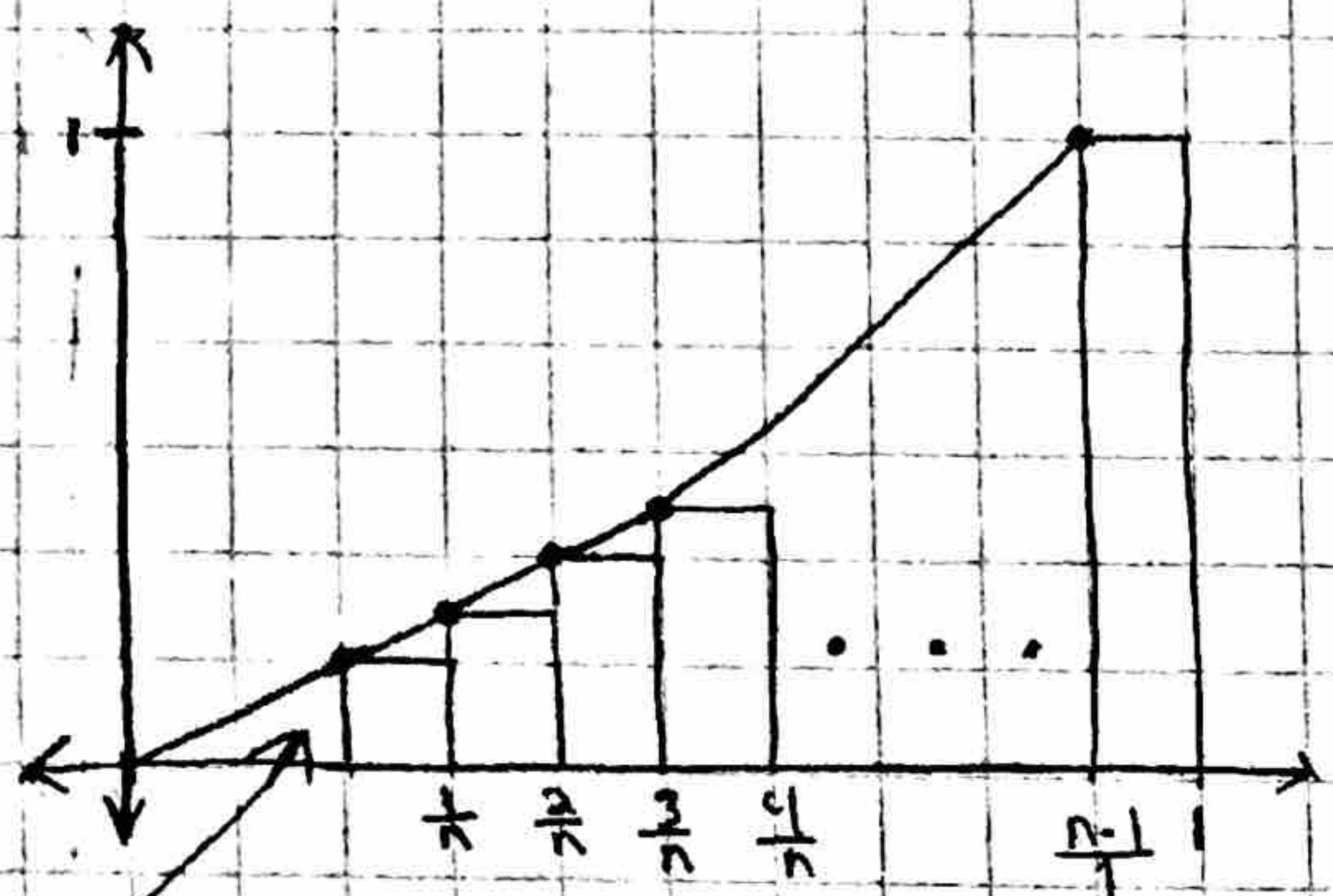
Left Endpoint Rule

Use n rectangles of equal width and the left endpoint rule to estimate the area A between the parabola $y = x^2$ and the x-axis over interval $[0, 1]$.

