1 Solutions

1.1 Problem 1

I throw two balls. What is the probability that both land in the bin on the far left? What is the probability that neither lands in the bin on the far right?

Answer The probability that both balls land in the bin on the far right is $1/5 \cdot 1/5 = 1/25$. The probability that neither lands in the bin on the far right is $19/20 \cdot 19/20 = 361/400$.

Let $c = 0$. From left to right, I number the bins $1, 2, \ldots, 17$. I throw 100 balls at the bins. For each ball, I figure out what bin it's in, and I add the bin's number to c. What is the expected value of c?

Answer We expect that 20 balls will land in the first bin and 5 balls will land in each of the other bins. Thus $c = 20 + 5 \cdot (2 + \ldots + 17) = 780$.

I throw 5 balls. What is the chance that one will land in the bin on the far left and no two balls will land in the same bin?

Answer To start, we calculate the probability that the *first ball thrown* lands in the bin on the far left and no two balls land in the same bin. There is a 1/5 chance that the first ball thrown lands in the bin on the far left. Then there is a 16/20 chance that the next ball will land in an OK bin, a 15/20 chance that the third ball will land in an OK bin and so on. This gives us a probability of $p = 1/5 \cdot 16/20 \cdot 15/20 \cdot 14/20 \cdot 13/20$. The value p is also the probability that the *second ball thrown* lands in the bin on the far left and no two balls land in the same bin. And the third. And so on. Thus the answer is $5p = 16/20 \cdot 15/20 \cdot 14/20 \cdot 13/20 \approx 27\%$.

1.2 Problem 2

We can easily solve a more general problem: What is

$$\sum_{i=0}^{n} s^i$$

for any s? Here's the cute trick. Suppose that $1 + s + \ldots + s^n = N$. Then we have

$$\begin{array}{ccccccc} 1+ & s & \ldots+ & s^n & & = & N \\ & s & \ldots+ & s^n+ & s^{n+1} & = & sN \end{array}$$

Now subtract those two lines from each other. This gives us $1 - s^{n+1} = (1 - s)N$, so

$$\sum_{i=0}^{n} s^i = \frac{1 - s^{n+1}}{1 - s}$$

(Note that this doesn't quite work when $s = 1$, but you know the answer for $s = 1$ anyway).

1.3 Problem 3

Take a look at the formula above. When n gets really really big, one of two things will happen to $c = s^{n+1}$. If $s < 1$ then c will get really really small (it will go to 0). If $s > 1$ then it will get even bigger (it will go to ∞). In our case $s = 1/2$ so the answer is

$$\frac{1-0}{1/2} = 2$$

1.4 Problem 4

Proof It is known that for $x, y \in \mathbb{R}$ the triangle inequality holds: $|x| + |y| \geq |x + y|$. We will prove by induction that for any n

$$|x_1| + \ldots + |x_n| \geq |x_1 + \ldots + x_n|$$

We already have the base case ($n = 2$), it is our given.
Suppose that the claim has been proven for $n = k$. Then we have

$$\begin{aligned} |x_1| + \ldots + |x_k| + |x_{k+1}| &\geq |x_1 + \ldots + x_k| + |x_{k+1}| \\ &\geq |x_1 + \ldots + x_k + x_{k+1}| \end{aligned}$$

Where the first step uses the induction hypothesis and the second step uses the given.

1.5 Problem 5

Proof Let F_n be the n^{th} Fibonacci number (the sequence is $1, 1, 2, 3, 5, 8, 13, \ldots$). We will prove by induction that $F_n \geq 2^{n/2}$ for $n \geq 8$. We have that $F_8 = 21 > 16$ so the claim is true for $n = 8$. We have that $F_9 = 34 > 22.627\ldots$ so the claim is true for $n = 9$.

Now suppose that the claim is true for $n = k$. We will prove that it is true for $n = k + 2$. We need to show that $F_{k+2} \geq 2F_k$. This is true because $F_{k+2} = F_{k+1} + F_k$ and $F_{k+1} \geq F_k$.

1.6 Problem 6

Proof Suppose, for the sake of contradiction, that $\sqrt{2}$ is rational. Then it can be expressed as a fraction p/q where p and q have no common factors. Thus $2q^2 = p^2$. This implies that p is even, because if p were odd then p^2 could not have a factor of 2. Thus $2q^2 = (2r)^2$ for some r. It follows that $q^2 = 2r^2$. By the same logic as before, q must be even. The fact that p and q are both even contradicts the fact that they have no common factors. Thus $\sqrt{2}$ is irrational. $\square$