CHAPTER 1

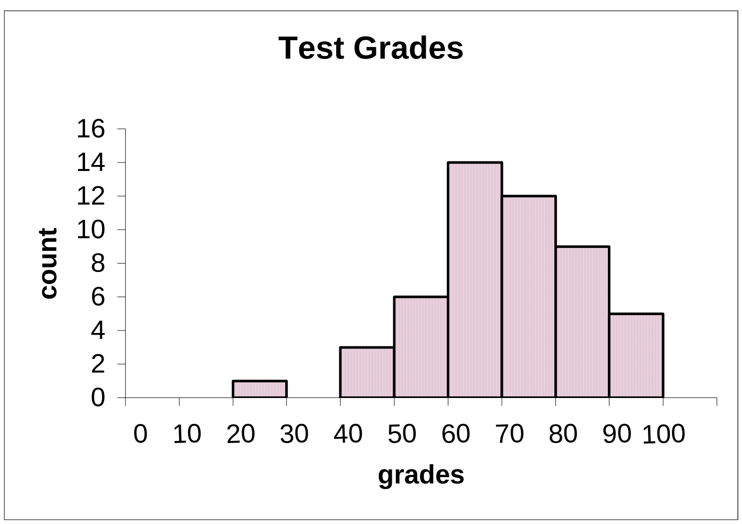
Data Presentation

After getting a grade back on a test, you must occasionally wonder how the rest of the class did. For example, if you thought the test was hard, you'd probably wonder if everyone else thought the same and your suspicions might be confirmed by hearing that no one got over 70 points out of 100. On the other hand, if your score was in the 70's you might be less than thrilled to discover that the rest of the class got scores in the 80's and 90's. So what is the best way of seeing how the rest of the class did?

One thing you can do is look at a list of the grades received by the students in the class, such as the following:

73	47	71	65	85	61	74	80	65	67
62	84	71	99	70	88	69	83	81	71
80	68	26	58	71	90	58	64	70	95
91	67	48	41	62	75	66	77	78	66
50	52	83	58	50	87	91	60	78	67

It is very difficult to get much of an idea of how the class did just by looking at these numbers. However, there are ways of presenting the data that make things much clearer. For Example, the diagram below, called a <u>histogram</u>, shows you at once how the grades are distributed.



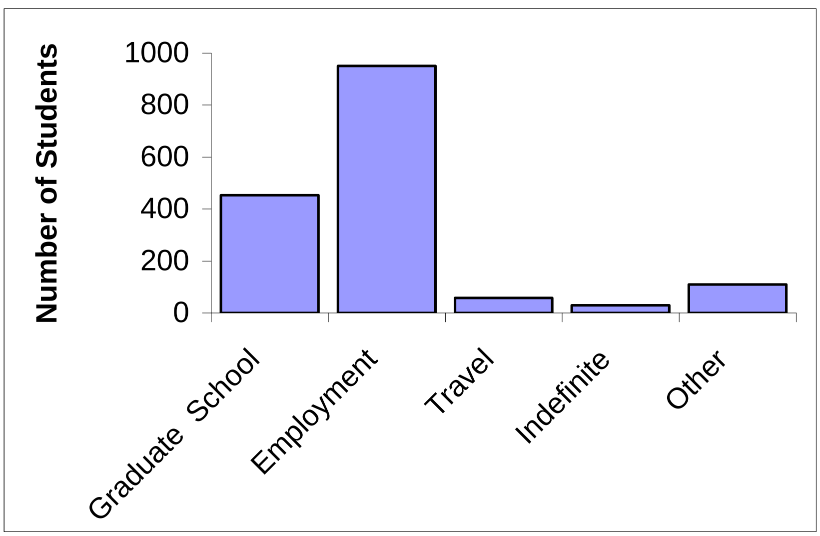
* Information and Graphs taken from Core Curriculum Quantitative Reasoning Requirement Data Text, © President and Fellows of Harvard College, 1991.

Different Ways of Graphing Data

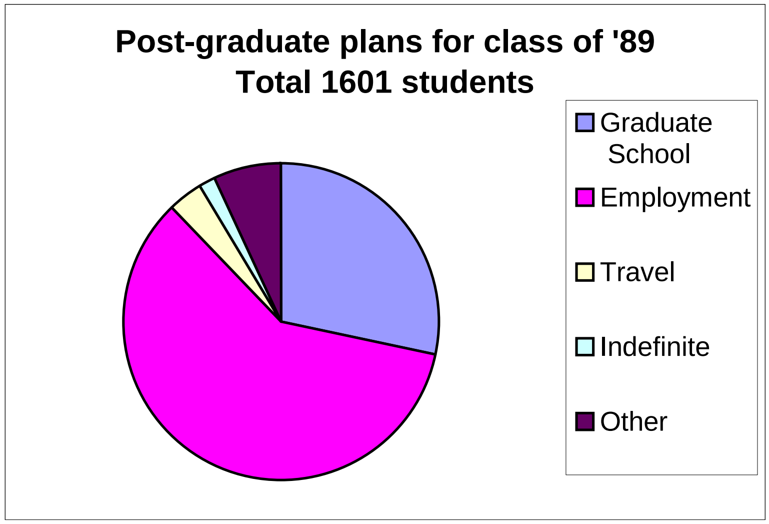
The following shows where graduating students from the class of 1989 planned to go after graduating.

