Calculus I Solutions - 5.3

1. Answer: $\int_0^2 1\,dx = 2$

   Detailed Solution:
   The definite integral represents the area between the horizontal line y = 1 and the x-axis, over the interval [0, 2], which is the area of a rectangle of width 2 and height 1.

   Therefore, $\int_0^2 1\,dx = (2)(1) = 2$.

2. $\int_0^4 2\,dx = 8$

3. Answer: $\int_{-4}^{-2} 1\,dx = 2$

   Detailed Solution:
   The definite integral represents the area between the horizontal line y = 1 and the x-axis, over the interval [-4, -2], which is the area of a rectangle of width 2 and height 1.

   Therefore, $\int_{-4}^{-2} 1\,dx = (2)(1) = 2$.

4. $\int_{-3}^{-1} 2\,dx = 4$

5. Answer: $\int_0^2 x\,dx = 2$

   Detailed Solution:
   The definite integral represents the area between the line y = x and the x-axis, over the interval [0, 2], which is the area of a triangle of width 2 and height 2.

   Therefore: $\int_0^2 x\,dx = \frac{1}{2}(2)(2) = 2$

6. $\int_0^3 2x\,dx = 9$

7. Answer: $\int_{-2}^{0} x\,dx = -2$

   Detailed Solution:

   The definite integral represents the negative area between the line y = x and the x-axis, over the interval [-2, 0], which is the negative area of a triangle of width 2 and height 2.

   Therefore, $\int_{-2}^{0} x\,dx = -\frac{1}{2}(2)(2) = -2$.

8. $\int_{-3}^{0} 2x\,dx = -9$

9. Answer: $\int_{2}^{4} x\,dx = 6$

   Detailed Solution:

   The definite integral represents the area between the line y = x and the x-axis, over the interval [2, 4], which is the area of a trapezoid of base lengths 2 and 4 with a height of 2.

   Using the formula ½(base1 + base2)height, we have:

   $\int_{2}^{4} x\,dx = \frac{1}{2}(2+4)(2) = 6$

10. $\int_{2}^{5} 2x\,dx = 21$

11. Answer: $\int_0^3 \sqrt{9-x^2}\,dx = \frac{9\pi}{4}$

Detailed Solution:

The definite integral represents the area between the curve $y = \sqrt{9-x^2}$ and the x-axis, over the interval [0, 3], which is one-fourth the area of a circle of radius 3.

Therefore: $\int_0^3 \sqrt{9-x^2}\,dx = \frac{1}{4}\pi(3)^2 = \frac{9\pi}{4}$

12. $\int_{-2}^{2} \sqrt{4-x^2}\,dx = 2\pi$

13. Answer: $\int_3^0 x\,dx = -\frac{9}{2}$

Detailed Solution:

Since $\int_3^0 x\,dx = -\int_0^3 x\,dx$, then the definite integral represents the negative area of a triangle with width 3 and height 3.

Therefore: $\int_3^0 x\,dx = -\frac{1}{2}(3)(3) = -\frac{9}{2}$

14. $\int_2^0 2x\,dx = -4$

15. Answer: $\int_{-2}^{2} x\,dx = 0$

Detailed Solution:

$$\int_{-2}^{2} x\,dx = \int_{-2}^{0} x\,dx + \int_{0}^{2} x\,dx$$

The definite integral represents the sum A_1 and A_2 where A_1 is the negative area of a triangle with width 2, height 2, and A_2 is the area of a triangle with width 2 and height 2.

$$\int_{-2}^{2} x\,dx = -\frac{1}{2}(2)(2) + \frac{1}{2}(2)(2)$$

$$= -2 + 2$$

$$= 0$$

16. $\int_{-4}^{2} 2x\,dx = -12$

17. Answer: $\int_0^2 1\,dx = 2$

Detailed Solution:

$$\int_0^2 1\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x = \lim_{n\to\infty}\sum_{i=1}^{n} f\left(\frac{2i}{n}\right)\frac{2}{n}$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n} 1\cdot\frac{2}{n}$$

$$= \lim_{n\to\infty}\frac{2}{n}\cdot\sum_{i=1}^{n} 1$$

$$= \lim_{n\to\infty}\frac{2}{n}\cdot n$$

$$= \lim_{n\to\infty} 2$$

$$= 2$$

18. $\int_0^4 2\,dx = 8$

19. Answer: $\int_{-4}^{-2} 1\,dx = 2$

Detailed Solution:

