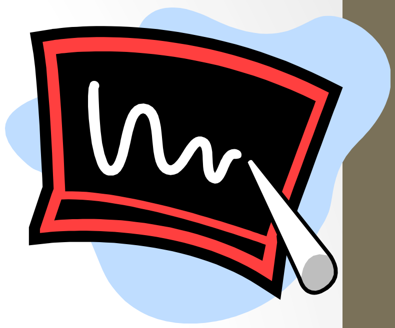
MATH E-3: Lecture 11

Quantitative Reasoning: Practical Math

MATH E 5

APRIL 26, 2016

Homework



- Assignment 9 grades available 4/30
- Assignment 10 is due 4/30
- Assignment 11 (**your last assignment!)** will be posted tomorrow, 4/27, and is due 5/7

Final Exam Information

- About 25 questions, mostly multiple choice. Topics to know:
  - **Confidence Intervals**
  - **Hypothesis Testing**
  - **Linear Growth**
  - **Scatterplots**
  - **Regression and Correlation**
  - **Exponential Growth and Decay**
  - **Percents**
  - **Summary Statistics (mean, median, mode)**
  - **No Excel questions on the final exam**

- Formulas you should know: **All** - except the first standard deviation formula

Final Exam Information

- **DATE:** Tuesday, May 10

- **TIME**: 7:40 pm – 9:40 pm (ET)*

  *Students living outside the 6 New England states will have a 24 hour window in which to complete the exam. You must arrange, with a proctor, any two hour period between May 10, 7:40 pm and May 11, 7:40 pm (ET) in which to complete your exam.

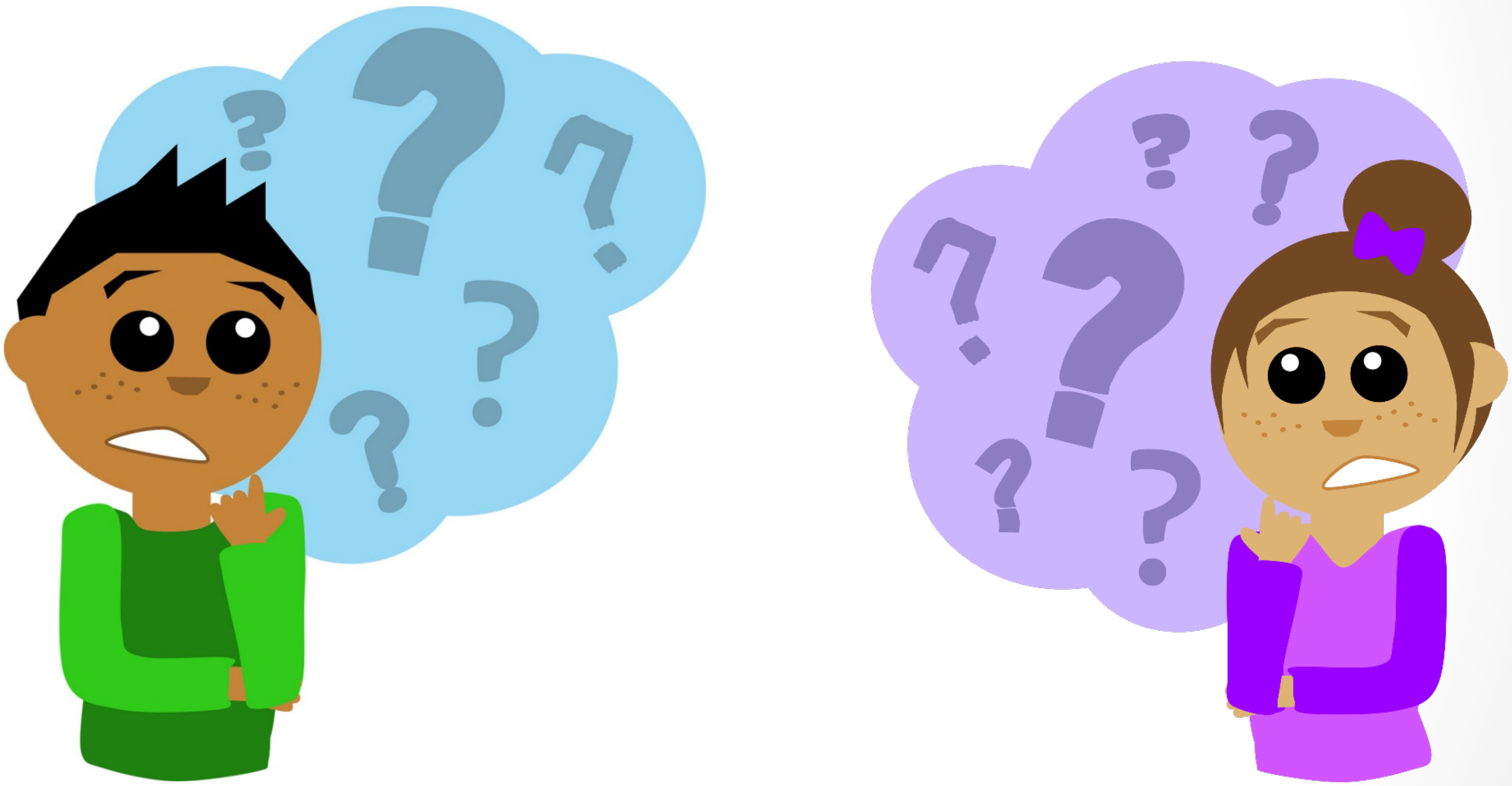
- **LOCATION**:
  - If you live within the 6 New England states: **Maxwell-Dworkin G115**
  - If you live outside the 6 New England states: You must arrange for a proctor:

- http://www.extension.harvard.edu/resources-policies/exams-grades-transcripts/exams-online-courses

- Proctor questions should be directed to: distance_exams@dcemail.harvard.edu or call (617) 495-0977 Monday through Friday, 9 am to 5 pm eastern time.

Final Exam Review Section

- When: Tuesday, May 3*, 7:40 pm (ET)
    - ***no class meeting**
- Where: Online, via Canvas Conferences
- Review session video and slides will be posted



Math in the news . . .

According to an article in Rolling Stone magazine online (actually from 2012), there are three key numbers which can help us understand the issue of climate change:

The First Number: 2° Celsius

The official position of planet Earth at the moment is that we can't raise the temperature more than two degrees Celsius – it's become the bottomest of bottom lines. Two degrees.

http://www.rollingstone.com/politics/news/global-warmings-terrifying-new-math-20120719

Math in the news . . .

Since we've increased the Earth's temperature by 0.8 degrees so far, we're currently less than halfway to the target. But, in fact, computer models calculate that even if we stopped increasing CO_2 now, the temperature would likely still rise another 0.8 degrees, as previously released carbon continues to overheat the atmosphere. That means we're already three-quarters of the way to the two-degree target.

http://www.rollingstone.com/politics/news/global-warmings-terrifying-new-math-20120719

Math in the news . . .

The Second Number: 565 Gigatons

Scientists estimate that humans can pour roughly 565 more gigatons of carbon dioxide into the atmosphere by midcentury and still have some reasonable hope of staying below two degrees. ("Reasonable," in this case, means four chances in five, or somewhat worse odds than playing Russian roulette with a six-shooter.)

In late May, the International Energy Agency published its latest figures – CO_2 emissions last year rose to 31.6 gigatons, up 3.2 percent from the year before.

Math in the news . . .

In fact, study after study predicts that carbon emissions will keep growing by roughly three percent a year – and at that rate, we'll blow through our 565-gigaton allowance in 16 years, around the time today's preschoolers will be graduating from high school.

Let's check that projection, based on the given figures of 31.6 gigatons in 2012, increasing at 3% per year.