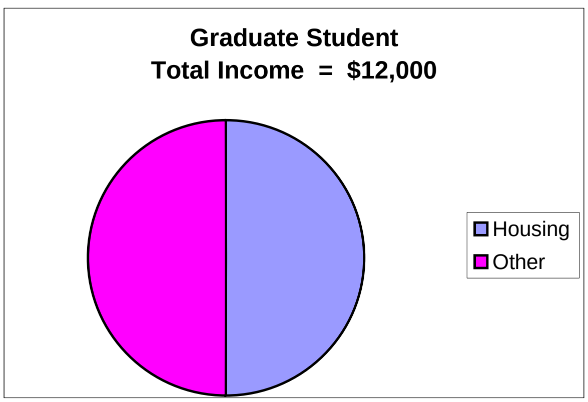
453	Graduate School
951	Employment
58	Travel
29	Indefinite
110	Other
1601	

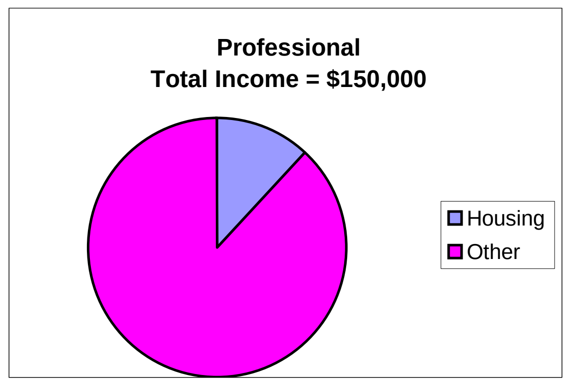
Here is another representation in graphical form.



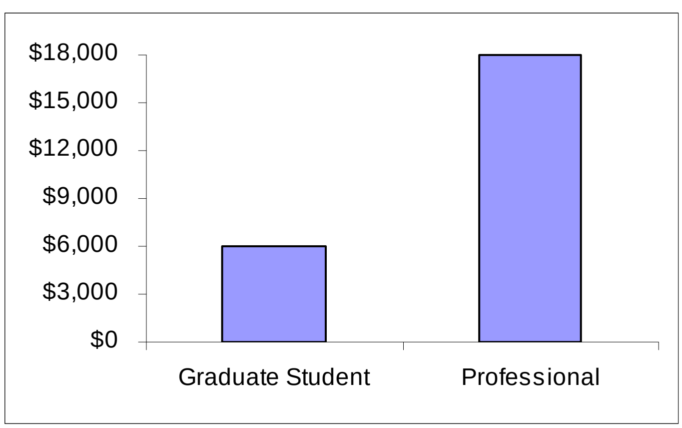
The following graphs depict the same data in yet another form. These charts are called Pie Charts. Pie Charts are excellent for showing relative percentages. When reading a Pie Chart, always refer to the 'Total' usually at the top of the chart.







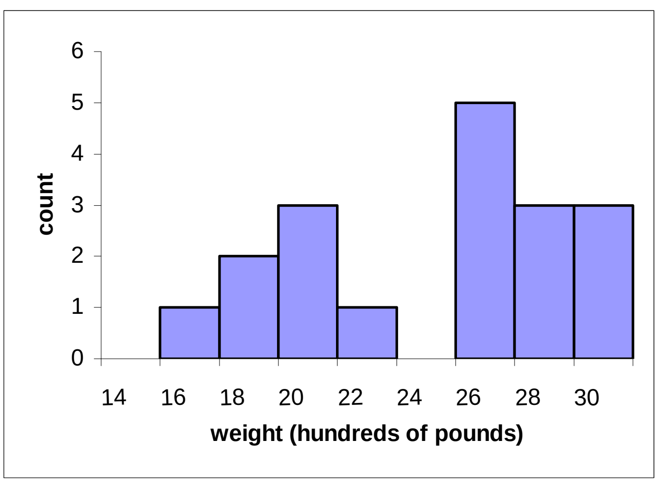
Here is another histogram showing just the amount of money each of the students in the above two charts spent on Housing.

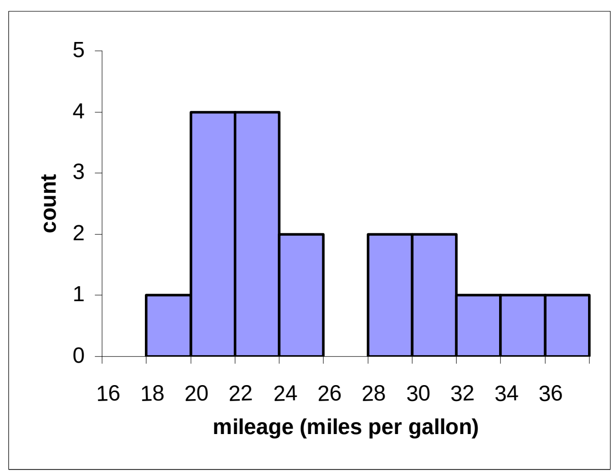


Consider the following data:

Car Model	Weight (pounds)	Miles per gallon
Honda Civic	1750	36
Datsun 310	1995	31
Honda Accord	2090	34
Mazda GLC	2000	29
Plymouth Champ	1950	33
Subaru	2015	30
Audi 4000	2250	28
Pontiac Phoenix	2550	24
Chevy Citation	2740	24
Saab 99	2530	21
Datsun 810	2660	21
Ford Pinto	2525	22
Mercury Bobcat	2600	22
Toyota Corona	2576	23
AMC Concord	2750	20
Buick Skyhawk	2800	19
Olds Starfire	2571	22
Plymouth Sapparo	2800	21

