The Black-Scholes Option Pricing Equation

Derivation and Application

by

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1 Abstract

The Black-Scholes option pricing equation is used to price financial derivatives contracts called options. These contracts entitle you to the right, but not the obligation to buy or sell a security at a given time in the future. We intend to show the equation's mathematical derivation, from the assumption that the partial-differential equation that is the solution to the heat equation is well-posed. Additionally, we will demonstrate an application in pricing a European-style call option. The difference between European and American-style options being that American options can be exercised at any time before the expiration date, while the original equation was designed for European-style options which can only be exercised on the expiration date. The model is based on the idea that the price of an option is dependent on a number of factors, including the price of the underlying asset, the strike price of the option, the time until expiration, and the volatility of the underlying asset, often referred to as the "Greeks". The Black-Scholes model made it possible to calculate the value of an option in a matter of seconds, which was a major breakthrough in financial modeling.

2 Acknowledgements

Special thanks to my advisor, Dr. Aida Kadic-Galeb. Who gave me the opportunity to work under her on a limb. She provided some much needed motivation and accountability. Without her guidance I would not have been able to produce this work. Her wisdom and patience exemplary of an ideal senior project advisor and professor.

Contents

3 History

The development of the Black-Scholes model was a major achievement in financial economics, and it led to the establishment of the field of quantitative finance. Prior to the development of the model, there was no reliable method for pricing options, and traders relied on intuition and experience to determine the fair value of an option. The Black-Scholes model changed this, and it has since become the standard model used to price options in financial markets, as well as a baseline tool to assess the valuation of new financial instruments.

The Black-Scholes model is not without its limitations, however. It assumes that the underlying asset follows a geometric Brownian motion, which may not be true in all cases. Brownian motion typically assumes a normal probability distribution, and it is most likely the case that financial markets enjoy changing probability distributions; at any rate, they are not typically normally distributed on any given day. Additionally, the model assumes that the environment is risk-neutral, there are no transaction costs, and no taxes. Despite these limitations, the Black-Scholes model remains an important tool in financial modeling, and it has been widely used in both academic research and practical applications.

Robert Merton and Myron Scholes were awarded the Nobel Prize in Economics in 1997 for their work on option pricing theory; Fisher Black was instrumental in their work, but passed away in 1995 in his mid-fifties. The Black-Scholes model revolutionized the field of finance and has had a significant impact on the way options are traded and priced. It has also been used to develop other financial models relating to the valuation of corporate liabilities, guarantees, insurance contracts, and investment decisions related to the flexibility of capital expenditure. Merton and Scholes are also credited for taking William Sharpe's Capital Asset Pricing Model (CAPM) for which he won the prize in 1990, and adapting it from static economic theory to dynamic market settings.

4 Black-Scholes Terminal Value Problem

Our intention is to start with the terminal value problem of the Black-Scholes Equation, and by change of variables reduce it to the heat equation. We assume that the initial value problem of the heat equation on the real line is well-posed, to avoid deriving everything back to calculus. Once there, we will use the known solution to the heat equation to represent the solution and change our variables back.

The terminal value problem for the value V(S,t) of a European call option on a security with price S at time t is:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

With V(0,t) = 0, V(S,t) $\sim$ S as S $\to \infty$ and

V(S,T) = max(S - K, 0).

The boundary conditions are such that at security price 0, the option is worth $0. V(S,t) is asymptotic to S as it approaches infinity, it may increase a lot over the duration but is fixed at expiration. K here is the strike price of the call option. The value of an option cannot be less than zero, so if the strike price is greater than the security price then max(S - K , 0) = 0.

5 Solution to the Black-Scholes Equation

Here we begin our changing of variables in order to reduce the terminal value problem into the heat equation. Then, we can use the known solution to heat equation to represent the solution. At which point, we can change the variables back.

We will use σ as volatility, the typical choice for standard deviations. K is again, the strike price of the option. We are attempting to price it at a final time T rather than an initial time T_0. The security price is still S. And x and τ will replace (S,t) in our change of variables, so we aren't lost when we switch back.

