CHAPTER 0

WELCOME TO CALCULUS

Calculus was first developed more than three hundred years ago by Sir Isaac Newton and Gottfried Leibniz to help them describe and understand the rules governing the motion of planets and moons. Since then, thousands of other men and women have refined the basic ideas of calculus, developed new techniques to make the calculations easier, and found ways to apply calculus to problems besides planetary motion. Perhaps most importantly, they have used calculus to help understand a wide variety of physical, biological, economic and social phenomena and to describe and solve problems in those areas.

The discovery, development, and application of calculus is a great intellectual achievement, and now you have the opportunity to share in that achievement. You should feel exhilarated. You may also be somewhat concerned, a common reaction of students just beginning to study calculus. You need to be concerned enough to work to master calculus and confident enough to keep going when you don't understand something at first.

Part of the beauty of calculus is that it is based on a few very simple ideas. Part of the power of calculus is that these simple ideas can help us understand, describe, and solve problems in a variety of fields. This book tries to emphasize both the beauty and the power.

In Section 0.1 (Preview) we will look at the main ideas which will continue throughout the book: the problems of tangent lines and areas. We will also consider a process that underlies both of those problems, the limiting process of approximating a solution and then getting better and better approximations until we finally get an exact solution.

Sections 0.2 (Lines), 0.3 (Functions), and 0.4 (Combinations of Functions) contain review material which you need to recall before we begin calculus. The emphasis in these sections is on material and skills you will need to succeed in calculus. You should have worked with most of this material in previous courses, but the emphasis and use of the material in these sections may be different than in those courses.

Section 0.5 (Mathematical Language) discusses a few key mathematical phrases. It considers their use and meaning and some of their equivalent forms. It will be difficult to understand the meaning and subtleties of calculus if you don't understand how these phrases are used and what they mean.

0.1 PREVIEW OF CALCULUS

Two Basic Problems

Beginning calculus can be thought of as an attempt, a historically successful attempt, to solve two fundamental problems. In this section we will start to examine geometric forms of those two problems and some fairly simple ways to attempt to solve them. At first, the problems themselves may not appear very interesting or useful, and the methods for solving them may seem crude, but these simple problems and methods have led to one of the most beautiful, powerful, and useful creations in mathematics: Calculus.

First Problem: Finding the Slope of a Tangent Line

Suppose we have the graph of a function $y = f(x)$, and we want to find the equation of the line which is **tangent** to the graph at a particular point P on the graph (Fig. 1). (We will give a precise definition of **tangent** in Section 1.0. For now, think of the tangent line as the line which touches the curve at the point P and stays close to the graph of $y = f(x)$ near P.) We know that the point P is on the tangent line, so if the x–coordinate of P is $x = a$, then the y–coordinate of P must be $y = f(a)$ and $P = (a, f(a))$. The only other information we need to find the equation of the tangent line is its slope, m_{tan}, and that is where the difficulty arises. In algebra, we needed two points in order to determine a slope, and so far we only have the point P. Lets simply pick a second point, say Q, on the graph of $y = f(x)$. If the x–coordinate of Q is b (Fig. 2), then the y–coordinate is $f(b)$, so $Q = (b, f(b))$. The slope of the line through P and Q is

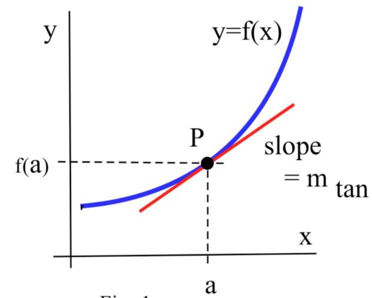


Fig. 1

$$m_{PQ} = \frac{\text{rise}}{\text{run}} = \frac{f(b) - f(a)}{b - a} .$$

If we drew the graph of $y = f(x)$ on a wall, put nails at the points P and Q on the graph, and laid a straightedge on the nails, then the straightedge would have slope m_{PQ} (Fig. 2). However, the slope m_{PQ} can be very different from the value we want, the slope m_{tan} of the tangent line. The key idea is that if the point Q is **close to** the point P, then the slope m_{PQ} is **close to** the slope we want, m_{tan}. Physically, if we slide the nail at Q along the graph towards the fixed point P, then the slope, $m_{PQ} = \frac{f(b) - f(a)}{b - a}$, of the straightedge gets closer and closer to the slope, m_{tan}, of the tangent line. If the value of b is very close to a, then the point Q is very close to P, and the value of m_{PQ} is very close to the value of m_{tan}. Rather than defacing walls with graphs and nails, we can calculate

