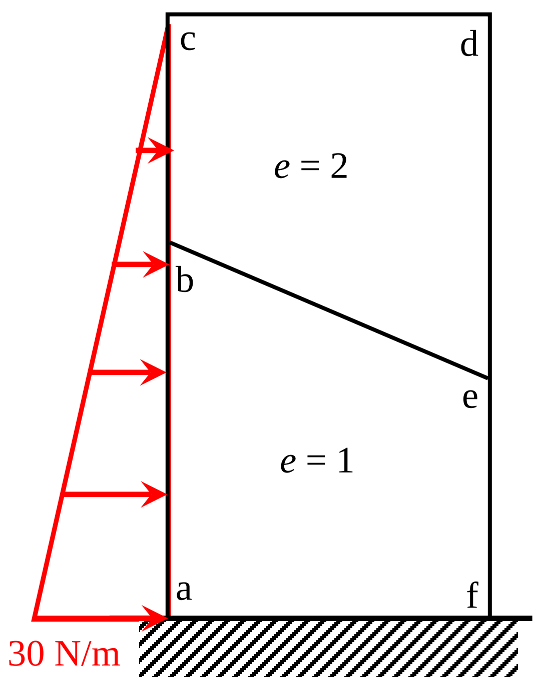
   b. 9-noded quadrilateral element
   Nodal coordinates: 1(–1,–1); 2(1,–1); 3(1,1);
   4(–1,1); 5(0,–1); 6(1,0);
   7(0,1); 8(–1,0); 9(0,0)



c. 8-noded brick element
Nodal coordinates: 1(–1,–1,–1); 2(1,–1,–1);
3(1,1,–1); 4(–1,1,–1);
5(–1,–1,1); 6(1,–1,1);
7(1,1,1); 8(–1,1,1)



2. Consider a thin plate with planar (in-plane) dimensions as shown in the following figure, and with Young's modulus $E = 3 \times 10^7$ Pa and Poisson's ratio $\nu = 0.3$. The left edge of the plate is loaded with a linearly varying load as shown. We consider this as a plane stress linear elasticity problem, and intend to solve it using 2D finite elements. The plate is discretized using 2 quadrilateral elements (e = 1, 2 in the figure). Find the in-plane nodal displacements.

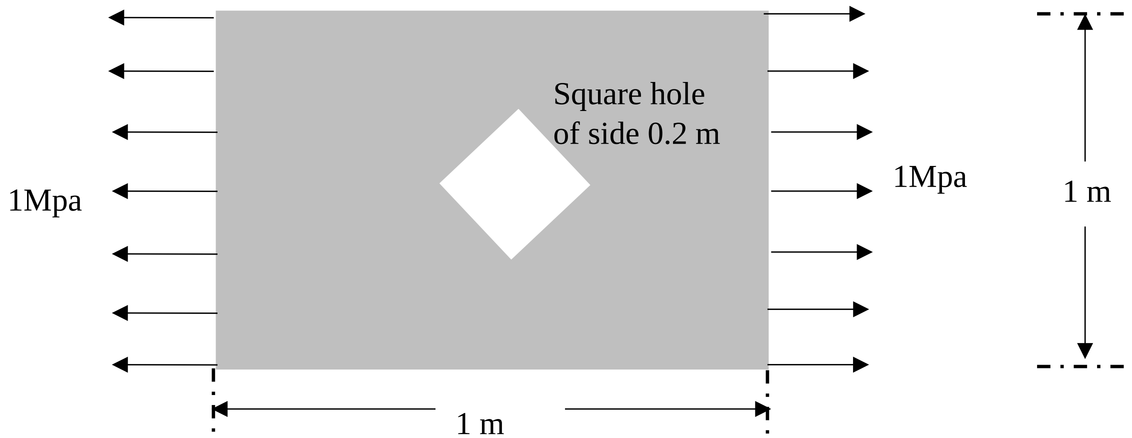


Dimensions:

ab = de = 2 m

bc = cd = ef = 1 m

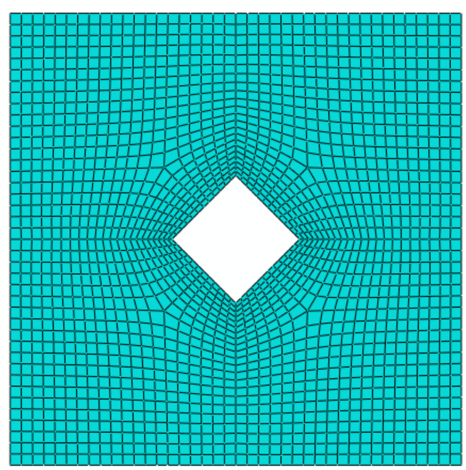
3. Consider the square plate of constant thickness (t = 0.01) with a square hole as shown in the following figure. The center points of the hole and the plate coincide. The hole is so oriented that its sides are at 45^0 to the sides of the plate. A uniform pressure of 1 MPa is applied on the boundary of the plate.



