Lecture 9: Linear Growth

Now we leave behind the "insecure" world of probability and begin looking at different types of growth, starting with **Linear Growth**. As we shall see over the next few weeks, linear (and other types of) growth can be quite useful when examining how certain things increase and decrease over time, such as invested money, population, costs, etc.

Line-ar growth, as the name suggests, has something to do with *lines*, and in particular, *straight* lines. (Usually when mathematicians talk about lines they mean straight lines.) It turns out that linear growth is very common. Examples from every day life include: renting a car, calculating your household bills such as telephone, light, gas bills; simple interest; taking a taxi ride; buying more than one item at a market; etc. You are (subconsciously at least) using linear growth to calculate the total costs.

In addition, linear growth is very often used to approximate other kinds of growth which would be much more difficult to calculate than the above everyday examples. For instance, although population (human, animal, bacteria, etc.) usually increases (or decreases) by means of *exponential* growth (lesson 12), over a short period of time, this type of growth is very close to linear growth. So oftentimes we will use linear growth to approximate and hence estimate population growth over a short period. However these estimates will be approximations, not exact answers.

Linear growth can be represented by a straight line, and so we shall be drawing lots of lines as part of the lesson. You will therefore want to use graph paper and definitely a ruler so that your lines are correctly proportioned and **straight**!

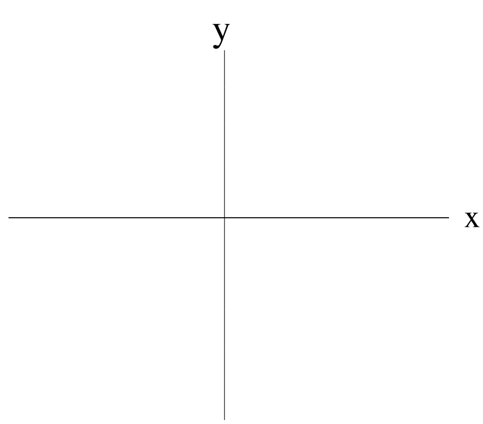
Lines - Definitions and Formulas

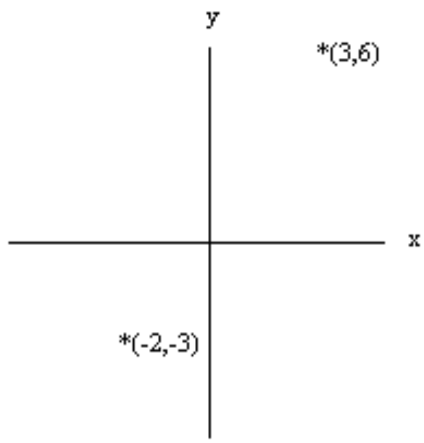
Axes: The horizontal axis is usually labeled 'x' and the vertical axis is labeled 'y'.

Coordinates of a point: - a point has two coordinates, an x-value and a y-value. The x-value comes first in a set of parentheses, then the y-value, separated by a comma. e.g. **(x, y)**

example: **(3, 6)** is the point that has an x-value of 3 and a y-value of 6.

Plotting the point: - to plot the point, start at the *origin*. The origin is the point where the two axes intersect, and has the coordinates, (0,0). Now, to plot a point, start at the origin and move horizontally the value of the **x-**coordinate. (If the x-value is positive then move right. If the x-value is negative then move left.) From this new position, move vertically the value of the **y**-coordinate. (Up if positive, down if negative.) Plot (3,6) and $(-2,-3)$ on the axes below along with me. Make sure you label the axes as we do on the board.





Linear Relationships:

When there is a relationship between two data variables that results in a *straight* line, we have what's called a linear relationship. That relationship is marked by a *constant growth rate*. That is, the growth is a constant *numerical* value. This constant is the *slope*.

Slope - there are several definitions of slope given in both words and as a formula. Also, when slope is used for specific everyday examples, we must put into words the *meaning* of slope. An example is given later. So, let's proceed with some definitions and formulas:

$$\text{Slope} = \frac{\text{Change in the y-coordinates}}{\text{Change in the x-coordinates}} \quad \text{or} \quad \text{Slope} = \frac{\text{rise}}{\text{run}} \quad \text{or 'How steep the line is.'}$$

In mathematical terms slope is usually designated by the letter "**m**" and the "change" is designated by the Greek letter D, or **Δ**, called Delta, indicating a *change*.