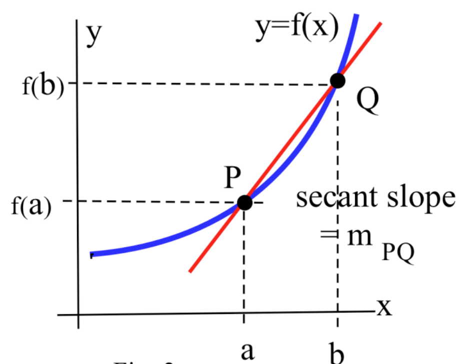


Fig. 2

$$m_{PQ} = \frac{f(b) - f(a)}{b - a}$$

and examine the values of m_{PQ} as b gets closer and closer to a. We say that m_{tan} is the limiting value of m_{PQ} as b gets very close to a, and we write

$$m_{tan} = \lim_{b \to a} \frac{f(b) - f(a)}{b - a} .$$

The slope m_{tan} of the tangent line is called the **derivative** of the function f(x) at the point P, and this part of calculus is called **differential calculus**. Chapters 2 and 3 begin differential calculus.

The slope of the tangent line to the graph of a function will tell us important information about the function and will allow us to solve problems such as:

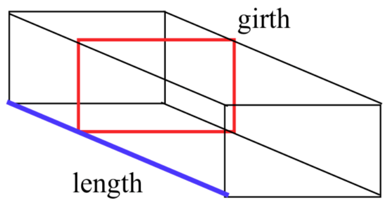


Fig. 3

"The US Post Office requires that the length plus the girth (Fig. 3) of a package not exceed 84 inches. What is the largest volume which can be mailed in a rectangular box?"

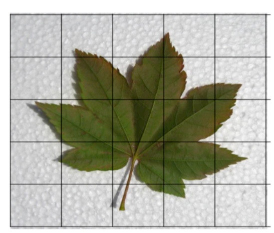
An oil tanker was leaking oil, and a 4 inch thick oil slick had formed. When first measured, the slick had a radius 200 feet and the radius was increasing at a rate of 3 feet per hour. At that time, how fast was the oil leaking from the tanker?

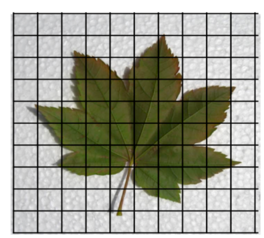
Derivatives will even help us solve such "traditional" mathematical problems as finding solutions of equations like $x^2 = 2 + \sin(x)$ and $x^9 + 5x^5 + x^3 + 3 = 0$.

Second Problem: Finding the Area of a Shape


Fig. 4

Suppose we need to find the area of a leaf (Fig. 4) as part of a study of how much energy a plant gets from sunlight. One method for finding the area would be to trace the shape of the leaf onto a piece of paper and then divide the region into "easy" shapes such as rectangles and triangles whose areas we could calculate. We could add all of the "easy" areas together to get the area of the leaf. A modification of this method would be to trace the shape onto a piece of graph paper and then count the number of squares completely inside the edge of the leaf to get a lower estimate of the area and count the number of squares that touch the leaf to get an upper estimate of the area. If we repeat this process with smaller squares, we have to do more counting and adding, but our estimates are closer together and closer to the actual area of the leaf. (This area can also be approximated using a sheet of paper, scissors and an accurate scale. How?)


Each square is 1 sq. cm
totally inside = 1
partially inside = 18
$1 \le$ number ≤ 19
1 sq.cm $\le$ area $\le$ 19 sq.cm


Each square is 1/4 sq/ cm
totally inside = 16
partially inside = 34
$16 \le$ number ≤ 50
4 sq.cm $\le$ area $\le$ 12.5 sq.cm

We can calculate the area A between the graph of a function y = f(x) and the x–axis (Fig. 5) by using similar methods. We can divide the area into strips of width w and determine the lower and upper values of y = f(x) on each strip. Then we can approximate the area of each rectangle and add all of the little areas together to get A_w, an approximation of the exact area. The key idea is that if w is small, then the rectangles are narrow, and the approximate area A_w is very

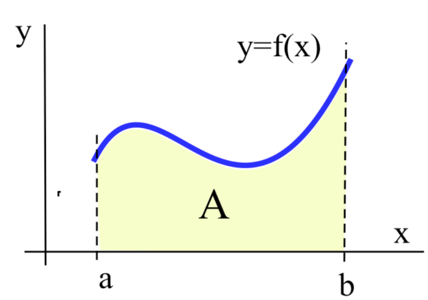


Fig. 5: What is the area of region A?

close to the actual area A. If we take narrower and narrower rectangles, the approximate areas get closer and closer to the actual area: $A = \lim_{w \to 0} A_w$.