Our first change of variables:

$$t = T - \frac{\tau}{(1/2)\sigma^2} \qquad \text{and} \qquad S = Ke^x$$

Gives us:

$$V(S,t) = Kv(x,\tau)$$

$$\tau = \frac{\sigma^2}{2}(T-t) \qquad \text{and} \qquad x = \log(\frac{S}{K})$$

Now we must find our first derivatives:

$$\frac{\partial V}{\partial t} = K(\frac{\partial v}{\partial \tau})(\frac{d\tau}{dt}) = K(\frac{\partial v}{\partial \tau})(\frac{-\sigma^2}{2})$$

and

$$\frac{\partial V}{\partial S} = K(\frac{\partial v}{\partial x})(\frac{dx}{dS}) = K(\frac{\partial v}{\partial x})(\frac{1}{S})$$

And the second derivative is:

$$\frac{\partial^2 V}{\partial S^2} = \frac{\partial}{\partial S}(\frac{\partial V}{\partial S})$$

$$= \frac{\partial}{\partial S}(K\frac{\partial v}{\partial S}\frac{1}{S})$$

$$= K\ (\frac{\partial v}{\partial x}) \cdot \frac{-1}{S^2} + K\frac{\partial}{\partial S}(\frac{\partial}{\partial x}) \cdot \frac{1}{S}$$

$$= K\ (\frac{\partial v}{\partial x}) \cdot \frac{-1}{S^2} + K\ \frac{\partial}{\partial x}\ (\frac{\partial v}{\partial x}) \cdot \frac{dx}{dS} \cdot \frac{1}{S}$$

$$= K\ \frac{\partial v}{\partial x} \cdot \frac{-1}{S^2} + K\ \frac{\partial^2 v}{\partial x^2} \cdot \frac{1}{S^2}$$

The terminal condition is:

$$V(S,T) = \max(S - K, 0) = \max(Ke^x - K, 0)$$

$$\text{but } V(S,T) = Kv(x,0) \text{ so } v(x, 0) = \max(e^x - 1, 0).$$

Now we must substitute our derivatives back into our terminal value problem.

Inputting our substitutions gives us:

$$\mathrm{K}\frac{\partial v}{\partial \tau}\cdot\frac{-\sigma^2}{2}+\frac{\sigma^2}{2}S^2(K\frac{\partial v}{\partial x}\cdot\frac{-1}{S^2}+K\frac{\partial^2 v}{\partial x^2}\cdot\frac{1}{S^2})+rS(K\frac{\partial v}{\partial x}\cdot\frac{1}{S})-rKv=0$$

And we can begin to simplify:

$$\Rightarrow K(\frac{\partial v}{\partial \tau}\cdot\frac{-\sigma^2}{2}+\frac{\sigma^2}{2}S^2(\frac{\partial v}{\partial x}\cdot\frac{-1}{S^2}+\frac{\partial^2 v}{\partial x^2}\cdot\frac{1}{S^2})+rS(\frac{\partial v}{\partial x}\cdot\frac{1}{S})-rv)=0$$
$$\Rightarrow \frac{\sigma^2}{2}S^2(\frac{\partial v}{\partial x}\cdot\frac{-1}{S^2}+\frac{\partial^2 v}{\partial x^2}\cdot\frac{1}{S^2})+rS(\frac{\partial v}{\partial x}\cdot\frac{1}{S})-rv)=\frac{\partial v}{\partial \tau}\cdot\frac{\sigma^2}{2}$$
$$\Rightarrow S^2(\frac{\partial v}{\partial x}\cdot\frac{-1}{S^2}+\frac{\partial^2 v}{\partial x^2}\cdot\frac{1}{S^2})+rS(\frac{\partial v}{\partial x}\cdot\frac{1}{S})-rv)=\frac{\partial v}{\partial \tau}$$

We need to take a breather to rename the remaining constant $\frac{r}{\frac{\sigma^2}{2}}$ as k, the measure between the risk-free rate and volatility.