Assuming 3% growth per year . . .

year	gigatons/yr	annual growth	cumulative gigatons	
2012	31.6	3%	31.6	
2013	32.5		64.1	
2014	33.5		97.7	
2015	34.5		132.2	
2016	35.6		167.8	
2017	36.6		204.4	
2018	37.7		242.1	
2019	38.9		281.0	
2020	40.0		321.0	
2021	41.2		362.3	
2022	42.5		404.7	
2023	43.7		448.5	
2024	45.1		493.5	
2025	46.4		539.9	
2026	**47.8**		**587.7**	**565-gigaton threshhold reached**
2027	49.2		637.0	

Math in the news . . .

So more like 14 years??

"The new data provide further evidence that the door to a two-degree trajectory is about to close," said Fatih Birol, the IEA's chief economist. In fact, he continued, "When I look at this data, the trend is perfectly in line with a temperature increase of about six degrees." That's almost 11 degrees Fahrenheit, which would create a planet straight out of science fiction.

OK, so the language is a bit hyperbolic (although I do like science fiction, but mainly in books); however the problem is both real and serious, so perhaps the language is justified.

Oh, and the third number?

The Third Number: 2,795 Gigatons

This number describes the amount of carbon already contained in the proven coal and oil and gas reserves of the fossil-fuel companies, and the countries (think Venezuela or Kuwait) that act like fossil-fuel companies. In short, it's the fossil fuel we're currently planning to burn. And the key point is that this new number – 2,795 – is higher than 565. Five times higher.

http://www.rollingstone.com/politics/news/global-warmings-terrifying-new-math-20120719

And now, for all you LOTR fans . . .

JRR Tolkien has all kinds of wondrous (and freaky) creatures in his books, including orcs, ents, and elves. He also has more than one dwarf . . . Which brings up the question: what is the correct plural of dwarf?

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Is it dwarfs or dwarves?

And now, for all you LOTR fans . . .

Apparently the “correct” form is actually *dwarfs*, although Tolkien preferred the “incorrect” form *dwarves*, and didn’t like editors “correcting” his manuscripts.

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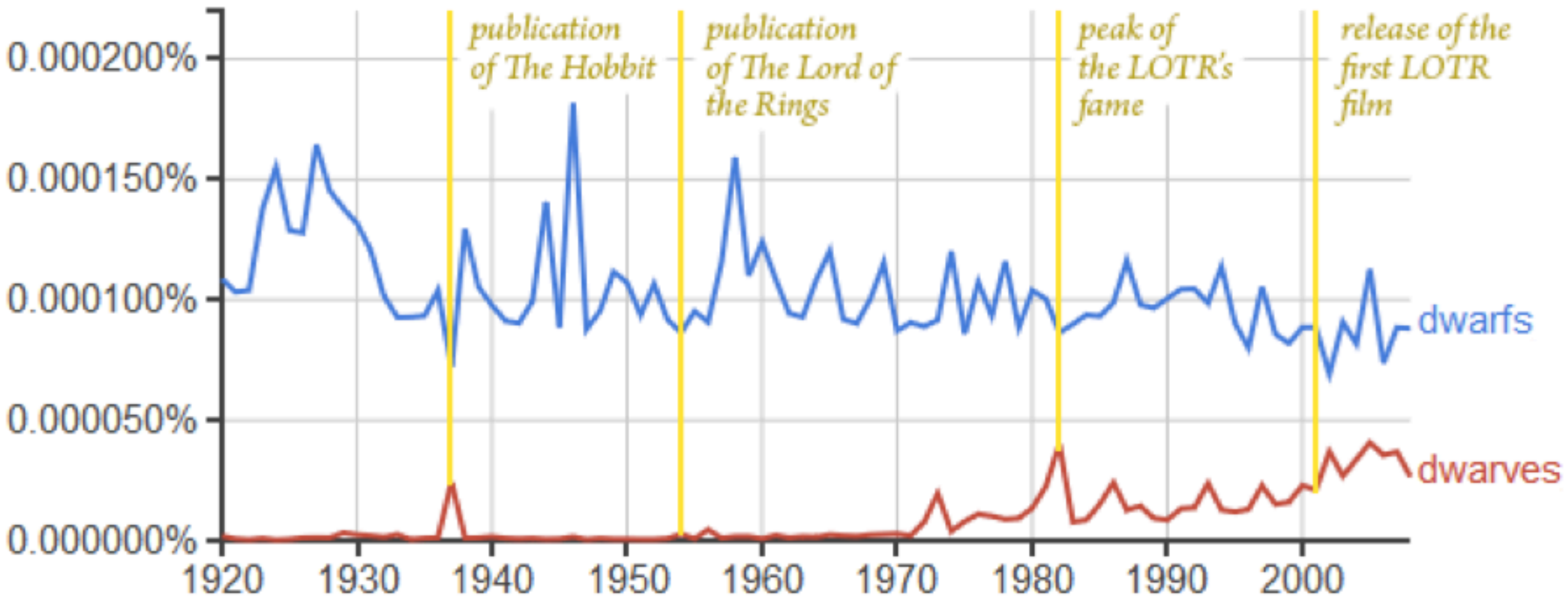
Apparently the “correct” form is actually *dwarfs*, although Tolkien preferred the “incorrect” form *dwarves*, and didn’t like editors “correcting” his manuscripts.

