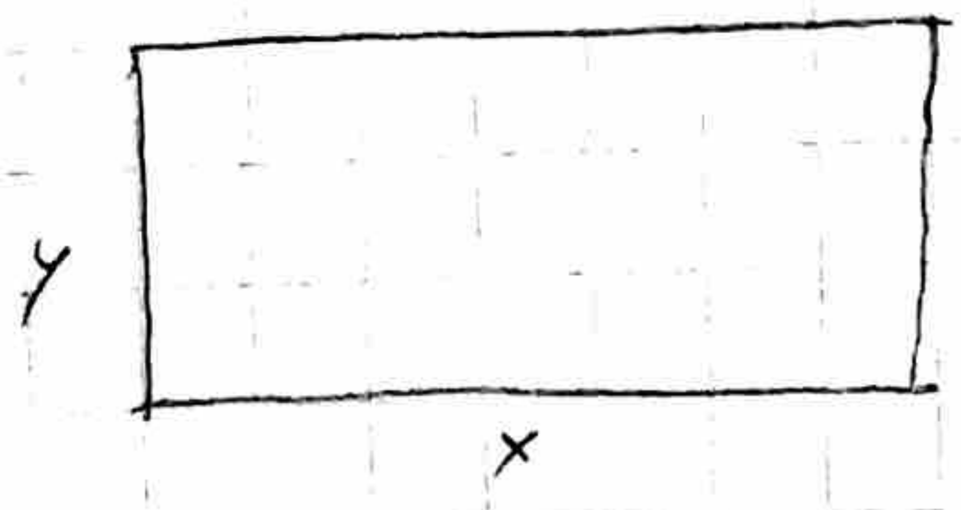
Example 1

A farmer has 3600 ft of fencing and wants to fence off a rectangular area for his cattle. What dimensions should the farmer use to maximize the area for grazing?

maximizing

①



Area = length x width, x = length

y = width

A = area

$\boxed{A = xy}$

② 3600 ft of fencing represents the perimeter

$$3600 = 2(x+y)$$

$$\boxed{3600 = 2x + 2y}$$

③ Set $A = xy$ in terms of x

④ Solve for y

$$3600 = 2x + 2y$$
$$-2x \quad -2x$$
$$\frac{3600 - 2x}{2} = \frac{2y}{2}$$

$$\frac{3600 - 2x}{2} = y$$

$$1800 - x = y$$

$$\boxed{y = 1800 - x}$$

⑤ Replace y

A

$A = xy$

$A = x(1800 - x)$

$A = 1800x - x^2$

$$\boxed{\begin{array}{c} A(x) \\ A(x) = xy \\ A(x) = x(1800-x) \\ A(x) = 1800x - x^2 \end{array}}$$

⑥ Domain Awareness

The smallest x can be is 0, x cannot be negative since we are working with a rectangular area.

⑦ Set $y = 0$ to get largest x value.

$$3600 = 2x + 2(0)$$

$$\frac{3600}{2} = \frac{2x}{2}$$

$$1800 = x \ , \ x = 1800$$

⑧ Express area A as a function of x

$$A(x) = 1800x - x^2 \ , \quad \text{where} \quad 0 \leq x \leq 1800$$

⑨ Get $A'(x)$

$$A'(x) = \frac{d}{dx}[1800x - x^2]$$

$$= \frac{d}{dx}[1800x] - \frac{d}{dx}[x^2]$$

$$= 1800 \cdot \frac{d}{x}[x] - 2x^{2-1}$$

$$= 1800 \cdot 1x^{1-1} - 2x$$

$$= 1800 \cdot 1 \cdot 1 - 2x$$

$$\boxed{A'(x) = 1800 - 2x}$$

⑩ Set $A'(x) = 0$, Get critical Number

$$1800 - 2x = 0$$

$$-1800 \qquad -1800$$

$$\frac{-2x}{-2} = \frac{-1800}{-2}$$

$$\boxed{x = 900}$$

(11) Get Domain Values for $A(x)$

$A(x) = 1800x - x^2$

$x(1800 - x)$

$\boxed{x = 0}$

$$\begin{array}{rcl} 1800 - x & = & 0 \\ -1800 & & -1800 \\ \hline \frac{-x}{-1} & = & \frac{-1800}{-1} \\ & \parallel & \end{array}$$

