CHAPTER 5

Probability

The chief aim of probability is to make **predictions** and **estimations** about the ***chances*** of certain things happening. We will be considering two types of probability. One type of probability involves choosing at random someone or something from a population of things. We will learn more about this type of probability in the next few weeks.

The other type of probability concerns the kind of situation where you do know in advance the chances of getting something. For example, if you are tossing a coin, say a hundred times, you may want to know how many tails you are ***likely*** to get. **You already know** the chance of getting a tail when you toss the coin once, (1/2 or 50%), so you use this knowledge to estimate how many tails you will get when you toss it a hundred times. Of course your ***estimation*** usually won't be exact, but that is the nature of probability or uncertainty. Winning megabucks also comes under this type of probability!!! This is the type we will study in this chapter.

Definitions

Event: Simply, an event is basically the experiment you want to do. For example, flipping a coin.

Outcome: 'What can happen' or the 'result' of an experiment. For example, in the above event of flipping a coin, the outcome could have been a head or a tail.

Equally Likely: Means that each event has the same chance of happening.

Dichotomous Variable: This is a result of an event that can occur in only two ways.

ex. a: Event: Flipping a coin and looking at the top of the coin after it has landed.
Results:
1. Head
2. Tail

ex. b: Event: The birth of a child.
Results:
1. Boy
2. Girl

ex. c: Event: Computer signal.
1. On
2. Off

An example of a variable that occurs in more than two ways, i.e. not a dichotomous variable: Throwing a single die. There are 6 possible results.

And / Or:

And: Usually means to multiply.

Or: Usually means to add.

At least / At Most:

At least: This is the smallest number you can have from a group.

ex. a: At least 5 from the group 1,2,3,4,5,6,7,8

This means 5, 6, 7, 8

At Most: This is the highest number you can have from a group.

ex. b: At most 5 from the group 1,2,3,4,5,6,7,8

This means 1, 2, 3, 4, 5

Factorial Notation - !

Factorial means to take the number and multiply it by each preceding whole number--not including zero.

ex. a: 7! **=** 7 x 6 x 5 x 4 x 3 x 2 x 1

ex. b: 15! = 15 x 14 x 13 x 12 x 11 x 10 x 9 x 8 x 7 x 6 x 5 x 4 x 3 x 2 x 1

Probability

In general, the probability of getting an outcome satisfying a particular condition can be computed as:

$$\textbf{Probability} = \frac{\text{number of outcomes satisfying the condition}}{\text{total number of possible outcomes}}$$

provided each outcome is equally likely to occur!

Another way to help understand this concept is to ask yourself the following question for each outcome of some event,

Will I be happy if this occurred?

Count up every 'yes' answer and this number is the numerator in your probability fraction.

This is similar to pulling out a name from a hat. If we put the 36 names of the students in this class into a hat with all the other students in the math department (say about 200 more students), we can ask the question, “If we pull out one name, what’s the chance it will be someone in our class?” the probability will be 36/236 or .1525 or about a 15% chance. Why? Because there are 36 outcomes that will satisfy the condition that the person is in our class. The total number of outcomes is 236 because there are 236 names in the hat.

Coins and Probability

First, we assume all problems with coins are 'fair' coins unless otherwise stated. That is, the coins are not weighted differently on each side. So, if you flip a fair coin, we basically know in advance that the chance of getting a head (or a tail) is 50% or ½. But what happens to the probability of getting a head if we flip 2 coins? How about 3 coins? Or 4?

Let's look at a few *Outcomes* when flipping coins. Each line below is ONE outcome.

One Coin	Two Coins	Three Coins
H	**H H**	**H H H**
T	**H T**	**H H T**
	T H	**H T H**
	T T	**H T T**
		T H H
		T H T
		T T H
		T T T

Notice the number of outcomes with each *additional* coin and see if you can see a pattern *emerging*.

Number of Coins	Number of Outcomes
1	2
2	4
3	8
4	?

It appears that the Number of Outcomes is equal to $2^{(\text{the number of coins})}$ For example, if we have 3 coins the number of outcomes is

$2^3 = 2 \cdot 2 \cdot 2 = 8$ Thus, there are 8 different ways (outcomes) three coins can fall.