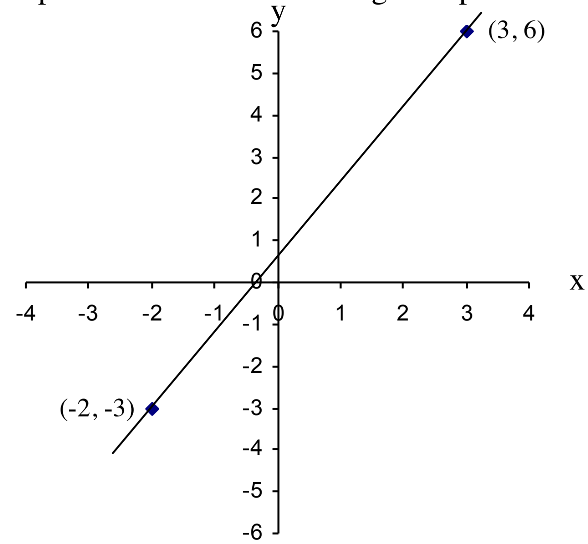
Given the two points, (X_1,Y_1) and (X_2,Y_2)

the formula for the slope of a line connecting these *two points* is:

$$\textbf{Slope} = \frac{\textit{Change in the y values}}{\textit{Change in the x values}}$$

$$m = \frac{\Delta Y}{\Delta X} = \frac{(Y_2) - (Y_1)}{(X_2) - (X_1)} \quad \text{or} \quad \frac{(Y_1) - (Y_2)}{(X_1) - (X_2)}$$

Example 1: Given the two points we used above: (3, 6) and (–2, –3) Find the slope of the line joining these two points. Notice that the line does not terminate at the points but continues through the points.



$$\textbf{Slope} = \frac{\textit{Change in the y values}}{\textit{Change in the x values}}$$

$$\mathbf{m} = \frac{\Delta Y}{\Delta X} = \frac{(Y_2) - (Y_1)}{(X_2) - (X_1)}$$

$$\mathbf{m} = \frac{(\quad) - (\quad)}{(\quad) - (\quad)}$$

Starting with (3, 6) as the first point and (-2, -3) the second and filling in the blanks we get:

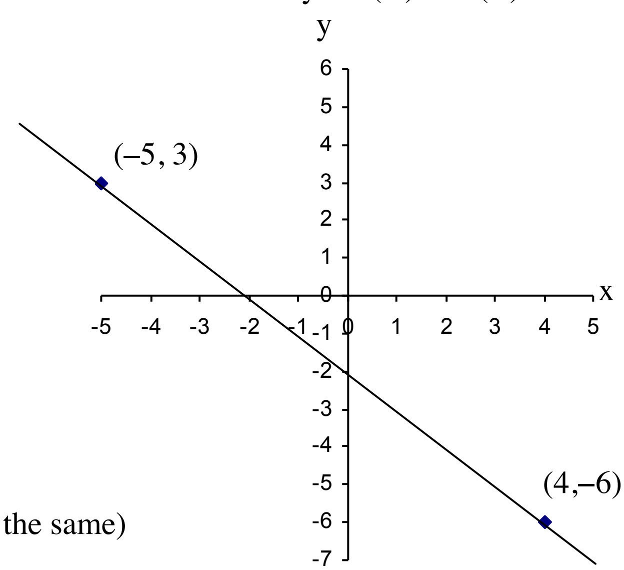
$$\mathbf{m} = \frac{(6) - (-3)}{(3) - (-2)} = \frac{9}{5}$$

Notice we have a fraction. The slope is a fraction. If we get a whole number, remember that a whole number can be written as a fraction.

Example 2: Given two points on a line: (–5, 3) and (4, –6) Find the slope of the line.

$$\textbf{Slope} = \frac{\textit{Change in the y values}}{\textit{Change in the x values}} \qquad \mathbf{m} = \frac{\Delta Y}{\Delta X} = \frac{(Y_2) - (Y_1)}{(X_2) - (X_1)}$$

Make sure you are consistent in your choice as to which is <u>the first point and which is the second point.</u> It's a good idea to leave the parentheses blank and just fill in the **Y's** and **X's**. *It does not matter* which point you start with as long as you are consistent. I have demonstrated both ways in (A) and (B) below.



$$\mathbf{m} = \frac{(\quad) - (\quad)}{(\quad) - (\quad)}$$

A) Starting with (–5, 3) as the first point and (4, –6) the second and filling in the blanks we get:

$$\mathbf{m} = \frac{(3) - (-6)}{(-5) - (4)} = \frac{9}{-9} = \mathbf{-1}$$

B) This time use (4,–6) first and (–5,3) second:

$$\mathbf{m} = \frac{(-6) - (3)}{(4) - (-5)} = \frac{-9}{\mathbf{9}} = \mathbf{-1} \quad \text{(exactly the same)}$$

Notice also that the slope is negative therefore '*falling*' when read from left to right.