E = 210GPa; v=0.3

The plate is discretized using quadrilateral elements, as shown below, where the mesh is generated using ABAQUS. The following files have been uploaded separately in the course website:

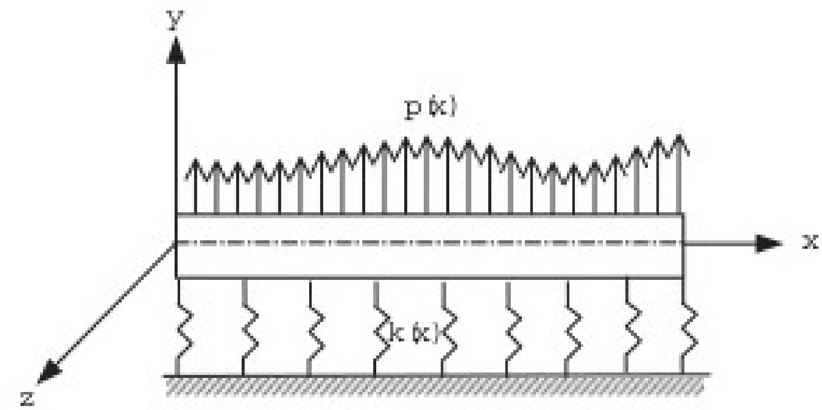
- *Elements_Square_Hole.txt*: This file contains the element connectivity (Column 1: element number; Columns 2 to 5: Node numbers for that element)
- *Nodes_Square_Hole.txt*: This file contains the nodal coordinates (Column 1: node number; Columns 2 and 3: x and y coordinates of that node)



Write a MATLAB code to perform a two-dimensional finite element analysis to determine the stress distribution in the plate. Plot the stress distribution. Also find the maximum stress and stress concentration factors around the hole.

Now consider the same plate with a circular hole of diameter 0.2 m, with the hole and plate centers coinciding. Redo the analysis, and compare the maximum stresses and stress concentration factors around the circular and square holes. The element connectivity and nodal coordinates for the plate with the circular hole are given in the files *Elements_Circular_Hole.txt* and *Nodes_Circular_Hole.txt*, respectively, which have been uploaded separately in the course website.

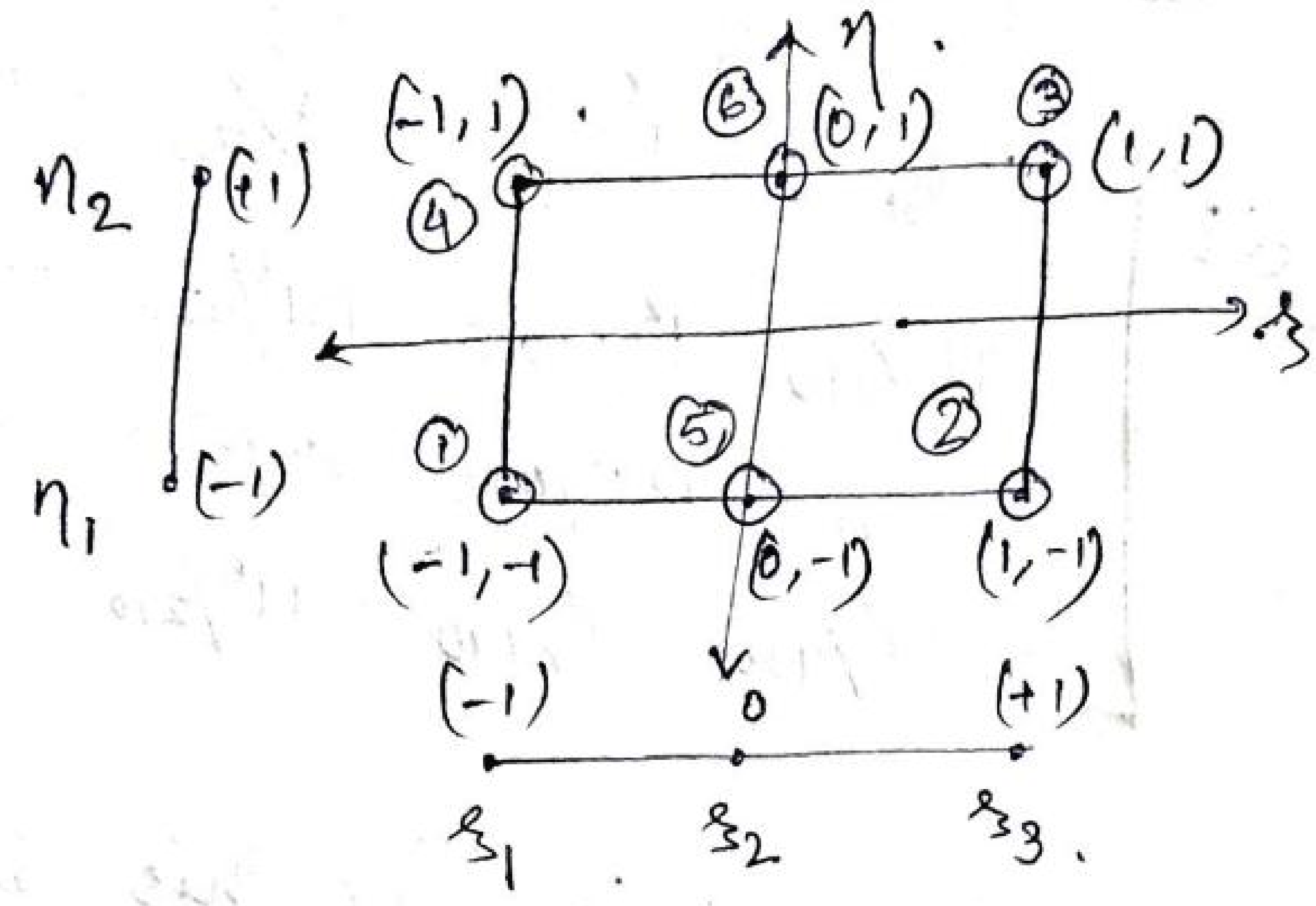
4. Consider a beam on an elastic foundation, as shown in the figure below. The foundation (bed of springs) has stiffness $k(x)$. The beam is subjected to the distributed load $p(x)$, has flexural rigidity $E(x)I(x)$, and length L. Note that the dimensions of $k(x)$ and $p(x)$ are force/length^2 and force/length, respectively.
   a. Formulate the strong and weak forms.
   b. Assuming k, E and I to be constants over the length of any element, derive the expression for the element stiffness matrix, including the effect of the foundation stiffness.