$$\int_{-4}^{-2} 1\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x = \lim_{n\to\infty}\sum_{i=1}^{n} f\left(-4+\frac{2(i-1)}{n}\right)\frac{2}{n}$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n} 1\cdot\frac{2}{n}$$

$$= \lim_{n\to\infty}\frac{2}{n}\cdot\sum_{i=1}^{n} 1$$

$$= \lim_{n\to\infty}\frac{2}{n}\cdot n$$

$$= \lim_{n\to\infty} 2$$

$$= 2$$

20. $\int_{-3}^{-1} 2\,dx = 4$

21. Answer: $\int_0^2 x\,dx = 2$

Detailed Solution:

$$\int_0^2 x\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x = \lim_{n\to\infty}\sum_{i=1}^{n} f\left(\frac{2i-1}{n}\right)\frac{2}{n}$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n}\frac{2i-1}{n}\cdot\frac{2}{n}$$

$$= \lim_{n\to\infty}\left[\frac{4}{n^2}\cdot\sum_{i=1}^{n} i - \frac{2}{n^2}\cdot\sum_{i=1}^{n} 1\right]$$

$$= \lim_{n\to\infty}\left[\frac{4}{n^2}\cdot\frac{n(n+1)}{2} - \frac{2}{n^2}\cdot n\right]$$

$$= \lim_{n\to\infty} 2$$

$$= 2$$

22. $\int_0^3 2x\,dx = 9$

23. Answer: $\int_{-1}^{2} x^2\,dx = 3$

Detailed Solution:

$$\int_{-1}^{2} x^2\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x = \lim_{n\to\infty}\sum_{i=1}^{n} f\left(-1+\frac{3i}{n}\right)\frac{3}{n}$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n}\left(-1+\frac{3i}{n}\right)^2\cdot\frac{3}{n}$$

$$= \lim_{n\to\infty}\frac{3}{n}\cdot\sum_{i=1}^{n}\left(1-\frac{6i}{n}+\frac{9i^2}{n^2}\right)$$

$$= \lim_{n\to\infty}\left[\frac{3}{n}\cdot\sum_{i=1}^{n}1-\frac{18}{n^2}\sum_{i=1}^{n}i+\frac{27}{n^3}\sum_{i=1}^{n}i^2\right]$$

$$= \lim_{n\to\infty}\left[\frac{3}{n}\cdot n-\frac{18}{n^2}\cdot\frac{n(n+1)}{2}+\frac{27}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}\right]$$

$$= \lim_{n\to\infty}\left[3+\frac{9}{2n}+\frac{9}{2n^2}\right]$$

$$= 3$$

24. $\int_{-2}^{3} 2x^2\,dx = \frac{70}{3}$

25. Answer: $\int_0^2 3x^4\,dx = \frac{96}{5}$

Detailed Solution:

$$\int_0^2 3x^4\,dx = 3\cdot\int_0^2 x^4\,dx$$

$$= 3\left(\frac{32}{5}\right)$$

$$= \frac{96}{5}$$

26. $\int_0^2 -2x^4\,dx = -\frac{64}{5}$

27. Answer: $\int_0^2 3x^3\,dx = 12$

Detailed Solution:

$$\int_0^2 3x^3\,dx = \frac{3}{2}\cdot\int_0^2 2x^3\,dx$$

$$= \frac{3}{2}(8)$$

$$= 12$$

28. $\int_0^2 -5x^3\,dx = -20$

29. Answer: $\int_2^0 2x^4\,dx = -\dfrac{64}{5}$

Detailed Solution:

$$\int_2^0 2x^4\,dx = -2\int_0^2 x^4$$

$$= -2\left(\frac{32}{5}\right)$$

$$= -\frac{64}{5}$$

30. $\int_2^0 -4x^4\,dx = -\dfrac{128}{5}$

31. Answer: $\int_0^2 \sin(x) + x^3\,dx = 5 - \cos(2)$

Detailed Solution:

$$\int_0^2 \sin(x) + x^3\,dx = \int_0^2 \sin(x)\,dx + \frac{1}{2}\int_0^2 2x^3\,dx$$

