Chapter 11: Exponential Growth

<u>First – a review of percents:</u>

1. We first review how to find Percent Increase or Decrease. We learned this concept as

$$\textbf{Percent Change} = \frac{\textbf{amount of change}}{\textbf{original amount}} \textbf{ x 100}$$

<u>So to find the percent increase or (decrease)</u>, divide the <u>change</u> (absolute value) <u>by the original</u> then multiply by 100.

$$\textbf{\% increase or decrease} = \frac{\Delta}{\textbf{original}} \textbf{*100} \quad \Delta \text{ (Delta means change)}$$

the division gives a decimal answer so to change to a % we multiply by 100. Another way to think of this formula is

$$\textbf{\% Change} = \frac{\textbf{New - Old}}{\textbf{Old}} \textbf{*100}$$

e.g. If the size of our class increases from 150 students to 162 students, then

$$\% \text{ increase } = \frac{162-150}{150}*100 = \frac{12}{150}*100 = 0.08*100 = 8\%$$

2. Now we study finding the actual <u>Final Amount</u> knowing the <u>Percent Increase</u>

Long way of getting the FINAL amount after the increase:

As you can see from the above example, we increased our 150 by a percent increase of **8%**

What we did was to multiply 150 by **8%** or **.08 = 12**

Then we added the 12 (amount of increase) to 150 to get the final amount: **150 + 12 =162**

Short way of getting the FINAL Amount after the increase:

What was actually done above was the following:

We took **100% of 150** and added **8% of 150 = 108% of 150 = 1.08 x 150 = 162**

Let's do another example:

ex. 1. A furniture store owner increases his price on his Chippendale Chairs by 35%. The current price of these beautiful chairs is $400 each. How much do they cost now?

Long Way: An increase of **35%** means multiply $400 by 35% and add that amount to $400

.35 x 400 = $140 now the final selling price is $400 + $140 = $540

Short Way: multiply original amount by 100% + % Increase = 135% = 1.35

$400 x 1.35 = $540

What about finding the Final Amount if we have a Percent Decrease?

ex. 2. Our furniture store owner has decided to give a discount on his Queen Anne Chairs. They too originally sold for \$400. He has decided to discount them by 35%.

Long Way: A decrease of 35% means multiply \$400 by 35% and SUBTRACT that amount from \$400.

400 x .35 = 140 so new price is 400 – 140 = \$260

Short Way: Let's look at what we REALLY did the Long Way

We took 100% of 400 and subtracted 35% of 400

Or we could say we took **100% – 35% of 400**

So let's try

100% – 35% = 65%

65% x 400 = .65 x 400 = \$260

So when we DECREASE by a percent, we really multiply by 100% – the % Decrease.

ex. 3. Here's another example. Reduce 200 by 8%.

Long way: 8% of 200 = 16 then 200 – 16 = 184

Short way: 100% – 8% = 92% and 92% of 200 = 184

Note: 100% as a decimal is a '1' Thus 100% – 8% = 1. – .08 = .92

MORE EXAMPLES

To find the increased amount if you **know the % change:**

ex. 4. to increase 40 by 15%

Long Way: multiply by the % and add it to the original. 0.15 * 40 + 40 = 6 + 40 = 46

Short Way: multiply by **(100% + % increase)**

Thus: multiply 40 by (1+ 0.15) or (1.15) **40*1.15 = 46**

ex. 5. You have \$2,000 and decide to put it in the bank instead of using it for bills. How much money will you have in the bank after one year if you deposit \$2,000 at an annual interest rate of 4.5%?

4.5% = 0.045

Long Way: .045 x \$2000 = 90 add 2,000 + 90 = \$2,090

Short Way: \$2000*(1+ 0.045) = **\$2000* (1.045)** = \$2,090

<u>To find the decreased amount</u> if you know the % decrease:

ex. 6. to decrease 40 by 15%

Long Way: multiply by the % and <u>subtract</u> it <u>from</u> the original. 40 – 0.15 * 40 = 40 – 6 = 34

Short Way: multiply by **(100% – % decrease)**

Thus: multiply 40 by **(1 – 0.15)** or (0.85) **40*0.85 = 34**

ex. 7. Let's say you owe $2,000 on your credit card. Your minimum monthly payment is 4.5% of your current bill. What is your balance after sending them the minimum payment?