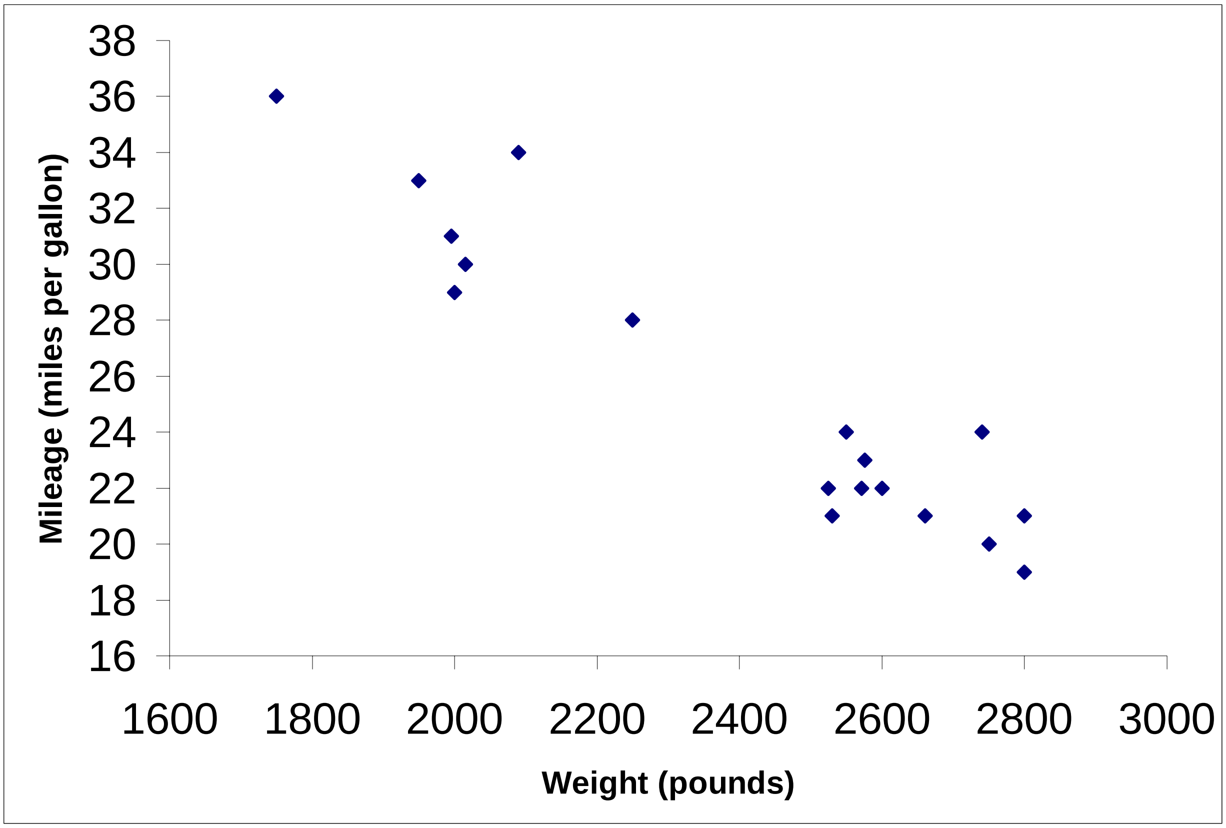
We could draw two histograms – one for the weights and one for the mileage. Each of these will show the distribution of values for one of the characteristics.