<u>DEFINITIONS:</u>

<u>Equation of a Line:</u>

$$Y = mX + b$$

m is the slope of the line and **b** is the y-intercept

<u>Meaning of Slope:</u> The slope (m) means that there is an *additional* change of ***m*** units in the Y-value for each unit (one) change in the X-value,. This will be seen more readily in the next example.

<u>Slant of the Line:</u> The slope can be thought of as the slant of the line. Slopes can be positive or negative.

If your line rises when looking at it from left to right, then the slope is positive.
If your line falls when looking at it from left to right, then the slope is negative.
Your calculations should confirm this.

<u>y-intercept</u>: The y-intercept (b) is the y-value of the point where the line CROSSES the y-axis. Since it is ON the y-axis, the X-value of this point is zero. More on the y-intercept a bit later.

Analyzing a line from its equation:

Example 3: Given the following equation of a line: $\mathbf{Y = 3X - 2.}$

Reading from the equation, the spot where 'm' or the 'slope' belongs is a '3.'
The y-intercept, or the spot where the 'b' belongs, is a –2.
Thus

slope = 3 and **y-intercept is –2**

The slope means that there is an additional change of 3 units in the y-value for each one unit change in the x-value.

The y-intercept means that the line will cross the y-axis at – 2**.** The coordinates of this point are (0,-2).

The line is drawn in the graph below but first we need to see how to draw the line.

In order to do so, we must find two points ON the line. That is, we need to 'draw' a straight line between two points that are **generated** from this particular equation. Before doing so, we generate the points and put them in a <u>table</u> in order to be better organized.

TABLES

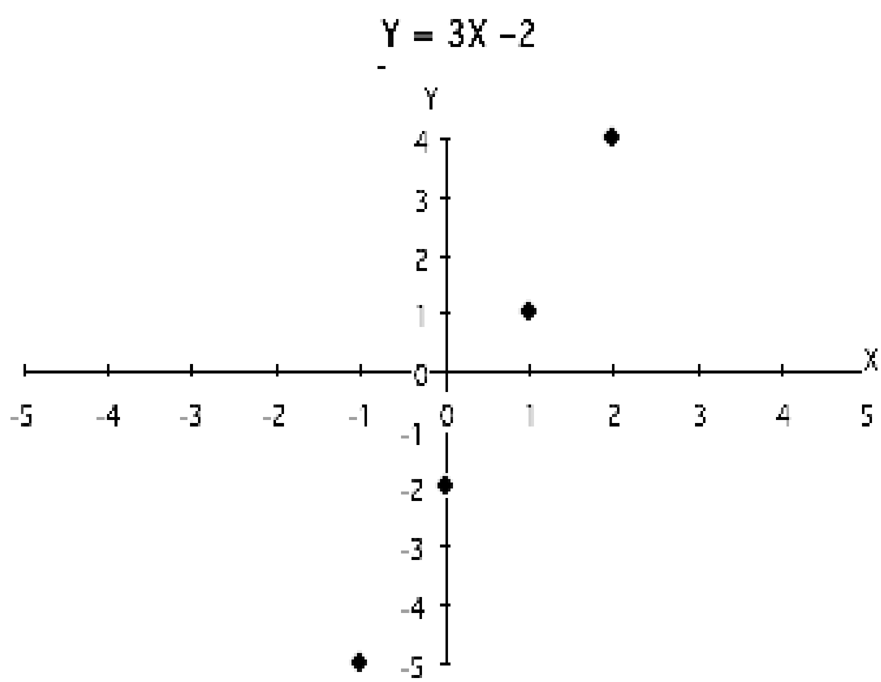
There are a few different ways to graph equations, but the easiest is to make a table of values. Each **pair** of values IS the coordinates of a point on the line. Replace the X with a specific value and calculate Y from your equation. This is the most difficult part to understand. What X value do I choose???? Remembering that a straight line has an infinite number of points on it, pick *any convenient* X-value.