Budhadityade
21103033

Question 1 :-

(a) 6-noded Quad element :-



$\xi_i, \eta_i \rightarrow$ are corresponding 1-D shape functions at the concerned node.

denoting the product of $\xi_i \eta_i$ as ψ_i to be the shape functions for this element.

Now, $\xi_1 = \frac{1}{2}\xi(\xi-1)$; $\xi_2 = (1-\xi^2)$; $\xi_3 = \frac{1}{2}\xi(\xi+1)$.

$$\eta_1 = \frac{1}{2}(1-\eta) \; ; \; \eta_2 = \frac{1}{2}(1+\eta) .$$

$$\therefore \psi_1 = \xi_1 \eta_1 = \frac{1}{4}\xi(\xi-1)(1-\eta) .$$

$$\psi_2 = \xi_3 \eta_1 = \frac{1}{4}\xi(\xi+1)(1-\eta)$$

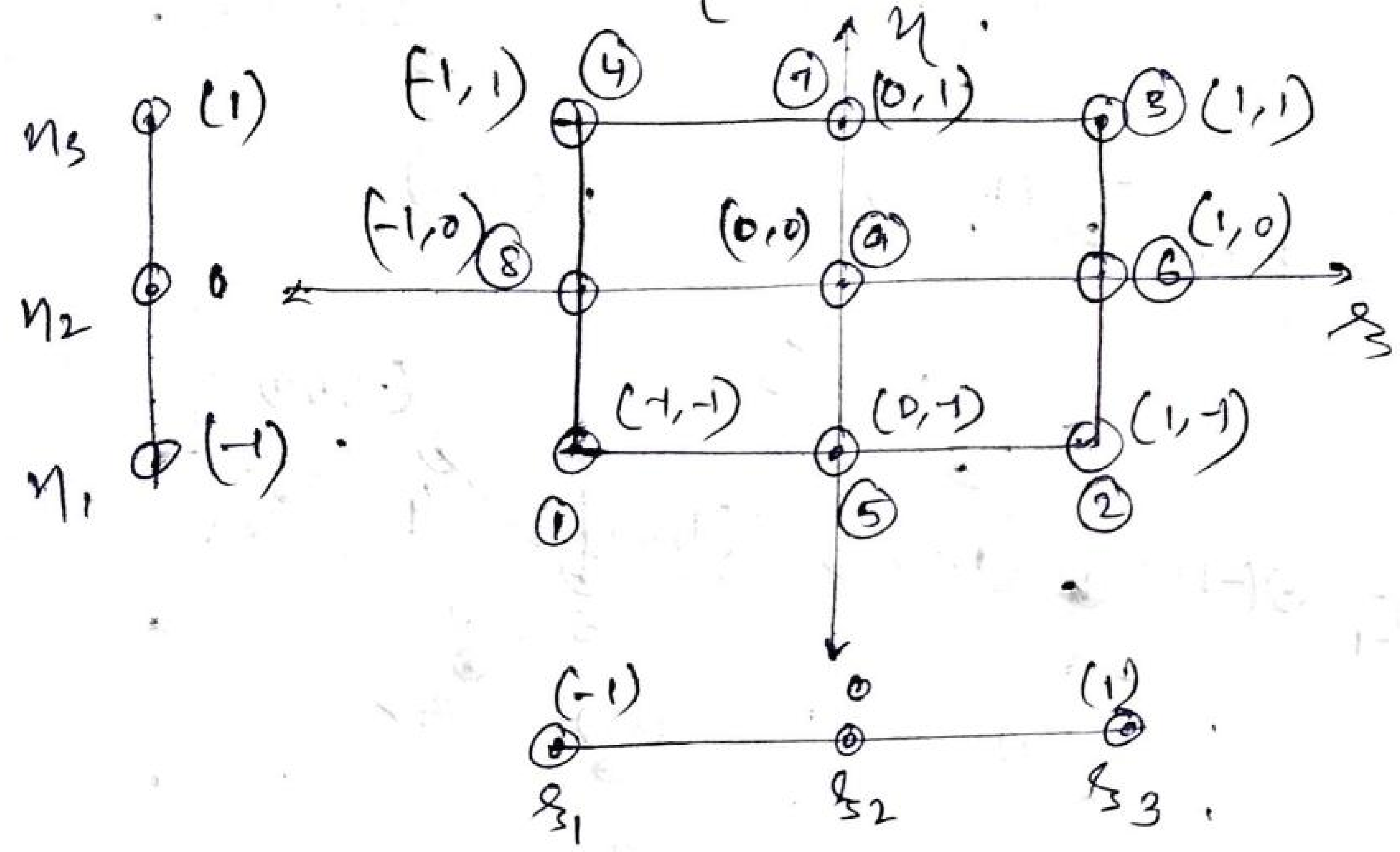
$$\psi_3 = \xi_3 \eta_2 = \frac{1}{4}\xi(\xi+1)(1+\eta) .$$