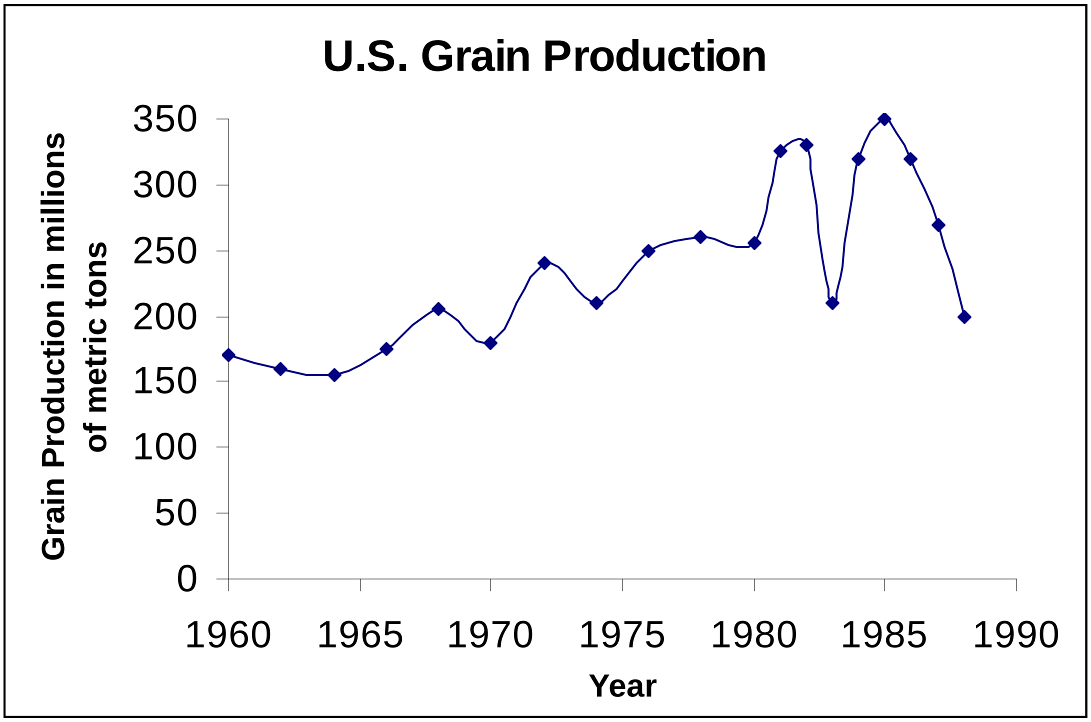


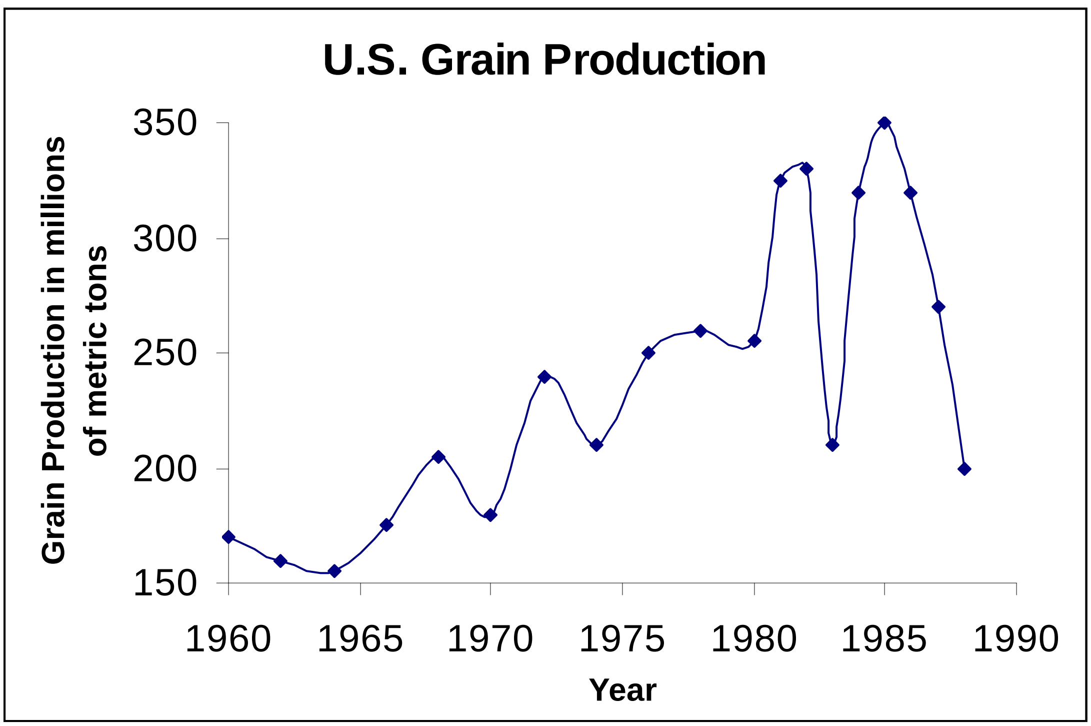
But what if we wish to compare the weight of the car with the mileage. Then we need to get a graph of 'paired' data as seen on the next page.

This is called a SCATTERPLOT. A scatter plot is useful for presenting paired data.



When you wish to look at how a quantity changes over time, a useful type of graph is a **line graph.** Look at the graph below.





Working with Numbers

In order to work with numbers, it's a good idea to review a bit about our number system and the numbers we use everyday. Thus, we will look at different 'kinds of numbers,' those that are familiar and others that are *unfamiliar*! Also, some operations may seem a bit new and perhaps you will never need to know them. But part of the course does attempt to familiarize the student with concepts that will indeed appear someplace in their lifetime either in books or in the media. Hopefully, when the student is confronted with one of these concepts, there will be a short Archimedean shout, "Eureka!"

The Complex Number System – This is the number system we use. However, it is composed of two types of numbers:

1. Real Numbers and
2. Those numbers that are not real – called Imaginary.

In everyday mathematics, we usually only utilize the 'Real' numbers. (More about 'imaginary' numbers later.) Let's look at the Real Number System.

The Real Number System (R) consists of 'SETS' of numbers as follows:

(N) **Natural** Numbers – Counting No's ex. 1, 2, 3, . . .

(W) **Whole** Numbers – Natural No's and Zero ex. **0**, 1, 2, 3, . . .

(Z) **Integers** – Whole No's and Negatives ex. . . . **–3**, **–2**, **–1**, 0, 1, 2, 3, . . .

(Q) **Rational** Numbers–Integers and Fractions ex. . . .**–2.5**, –2, –1, 0, $\mathbf{\frac{1}{3}}$, 1, 2, **3.625**, $\sqrt{\mathbf{9}}$, $\mathbf{\frac{4}{1}}$

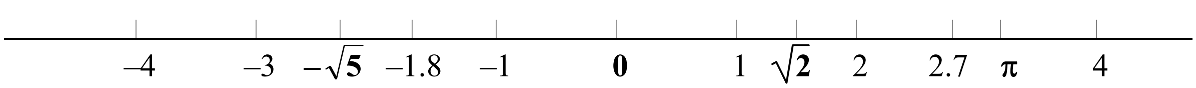
i.e. numbers expressed as a ratio, a fraction or a decimal. Decimals must be terminating or repeating.

(H) **Irrational** Numbers – numbers that **<u>cannot</u>** be expressed as a ratio. ex. . ..0.101001000100001. . ., $\sqrt{\mathbf{2}}$, **e,** π, 4. . (please see footnote below on 'e')[1]

Thus, the real numbers consists of all the Rational and Irrational numbers.

(R) **Real Numbers** = **Rational and Irrationals** ex. . . .–3, $-\sqrt{\mathbf{5}}$, –2, –1.8, 0, 1, $\sqrt{\mathbf{2}}$, 2.7, 3, π

The Real Number Line: we can 'graph' the real numbers on a number line. Here are a few.



[1] 'e' is a number used in compounding interest as well as in other areas of higher mathematics.

Problem -- Name the SET in the Real Number System that contains the following numbers. To find the answer, look at each set and see if the number is contained in that set. There can be more than one set. If the number is not a real number, then say so.

ex. 1. **The number 5**: 5 is a Natural Number, it is also a Whole Number, an Integer, and a Rational Number.

ex. 2. **The number -6**: -6 is an Integer, and a Rational Number.

ex. 3. **The number $\sqrt{5}$** : $\sqrt{5}$ is an Irrational Number

ex. 4. **The number** $\sqrt{-4}$: this is the square root of a negative number. It is neither rational nor irrational. In fact, this type of number is NOT anywhere in the set of real numbers. So it is NOT a real number.