Once we have the total number of outcomes, we can find the probability of some outcome occurring. Let's do an example:

ex. 1. If you flip 2 coins, what is the probability of getting all tails? If you look at the above table, we see that there is only one way (outcome) to get all tails. We now know there are 4 total possible outcomes. Using our definition of probability we get:

$$\textbf{Probability} = \frac{\text{number of outcomes satisfying the condition}}{\text{total number of possible outcomes}} = \frac{1}{4} = 0.25$$

ex. 2. If you flip 2 coins, what is the probability of getting just one head? Again, looking at the table, there are two outcomes with one head. (Note: one head means exactly one head.)

$$\textbf{Probability} = \frac{\text{number of outcomes satisfying the condition}}{\text{total number of possible outcomes}} = \frac{2}{4} = 0.50$$

QUESTION: How can we find the number of outcomes if we do not have a table like the one above? To answer this question, we will study ***Pascal's Triangle***.

PASCAL'S TRIANGLE:

With coins or anything else that can occur in two different ways (*dichotomous variable*), we have an easy way of getting the number of outcomes if we do not have a chart. We use ***Pascal's Triangle.*** Before we can use the triangle, we need to *construct* it.

Constructing Pascal's Triangle

To build this triangle, think of a pyramid (or if you know the sport of bowling, think of the pin set-up). Each row in the triangle will have one more element than the previous row, similar to the set up of pins in a bowling alley! To begin, start with two 'ones' in the first row. The second row still has two ones on the outside of the row. The third element is the sum of the two numbers directly above its space, again in a triangle (demonstrated in class). Each row represents the number of coins we are dealing with. For example, the line with a 5 as the second number in the row, represents working with 5 coins. (or 5 of any dichotomous variable - e.g. the on/off switch on a computer; or true/false answers on a quiz!)

Look at the triangle:

PASCAL'S TRIANGLE

1 1 ← The two ones represent a head and a tail on one coin

1 2 1

This line gives the probabilities For flipping five coins.

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

5h 4h 3h 2h 1h 0h

Once we have constructed the triangle, we will learn how each element of the triangle is used. This will be demonstrated *in class* and in **EXAMPLE 2** a bit later in this handout.

<u>Dice or "working with many-sided things!"</u>

We will now look at probabilities concerning objects that have more than two possibilities on one 'toss.' To demonstrate, let's talk a bit about dice. We know that on one 'die' there are 6 possibilities. The idea here is to see if we can project the outcomes if we have more than one die. A helpful tool for dice is the **chart** given below. This is only good for ***two*** of something with many sides like a pair of 6-sided dice. The chart below helps to give the outcomes when two dice are thrown. The outcome is the sum of the numbers that appear on the top of each die.

The horizontal row at the top of the chart (above the double line) represents the six possibilities on the first die. The vertical row, (on the left of the double line) represents the six possibilities on the second die. To get the sum of the two dice, go along the horizontal and down the vertical.

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

For example, if we are looking for the number of ways to get a sum of '**9**' when you throw two dice, just look for the number of 9's that appear in the **<u>area of the chart under and to right of the double lines. (The 'nines' have been boxed in.)</u>** Notice there are 'four' 9's. That means there are **<u>4 ways</u>** to get a total of a nine:

3 + 6, 6 + 3, 4 + 5, 5 + 4

Note: We are learning how to 'chart' or make a 'dice' table. Later, we will learn how to use this table to find the probability of certain events when we throw two dice.

The following are **EXAMPLES** demonstrating how to find **PROBABILITIES.** These examples will assist you when doing an assignment.

EXAMPLE 1. This example explains what is meant by equally likely outcomes.

Event: Two coins are flipped.

a) Are the outcomes no heads, one head, and two heads equally likely? Why or why not?

b) What is a set of equally likely outcomes?

c) If you toss a fair coin 10 times, which sequence of heads and tails is more likely?