$\boxed{x = 1800}$

(12) Compare Critical Numbers

$x = 0, \quad x = 900, \quad x = 1800$

(13) Get Maximum

$A(x) = 1800x - x^2$

$A(0) = 1800(0) - (0)^2$

$\parallel$

$\boxed{A(0) = 0}$

$A(900) = 1800(900) - (900)^2$

$\parallel$

$\boxed{A(900) = 810000}$

$A(1800) = 1800(1800) - (1800)^2$

$\parallel$

$3.24 \times 10^6 - 3.24 \times 10^6$

$\parallel$

$\boxed{A(1800) = 0}$

$\boxed{\text{Maximum Value for } A(x) = 1800x - x^2 \text{ is } 810000, \text{ where } x = 900}$

(14) Get y value

$y = 1800 - x, \quad x = 900$

$y = 1800 - 900$

$\boxed{y = 900}$

(15) Summary

$x = 900, \quad y = 900$

Dimensions of the rectangle that maximize grazing area is 900 ft wide by 900 ft long.

Example 2

Find two numbers who difference is 40 and whose product is a minimum.

minimizing

① Set up equations

$40 = x - y$, where x and y are the two unknown numbers

$P = xy$, where P is the product and x and y are the two unknown numbers

② Set $P = xy$ in terms of x

③ Solve for y

$$40 = x - y$$
$$-x \quad -x$$
$$\frac{40 - x}{-1} = \frac{-y}{-1}$$

$$-40 + x = y, \quad y = x - 40$$

④ Replace y

$$P(x) = xy$$
$$P(x) = x(x - 40)$$
$$P(x) = x^2 - 40x$$

⑤ Domain Awareness

Difference of 40, no restrictions on domain.

Domain of x $(-\infty, \infty)$

⑥ Get $P'(x) = \frac{d}{dx}[x^2 - 40x]$

$$\frac{d}{dx}[x^2] - \frac{d}{dx}[40x]$$

$$2x^{2-1} - 40 \cdot \frac{d}{dx}[x]$$

$$2x - 40 \cdot 1x^{1-1}$$

$$2x - 40 \cdot 1$$

$$\boxed{P'(x) = 2x - 40}$$

⑦ Get Critical Number .

$$P'(x) = 0$$

$$2x - 40 = 0$$

$$\underline{\quad +40 \quad +40}$$

$$\frac{2x}{2} = \frac{40}{2}$$

$\boxed{x = 20}$ ← 1st number

⑧ Get $P''(x)$ Second Derivative

$$P''(x) = \frac{d}{dx}[2x - 40]$$

$$= \frac{d}{dx}[2x] - \frac{d}{dx}[40]$$

$$= 2 \cdot \frac{d}{dx}[x] - 0$$

$$= 2 \cdot 1x^{1-1}$$

$$= 2 \cdot 1 \cdot 1$$

$$= \boxed{P''(x) = 2} \rightarrow$$

⑨ Apply Second Derivative Test

$$P'(x) = 2x - 40$$

$$P'(20) = 2(20) - 40$$

$$40 - 40$$

$$\boxed{P'(20) = 0}$$

$$P''(x) = 2$$

$$\boxed{2 > 0}$$

$P'(x) = 0$ and $P''(x) > 0$, $P(x) = x^2 - 40x$ have a relative minimum at $x = 20$

(10) Get y value

$y = x - 40$, $x = 20$

$y = 20 - 40$

$\boxed{y = -20}$ ← Second Number

(11) Summary

First Number

$x = 20$

Second Number

$y = -20$

20 and -20 have a difference of 40.

$P(20) = (20)^2 - 40(20)$

$400 - 800$

$\boxed{P(20) = -400}$

$20 \cdot -20 = -400$

Product of 20 and -20 is the minimum of $P(x) = x^2 - 40x$