Long Way: .045 x $2000 = 90 subtract: 2,000 – 90 = $1,910

Short Way: 100% – 4.5% = 95.5% = .955 or
1 - .045 = .955
$2,000* (.955) = $1,910

<u>Now for the new material: EXPONENTS</u>

$\mathbf{a}^n$ In general, '**a**' is the base and '**n**' is the exponent.

6^2 read, 'six squared,' means 6 x 6 = 36. Here, 6 is the base and 2 is the exponent.
It is also read as "6 to the power of 2."

$2^3 = 2*2*2 = 8$ be careful here to multiply 2 by itself three times. The answer is 8 not 6.

$(-3)^4 = (-3)(-3)(-3)(-3) = 81$ Here, the number (–3) is raised to the 4th power. But, <u>notice</u> that $-(3)^4 = -3x3x3x3 = -81$. The negative sign is outside the parentheses. The exponent only applies to what's inside the parentheses.

Using letters instead of numbers is the same.

x^4 is read as "x to the power of 4" "x" is the base and "4" is the exponent

x^4 means multiply x by itself 4 times i.e. x*x*x*x

*<u>**ZERO - AGAIN.**</u> **ANY NUMBER RAISED TO THE ZERO POWER = 1 (exception?)**

examples: $5^0 = 1$, $397^0 = 1$, $(q^7xyz)^0 = 1$ $\mathbf{(-42z)^0 = 1}$, **but** $\mathbf{-(42z)^0 = -1}$

RULES FOR MANIPULATION OF EXPONENTS

Multiplying numbers with the same base that have an exponent - add the exponents.

ex. $5^4 \times 5^2 = 5^{(4+2)} = 5^6$

ex. $a^b \times a^c = a^{b+c}$

Dividing numbers with the same base that have an exponent - subtract the exponents.

ex. $5^4 \div 5^2 = \dfrac{5^4}{5^2} = 5^{4-2} = 5^2$

ex. $\dfrac{5^3}{5^5} = 5^{3-5} = 5^{-2}$

ex. $a^b \div a^c = \dfrac{a^b}{a^c} = a^{b-c}$

Raising a number with an exponent to a power - multiply exponents.

ex. $(5^2)^4 = 5^8$

A number with a negative exponent means: 1 divided by that number with the positive of the exponent.

ex. $5^{-2} = \dfrac{1}{5^2} = \dfrac{1}{25}$

<u>Scientific Notation</u>

Scientific Notation is used extensively in everyday mathematics as well as in many other disciplines as economics, the natural sciences, and astronomy. It is a convenient way to write very large or very small numbers. ***The rules utilize decimals and exponents:***

<u>Rule 1:</u> Only one (1) digit appears to the left of the decimal point. That is called the 'significant' digit. There can be as many numbers to the right of the decimal as you may want. e.g. $6.82147\text{x}10^3$. The '6' is to the left of the decimal point. It is the significant digit.

<u>Rule 2:</u> Following the numbers, we multiply by a power of 10. e.g. $3500 = 3.5\text{x}10^3$
If the power is positive, we have a large number. To get to the <u>expanded</u> form, move the decimal point to the RIGHT the number of places indicated by the magnitude of the power and fill in with zeroes if necessary. $3.5\text{x}10^3$ means move the decimal point 3 places to the right, i.e. 3500. If the power is negative, move the decimal that number of places to the left. $7.2583\text{x}10^{-4} = .00072$

<u>Example</u>:The distance from the earth to the sun is $9.3\text{x}10^7$ miles. That is, 93 million miles, or moving the decimal place to the right 7 places, we get 93,000,000 miles.