Solutions:

There are four possible outcomes when you flip two coins. Listed are the outcomes and the probabilities associated with these outcomes

Outcome	Probability
H H	**¼ = .25**
H T	**¼ = .25**
T H	**¼ = .25**
T T	**¼ = .25**

To answer the questions in Part a) above, we summarize below:

No heads = TT Probability is: ¼ = .25
One head = HT and TH Probability is: 2/4 = 0.5 Because we have ¼ + ¼ .
Notice here that one head could be either HT or TH.
Since there are 2 possibilities of getting one head, the probability = .5
Two Heads = HH Probability is: ¼ = .25

All of the above are not equally likely because the probability of getting one head when you flip two coins is 0.5 whereas the probability of getting two heads or zero heads (two tails) is 0.25.

b) **Two heads and zero heads (2Tails) are equally likely outcomes since they have equal probabilities** (i.e. 0.25).

c) If we are looking for a **specific sequence** of heads and tails, one is **not** more likely than another. For example, look above at the four possibilities. Each *individual outcome* only appears one time. They are each have an equal likelihood of occurring. The *sequence* (H T) and the *sequence* (H H) both have an equal chance of happening. Similarly the specific ***sequences***

HHHHHHHHHH and HTHTTHTHHT are **equally likely:**

However, if we ask the question: What is the probability of getting 5 heads if you flip 10 coins, then that probability will be much greater than the probability of getting all 10 heads. Why?

Answer: There is only one way to get all heads. There are 252 ways of getting 5 heads. (The probability is 252/1024) We will see how to get this answer in a short time.

EXAMPLE 2. Learning how to use Pascal's Triangle

Find the probability of getting 3 heads when you flip 4 coins.

$$\text{Probability} = \frac{\text{\# outcomes that have 3 heads in them}}{\text{total possible outcomes}}$$

Solution:

We have learned in class that the total possible outcomes when you flip 4 coins is $2^4 = 16$. This is going to be the denominator of our fraction. Another way to get the TOTAL POSSIBLE OUTCOMES is to use Pascal's Triangle: We look at the row that has the second number equal to a 4. This is the row for 4 coins. Let's look at the triangle again. To get the TOTAL POSSIBLE OUTCOMES when you flip 4 coins, we simply ADD the row across: 1 + 4 + 6 + 4 + 1 = 16. This is our denominator in the formula above for Probability.

Now we need to know how many of these 16 outcomes contain 3 heads. The easy way to see this *outside of writing out all the outcomes,* is to look again at Pascal's Triangle.

1 1

1 2 1

1 3 3 1

This row represents 4 coins.

1 4 6 4 1

4h 3h 2h 1h 0h

The numbers in the last row represent the number of ways to get a certain number of heads (or tails). The '1' is the number of ways to get all heads. That's 4 heads. The '4' is the number of ways to get 3 heads, i.e. there are 4 different outcomes containing exactly three heads. For example, one outcome might be a HTHH, another HHHT. This number in the 4 coin row is the numerator in our formula for Probability. So, we have our numerator and our denominator. Therefore,

Probability of getting 3 heads when flipping four coins is

$$\mathbf{P(3h,4c) = \frac{4}{16} = 0.25}$$

For your information, I have written out all 16 total possible outcomes when you flip 4 coins. Notice the number of ways of getting only three heads.

One way to get 4 Heads

H H H H

Four ways to get 3 Heads

H H H T
H H T H
H T H H
T H H H

Six ways to get 2 Heads

H H T T
T H H T
T T H H
H T H T
T H T H
H T T H

Four ways to get 1 Head

H T T T
T H T T
T T H T
T T T H

One way to get 0 Heads

T T T T

EXAMPLE 3.

Event: One die is rolled.

a) What is the probability of obtaining a 2 or a 3?

b) What is the probability of obtaining an odd number?

Solutions:

There are six total outcomes on the roll of one die: you could roll a 1, 2, 3, 4, 5, or 6.

a) The probability of rolling a 2 **or** a 3 is the **sum** of the probability of rolling a 2 plus the probability of rolling a 3.

Probability of rolling a 2 = 1/6
Probability of rolling a 3 = 1/6
2/6 = 1/3 = 0.333...= 0.33 (2 d.p.)

b) There are 3 possible odd outcomes (1 or 3 or 5) out of a total of 6 outcomes. Therefore the probability is: 3/6 = 1/2 = **0.5**

EXAMPLE 4.

What is the probability of getting a total of 4 or less when you throw 2 dice?

Solution:

Let's look at the table for two dice and find how many fours, threes, and twos appear.