The Imaginary Numbers (I)

An 'imaginary' number is just what you expect: it is a number that we cannot 'see.' The imaginary numbers are very important in mathematics and physics especially in the study of electric circuits. (We utilize electricity although we cannot 'see' it. Thus imaginary numbers seem appropriate here!) We will not 'use' imaginary numbers in this course, but you should be able to recognize them. The best way to understand the concept is via an example. But first, we need to do a little review of certain 'real' numbers.

Let's review what a square root is first. The square root of 9, written $\sqrt{9}$, means find a number such that when you multiply it by itself it gives 9. So that number is 3, since 3 x 3 = 9. 3 is a nice rational number. (Note: Although -3 x -3 = 9, we define the square root of a number to be the positive square root. This is called the 'Principal Square Root.') Another example: The square root of 36 is 6. Why? Because 6 x 6 = 36.

Now let's go back to ex. 4 above: To what Set of numbers does $\sqrt{-4}$ belong?

ex. 4. **The number** given was $\sqrt{-4}$ Now that we know about square roots, this means: find a number such that when you multiply it by itself you would get $\mathbf{-4}$. Can we find such a number? Let's see,

What if we try 2? 2 x 2 = 4 NOT $\mathbf{-4}$

What if we try -2, -2 x -2 = 4 still not -4

(Later, we will review the fact that when you multiply two negative numbers together, the answer is a positive number.)

Thus we cannot find such a number. Here's where mathematicians INVENT a number that would help to solve this problem. Hence the introduction of IMAGINARY numbers.

Imaginary Numbers - Definition: We define the basic imaginary number designated by the letter, *i* , to be equal to the square root of negative 1. That is

$$i = \sqrt{-1}$$

Thus, whenever we *cannot* find a square root of a *negative* number, we can assume it is imaginary. Going back to the $\sqrt{-4}$, we now can write this number as the product of two square roots, one of which we can find, i.e. since $-4 = 4$ times -1 then

$$\sqrt{-4} = \sqrt{4} \text{ x } \sqrt{-1} = 2i \quad \textit{(since the} \sqrt{4} = 2)$$

$\sqrt{-4}$ is an imaginary number which can be written as 2*i*.

Look at another example:

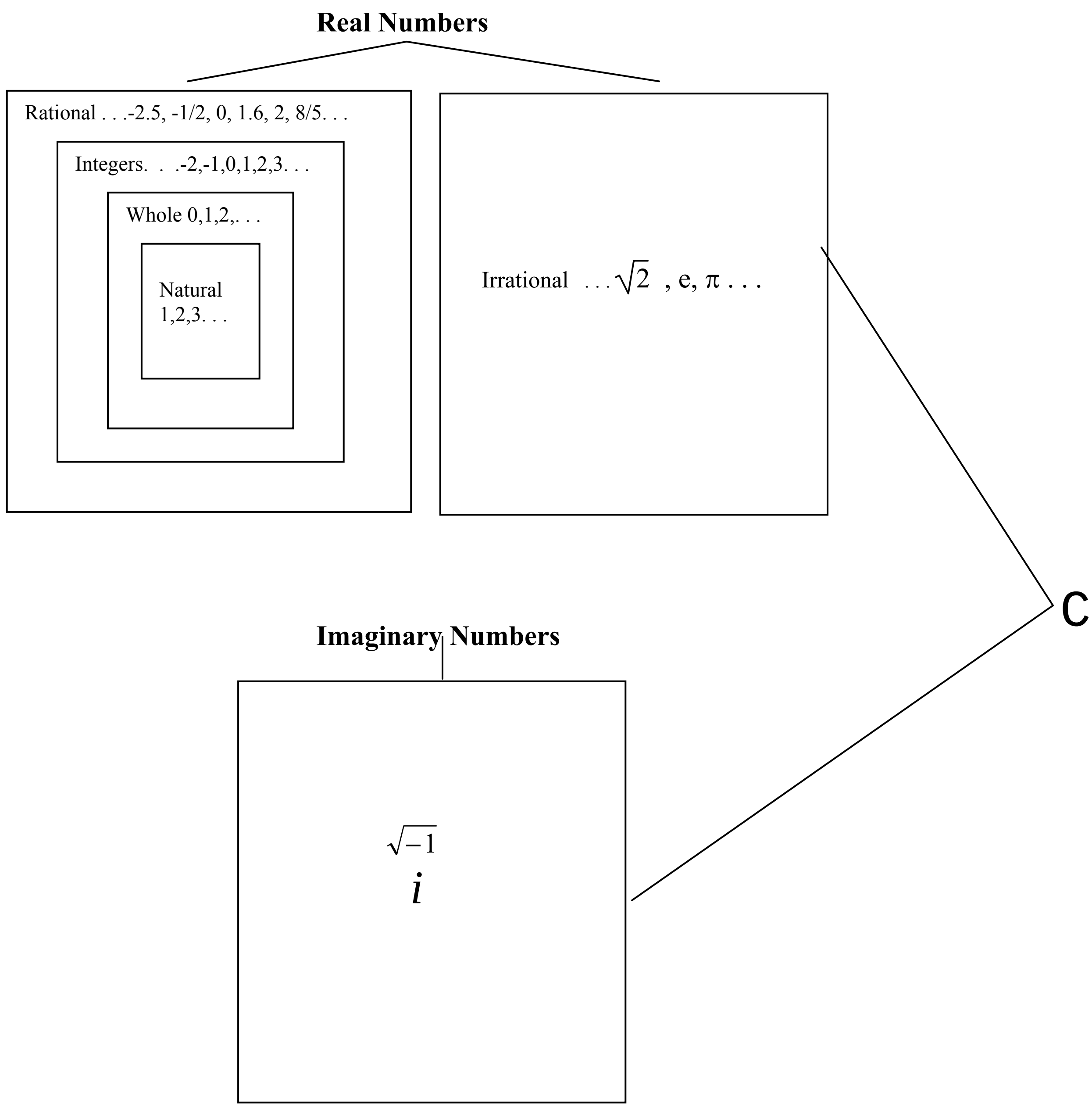
$\sqrt[3]{-8}$ This is NOT imaginary. It is a real number because we can find the 'cube root.'

