## Example 3

### Mean Value Theorem Logic

$f(x) = x^2 + x \, , \, [-1, 1]$

Based on Theorem 3.2.1, $x^2 + x$ is continuous and differentiable everywhere.

There exist a c on interval $[-1, 1]$ where $f'(c) = \dfrac{f(b) - f(a)}{b - a}$

Slope of Secant Line

### Get Secant Slope

$a = -1 \, , \, b = 1 \, , \, f(\overset{a}{-1}) = (-1)^2 + (-1) = 1 - 1$

$f(\overset{a}{-1}) = 0$

$f(a) = 0$

$f(\overset{b}{1}) = (1)^2 + 1 = 1 + 1$

$f(\overset{b}{1}) = 2$

$f(b) = 2$

$$f'(c) = \frac{f(b) - f(a)}{b - a} = \frac{2 - 0}{1 - (-1)} = \frac{2}{1 + 1} = \frac{2}{2} = 1$$

$f'(c) = 1$

### Get Tangent Slope

$$f'(x) = \frac{d}{dx}[x^2 + x] = \frac{d}{dx}[x^2] + \frac{d}{dx}[x] = 2x^{2-1} + 1x^{1-1} = 2x^1 + 1x^0 = 2x + 1 \cdot 1$$

Example 3 cont

$f'(x) = 2x + 1$

$\shortparallel$

$f'(c) = 2c + 1$, where $x = c$

$\shortparallel$

$c$ is within $[-1, 1]$ $\quad c = \frac{(-1+1)}{2} = \frac{0}{2} = 0$

$c = 0$

$\shortparallel$

$f'(0) = 2(0) + 1$

$\shortparallel$

$0 + 1$

$\shortparallel$

$f'(0) = 1$, where $c = 0$

$\boxed{f'(0) = 1}$, slope of tangent

Example 5

Suppose that $f(0) = -2$ and $f'(x) \leq 4$ for all values of $x$. How large can $f(1)$ be?

$f'(x) \leq 4$ for all values of $x$

$f$ is differentiable and continuous everywhere

Mean Value Theorem: there exists a $c$ in interval $(0, 1)$, where $f'(c) = \dfrac{f(1) - f(0)}{1 - 0}$

$$= \frac{f(1) - (-2)}{1}$$

$$= f'(c) = \frac{f(1) + 2}{1}$$

$$= \boxed{f'(c) = f(1) + 2}$$

$c = x$

$f'(x) \leq 4$, then $f'(c) \leq 4$

or

$f(1) + 2 \leq 4$

$$\begin{array}{r} f(1) + 2 \leq 4 \\ -2 \quad -2 \\ \hline \end{array}$$

$f(1) \leq 2$ ← This is how large $f(1)$ can be.

Example 6

Suppose that $f(-1) = -1$ and $f'(x) \leq 3$ for all values of $x$.
How large can $f(2)$ be?

$f'(x) \leq 3$ for all values of $x$

$f$ is continuous and differentiable everywhere

Mean Value Theorem: $c$ exists in interval $(-1, 2)$
where $f'(c) = \dfrac{f(2) - f(-1)}{2-(-1)}$

$$= \frac{f(2) - f(-1)}{2-(-1)}$$

$$= \frac{f(2) - (-1)}{2+1}$$

$$f'(c) = \frac{f(2)+1}{3}$$

$$3 \cdot f'(c) = \frac{f(2)+1}{3} \cdot 3$$

$$\boxed{3f'(c) = f(2) + 1}$$

$f'(x) \leq 3$, where $f'(x) = 3$

$f'(c) = f'(x)$

$3f'(c) \leq 9$, where $f'(c) = 3$

$3f'(c) = f(2) + 1$

$$\begin{array}{r} f(2) + 1 \leq 9 \\ -1 \quad -1 \\ \hline \end{array}$$

$\boxed{f(2) \leq 8}$ ← This is how large $f(2)$ can be.

SI #3

$f(x) = \sin(x) + x$

$\sin(x) + x$ is continuous and differentiable everywhere

$f'(x) = \frac{d}{dx}[\sin(x) + x]$

$\parallel$

$f'(x) = \frac{d}{dx}[\sin(x)] + \frac{d}{dx}[x]$

$\parallel$

$\cos(x) + 1x^{1-1}$

$\parallel$

$\cos(x) + 1 \cdot x^0$

$\parallel$

$\cos(x) + 1 \cdot 1$

$\parallel$

$\boxed{f'(x) = \cos(x) + 1}$ ← Slope of Tangent

$f'(c) = \cos(c) + 1$

c is between $[0, \pi]$

$$\boxed{c = \frac{(0 + \pi)}{2} \approx 1.5708}$$