But let’s look at some numbers, since this is after all a math course . . .

Dwarfs or Dwarves?



https://jakubmarian.com/dwarves-or-dwarfs-which-spelling-is-correct/

OK, take a deep breath . . . preferably of clean air

Exponential Growth and Decay

A review of “exponents”

3^5 means:

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“3 to the power of 5”

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3 * 3 * 3 * 3 * 3 = ?

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Check this on your calculator!

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If you got 125 then what happened?

Two ways of doing % change

A furniture-store owner increases his price on his Chippendale Chairs by 35%. The current price of these beautiful chairs is $400 each. How much do they cost now?

Long Way:

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Long Way: An increase of **35%** means multiply $400 by 35% and add that amount to $400

.35 x 400 = $140   now the final selling price is $400 + $140 = **$540**

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Short Way: multiply original amount by 100% + % Increase = 135% = 1.35

$400 x 1.35 = $540

Percent decrease . . .

Our furniture store owner has decided to give a discount on his Queen Anne Chairs. They too originally sold for $400. He has decided to discount them by 35%.

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Long Way: A decrease of 35% means multiply $400 by 35% and SUBTRACT that amount from $400.

400 x .35 = 140   so new price is 400 – 140 = **$260**

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400 x .35 = 140 so new price is 400 – 140 = **$260**

Short Way: Let's look at what we REALLY did the Long Way

We took 100% of 400 and subtracted 35% of 400

Or we could say we took **100% – 35% of 400**

100% – 35% = 65%

So let's try

65% x 400 = .65 x 400 = **$260**

So when we DECREASE by a percent, we really multiply by 100% – the % Decrease.

The power of exponential growth . . .

You invest $1,000 in a money market account for 6 years, at an annual interest rate of 6% (compounded annually).

How much will you have after 6 years?

A table is a helpful way to get started:

The power of exponential growth . . .

year	balance
0	\$1,000
1	\$1,000 * 1.06 = \$1,060
2	\$1,000 * 1.06 * 1.06 = \$1,123.60
3	\$1,000 * 1.06 * 1.06 * 1.06 = \$1,191.02

Too much writing . . .!!

Remember – this is called “exponential” growth.

The power of exponential growth . . .

$1.06 * 1.06 = 1.06^2$

$1.06 * 1.06 * 1.06 = 1.06^3$

$1.06 * 1.06 * 1.06 * 1.06 = 1.06^4$

Etc.

The power of exponential growth . . .

year	balance
0	\$1,000
1	\$1,000 * 1.06^1 = \$1,060
2	\$1,000 * 1.06^2 = \$1,123.60
3	\$1,000 * 1.06^3 = \$1,191.02
4	\$1,000 * 1.06^4 = \$1,262.48
5	\$1,000 * 1.06^5 = \$1,338.23
6	**\$1,000 * 1.06^6 = \$1,418.52**

Of course we could go straight to year 6 . . .