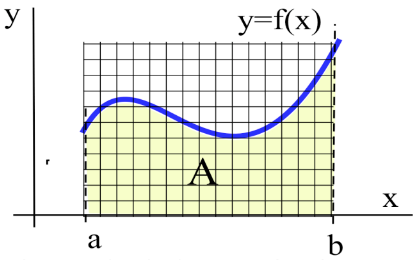


Fig. 5: What is the area of region A?

The process we used is the basis for a technique called **integration**, and this part of calculus is called **integral calculus**. Integral calculus and integration will begin in Chapter 4.

The process of taking the limit of a sum of "little" quantities will give us important information about the function and will also allow us to solve problems such as:

"Find the length of the graph of $y = \sin(x)$ over one period (from $x = 0$ to $x = 2\pi$)."

"Find the volume of a torus ("doughnut") of radius 1 inch which has a hole of radius 2 inches. (Fig. 6)"

"A car starts at rest and has an acceleration of $5 + 3\sin(t)$ feet per second per second in the northerly direction at time t seconds. Where will the car be, relative to its starting position, after 100 seconds?"

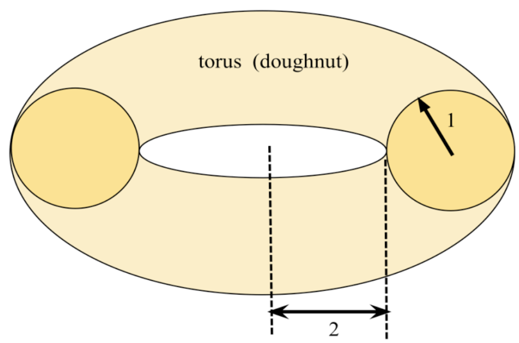


Fig.6: What is the volume of the torus?

A Unifying Process: Limits

We used a similar processes to "solve" both the tangent line problem and the area problem. First, we found a way to get an approximate solution, and then we found a way to improve our approximation. Finally, we asked what would happen if we continued improving our approximations "forever", that is, we "took a limit." For the tangent line problem, we let the point Q get closer and closer and closer to P, the limit as b approached a. In the area problem, we let the widths of the rectangles get smaller and smaller, the limit as w approached 0. Limiting processes underlie derivatives, integrals, and several other fundamental topics in calculus, and we will examine limits and their properties in Chapter 1.

Two Sides Of The Same Coin: Differentiation and Integration

Just as the set–up of each of the two basic problems involved a limiting process, the solutions to the two problems are also related. The process of differentiation for solving the tangent line problem and the process of integration for solving the area problem turn out to be "opposites" of each other: each process undoes the effect of the other process. The Fundamental Theorem of Calculus in Chapter 4 will show how this "opposite" effect works.

Extensions of the Main Problems

The first 5 chapters present the two key ideas of calculus, show "easy" ways to calculate derivatives and integrals, and examine some of their applications. And there is more. In later chapters, new functions will be examined and ways to calculate their derivatives and integrals will be found. The approximation ideas will be extended to use "easy" functions, such as polynomials, to approximate the values of "hard" functions such as sin(x) and e^x. And the notions of "tangent lines" and "areas" will be extended to 3–dimensional space as "tangent planes" and "volumes".

Success in calculus will require time and effort on your part, but such a beautiful and powerful field is worth that time and effort.

PROBLEMS

(Solutions to odd numbered problems are given at the back of the book.)

Problems 1 – 4 involve estimating slopes of tangent lines.

1) Sketch the lines tangent to the curve shown in Fig. 7 at x = 1, 2 and 3. Estimate the slope of each of the tangent lines you drew.