$$\Rightarrow S^2(\frac{\partial v}{\partial x}\cdot\frac{-1}{S^2}+\frac{\partial^2 v}{\partial x^2}\cdot\frac{1}{S^2})+kS(\frac{\partial v}{\partial x}\cdot\frac{1}{S})-kv)=\frac{\partial v}{\partial \tau}$$
$$\Rightarrow (-\frac{\partial v}{\partial x}+\frac{\partial^2 v}{\partial x^2})+k(\frac{\partial v}{\partial x})-kv=\frac{\partial v}{\partial \tau}$$
$$\Rightarrow -\frac{\partial v}{\partial x}+\frac{\partial^2 v}{\partial x^2}+k(\frac{\partial v}{\partial x})-kv=\frac{\partial v}{\partial \tau}$$
$$\Rightarrow \frac{\partial v}{\partial \tau}=\frac{\partial^2 v}{\partial x^2}+(k-1)\frac{\partial v}{\partial x}-kv$$

We now simplify further by changing the dependent variable scale again:

$$\mathrm{v}=e^{\alpha x+\beta\tau}u(x,\tau)$$

By the product rule:

$$v_\tau=\beta e^{\alpha x+\beta\tau}u+e^{\alpha x+\beta\tau}u_\tau$$

and

$$v_x=\alpha e^{\alpha x+\beta\tau}u+e^{\alpha x+\beta\tau}u_x$$

and

$$v_{xx}=\alpha^2e^{\alpha x+\beta\tau}u+2\alpha e^{\alpha x+\beta\tau}u_x+e^{\alpha x+\beta\tau}u_{xx}$$

Now, we can plug our derivatives into our constant coefficient equation that we simplified and factor out our common $e^{\alpha x+\beta\tau}$ throughout:

$$\beta u + u_\tau = \alpha^2 u + 2\alpha u_x + u_{xx} + (k-1)(\alpha u + u_x) - ku.$$

Simplify by combining like terms:

$$\rightarrow u_\tau = u_{xx} + [2\alpha + (k-1)]u_x + [2\alpha + (k-1)\alpha - k - \beta]u.$$

Let $\alpha = \frac{k-1}{2}$, to make u_x zero, and let $\beta = \alpha^2 + (k-1)\alpha - k = -\frac{(k+1)^2}{4}$ so the u coefficient is zero as well. This lets us reduce neatly to:

$$u_\tau = u_{xx}$$

Now, we have a basis by which we can transform the initial condition:

$$\begin{aligned} u(x,0) &= e^{-(-\frac{(k-1)^2}{2})x-(-\frac{(k+1)^2}{4})\cdot 0}v(x,0) \\ &= e^{\frac{(k-1)}{2}x}max(e^x - 1, 0) \\ &= max(e^{\frac{(k+1)}{2}x} - e^{\frac{(k-1)}{2}x}, 0) \end{aligned}$$