$$\psi_4 = \xi_1 \eta_2 = \frac{1}{4}\xi(\xi-1)(1+\eta) .$$

$$\psi_5 = \xi_2 \eta_1 = \frac{1}{2}(1-\xi^2)(1-\eta)$$

$$\psi_6 = \xi_2 \eta_2 = \frac{1}{2}(1-\xi^2)(1+\eta) .$$

(b) Similarly for the 9-noded Quad element we require parabolic shape functions along both ξ and η axes -



$$\xi_1 = \frac{1}{2}\xi(\xi-1) \; ; \quad \xi_2 = (1-\xi^2) \; ; \quad \xi_3 = \frac{1}{2}\xi(\xi+1).$$

$$\eta_1 = \frac{1}{2}\eta(\eta-1) \; ; \quad \eta_2 = (1-\eta^2) \; ; \quad \eta_3 = \frac{1}{2}\eta(\eta+1).$$

$$\therefore \psi_1 = \xi_1\eta_1 = \frac{1}{4}\xi(\xi-1)\eta(\eta-1).$$

$$\psi_2 = \xi_3\eta_1 = \frac{1}{4}\xi(\xi+1)\eta(\eta-1).$$

$$\psi_3 = \xi_3\eta_3 = \frac{1}{4}\xi(\xi+1)\eta(\eta+1).$$

$$\psi_4 = \xi_1\eta_3 = \frac{1}{4}\xi(\xi-1)\eta(\eta+1).$$

$$\psi_5 = \xi_2\eta_1 = \frac{1}{2}(1-\xi^2)\eta(\eta-1).$$

$$\psi_6 = \xi_3\eta_2 = \frac{1}{2}\xi(\xi+1)(1-\eta^2).$$

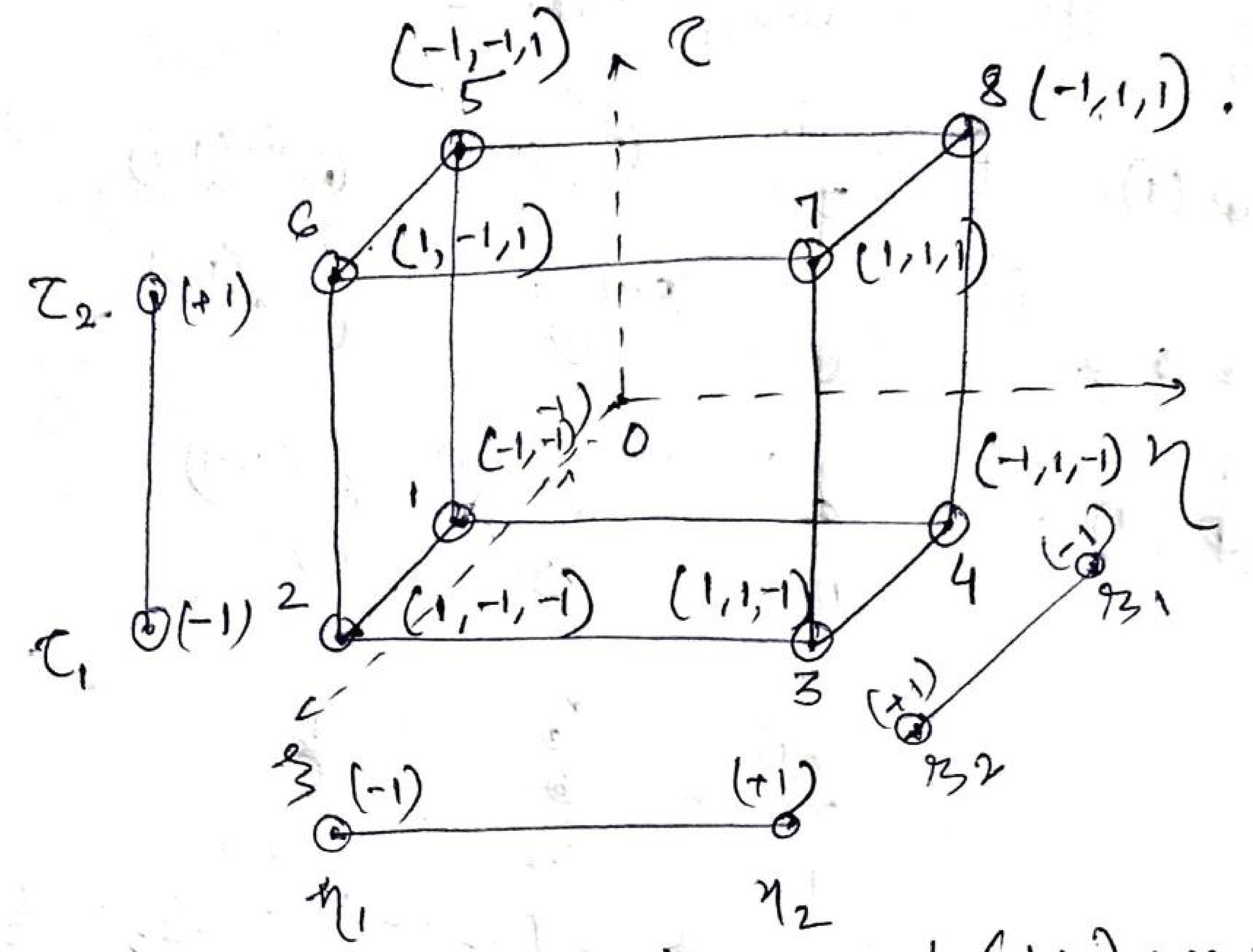
$$\psi_7 = \xi_2\eta_3 = \frac{1}{2}(1-\xi^2)\eta(\eta+1).$$