$$= 1 - \cos(2) + \frac{1}{2}(8)$$

$$= 5 - \cos(2)$$

32. $\int_0^2 x^4 - \sin(x)\,dx = \dfrac{27}{5} + \cos(2)$

33. Answer: $\int_{-1}^{0} x^4\ dx = \frac{1}{5}$

Detailed Solution:

$$\int_{-1}^{2} x^4\, dx = \int_{-1}^{0} x^4\, dx + \int_{0}^{2} x^4\, dx$$

$$\frac{33}{5} = \int_{-1}^{0} x^4\, dx + \frac{32}{5}$$

Therefore: $\int_{-1}^{0} x^4\, dx = \frac{1}{5}$

34. $\int_{-1}^{0} x^4 + 6\, dx = \frac{31}{5}$

35. Answer: $4 \le \int_{4}^{6} \sqrt{x}\, dx \le 2\sqrt{6}$

Detailed Solution:

$$\sqrt{4}(6-4) \le \int_{4}^{6} \sqrt{x}\, dx \le \sqrt{6}(6-4)$$

$$2(2) \le \int_{4}^{6} \sqrt{x}\, dx \le \sqrt{6}(2)$$

$$4 \le \int_{4}^{6} \sqrt{x}\, dx \le 2\sqrt{6}$$

36. $3\sqrt{5} \le \int_{5}^{8} \sqrt{x}\, dx \le 3\sqrt{8}$

37. Answer:
Detailed Solution:

$$\ln(e)(e^2 - e) \le \int_e^{e^2} \ln(x)\,dx \le \ln(e^2)(e^2 - e)$$

$$(1)(e^2 - e) \le \int_e^{e^2} \ln(x)dx \le (2)(e^2 - e)$$

$$e^2 - e \le \int_e^{e^2} \ln(x)dx \le 2(e^2 - e)$$

38. $2\ln(2) \le \int_2^4 \ln(x)\,dx \le 2\ln(4)$

39. Let g(x) = kf(x).
Divide the interval [a, b] into n subintervals of equal width Δx.
Let x_i^* be an arbitrary point in the ith subinterval. Then:

$$\int_a^b kf(x)\,dx = \int_a^b g(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} g(x_i^*)\Delta x$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} kf(x_i^*)\Delta x$$

$$= k \cdot \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

$$= k \cdot \int_a^b f(x)\,dx$$

40. Divide the interval [b, a] into n subintervals of equal width $\Delta x = (a-b)/n$. Let x_i^* be an arbitrary point in the ith subinterval. Then:

$$\int_b^a f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\frac{a-b}{n}$$

$$= -\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\frac{b-a}{n}$$

$$= -\int_a^b f(x)\,dx$$

41. Let g(x) = f(x) + g(x). Divide the interval [a, b] into n subintervals of equal width Δx. Let x_i^* be an arbitrary point in the ith subinterval. Then:

$$\int_a^b f(x)+g(x)\,dx = \int_a^b h(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} h(x_i^*)\Delta x$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n}\left[f(x_i^*)+g(x_i^*)\right]\Delta x$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n}\left[f(x_i^*)\Delta x+g(x_i^*)\Delta x\right]$$

$$= \lim_{n\to\infty}\left[\sum_{i=1}^{n} f(x_i^*)\Delta x+\sum_{i=1}^{n} g(x_i^*)\Delta x\right]$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x+\lim_{n\to\infty}\sum_{i=1}^{n} g(x_i^*)\Delta x$$

$$= \int_a^b f(x)\,dx+\int_a^b g(x)\,dx$$

42. Let g(x) = f(x) - g(x). Divide the interval [a,b] into n subintervals of equal width Δx. Let x_i^* be an arbitrary point in the ith subinterval. Then:

$$\int_a^b f(x) - g(x)\, dx = \int_a^b h(x)\, dx = \lim_{n\to\infty} \sum_{i=1}^{n} h(x_i^*)\Delta x$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left[f(x_i^*) - g(x_i^*) \right] \Delta x$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left[f(x_i^*)\Delta x - g(x_i^*)\Delta x \right]$$

$$= \lim_{n\to\infty} \left[\sum_{i=1}^{n} f(x_i^*)\Delta x - \sum_{i=1}^{n} g(x_i^*)\Delta x \right]$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x - \lim_{n\to\infty} \sum_{i=1}^{n} g(x_i^*)\Delta x$$

$$= \int_a^b f(x)dx - \int_a^b g(x)dx$$