The Cube Root of $-8 = -2$ Why? Because -2 times -2 times $-2 = -8$

(Again, we will review multiplication of negative numbers later. But for now, a negative number times a positive number = a negative number. e.g. 3 x −2 = − 6)

*Do not worry if you cannot understand this at first as you will not be asked to do any calculations with imaginary numbers. The purpose of introducing it is to expose you to as many types of numbers as we can. Let's review what we have so far in the form of a graphic on the next page.

The Complex Number System



Mathematical Notations

To indicate that one value is greater than another, we use the *inequality* symbol for greater than, $>$

$$\mathbf{8 > 5} \qquad\qquad \mathbf{0 > -4} \quad \text{watch the negative sign!}$$

Similarly, to indicate one value is less than another, use the 'less than' symbol, $<$

$$\mathbf{3 < 7} \qquad\qquad \mathbf{-6 < -2}$$

This notation applies to all numbers including fractions, decimals, percents, radicals, etc.

$$\frac{2}{3} > \frac{1}{2} \qquad -2.75 < 0.1 \qquad 1.25\% > 0.3\% \qquad \sqrt{2} < \pi$$

There is also a sign that indicates an equality **or** an inequality.

ex. 1. a number, x, that is less than **or** equal to 5 is written $\mathbf{x \le 5}$

ex. 2. an integer, x, that is *between* -4 and $+4$ that does NOT include -4 but does $+4$ is written as $\mathbf{-4 < x \le 4}$ These integers are $-3, -2, -1, 0, 1, 2, 3, 4$.

Absolute Value The absolute value of a number is simply the numeral by itself without any sign on it. This is called the 'magnitude' of the number. The positive is understood. To indicate we want an absolute value, we enclose the numeral within two straight lines. The absolute value of -12 is 12 written: $|-12| = 12$. Also, the absolute value signs are like parentheses. We do what's inside them first. For example:

$$|15-20| = |-5| = 5.$$

Other examples: a. $|-7| = 7$ b. $|99| = 99$ c. $|11 + -2| = |9| = 9$

Other useful manipulations

Rounding = In everyday math, we round numbers by utilizing the number 5. To round:

a. look at the last number we wish to end our rounding.
b. look at the number following this one
 if this number is a 5 or larger then add one to the last number you wish to end with.
 If this number is less than a 5, then truncate from here

Ex. 1.: 3948.1263 Say we wish to round to two decimals after the decimal point. That is we want to end at the '2' So, a. look at the 2 b. look at the number after the 2. It is a 6. Since the 6 is larger than a 5, we round up the 2 to a 3. The final number is 3948.13

Ex. 2. 3948.1263 Let's round this number to a whole number. So we want to end at the 8. Since the next number is a 1 and less than a 5 we just 'truncate' the rest of the numbers so the final rounded number is 3948.

CHAPTER 2

Algebra – an introduction

Algebra is simply the manipulation of numbers using letters instead of numbers. Here are some things to keep in mind when doing algebraic manipulation.

1) Think of a letter as a number when confusion arises.
2) An *equation* must have an equal sign in it.

   $x + 5 = 8$ is an equation.
3) An algebraic *expression* does NOT contain an equal sign. $x + 5$ is an expression.
4) To 'solve' an equation means to find the numeric value of the 'letter,' called the ‘variable.’
5) We can 'move' letters and numbers around from one side of the equal sign to the other in order to 'solve' an equation, but we must use certain rules to do so in order to always keep the equation in 'balance.'

Solving Equations – Isolate the 'variable' (the letter)

Let's assume you cannot tell that $x = 3$ in the equation $x + 5 = 8$. We can find 'x' by trying to isolate it. If we move the 5 to the other side of the equal sign then the 'x' will be isolated and we will have its value. You must always keep the equation in balance by doing the same thing to both sides.

To move a value (or a letter) you must do the *opposite* operation to it. Here we have 5 ADDED to the x. We need to SUBTRACT the 5 from the left side so we get just an 'x' by itself on the left. However, to keep the equation in balance, we MUST do the same thing to the other side of the equal sign. Therefore SUBTRACT 5 from the right side. Let's do it:

ex. 1. $x + 5 = 8$

$x + 5 - 5 = 8 - 5$

$x = 3$ because $x + 5 - 5 = x + 0$ the 5's cancel!

Other examples:

ex. 2. $Y - 7 = 6$

$Y - 7 + 7 = 6 + 7$ Add 7 to both sides.

$Y = 13$

ex. 3. $3A - 4 = 11$ We still try to isolate the variable, in this case, 'A'.
Hint: ALWAYS get rid of the stuff furthest from the letter first.
Therefore, move the '4' to the right side.

$3A - 4 + 4 = 11 + 4$

$3A + 0 = 15$ Now we are almost there. We must somehow get rid of the '3'

$3A = 15$ DO THE OPPOSITE OPERATION TO BOTH SIDES.
3A means 3 times A. so the opposite of multiplication is division.

$\frac{3A}{3} = \frac{15}{3}$ Divide both sides by 3.