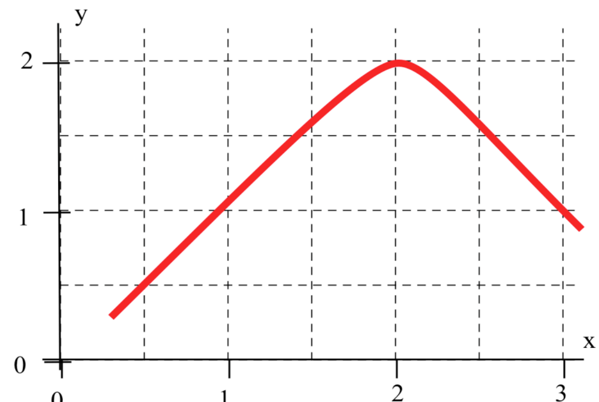


Fig.7

2) Fig. 8 shows the weight of a "typical" child from age 0 to age 24 months. (Each of your answers should have the units "kilograms per month.")
   (a) What was the average rate of weight gain from month 0 to month 24?
   (b) What was the average weight gain from month 9 to month 12? from month 12 to month 15?
   (c) Approximately how fast was the child gaining weight at age 12 months? at age 3 months?

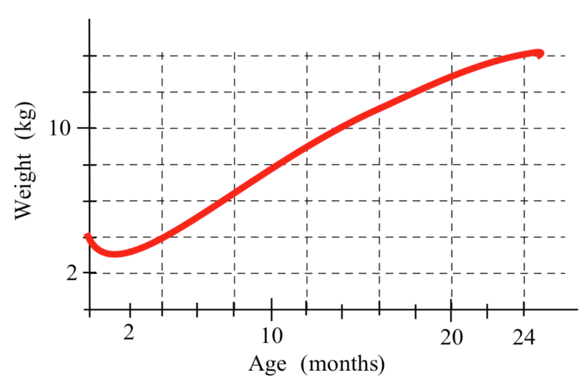


Fig.8

3) Fig. 9 shows the temperature of a cup of coffee during a ten minute period. (Each of your answers in (a) – (c) should have the units "degrees per minute.")

(a) What was the average rate of cooling from minute 0 to minute 10?

(b) What was the average rate of cooling from minute 7 to minute 8? from minute 8 to minute 9?

(c) What was the rate of cooling at minute 8? at minute 2?

(d) When was the cold milk added to the coffee?

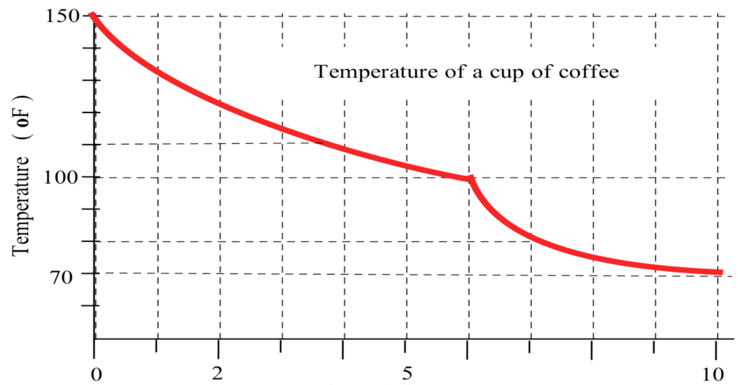


Fig. 9

4) Describe a method for determining the slope at the middle of a steep hill on campus

(a) using a ruler, a long piece of string, a glass of water and a loaf of bread.

(b) using a protractor, a piece of string and a helium–filled balloon.

Problems 5 and 6 involve approximating areas.

5) Approximate the area of the leaf in Fig. 4 .

6) Fig. 10 shows temperatures during the month of November.

(a) Approximate the shaded area between the temperature curve and the 65° line from Nov. 15 to Nov. 25.

(b) The area of the "rectangle" is (base)(height) so what are the units of your answer in part (a)?

(c) Approximate the shaded area between the temperature curve and the 65° line from Nov. 5 to Nov. 30.

(d) Who might use or care about these results?

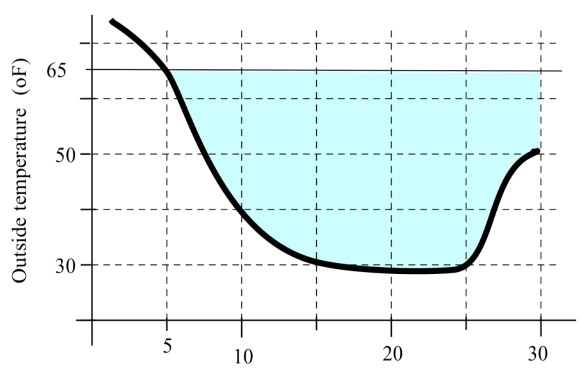


Fig. 10

7) Describe a method for determining the volume of a standard incandescent light bulb using a ruler, a tin coffee can, a scale, and a jug of wine.