$$\psi_8 = \xi_1\eta_2 = \frac{1}{2}\xi(\xi-1)(1-\eta^2).$$

$$\psi_9 = \xi_2\eta_2 = (1-\xi^2)(1-\eta^2).$$

(c) For 8-noded Brick element, we use linear shape functions along 3-mutually ⊥'lar axes - ξ, η and τ.

P.T.O.



$\xi_1 = \frac{1}{2}(1-\xi)$; $\xi_2 = \frac{1}{2}(1+\xi)$; $\eta_1 = \frac{1}{2}(1-\eta)$; $\eta_2 = \frac{1}{2}(1+\eta)$

$\tau_1 = \frac{1}{2}(1-\tau)$; $\tau_2 = \frac{1}{2}(1+\tau)$.

$$\therefore \psi_1 = \xi_1 \eta_1 \tau_1 = \frac{1}{8}(1-\xi)(1-\eta)(1-\tau)$$

$$\psi_2 = \xi_2 \eta_1 \tau_1 = \frac{1}{8}(1+\xi)(1-\eta)(1-\tau).$$

$$\psi_3 = \xi_2 \eta_2 \tau_1 = \frac{1}{8}(1+\xi)(1+\eta)(1-\tau).$$

$$\psi_4 = \xi_1 \eta_2 \tau_1 = \frac{1}{8}(1-\xi)(1+\eta)(1-\tau).$$

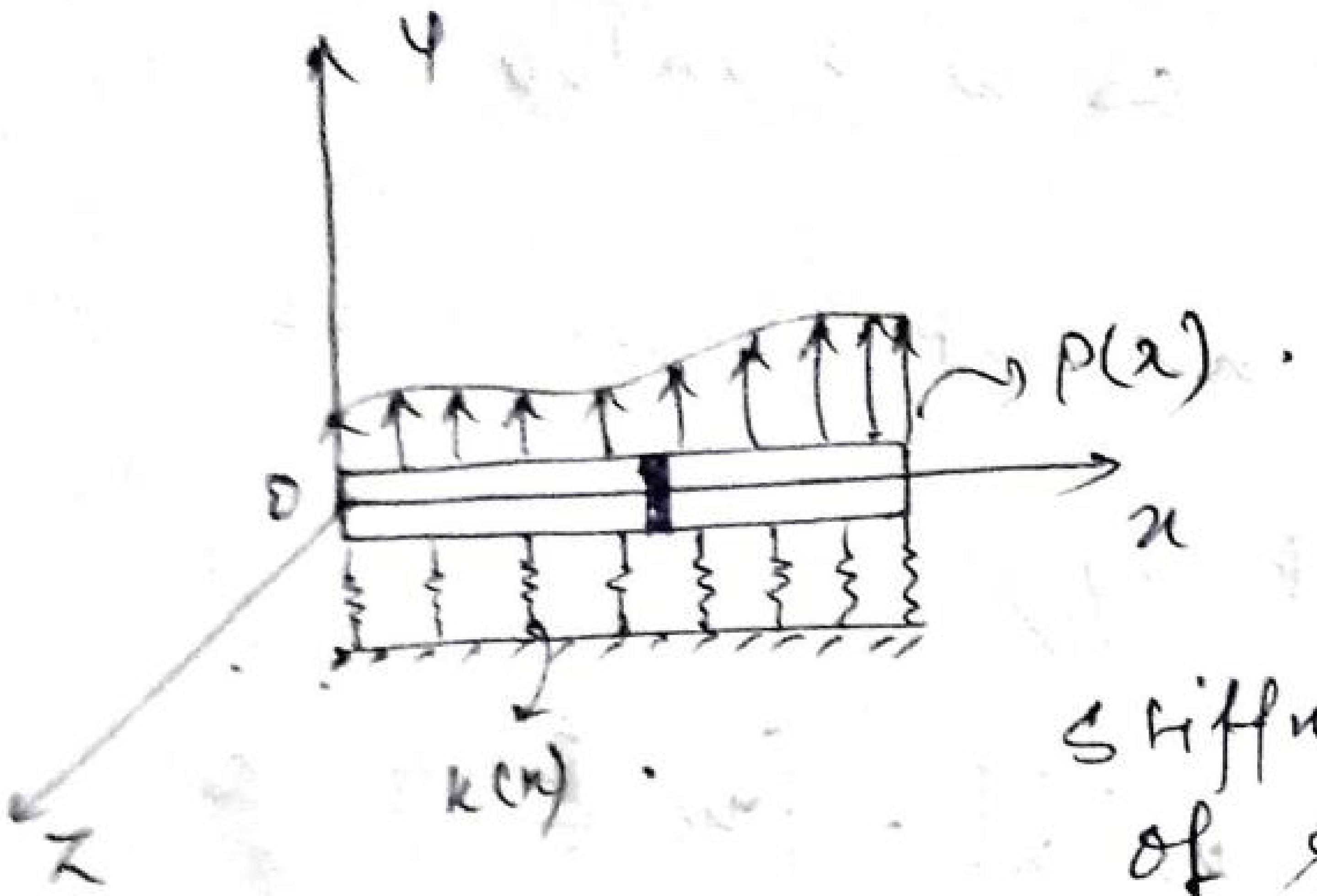
$$\psi_5 = \xi_1 \eta_1 \tau_2 = \frac{1}{8}(1-\xi)(1-\eta)(1+\tau).$$