<u>Calculators and Scientific Notation:</u> -- Most calculators display a number like $\mathbf{7.2583x10^{-4}}$ by putting the exponent as a two digit number a few blank spaces to the right of the digits, or by putting an 'E' in front of the exponent.
$\boxed{\mathbf{7.2583\quad -04}}$ or $\boxed{\mathbf{7.2583\ E{-}04}}$ 93 million as $\boxed{\mathbf{9.3000\quad 07}}$ Check your calculators!!

Using Exponents: EXPONENTIAL GROWTH and DECAY

Over the last few weeks we have looked at Linear Growth - things which grow at a constant rate - and how this growth can be represented by a straight line on a graph. We also discussed how to estimate in-between values and projected values - as well as Regression - estimating the linear relationship when it is not exact.

Today, we look at **exponential growth** - things which grow at a faster and faster rate (or decrease at a slower and slower rate). Examples of exponential growth are:

- money in the bank at a compound interest rate,
- bacteria growing in a laboratory,
- chain letters.

Examples of exponential decrease:

- a radioactive element which loses half of its mass during a given period of time – called the "half-life."
- endangered species of animals.

Exponential growth is much more ***spectacular*** than linear growth - to your advantage for money in the bank, but to your disadvantage for diseases from bacteria!

In class we will show a comparison between simple interest (linear growth) and compound interest (exponential growth). You'll see quite a difference, even though it sometimes takes a while to become obvious. Remember, a *dollar* increase is linear, whereas a *percent* increase is exponential, and far better in the **long run.**

Comparison of two kinds of growth

Linear Growth	**Exponential Growth**
Characterized by a *constant* number indicating a *steady* rate. This constant is the growth rate is called the slope In linear growth we ADD this amount each time	Characterized by a constant percent This constant percent is the growth rate In exponential growth we MULTIPLY by a *growth factor* each time. This growth factor is 100%±the growth rate.
Equation looks like y = (**growth rate**)x + initial amount	Equation looks like y = initial amount(**growth factor**)^n
$\mathbf{y = mx + b}$	$\mathbf{y = A(1 \pm r)^n}$

Difference between Growth Rate and Growth Factor.

WHY A GROWTH FACTOR

If something grows by a percent, e.g. 5%, that percent is called the '**Rate**' of growth.

However, when we want to increase by 5%, we have learned that we multiply by **1.05,** i.e. **100% + 5%.** This sum is called the '**Growth Factor.**'

If we decrease by a percent, say 8%, we multiply by **100% – 8% = 92%**
This difference becomes our **growth factor. (Note: It's called a growth factor even though it is decreasing and not 'growing.' It is also called a 'decay' factor.)**

In general:

Growth Factor = 100% plus or minus the Growth Rate

i.e.

Growth Factor = 100% ± Growth Rate

Examples:

Growth Rate		Growth Factor	
Rate as a %	**As a Decimal**	**Increase**	**Decrease-Decay**
15%	0.15	1.15	0.85
3%	0.03	1.03	0.97
$7\frac{1}{2}$ %	0.075	1.075	0.925
100%	1.00	2.00	---
200%	2.00	3.00	---

Notice from the above table: doubling is the same as a 100% increase. Why? Because if we increase by a percent we get:

100% + percent increase.

100% + 100% = (1. + 1.) = 2 so we are doubling!

Question: If we triple something, what is the percent increase? Let's go back and look at the formula once again for percent increase:

100% + percent increase = 1.+ percent increase as a decimal. (1. + 2.) = 3

Thus if we triple we INCREASE by 200% since 2. = 200%!