X	**Y**
–1	–5
0	–2
1	1
2	4

The equation is **Y = 3X – 2.**

Here, I let X = 0 and thus Y = –2 or when X = 2 we get Y = 3(2) – 2 = 6 – 2 = 4

Continuing, we get the other points. Now all we need is to plot these points. In actuality we only need two points to draw a straight line. Here we have four. However, all the points should land ON the line. Let's plot them from the table we generated and then draw the line through them.



Example 4: Given the equation of a line below, identify the slope and y-intercept. Give the meaning of each.

$$\mathbf{Y = \frac{3}{2}X + 4}$$

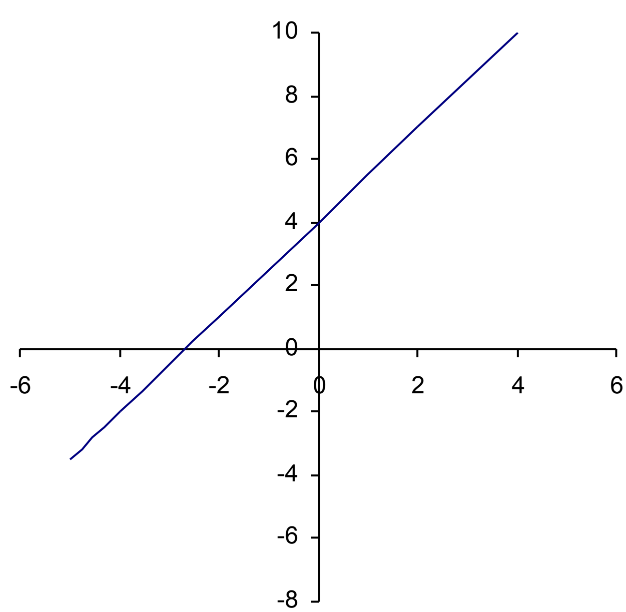
Slope = $\frac{3}{2}$ This means that there are an additional 3 units in the y-value for each 2 units in the x-value. Or if you divide 3 by 2, we see that there are an additional 1.5 units in the y-value for each 1 unit in the x-value.

Y-intercept is 4 This means the line crosses the y-axis at the point (0, 4)

Let's graph this line. Since we are working with fractions, it's easier to use multiples of the denominator. For example, in our case the denominator of the slope is a '2', so it is easier to pick x-values like 2, 4, 6. (If the denominator was a 3, then choose numbers like 3, 6, 9 for x-values.)

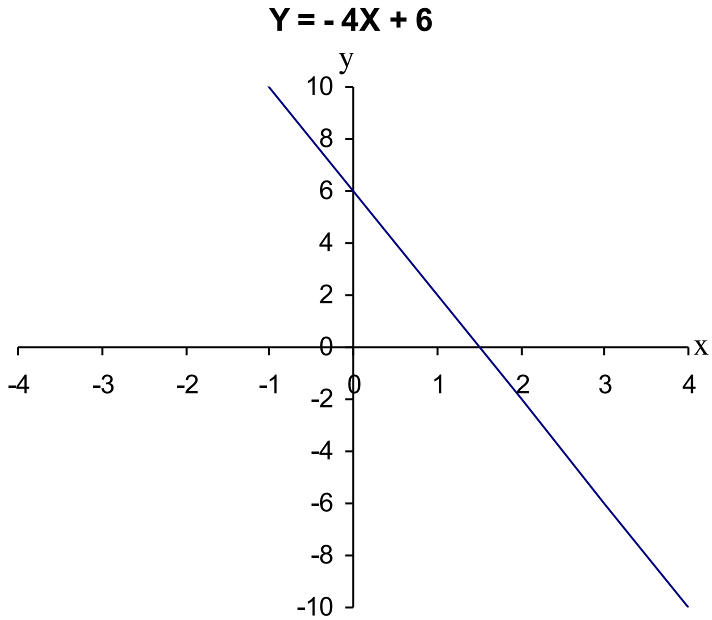
X	**Y**
–4	–2
–2	1
0	4
2	7
4	10

$$\mathbf{Y = \frac{3}{2}X + 4}$$



Example 5: Given the slope of a line = –4 and the y intercept is 6. Write the equation. Then graph the line.

Since the General Equation is Y = mX + b, simply REPLACE the ‘m’ with the given slope and the ‘b’ with the given y-intercept. We get **Y= –4X + 6**



Notice two things:
1. The y-intercept has coordinates (0, 6). Remember, the y-intercept is always when the x-value equals zero.