$$\psi_6 = \xi_2 \eta_1 \tau_2 = \frac{1}{8}(1+\xi)(1-\eta)(1+\tau)$$

$$\psi_7 = \xi_2 \eta_2 \tau_2 = \frac{1}{8}(1+\xi)(1+\eta)(1+\tau)$$

$$\psi_8 = \xi_1 \eta_2 \tau_2 = \frac{1}{8}(1-\xi)(1+\eta)(1+\tau).$$

Question 4 :- Beam on Elastic foundation.

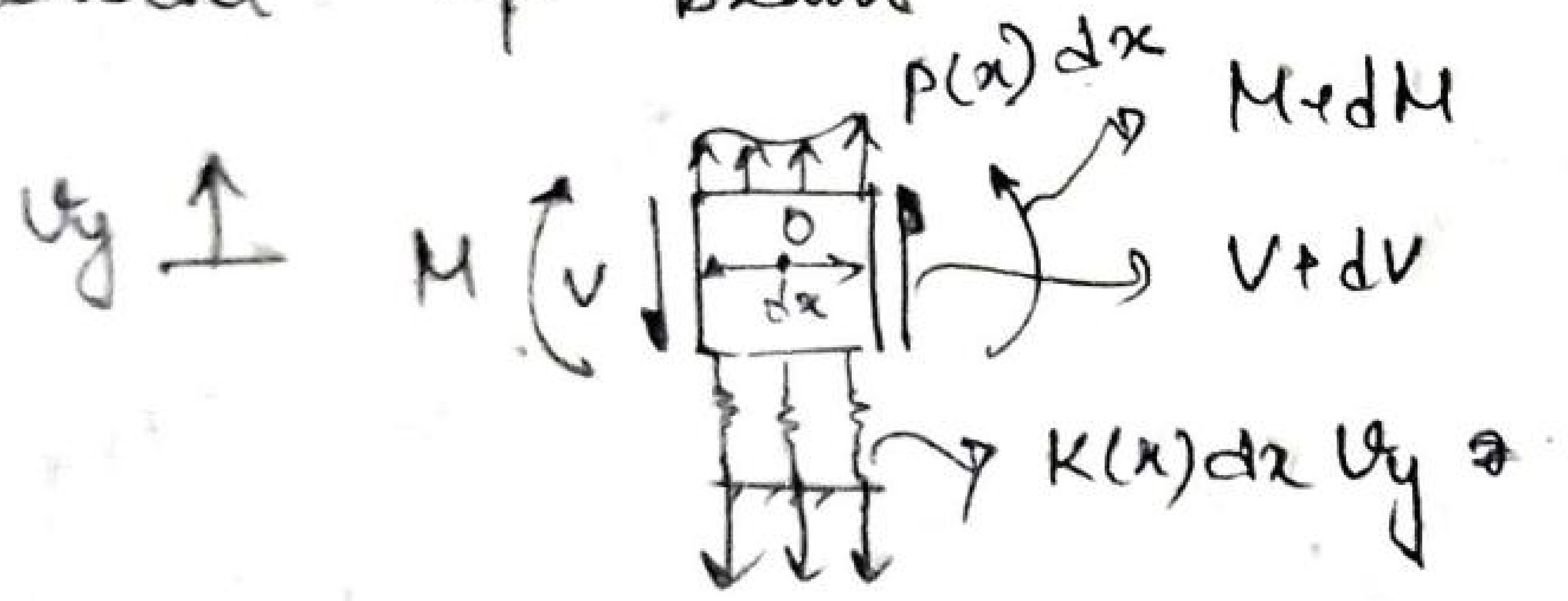


Stiffness per unit length of spring = $k(x)$ N/m/m

Distributed load = $p(x)$ N/m.

Flexural rigidity = $E(x)I(x)$.