Another couple of examples . . .

a) You invest \$2,500 for 5 years at an a.p.r. (annual percentage rate) of 4%. How much will you have in 5 years?

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$6,000 * 1.075^8 = $10,700.87

An exponential decrease example

An endangered species population currently numbers 7,500 and is declining at an annual rate of 4%. How many of this species will be left in 10 years (assuming the rate of decline remains the same)?

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An endangered species population currently numbers 7,500 and is declining at an annual rate of 4%. How many of this species will be left in 10 years (assuming the rate of decline remains the same)?

This time each year we are left with **100% - 4% = 96%** of the previous year's population.

So . . . $\mathbf{7{,}500 * 0.96^{10} = 4{,}986}$ **left in 10 years.**

Some theory behind all this . . .

General formula for exponential growth/decay:

$$Y = A(1 \pm r)^n$$

Each letter stands for something:

"Y" is the value of the thing you are measuring ($, population, etc.)
"A" is the initial value
1 is the 100% you start with
"r" is the growth (or decay) rate as a decimal
"n" is the number of time periods (years, days, etc.)
"±" because it can be either an increase or a decrease.

Using the formula

General formula for exponential growth/decay:

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A city population is currently 350,000 and is growing at a rate of 2.5% per year. What will the population be in 10 years? 25 years? 50 years?

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a) $Y = 350{,}000 * (1.025)^{10} =$ **448,030**

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b) $Y = 350{,}000 * (1.025)^{25} = 648{,}880$

c) $Y = 350{,}000 * (1.025)^{50} =$ **1,202,988**

A bit more "theory" . . .

General formula for exponential growth/decay:

$$Y = A(1 \pm r)^n$$

We call "r" the "growth rate" (or "decay rate")

We call "1 + r" the "growth factor"; "1 – r" the "decay factor"

An example of extreme exponential growth

You have one bacterium in a test-tube, and it doubles every 2 minutes.

How many bacteria will you have in 30 minutes? 1 hour? 1 day?

Start with a table:

Minutes	# bacteria
0	1 . . .

An example of extreme exponential growth

Start with a table:

Minutes	# bacteria
0	1
2	2
4	4
6	8
8	16
10	32

Can you see a pattern?

An example of extreme exponential growth

Minutes	# bacteria	
0	1	
2	2	
4	4	$= 2^2$
6	8	$= 2^3$
8	16	$= 2^4$
10	32	$= 2^5$
…	…	
n	?	=

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30	?	$= 2^{15}$	=
60	?	$= 2^{30}$	=
1440	?	$= 2^{720}$	=

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1440	?	$= 2^{720}$	= **5.5157E+216**

2^{720} = 5.5157E+216?!

This last number, 2^{720} = **5.5157E+216,** is the same as moving the decimal place to the right 216 places!! It is so big that it may have crashed your calculator! It is also bigger than the number of electrons in the known universe. So what happened?

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Fortunately "most" (i.e. virtually all) of the bacteria die fairly quickly!

Estimating when a goal will be met

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Try n = 30, $Y = 700{,}000 * (1.015)^{30} = 1{,}094{,}156$; so back a little.

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Try n = 30, $Y = 700{,}000 * (1.015)^{30} = 1{,}094{,}156$; so back a little.

Try n = 24, we get $Y = 700{,}000 * (1.015)^{24} = 1{,}000{,}652$. So 24 years, to the closest year.

Compounding interest more than once per year

Our original example: \$1,000 invested at 6% for 6 years, interest compounded annually. The final amount was **\$1,418.52**

How about if the interest is compounded semi-annually?

Now we are getting our interest twice per year – so 3% on June 30 and 3% on December 31. And there are twice as many compounding periods as previously.

So . . . $Y = 1{,}000 * (1 + 6\%/2)^{2*6}$

=

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So . . . $Y = 1{,}000 * (1 + 6\%/2)^{2*6}$

$= 1{,}000 * 1.03^{12}$ = **\$1,425.76**

So a few dollars more than with annual compounding.