6 Market Dynamics: Why the heat-equation?

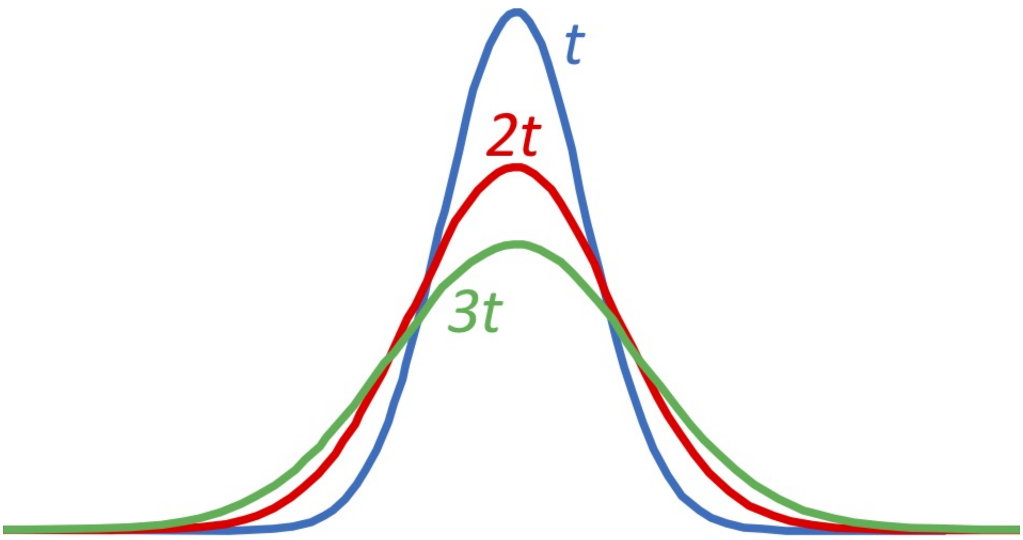
A financial market is composed of buyers and sellers. Buyers bid the price which they would like to pay, sellers ask the price they wish to receive for a given contract. This becomes quite a noisy function of market supply and demand for markets of even modest sizes. We can compare heat flux to buying demand in a financial market. Imagine we are given a 1-D metal rod with heat flux described by:

$$q(x,t) = -\kappa(\frac{\partial u}{\partial x})$$

A heat flux from left to right. $(\frac{ThermalEnergy}{Time})$

When heat fluxes in from left to right it passes to the adjacent section of the rod transferring and dispersing the heat. This is the manner in which a security price rises due to buying demand. When sellers become aware of a flux of buyers bidding and hitting the ask, they either refuse to sell or hold out for a higher price. 1-D number line is all we need, buyers flux in from the left and there is a transference of buying pressure and distribution of said pressure increasing the security price towards infinity.

Of course prices do not always go up. We are also often considering somewhere in the middle of our number line, left towards zero and right towards infinity. Each individual buyer and seller coming to the marketplace with their own unique ideas and motivations. It is easy to see why we may want to treat this as a random variable with a normal distribution. Why a normal distribution? Most prices are going to tend around the bid-ask spread in the middle of the distribution where most buyers and sellers are trading in order to get their orders filled. Fittingly, when we find the solution to the heat equation, what we'll get is a diffusion of a Gaussian kernel after performing the Fourier transform (side note: Fourier came up with the transform to specifically solve the heat equation).



Diffused Gaussian kernel

Now we need to use the Fourier transform to turn our partial-differential equation into an ordinary-differential equation:

$$u_\tau = u_{xx} \longrightarrow F(u_{xx}) = -\omega^2 \hat{u}(\omega, t)$$

$\rightarrow \frac{d}{dt}\hat{u} = -\omega^2 \hat{u}$

$\rightarrow \hat{u}(\omega, t) = e^{-\omega^2 t}\hat{u}(\omega, 0)$

Mapping back onto $u(x, \tau)$:

$u(x, \tau) = F^{-1}(\hat{u}(\omega, \tau))$

$\rightarrow F^{-1}(e^{-\omega^2 \tau}) * u(x, 0)$ *Our initial condition*

$\uparrow$ Gaussian: $\rightarrow F^{-1}(e^{-\omega^2 \tau}) = \frac{1}{2\sqrt{\pi\tau}} e^{\frac{-x^2}{4\tau}}$

Our initial condition convolved with the Gaussian

to give us the solution to our heat-equation.

$$u(x, \tau) = \frac{1}{2\sqrt{\pi\tau}} \int_{-\infty}^{\infty} u_0(s) e^{-\frac{(x-s)^2}{4\tau}} ds$$

6.1 Heat-Equation Solution Representation

We are now in a position to apply the solution to the heat-equation as it relates to the Black-Scholes formula:

$$u(x, \tau) = \frac{1}{2\sqrt{\pi\tau}} \int_{-\infty}^{\infty} u_0(s) e^{-\frac{(x-s)^2}{4\tau}} ds$$

First, we need to make another change of variables, how did you guess? We will let z $=\frac{(s-x)}{\sqrt{2\tau}}$, which means dz $=-(\frac{1}{\sqrt{2\tau}})dx.$

Giving us a new integral:

$$u(x,\tau) = \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} u_0(z\sqrt{2\tau} + x) e^{-\frac{z^2}{2}} dz$$

We are only interested in the domain of u_0 where $u_0 > 0$ for $z > -\frac{x}{\sqrt{2\tau}}$.