<u>MAKE A TABLE</u> - Making a table as we did with linear growth facilitates doing exponential problems. Because we are often working with growth over time, we usually put **<u>time on the x-axis.</u>** *Time* is the <u>independent</u> variable so it is the first column of our table. We often denote time by the letter '**n**' or the letter **'t'** for number of years, months, days, etc. The *amount* we have left, after a certain amount of time, goes on the y-axis. It is the dependent variable and is the second column of our table. (We will be graphing exponential tables and equations next week.)

ex. If we put money, $500, in the bank at, say 6% interest compounded annually, and we wish to know how much we have after 3 years, then our table is as follows:

Time in years	**Amount money**
0	$500
1	500x1.06=530
2	530x1.06=561.80
3	561.80x1.06=595.51

a) What is the **Growth Rate**? The <u>6% is the growth rate</u>.

b) What is the **Growth Factor?** <u>1.06 is the growth factor</u> since 100% + 6% = 1.06 as a decimal
Note: we begin with zero years because we just put our money in and no time has passed.

Exponential Equations

Let's try to get the <u>equation</u> of this exponential problem. In general, our equation should look like:

amount of money = **(initial amount)** times **(the growth factor)** raised to some **power**. Or

$$\mathbf{y = A(1 \pm r)^n}$$

Looking at the table, we could rewrite by leaving the initial amount intact as follows:

Time in years	**Amount money**
0	500
1	500x1.06=530
2	(500x1.06)x1.06=561.80
3	(500x1.06x1.06)x1.06=595.51

Notice that the number of times we multiply the initial amount by 1.06 is equal to the time in years! Rewrite one more time using **<u>exponents</u>**.

Time in years	**Amount money**
0	500
1	$500 \times 1.06^{\mathbf{1}} = 530$
2	$500 \times 1.06^{\mathbf{2}} = 561.80$
3	$500 \times 1.06^{\mathbf{3}} = 595.51$

We try to find a relationship between the <u>exponent</u> in the <u>right hand column</u> with the <u>number</u> in the <u>left hand column</u>! We will see how this works in class with other problems. So, now we can write the equation by substituting our values into the general equation for any exponential.

General equation: $\mathbf{y = A(1 \pm r)^n}$

Our equation: $\mathbf{y = 500(1.06)^n}$ where: **A = 500** and **Growth Factor = 1.06**

Other Examples

Ex. 1. The human embryo begins as a single fertilized cell. At first, each cell in the embryo divides approximately once each day.

a) A newborn child has on the order of a trillion (1,000,000,000,000) cells. Assume that the cell in the embryo continues to divide once each day, that is it doubles itself. How long after conception would the fetus have as many cells as a newborn child?

b) how does your answer to part (a) compare with the normal human gestation period of nine months (use 270 days)? What might be the cause of any disparity?

Partial Solution.

Time in days	**Amount cells**
0	1
1	1x2 = 2
2	2x2 = 4 or $1x2^2$
3	4x2 = 8 or $1x2^3$
4	8x2 = 16 or $1x2^4$

In this case it does not look like we have a 'percent' involved. But we notice that the '2' is the growth factor. When something doubles, we increase it by 100%. So using the same theory that the growth factor is 100% + the growth rate, we get 100% + 100% = 200% which is a '2.0' as a decimal.

We need to continue this table until the right hand side gets to 1 trillion. By trial and error we will find the number of days that approximates the 1 trillion. We will do this in class. (Note: if you know how to use logarithms, of course you may do so. However, there is nothing wrong with trial and error!)

ANOTHER DEFINITION: HALF-LIFE - A half life is the time it takes to end up with 'half' of what you started with.

Ex. 2. A barrel contains 100 pounds of radio-active material. The material has a 'half-life' of one month.

a) How much of the material will be left after one year? 5 years?

b) How long will it take for there to be only one (1) ounce left? (16 ounces = 1 pound)

Partial Solution.

Time in months	**Amt radioactive material**
0	100
1	100x(0.5)=50
2	$100x(0.5)^2 = 25$
3	$100x(0.5)^3$
4	$100x(0.5)^4$
. . .	
. . .	
n	$100x(0.5)^n$

Hint: Solution a) - Notice that one year = 12 months.
Hint: Solution b) – Notice the units, pounds and ounces.