More frequent compounding

How about if our interest is compounded monthly?

$Y = 1{,}000 * (1 + 6\%/12)^{12*6}$

More frequent compounding

How about if our interest is compounded monthly?

$Y = 1{,}000 * (1 + 6\%/12)^{12*6}$

$= 1{,}000 * (1.005)^{72}$

= **$1,432.04**

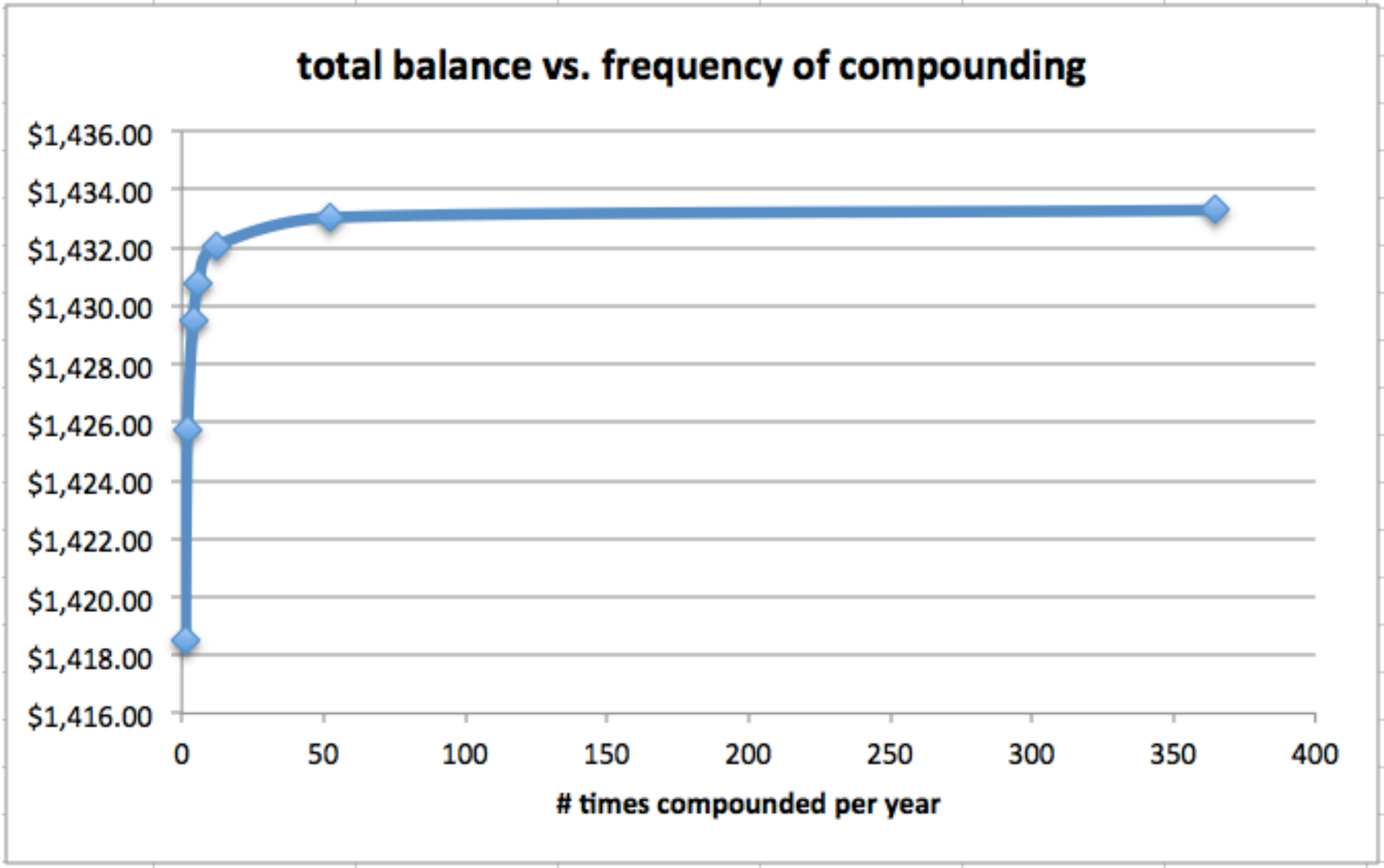
So even a few dollars better than semi-annual compounding.