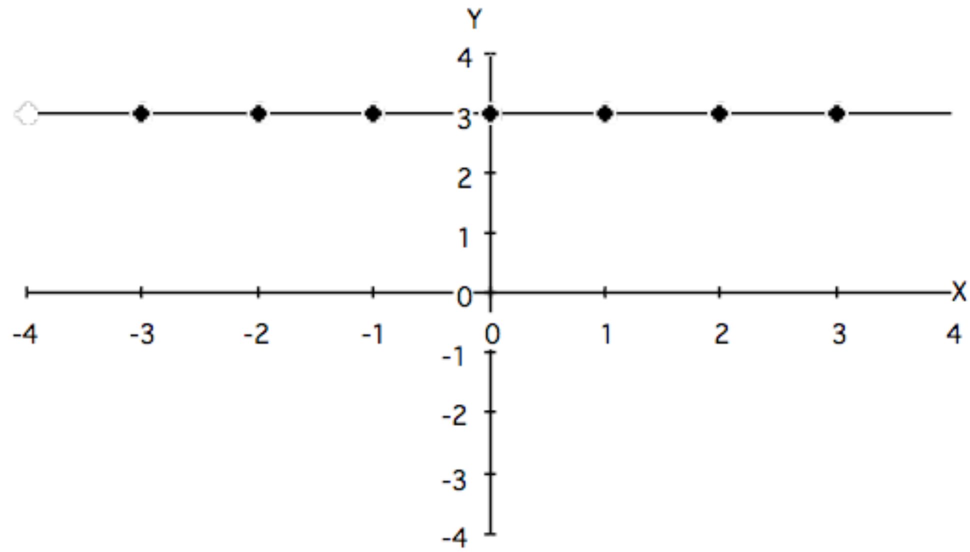
2. The scales on each axis are slightly different. The x-axis scale is larger than that on the y axis. This is acceptable. The scales do NOT need to be the same. However, if they are not, the slope of the line sometimes does not look as steep or as shallow as it would be if the scales were equal.

More Notes on Lines

Horizontal Lines: Horizontal lines have a **slope = 0.**

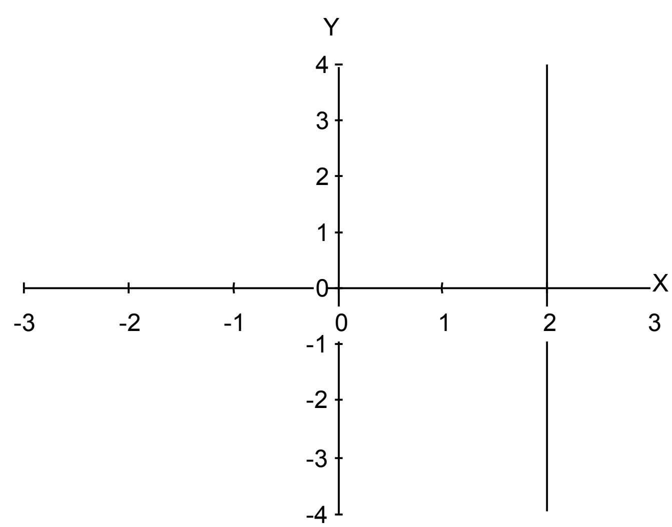
Why? Every point on a horizontal line has the same y-coordinate. Therefore, the numerator in the slope equation is always zero. Look at the graph below:

Graph of Horizontal Line Y = 3



Vertical Lines: Vertical lines have **NO SLOPE**. Why? Every point on a vertical line has the same x-coordinate. Therefore, the denominator in the slope equation is always zero and we cannot divide by zero!

Graph of Vertical Line X = 2



APPLICATIONS OF LINES

Example 6: In a small company, the cost to manufacture 'tee shirts' is $4.00 per shirt. The costs for the initial machinery, rent, insurance, etc. stays the same at a fixed number of $7.00. This can be written as an equation letting C represent the Cost. In our case, the 'x' will stand for the number of 't-shirts' we wish to manufacture. Notice that the 'C' takes the place of our 'y' in the general equation.

$C = 4x + 7$

Thus to manufacture 5 shirts it would cost \$27.00 or $C = 4(5) + 7$.

Meaning of Slope:

We know the slope = 4 since it the number that precedes the ‘x.’ We know that in general the slope means that there is an additional change in y for each unit change in x. Thus, in this *particular* case, the slope of ‘4’ indicates that:

There is an additional change of \$4.00 in the Cost (y) for each one t-shirt (x) manufactured.