	1	2	3	4	5	6
1	**2**	**3**	**4**	5	6	7
2	**3**	**4**	5	6	7	8
3	**4**	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

As you can see, there are six different possibilities. This is our numerator. There are 36 total possibilities when you throw two dice. Therefore

$$P(\text{4 or less, two dice}) = \frac{6}{36} = \frac{1}{6} = .1666666\ldots = .17 \text{ (2d.p.)}$$

EXAMPLE 5.

Each member of a 5-person family buys a lottery ticket for which 128 tickets were sold. Only one prize is given . In the type of lottery, the tickets are put into a hat and one is drawn out. This is **NOT** like the megabucks or Mass Millions.

Solution:

This is a good example of where you can answer the hypothetical question, "*Will I be happy if . . .?"*

We don't care who in the family wins the money. We will be <u>happy</u> if any one of the five members wins. Thus, there are five chances to win out of 128. So the probability of winning is:

$$\frac{5}{128} = \mathbf{0.03906\,..} = \mathbf{0.04} \text{ (2 d.p.)}$$

M**EGABUCKS** – Modeled after the Massachusetts State Lottery

EXAMPLE 6.

What is the probability of winning megabucks if you buy one ticket?

First of all, you have to pick 6 numbers out of a total of 48 numbers. You cannot repeat any of these numbers. One way to look at YOUR CHOICE is to line up your choice of numbers in a row. Let's assume I chose **8 11 26 35 40 47**. What is my *chance* of winning??

$$\textbf{Probability} = \frac{\text{number of outcomes satisfying the condition}}{\text{total number of possible outcomes}}$$

The 'condition' to be satisfied is winning on one ticket. So the <u>numerator</u> = 1. Now we must find the <u>denominator</u> of this fraction. We need the total number of possible outcomes i.e. the total number of combinations similar to the number I chose. Let's look at the number I chose

8 11 26 35 40 47 this is the same number as **26 40 8 47 11 35**

This order of numbers counts as two possibilities. However, since it's the same number, we only need to count it once. There are other ways to list this same number. e.g.

47 35 11 26 8 40 etc. So you see we have a lot of numbers that are really the same.

Thus, the <u>order</u> of the number does NOT make a difference in this game.

(continued on next page)

Secondly, we cannot have any repeated numbers.

For example: we cannot pick a number like 8 8 11 11 11 26 so we have a few other combinations that are extraneous. An easy way to look at this type of repeat is to line up 6 boxes. In each box, we pick a number, but once we pick a number, we cannot use it again. So, there is one less number to choose from each time. Here are the number of choices for each box. We multiply these numbers together to get the total.

$$\boxed{48} \times \boxed{47} \times \boxed{46} \times \boxed{45} \times \boxed{44} \times \boxed{43}$$

This gives us the total possibilities including numbers like 8 11 26 35 40 47 and 11 26 40 35 47 8. This 'BOX TOTAL' is a large number. (Try it on your calculator. You should get 8,835,488,640.)

Now, since the numbers like the two above are the same, we need to eliminate these 'repeats.' In order to eliminate these 'repeats' we must DIVIDE the above total (what we get by multiplying the 48 x 47 x 46 x 45 x 44 x 43) by all of these 'repeats.'

This number of 'repeats' is found using the factorial notation. Since there are 6 numbers that can be repeated, the TOTAL number of just the repeat is 6! or 6 Factorial. The exclamation point is called a factorial. That's the same as saying

6 x 5 x 4 x 3 x 2 x 1 =720 or there are 720 numbers that are basically the same.

We must get rid of all these 'same' numbers so we DIVIDE the box total by 6 factorial.

(A) The total possibilities $= \dfrac{\boxed{48} \times \boxed{47} \times \boxed{46} \times \boxed{45} \times \boxed{44} \times \boxed{43}}{6 \times 5 \times 4 \times 3 \times 2 \times 1} =$

$$= \frac{8835488640}{720} = 12{,}271{,}512$$

This is the **DENOMINATOR** of our original probability equation.

so my chance of winning megabucks is one out of about 12 million

$$\text{Probability} = \frac{1}{12271512}$$

NB: In equation (A) above where we are finding the total possibilities, notice the denominator: 6x5x4x3x2x1. This can be written in factorial notation as **6!** Now, look at the boxed in numbers in the numerator. Do you see anything similar in the numerator? It is NOT quite 48! in fact, 42! is actually missing from 48! This is because we only have 6 places to fill!