Rule of 70

For small percents, it's possible to **estimate** how long something will take to double (or decrease by half) using the *rule of 70*. This says that:

The **Doubling Time** $= \dfrac{70}{\text{growth rate}}$ Note: here the growth rate is written as a whole number. e.g. 5% use 5 14% use 14

For Example. If we wish to know how long it will take for our $500 above to double at 6% interest, we say

$$\textbf{Doubling Time} = \frac{70}{6} = 11.666\ldots \text{ or approximately 12 years!!}$$

This also works for Halving: The "halving time" is the time it will take for something to lose one-half its original amount. This is called the **half-life.**

$$\textbf{Half-life} = \frac{70}{\text{growth rate}}$$

(Again, this formula works if the rate is fairly small – approximately less than 20)

Formula for Compounding Interest

$\mathbf{Q = A(1 + \frac{r}{n})^{nt}}$ where **A** = the ***initial*** amount of money invested

r = the interest rate in decimals e.g. 8% = .08
t = number of years invested
n = number of compounding periods in year e.g. monthly n = 12
Q= amount of money at the end of the time period (or use a **Y**)

e.g. If we invest $1000 in an account that earns 8% interest compounded annually, at the end of 25 years we would have $6,848.50. i.e. P = $1000, r = 8%, t = 25 years, n = 1 (annually).

$$\text{Amount} = \$1000*(1 + .08)^{25} = \$6{,}848.48 \quad \text{notice that } t = 25.$$

Now, what if the interest is *compounded semi-annually*, n = 2? or monthly, n = 12? What do we do? Let's see what happens to our $1,000 at different compounding periods.

Semi-annually, n = 2 $\text{Amount} = \$1000*(1 + \frac{.08}{2})^{2*25} = \$7{,}106.68$

Remember to divide r by n, the # of periods.

Monthly, n = 12 $\text{Amount} = \$1000*(1 + \frac{.08}{12})^{12*25} = \$7{,}340.18$

Try daily, n = 365 $\text{Amount} = \$1000*(1 + \frac{.08}{365})^{365*25} = \$\ 7{,}387.43$

Example to follow while doing homework – with solutions:

Example: Suppose the population of a small country is 450,000. If the population grows at an annual rate of 2%, do the following:.

Write down the Growth Rate and Growth Factor where we can see them.

Give a general equation for this problem. That is write your answer as

$\mathbf{y = A(1 \pm r)^n}$ by giving the value of the initial amount, A, and the factor, $1\pm r$.

a) What will the population be in 10 years?

b) What will it be in 25 years?

c) How long will it take for the population to reach 1 million?

Solution: - If the annual growth rate is 2%, we are increasing the population by 2% per year. To calculate the new increased amount, multiply by 100% + 2% = 1.02. You may leave out the table if you feel comfortable doing so.

Growth **Rate** = 2%   Growth **Factor** = 1.02

Years	Population
0	450,000
1	450,000(1.02)
2	$450{,}000(1.02)(1.02) = 450{,}000(1.02)^2$
3	$450{,}000(1.02)^3$
.	
.	
n	$\mathbf{450{,}000(1.02)^n}$

$$\boxed{\textbf{General Equation: } \mathbf{Y = 450{,}000(1.02)^n}}$$

a) In 10 years, the population will be $450{,}000(1.02)^{10} = 450{,}000(1.21899\ldots) = \mathbf{548{,}547}$

b) In 25 years, the population will be $450{,}000(1.02)^{25} = \mathbf{738{,}273}$

c) For the population to reach 1 million we want to find the "n" in the equation

$$450{,}000(1.02)^n = 1{,}000{,}000$$

To do this, take a guess at what you think "n" might be based on your previous calculations.

Try n= 30 since when n was 25 the population was about 3/4 of a million.

$450{,}000(1.02)^{30} = 815{,}113$ (too small)

Try n= 50

$450{,}000(1.02)^{50} = 1{,}211{,}215$ (a bit too large)

Try n=40

$450{,}000(1.02)^{40} = 993{,}618$ You will find that $\boxed{\mathbf{40\frac{1}{3}}}$ years is just about right.