$A = 5$ <u>Check</u>: You should always check your answer. If we replace the A by '5' in the original equation, then indeed our equation works. $3(5) - 4 = 11$ ✔

Operations with Everyday Numbers

Signed Numbers – all numbers have a sign attached to the front of the number. It is either positive or negative. If the sign does not show, it is positive.

e.g. 8 , + 8, – 10

Multiplication and Division of numbers

Notice the multiplication sign between two numbers can be an 'x' as in $a \times b$, an asterisk *, $a * b$, a dot in the center $a \bullet b$, or parentheses $(a)(b)$. The best way to understand the multiplication or division rules is by example. Here are the rules in symbol form with a specific example.

RULES FOR MULTIPLICATION AND DIVISION

$+ \cdot + = +$	**3 x 9 = 27**	$+ \div + = +$	**8/4 = 2**
$+ \cdot - = -$	**3 x –9 = –27**	$+ \div - = -$	**8/–4 = –2**
$- \cdot + = -$	**–3 x 9 = –27**	$- \div + = -$	**–8/4 = –2**
$- \cdot - = +$	**–3 x –9 = 27**	$- \div - = +$	**–8/–4 = 2**

ZERO - Because the number, Zero, has so many different properties, we will discuss each one within the concept we are learning.

Multiplication by zero: Any number times zero = zero

ex. 5 x 0 = 0

ex. a x 0 = 0

Division of zero by a number: Zero divided by any number = zero

ex. $\frac{0}{6} = 0$ Why? because 6 x 0 = 0 $\frac{0}{6} = 0$

Consider the following example. $\frac{8}{4} = 2$ Why? because 4 x 2 =8. But what about $\frac{3}{0}$?

Does that mean $\frac{3}{0} = 0$ because 0 x 0 = 3? Hmmmmmmmm! This doesn't make sense.

Here's another Contradiction: Some people say that $\frac{3}{0} = 3$ because if you take 3 things and divide them between 0 people still have 3 things left. Well okay. But that means that each of the 0 people gets 3 things! What a mess! Let's use the multiplication rule: $\frac{3}{0} = 3$ because 0 x 3 = 3.
Hmmmmmmmm! Egad! With so many contradictions, it appears that any number divided by zero has no meaning.

Thus we say any number divided by zero is **Undefined**. ex. $\mathbf{\frac{3}{0}}$ **= Undefined**

Addition and Subtraction in the Real Number System

In order to add or subtract numbers as we do everyday, we must first start by looking at two numbers to check their signs and the ***operation*** between them. For example:

ex. a. $7 + -3$ We have a positive 7 and a negative 3 and the <u>operation</u> is addition.

$7 \boxed{+} -3$ Here, I have put the operation in a box so we can see the addends better.

ex. b. $-8 - 2$ We have a negative 8 and a positive 2 and the <u>operation</u> is subtraction.

$-8 \boxed{-} 2$ (the sign on the 2 is invisible. When a number is positive we do not need to show the plus sign. This could have been written as $-8 - +2$)

- Process is as follows. When we <u>**add**</u>, we use the rules for addition below. When we <u>**subtract**</u>, it's much easier to *change the subtraction into an addition and use the rules for addition. Let's see what that means.*

In actuality, subtraction is simply "*adding the opposite."* So, before we give the rules on addition, let's demonstrate how to make a subtraction into an addition.

Changing an operation sign to the opposite operation.

We must do <u>two</u> things: You cannot do one WITHOUT doing the other at the same time.

1. CHANGE the *subtraction* <u>operation</u> sign (a minus sign) to an addition (plus) sign.
2. CHANGE the <u>sign</u> of the number **after** the operation sign to its opposite sign. Say aloud,

"ADD THE OPPOSITE"

Using example **b.** from above, **ex. b. $-8 - 2$**: immediately, say to yourself, "add the opposite."

Thus, change the operation sign (minus) to a plus, and the invisible + sign on the 2 to a negative. Here's what we get so far:

ex. b. $\mathbf{-8 \boxed{-} 2}$
$\mathbf{-8 \boxed{+} -2}$

Once we know how to change the operation from subtraction to addition, we can <u>utilize the rules for addition</u>. Let's do one more example of a <u>change</u> first.

ex. c. $\mathbf{6 \boxed{-} -2}$
$\mathbf{6 \boxed{+} +2}$

Say, 'add the opposite' and as you do, change the signs.

For both examples, b. and c., we can now 'add.' Of course, we can do ex. c. as this is a simple addition. But what about ex. b. with the two negatives? Here, we need the <u>rules for addition.</u>

THE RULES FOR ADDITION

(1) If the signs on the two numbers are the same, just <u>add</u> the two numerals*, use the sign that's there.

(2) If the signs are different, find the <u>difference</u> of the numerals*, use the sign of the larger one.

***Note: The <u>numeral</u> is simply the number *without* the sign attached. This is often designated as the 'absolute' value or the 'magnitude' of the number.**

Addition and Subtraction Examples

ex. 1. $\mathbf{13 - -8}$ (the 'numerals' are 13 and 8)

Step One: "add the opposite" $\mathbf{13 - -8 = 13 + +8}$
Step Two: Since the signs are the same just add

$$\mathbf{13 + 8 = 21}$$

ex. 2. $\mathbf{-8 - 2}$ (ex. b. from previous page)

Step One: "add the opposite" $\mathbf{-8 - 2 = -8 + -2}$
Step Two: Signs are the same on both numerals so add numerals, use sign that's there

$$\mathbf{-8 + -2 = -10}$$

ex. 3. $\mathbf{10 - 6}$ this is a normal subtraction as we know it, but let's see if the rules work.