On this domain, $u_0 = e^{\frac{(k+1)}{2}(x+z+\sqrt{2\tau})} - e^{\frac{(k-1)}{2}(x+z+\sqrt{2\tau})}$, which gives us a pair of integrals we can call I_1 and I_2:

$$\frac{1}{2\sqrt{\pi}} \int_{-x/\sqrt{2}}^{\infty} e^{\frac{(k+1)}{2}(x+z+\sqrt{2\tau})} dz - \frac{1}{2\sqrt{\pi}} \int_{-x/\sqrt{2\tau}}^{\infty} e^{\frac{(k-1)}{2}(x+z+\sqrt{2\tau})} dz \qquad (1)$$

We can evaluate I_1 by completing the square in the exponent. So, completing the square gives us:

$\frac{k+1}{2}(x + z\sqrt{2\tau}) - \frac{z^2}{2}$
$= (\frac{-1}{2})(z^2 - \sqrt{2\tau}(k+1)z) + (\frac{k+1}{2})x$
$= (\frac{-1}{2})(z^2 - \sqrt{2\tau}(k+1)z + \tau\frac{(k+1)^2}{2}) + (\frac{k+1}{2})x + \tau\frac{(k+1)^2}{4}$
$= (\frac{-1}{2})(z - \sqrt{\tau/2}(k+1))^2 + \frac{(k+1)x}{2} + \frac{\tau(k+1)^2}{4}.$

So, our integral I_1 as we last remember it: $\frac{1}{2\sqrt{\pi\tau}} \int_{-x/\sqrt{2\tau}}^{\infty} e^{\frac{(k+1)}{2}(x+z+\sqrt{2\tau})} e^{\frac{-z^2}{2}} dz.$

Now becomes:

$$\frac{e^{\frac{(k+1)x}{2} + \frac{\tau(k+1)^2}{4}}}{\sqrt{2\pi}} \int_{-x/\sqrt{2\tau}}^{\infty} e^{\frac{-1}{2}(z-\sqrt{\tau/2}(k+1))^2} dz \qquad (2)$$

We can clean this up a little with another change of variables, letting $y = z - \sqrt{\tau/2}(k+1)$ and adjusting the limits of integration:

$$\frac{e^{\frac{(k+1)x}{2}+\frac{\tau(k+1)^2}{4}}}{\sqrt{2\pi}} \int_{-x/\sqrt{2\tau}-\sqrt{\tau/2}(k+1)}^{\infty} e^{(-\frac{y^2}{2})} dz \tag{3}$$

This integral can conveniently be represented as a cumulative distribution function of a normal random variable since it represents our diffusion of Gaussian functions. So we get to put a nice bow on it with Φ.

$$\Phi(d) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d} e^{(-\frac{y^2}{2})} dy \tag{4}$$

So now we have:

$$I_1 = e^{\frac{(k+1)x}{2}+\frac{\tau(k+1)^2}{4}} \Phi(d_1) \tag{5}$$

Where $d_1 = \frac{x}{\sqrt{2\tau}} + \sqrt{\frac{\tau}{2}}(k+1)$.

At last, we have the solution of the transformed heat equation initial value problem:

$$u(x,\tau) = e^{\frac{(k+1)x}{2}+\frac{\tau(k+1)^2}{4}} \Phi(d_1) - e^{\frac{(k-1)x}{2}+\frac{\tau(k-1)^2}{4}} \Phi(d_2) \tag{6}$$

Where $d_2 = \frac{x}{\sqrt{2\tau}} + \sqrt{\frac{\tau}{2}}(k-1)$.