(can it go on for ever??)

Balance with more frequent compounding…

	comp freq	balance	change
annually	1	$1,418.52	
semi-annually	2	$1,425.76	$7.24
quarterly	4	$1,429.50	$3.74
bi-monthly	6	$1,430.77	$1.27
monthly	12	$1,432.04	$1.28
weekly	52	$1,433.03	$0.99
daily	365	$1,433.29	$0.26

What's the shape of this graph?



Scientific Notation

Scientific Notation is used extensively in everyday mathematics as well as in many other disciplines such as economics, the natural sciences, and astronomy. It is a convenient way to write very large or very small numbers.

The rules utilize decimals and exponents:

Scientific Notation

Rule 1: Only one (1) digit appears to the left of the decimal point. That is called the 'significant' digit. There can be as many numbers to the right of the decimal as you may want.

e.g. 6.82147×10^3. The '6' is to the left of the decimal point. It is the “significant digit.”

Scientific Notation

<u>Rule 2:</u> Following the numbers, we multiply by a power of 10.
e.g. $3500 = 3.5 \times 10^3$

If the power is positive, we have a large number. To get to the <u>expanded</u> form, move the decimal point to the RIGHT the number of places indicated by the magnitude of the power and fill in with zeroes if necessary. 3.5×10^3 means move the decimal point 3 places to the right, i.e. 3500. If the power is negative, move the decimal that number of places to the left.
$7.2583 \times 10^{-4} = .00072$

Examples . . .

Convert to ordinary numbers:

a) 3.5162×10^6 =

b) 1.24837×10^{-8} =

Convert to scientific notation:

c) 15,277,000,000 =

d) 0.000000007129 =

Examples . . .

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a) 3.5162×10^{6} = **3,516,200**

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a) 3.5162×10^6 = 3,516,200

b) 1.24837×10^{-8} = 0.0000000124837

Convert to scientific notation:

c) 15,277,000,000 = 1.5277×10^{10}

d) 0.000000007129 = $\mathbf{7.129 \times 10^{-9}}$

RULES FOR MANIPULATION OF EXPONENTS

Multiplying numbers with the same base that have an exponent - add the exponents.

ex. $5^4 \times 5^2 = 5^{(4+2)} = 5^6$

ex. $a^b \times a^c = a^{b+c}$

Dividing numbers with the same base that have an exponent - subtract the exponents.

ex. $5^4 \div 5^2 = \frac{5^4}{5^2} = 5^{4-2} = 5^2$

ex. $\frac{5^3}{5^5} = 5^{3-5} = 5^{-2}$

ex. $a^b \div a^c = \frac{a^b}{a^c} = a^{b-c}$

Raising a number with an exponent to a power - multiply exponents.

ex. $(5^2)^4 = 5^8$

A number with a negative exponent means: 1 divided by that number with the positive of the exponent.

ex. $5^{-2} = \frac{1}{5^2} = \frac{1}{25}$

A possible extra topic . . .

Although "trial and error" is a method that works quite well – provided you don't have too many trials and errors! There is a somewhat more "elegant" method. It is not part of the course, but if we have time I will demonstrate it in class.

Either way . . .

This is our last class for the semester. I hope you have enjoyed our exploration of some of the useful and practical aspects of mathematics. We know that math is not always the easiest of subjects; but if you have been persistent and diligent with the assignments, and have done your best on the two quizzes, you should not feel too apprehensive about the upcoming final exam. (Although a little adrenalin isn't a bad thing!)

Either way . . .

We wish you all the best for the final, and for your future after the course is over – what I like to call the "aftermath"! For those graduating this May, we hope to see you at the Commencement ceremonies!

Graeme, Sue, Jessica, Lori, and Nina