Let us take a small differential element of beam -



$\Sigma F_y = 0$,

$$(\cancel{V} + dV) + P(x)dx - \cancel{V} - K(x)dx\,v_y = 0$$

$$\Rightarrow \quad dV = -P\,dx + K v_y\,dx.$$

$$\Rightarrow \quad \boxed{\frac{dV}{dx} = K v_y - P} \quad \text{-(i)}.$$

$\Sigma M_0 = 0$,

$$(\cancel{M} + dM) - \cancel{M} + (V + dV)\frac{dx}{2} + V\frac{dx}{2} = 0$$

$$\Rightarrow \quad dM + V\frac{dx}{2} + \cancel{dV\frac{dx}{2}}^{\,0} + V\frac{dx}{2} = 0$$

(neglect)

$$\Rightarrow \quad \boxed{\frac{dM}{dx} = -V} \quad \text{-(ii)}.$$

From strength of materials concept, we know –

from Euler-Bernoulli beam theory $\boxed{M = EI \frac{d^2 v_y}{dx^2}}$. $\left(\frac{M}{I} = \frac{E}{R} \text{ and } R = \frac{d^2 v_y}{dx^2}\right)$ curvature

$\therefore$ from (ii) $\frac{d}{dx}\left(EI \frac{d^2 v_y}{dx^2}\right) = -V$.

$$\Rightarrow \frac{d^2}{dx^2}\left(EI \frac{d^2 v_y}{dx^2}\right) = -\frac{dV}{dx}$$

from (i) $\rightarrow$

$$\frac{d^2}{dx^2}\left(EI \frac{d^2 v_y}{dx^2}\right) = -(Kv_y - P)$$

$$\Rightarrow \boxed{\frac{d^2}{dx^2}\left(EI \frac{d^2 v_y}{dx^2}\right) + Kv_y - P = 0}$$

which is the required strong form of the problem –

subjected to the following Boundary conditions –

$$\left.\begin{aligned} v_y &= \bar{v}_y \quad \text{at } \Gamma_{vy} \\ v_\theta &= \bar{v}_\theta \quad \text{at } \Gamma_{v\theta} \end{aligned}\right\} \text{Disp. BCs.}$$

and

Natural B.Cs. $\left\{\begin{aligned} -\frac{d}{dx}\left(n\, EI \frac{d^2 v_y}{dx^2}\right) &= \bar{F}_y \text{ at } \Gamma_{Fy} \\ \left(n\, EI \frac{d^2 v_y}{dx^2}\right) &= \bar{F}_\theta \text{ at } \Gamma_{F\theta} \end{aligned}\right.$

Next we derive the weak form –

$$\int_\Omega w \frac{d^2}{dx^2}\left(EI\frac{d^2 v_y}{dx^2}\right)dx + \int_\Omega w K(x) v_y\, dx - \int_\Omega w P\, dx = 0.$$

$$\underbrace{\int_\Omega w \frac{d^2}{dx^2}\left(EI\frac{d^2 v_y}{dx^2}\right)dx}_{\text{I}}$$

Integrating I by parts –

$$I = \int_\Omega w \left(EI v_y''\right)'' dx$$

$$= \left[w\left(EI v_y''\right)' n\right]_\Gamma - \int_\Omega w'\left(EI v_y''\right)' dx$$

Again by parts –

$$= \left[w\left(EI v_y''\right)' n\right]_\Gamma - \left\{\left[w'\left(EI v_y''\right)n\right]_\Gamma - \int_\Omega w''\left(EI v_y''\right)dx\right\}$$

Now breaking up the boundary terms into disp. as well as traction components.

$$= \int_\Omega w''\left(EI v_y''\right)dx + \left[\cancel{w\left(EI v_y''\right)' n}\right]_{\Gamma_{vy}}^{0} + \left[w\left(EI v_y''\right)' n\right]_{\Gamma_{Fy}}.$$

$$- \left[\cancel{w'\left(EI v_y''\right) n}\right]_{\Gamma_{v\theta}}^{0} - \left[w'\left(EI v_y''\right) n\right]_{\Gamma_{F\theta}}$$

w's are arbitrary weight functions which have the property of vanishing at the disp. BCs (both w and w').

$$\therefore I = \int_\Omega w''\left(EI v_y''\right)dx - \left[w \bar{F}_y\right]_{\Gamma_{Fy}} - \left[w' \bar{F}_\theta\right]_{\Gamma_{F\theta}}$$

Using the traction BCs previously stated.

$\therefore$ the W-form becomes –

$$\underbrace{\int_\Omega w''(EI\, v_y'')\,dx + \int_\Omega w\, k\, v_y\, dx}_{K\text{-term}} - \left[w F_y\right]_{\Gamma_{Fy}} - \left[w' F_\theta\right]_{\Gamma_{F\theta}} - \int_\Omega w p\, dx = 0.$$

(b) Now following the Galerkin's approach the w's are composed of the same shape func. as that of the v_y's –

$$\underset{=}{v_y^e} = \underset{\sim}{N^e}\, \underset{=}{d^e} \quad \text{and} \quad \underset{=}{w} = \underset{\sim}{N^e}\, \bar{\underset{=}{w}} \rightsquigarrow \text{arbitrary}.$$

Since w is a scalar, we can write it in transpose form also.

$$\underset{=}{w} = \bar{\underset{=}{w}}^T \underset{\sim}{N^{eT}}.$$

and denoting $\underset{\sim}{N^{e''}} = \underset{\sim}{B^e} \longrightarrow$ we get

$$\int_\Omega \bar{w}^T \left(\underset{\sim}{B^{eT}} EI\, \underset{\sim}{B^e}\, dx\right) \underset{=}{d^e} + \int_\Omega \bar{\underset{=}{w}}^T \underset{\sim}{N^{eT}} k\, \underset{\sim}{N^e}\, dx\, \underset{=}{d^e}$$

$$- \left[\bar{w}^T \underset{\sim}{N^{eT}} \bar{F}_y\right]_{\Gamma_{Fy}} - \left[\bar{w}^T (\underset{\sim}{N^{eT}})' F_\theta\right]_{\Gamma_{F\theta}} - \int \bar{\underset{=}{w}}^T \underset{\sim}{N^{eT}} p\, dx = 0.$$