7 The Black-Scholes Formula

We now get to unravel our prize from all the changes in variables. Recall, our small v from earlier in $v(x,\tau)$; $u(x,\tau)$ was our heat equation solution representation.

$$v(x,\tau) = e^{\frac{-(k-1)x}{2}+\frac{\tau(k+1)^2}{4}} u(x,\tau) \tag{7}$$

$$v(x,\tau) = e^{\frac{-(k-1)x}{2}+\frac{\tau(k+1)^2}{4}}(e^{\frac{(k+1)x}{2}+\frac{\tau(k+1)^2}{4}}\Phi(d_1) - e^{\frac{(k-1)x}{2}+\frac{\tau(k-1)^2}{4}}\Phi(d_2)) \quad (8)$$

$$\Rightarrow v(x,\tau) = e^{\frac{2kx+k^2\tau+\tau}{2}}\Phi(d1) - e^{\frac{2kx+k^2\tau-2k\tau+\tau-2x}{2}}\Phi(d2) \quad (9)$$

Set x = log(S/K), $\tau = (\frac{1}{2})\sigma^2(T-t)$ and $V(S,t) = Kv(x,\tau)$.

$$\begin{aligned} V(S,t) = S\Phi(&\frac{log(S/K)+(r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}) \\ -Ke^{-r(T-t)}\Phi(&\frac{log(S/K)+(r-\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}) \end{aligned} \quad (10)$$

The full closed-form solution of the Black-Scholes Equation.

Where S = Security Value, K = Strike Price, r = Risk-free Interest Rate, σ = Volatility, T = Time of Expiration, t = Current Time.

To make it more digestible, we can let

$$d_1 = (\frac{log(S/K)+(r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}})$$

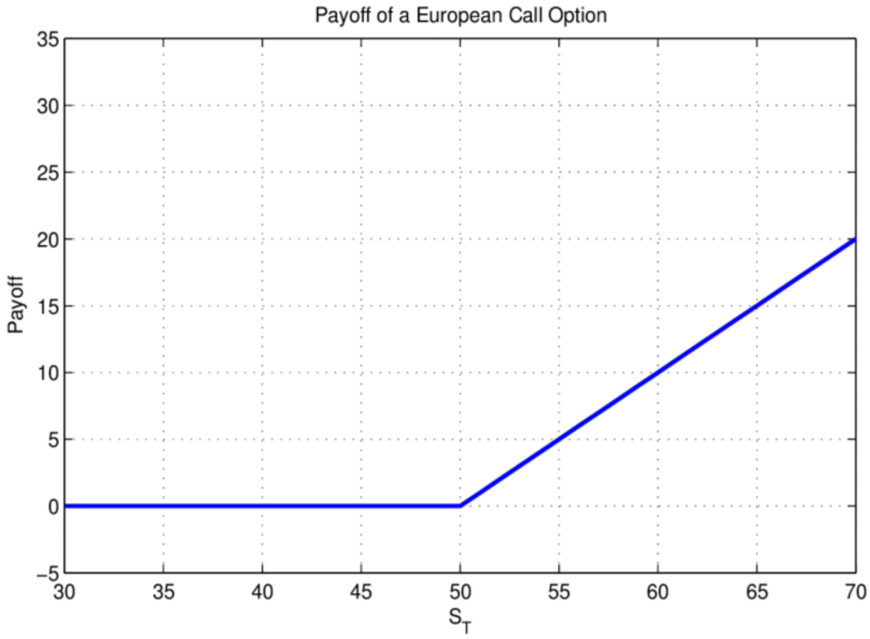
and

$$d_2 = (\frac{log(S/K)+(r-\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}})$$