More simply,

It costs an additional \$4.00 for each t-shirt manufactured.

Meaning of Y–intercept:

Now let's look at the meaning of the y-intercept as it applies to an application. We know that the y-intercept is the y-value of where our line crosses the y-axis. We learned that the x-portion of the coordinates of the y-intercept is zero. In the problem above, the y-intercept is 7.
The coordinates are (0,7).

So what does that mean? In our case, since the x-value applies to t-shirts, when x = 0, then that's <u>when we do not manufacture any t-shirts at all. Thus our</u>

<u>y-intercept means</u>: It costs $7.00 even though we do not manufacture any t-shirts.

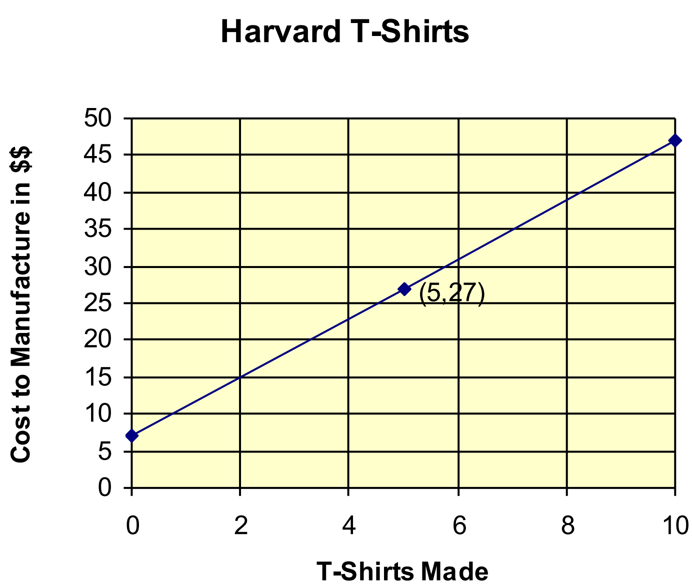
This, of course, is the initial or 'fixed' cost. In any equation, we know that the y-intercept is when x = 0.

Let's graph this equation. Make sure you use the correct labels. In this case, t-shirts and dollars.

X t-shirts	Y Cost	Coordinates
0	7	(0, 7)
5	27	(5,27)
10	47	(10,47)

Notice: We have larger numbers since we are dealing with an application so we may need to change our scales on the axes. Since we are dealing with an application where there are <u>no negative values, we really only need to use the **first quadrant**</u> of our coordinate axes as shown in the following graph.

***NB: We refer to x as the INDEPENDENT VARIABLE. Usually, the y-value DEPENDS on the x-value. Thus, y is the DEPENDENT VARIABLE.**



ALGEBRA EXTRAS

<u>How to find the point of intersection of two lines.</u> If two lines intersect, then at that point, the lines are equal.

Example 7: Find the point of intersection of the two lines

$$\mathbf{Y = 3X - 2} \quad \text{and} \quad \mathbf{Y = -5X + 4}$$

Set **<u>equal</u>** to each other – i.e. set the '**X**' part of the equations equal to each other.

$3X - 2 = -5X + 4$ get all X's on one side and all numbers on the other side

$3X + 5X = 4 + 2$

$8X = 6$ divide both sides by 8

$X = \frac{6}{8} = .75$ when X = .75 find Y by replacing the X in either equation with .75

thus $Y = 3X - 2$

$Y = 3(.75) - 2$

$Y = .25$ so point of intersection is (.75, .25) or $\left(\frac{3}{4}, \frac{1}{4}\right)$

Graph these two equations to visually 'see' the point of intersection. (Left for an exercise!)

<u>How to find the Y-Intercept from two points given:</u> Here again, we use algebra:

Example 8: Find the equation of a line through (8,2) and (-8,-6)

Step 1. Find the Slope: e.g. (8,2) and (-8, -6) Slope = $\frac{(2)-(-6)}{(8)-(-8)} = \frac{8}{16} = \frac{1}{2}$

Step 2: Fill in the general equation with the slope. Leave the y-intercept as the letter 'b'

$$\mathbf{Y = \frac{1}{2}x + b}$$

Step 3: Replace the X and Y with values of one of the Given Points and solve for 'b'. I will use (8,2)

$$\mathbf{2 = \frac{1}{2}(8) + b}$$

$2 = 4 + b$ so solving, we get b = -2 Now rewrite the entire equation.

$$\mathbf{Y = \frac{1}{2}x - 2}$$