since $\bar{w}^T$ is arbitrary, cancel it from the equation –

$$\left[\int_\Omega \underset{\sim}{B^{eT}} EI\, \underset{\sim}{B^e}\, dx + \int_\Omega \underset{\sim}{N^{eT}} k\, \underset{\sim}{N^e}\, dx\right] \underset{=}{d^e} - \left[\underset{\sim}{N^{eT}} \bar{F}_y\right]_{\Gamma_{Fy}}$$

$$- \left[(N^{eT})' F_\theta\right]_{\Gamma_{F\theta}} - \int_\Omega \underset{\sim}{N^{eT}} p\, dx = 0.$$

Now the stiffness matrix will be formed from the equation –

$$K^e = \int_\Omega \underbrace{\underset{\sim}{B^{eT}} EI\, \underset{\sim}{B^e}\, dx}_{①} + \int_\Omega \underbrace{\underset{\sim}{N^{eT}} k\, \underset{\sim}{N^e}\, dx}_{②}.$$

Now recalling the beam shape functions

$$N_1^e(x^e) = 1 - \frac{3x^{e^2}}{l^{e^2}} + 2\left(\frac{x^e}{l^e}\right)^3$$

$$N_2^e = x^e - \frac{2x^{e^2}}{l^e} + \frac{x^{e^3}}{l^{e^2}}.$$

$$N_3^e = 3\left(\frac{x^e}{l^e}\right)^2 - 2\left(\frac{x^e}{l^e}\right)^3$$

$$N_4^e = -\frac{x^{e^2}}{l^e} + \frac{x^{e^3}}{l^{e^2}}$$

and $$\underset{\sim}{B}^e = (N^e)'' = \left[\left(-\frac{6}{l^{e^2}} + \frac{12x^e}{l^{e^3}}\right), \left(-\frac{4}{l^e} + \frac{6x^e}{l^{e^2}}\right), \left(\frac{6}{l^{e^2}} - \frac{12x^e}{l^{e^3}}\right), \left(-\frac{2}{l^e} + \frac{6x^e}{l^{e^2}}\right)\right].$$

Now the 1st stiffness term –

$\int_0^{l^e} \underset{\sim}{B}^{e^T} EI \underset{\sim}{B}^e \, dx$ and given that EI is const over the length of the beam, performing symbolic matrix multiplication on MATLAB, we end up getting –

$$K_1 = \frac{EI}{l^{e^3}} \begin{bmatrix} 12 & 6l^e & -12 & 6l^e \\ 6l^e & 4l^{e^2} & -6l^e & 2l^{e^2} \\ -12 & -6l^e & 12 & -6l^e \\ 6l^e & 2l^{e^2} & -6l^e & 4l^{e^2} \end{bmatrix}_{4\times4}$$

which exactly same as the element stiffness matrix we derived in the direct stiffness approach.

For the 2nd term –

$$K_2 = \int_0^{l^e} \underset{\sim}{N}^{e^T} k \underset{\sim}{N}^e \, dx$$

and given k is also const. over the length we perform symbolic multiplication yet again and get the following matrix –

$$K_2 = K l^e \begin{bmatrix} 13/35 & 11 l^e/210 & 9 l^e/70 & -13 l^{e2}/420 \\ 11 l^e/210 & l^{e2}/105 & 13 l^e/420 & -l^{e2}/140 \\ 9/70 & 13 l^e/420 & 13/35 & -11 l^e/210 \\ -13 l^e/420 & -l^{e2}/140 & -11 l^e/210 & l^{e2}/105 \end{bmatrix}_{4\times4}.$$

∴ The overall stiffness matrix for this beam becomes –

$$K^T = K_1 + K_2.$$

$$\Rightarrow K^T = \left[\begin{array}{c|c|c|c} \frac{12EI}{l^{e3}} + \frac{13Kl^e}{35} & \frac{6EI}{l^{e2}} + \frac{11Kl^{e2}}{210} & -\frac{12EI}{l^{e3}} + \frac{9l^e K}{70} & \frac{6EI}{l^{e2}} * -\frac{13Kl^{e2}}{420} \\ \hline \frac{6EI}{l^{e2}} + \frac{11Kl^{e2}}{210} & \frac{4EI}{l^e} + \frac{Kl^{e3}}{105} & -\frac{6EI}{l^{e2}} + \frac{13Kl^{e2}}{420} & \frac{2EI}{l^e} * -\frac{Kl^{e3}}{140} \\ \hline -\frac{12EI}{l^{e3}} + \frac{9Kl^e}{70} & -\frac{6EI}{l^{e2}} + \frac{13Kl^{e2}}{420} & \frac{12EI}{l^{e3}} + \frac{13Kl^e}{35} & -\frac{6EI}{l^{e2}} - \frac{11Kl^{e2}}{210} \\ \hline \frac{6EI}{l^{e2}} * -\frac{13Kl^{e2}}{420} & \frac{2EI}{l^e} - \frac{Kl^{e3}}{140} & -\frac{6EI}{l^{e2}} - \frac{Kl^{e2} \times 11}{210} & \frac{4EI}{l^e} + \frac{l^{e3} K}{105} \end{array}\right]_{4\times4}$$