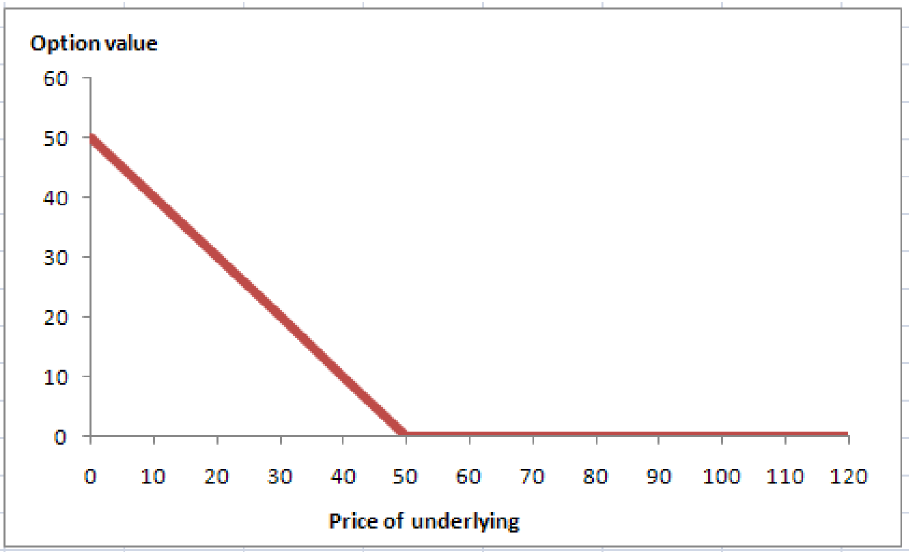
For this more elegant looking version that we derived out of that hornet's nest.

$$V_C(S,t) = S\cdot\Phi(d_1) - Ke^{-r(T-t)}\cdot\Phi(d_2) \quad (11)$$

8 European Option Contract Examples



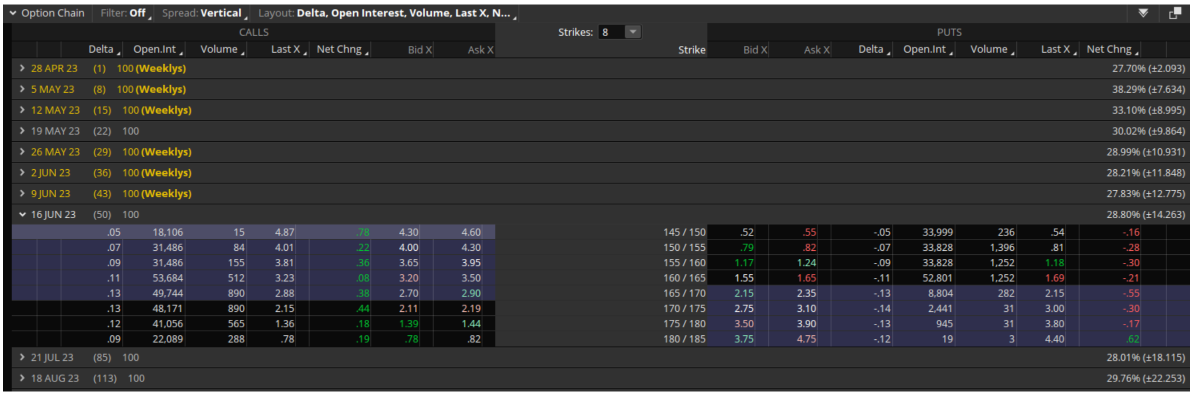
Here the image depicts the fact that a call is worthless unless it is at or above the security price S at time T. S_T on the horizontal axis. Of course, the appeal of the call option is there is no cap on the potential profit should the security dramatically increase in value.



This is a representation of the value of a put option relative to its underlying security. That is the right, but not the obligation to sell a security at a given price at some point in the future. Which can similarly be priced by the Black-Scholes model. This is a contract often used to hedge a long position against unforeseeable loss.

9 Pricing a European Call Option

We're going to price a call option for Apple, stock ticker (AAPL). While this isn't, a European company, it's much easier to access American market data in the United States. Also. the broad familiarity of the public with the Apple corporation makes for a good example.