Step One: "add the opposite" $10 + -6$
Step Two: Look at the signs. Signs are different so find the difference of just the numbers themselves, i.e. the 10 and the 6. Now use the sign of the larger numeral.

$\mathbf{10 - 6 = 10 + -6 = 4}$ Since 10 is positive and largest numeral.

ex. 4. $\mathbf{-9 - (-6)}$ (You may remove parentheses if there is nothing left to do inside them!)
$\mathbf{-9 - -6}$ (Be careful you don't lose the two negative signs.)

Step One: "add the opposite" $\mathbf{-9 - -6 = -9 + +6}$
Step two: Signs are different so find the difference. (We want the difference of just the numerals themselves; in this case, find the difference between 9 and 6.) Then use sign of larger numeral. Thus

$\mathbf{-9 + +6 = -3}$ We can also write this as $-9 + 6$.

ex. 5. **– 7 – 1**

Step One: "add the opposite" **– 7 – 1 = – 7 + – 1**
Step two: Signs same so add the numerals and use sign that's there.

– 7 + – 1 = – 8

ex. 6. **4 – 11**

Step One: "add the opposite" **4 – 11 = 4 + – 11**
Step two: Signs different so find the difference between 4 and 11. Use sign of larger.

4 + – 11 = – 7

Zero – the elusive number – once again.

As previously mentioned, because the number, Zero, has so many different properties, we will discuss each one within the concept we are learning.

In this section on operations with signed numbers, it should be noted that zero has NO SIGN associated with it. It is neither positive nor negative. So right at the start of our discussion of zero, it appears to be 'elusive' as the title implies.

However, in everyday mathematics, we need not worry about the properties of zero as we are so used to working with this unusual number. A few examples are given below just as a reminder.

First: we know that if we have one number and subtract that number from itself, we have nothing left, thus zero. Similarly, taking a negative number and adding the same amount to it will give us 'nothing' again. Here are a few examples:

ex. 7. 2 – 2 = 0 and −2 + 2 = 0

ex. 8. **– 5 – – 5 = 0 Why?**

Step One: "add the opposite" **– 5 $\boxed{-}$ – 5 = –5 + + 5 = –5 + 5**
Step two: Signs different so find the difference between 5 and 5 which is zero.

so **–5 + 5 = 0**

PERCENT INCREASE OR DECREASE

$$\text{PERCENT CHANGE} = \frac{\text{AMOUNT OF CHANGE}}{\text{ORIGINAL AMOUNT}} \times 100$$

To find the percent increase or (decrease), divide the change (absolute value) by the original, then multiply by 100.

$$\textbf{\% increase or decrease} = \frac{\Delta}{\textbf{original}} * 100$$

Δ (Delta means change)

The division gives a decimal answer so in order to change to a % we multiply by 100 i.e. move the decimal point two places to the right.

Another way to think of this formula is

$$\textbf{\% change} = \frac{\textbf{New - Old}}{\textbf{Old}} * 100$$

ex. If the size of our class increases from 10 students to 18 students, then

$$\% \text{ increase} = \frac{18 - 10}{10} * 100 = \frac{8}{10} * 100 = 0.8*100 = 80\%$$

To find the increased total amount if you know the % change, multiply by the % and add it to the original.

ex. to increase 40 by 15% $40 + 0.15 * 40 = 40 + 6 = 46$

****The shorter way** is to **multiply by (100% + % increase).** Why the 100? because 40 is100% of itself.

ex. to do the problem multiply 40 by (1+ 0.15) or (1.15) (100% as a decimal = 1)
40*1.15 = 46

ex. How much money will you have in the bank after one year if you deposit $2000 at an annual interest rate of 8.5%? 8.5% is really the % increase, therefore since 8.5% = 0.085 and using the shorter method,

$2000*(1+0.085) = **$2000*(1.085)** = $2170

Of course, if you only wanted the 'increased' amount itself, then just multiply by .085, i.e.
$2000*.085=$170

To find the decreased total amount if you know the % change, multiply by the % and subtract this product from the original amount.

ex. to decrease 40 by 15% $40 – 0.15 * 40 = 40 – 6 = 34$

****The shorter way** is to **multiply by (100% – % decrease)**

ex. to decrease 40 by 15% multiply 40 by (1– 0.15) or (0.85) **40*0.85 = 34**

More Examples with Solutions:

ex. What is 85 increased by 40%

	Long Way	Short Way
Step 1:	85 * .40 = 34	85 * 1.40 = 119
Step 2:	85 + 34 = 119	

ex. What is 250 decreased by 60%

	Long Way	Short Way
Step 1:	250 * .60 = 150	250 * .40 = 100
Step 2:	250 – 150 = 100	

ex. Find the percent change if the Dow Jones Average drops from 10,200 to 9,400

$$\textbf{Percent change} = \frac{\textbf{New - Old}}{\textbf{Old}} \ \textbf{*100}$$

$$\textbf{\% Change} = \frac{9400-10200}{10200} * 100 = \frac{-800}{10200} * 100 = -.07843 * 100 = -7.8\% \text{ rounded to 1 dec.}$$

You can also call this a 7.8% decrease. If you do not use the negative sign, make sure the reader knows it's a decrease.

ex. From the Historical Tables of the Whitehouse's Office of Budget and Management, it is given that the amount spent on Defense Increased from 388.870 Billion in 2003 to 435.674 Billion in 2004[2].

By what percent did Defense Spending increase from 2003 to 2004?

Amount Spent 2003 $388.870 Billion
Amount Spent 2004 $435.674 Billion

$$\% \text{ Change} = \frac{435.674-388.870}{388.870} * 100 = \frac{46.804}{388.870} * 100 = 0.12035 * 100 = 12.035\%$$

[2] http://www.whitehouse.gov/omb/budget/fy2005/pdf/hist.pdf