Option Chain Filter: Off Spread: Vertical Layout: Delta, Open Interest, Volume, Last X, N... Strikes: 8

	CALLS								PUTS							
	Delta	Open.Int	Volume	Last X	Net Chng	Bid X	Ask X	Strike	Bid X	Ask X	Delta	Open.Int	Volume	Last X	Net Chng	
28 APR 23 (1) 100 (Weeklys)																27.70% (±2.093)
5 MAY 23 (8) 100 (Weeklys)																38.29% (±7.634)
12 MAY 23 (15) 100 (Weeklys)																33.10% (±8.995)
19 MAY 23 (22) 100																30.02% (±9.864)
26 MAY 23 (29) 100 (Weeklys)																28.99% (±10.931)
2 JUN 23 (36) 100 (Weeklys)																28.21% (±11.848)
9 JUN 23 (43) 100 (Weeklys)																27.83% (±12.775)
16 JUN 23 (50) 100																28.80% (±14.263)
	.05	18,106	15	4.87	.78	4.30	4.60	145 / 150	.52	.55	-.05	33,999	236	.54	-.16	
	.07	31,486	84	4.01	.22	4.00	4.30	150 / 155	.79	.82	-.07	33,828	1,396	.81	-.28	
	.09	31,486	155	3.81	.36	3.65	3.95	155 / 160	1.17	1.24	-.09	33,828	1,252	1.18	-.30	
	.11	53,684	512	3.23	.08	3.20	3.50	160 / 165	1.55	1.65	-.11	52,801	1,252	1.69	-.21	
	.13	49,744	890	2.88	.38	2.70	2.90	165 / 170	2.15	2.35	-.13	8,804	282	2.15	-.55	
	.13	48,171	890	2.15	.44	2.11	2.19	170 / 175	2.75	3.10	-.14	2,441	31	3.00	-.30	
	.12	41,056	565	1.36	.18	1.39	1.44	175 / 180	3.50	3.90	-.13	945	31	3.80	-.17	
	.09	22,089	288	.78	.19	.78	.82	180 / 185	3.75	4.75	-.12	19	3	4.40	.62	
21 JUL 23 (85) 100																28.01% (±18.115)
18 AUG 23 (113) 100																29.76% (±22.253)

This is the options chain for AAPL as of 4/27/2023. We will price the call option at the bottom of the June 16 2023 expiration pull-down with a strike price of $180.

```
import numpy as np
from scipy.stats import norm

N = norm.cdf

def BS_CALL(S, K, T, r, sigma):
    d1 = (np.log(S/K) + (r + sigma**2/2)*T) / (sigma*np.sqrt(T))
    d2 = d1 - sigma * np.sqrt(T)
    return S * N(d1) - K * np.exp(-r*T)* N(d2)

def BS_PUT(S, K, T, r, sigma):
    d1 = (np.log(S/K) + (r + sigma**2/2)*T) / (sigma*np.sqrt(T))
    d2 = d1 - sigma* np.sqrt(T)
    return K*np.exp(-r*T)*N(-d2) - S*N(-d1)
```

Here I've coded both a put and call option in Python with the help of two libraries, Numpy and Scipy for additional functionality. You can see we can use the continuous distribution function norm from the Scipy library.

Finally the outcome:

```
[1]: import numpy as np
     from scipy.stats import norm

     N = norm.cdf

     def BS_CALL(S, K, T, r, sigma):
         d1 = (np.log(S/K) + (r + sigma**2/2)*T) / (sigma*np.sqrt(T))
         d2 = d1 - sigma * np.sqrt(T)
         return S * N(d1) - K * np.exp(-r*T)* N(d2)
```

```
[10]: BS_CALL(166.84, 180, .136, .05, .3694)
```

```
[10]: 4.59473589195904
```

For a price of $4.59 per option. You may say wow! That option was drastically mispriced! It was selling for 90 cents! We can make a fortune! Alas, remember that we have priced a European style option that can only be exercised on the expiration date. As opposed to an American option, which can be exercised at any point prior to the expiration date.

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