Find the limit of the sum of the areas of the n rectangles as n goes to infinity.



$i = 1, 2, 3, \ldots n$, where i is the rectangle position along x-axis and n is the total number of rectangles.

$\underline{i = 1}$

width of rectangle is $\frac{1}{n}$

Height of rectangle is $f((i-1)/n) = f((1-1)/n)$

$= f(0/n)$

$= f(0) = 0^2 = 0$

Height of Rectangle is 0

$\underline{i = 2}$

width of rectangle is $\frac{1}{n}$

Height of rectangle is $f((2-1/n) = f(1/n)$

$= (1/n)^2$

$\underline{i = n}$

Width of rectangle is $\frac{1}{n}$

Height of rectangle is $f((n-1/n) = [(n-1)/n]^2$

$L_n = A_1 + A_2 + \ldots + A_n$, where L_n is the sum of the rectangle area.

$$\left(\frac{1}{n}\right)\left(\frac{0}{n}\right)^2 + \left(\frac{1}{n}\right)\left(\frac{1}{n}\right)^2 + \ldots + \left(\frac{1}{n}\right)\left[(n-1)/n\right]^2$$

$$= \frac{1}{n}\left[\left(\frac{0}{n}\right)^2 + \left(\frac{1}{n}\right)^2 + \ldots + \left(\frac{n-1}{n}\right)^2\right]$$

$$= \frac{1}{n}\left[\frac{0^2 + 1^2 + \ldots + (n-1)^2}{n^2}\right]$$

$$\downarrow$$

Set $1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$

$$-n^2 \qquad -n^2$$

$$1^2 + 2^2 + \ldots + (n-1)^2 = \boxed{\frac{n(n+1)(2n+1)}{6} - n^2}$$

$$\downarrow$$

$$L_n = \frac{1}{n}\left[\frac{0^2 + 1^2 + 2^2 + \ldots + (n-1)^2}{n^2}\right]$$

$$= \frac{1}{n}\left[\frac{\frac{n(n+1)(2n+1)}{6} - n^2}{n^2}\right]$$

$$= \frac{\frac{1}{n}\cdot\frac{n(n+1)(2n+1)}{6} - \frac{1}{n}\cdot n^2}{\frac{1}{n}\cdot n^2}$$

$$= \frac{\frac{(n+1)(2n+1)}{6} - \frac{n^2}{n}}{\frac{n^2}{n}}$$

$$= \frac{\frac{(n+1)(2n+1)}{6} - n}{n}$$

$$\frac{(n+1)(2n+1)}{6} - n \div \frac{n}{1}$$

$$L_n = \frac{1}{n}\left[\frac{\frac{n(n+1)(2n+1)}{6} - n^2}{n^2}\right]$$

$$(n^2+n)(2n+1)$$
$$=$$
$$= 2n^3 + n^2 + 2n^2 + n$$
$$=$$
$$\frac{\frac{2n^3+3n^2+n}{6} - n^2}{n^2}$$

$$=$$

$$\frac{2n^3+3n^2+n}{6} - \frac{n^2}{1}\cdot\frac{6}{6}$$
$$=$$
$$\frac{2n^3+3n^2+n}{6} - \frac{6n^2}{6}$$
$$=$$
$$\frac{2n^3-3n^2+n}{6} \div \frac{n^2}{1}$$
$$=$$
$$\frac{2n^3-3n^2+n}{6}\cdot\frac{1}{n^2}$$
$$=$$
$$\frac{1}{n}\left(\frac{2n^3-3n^2+n}{6n^2}\right)$$

$$\frac{\frac{1}{n}(2n^3) - \frac{1}{n}(3n^2) + \frac{1}{n}(n)}{6n^2}$$
$$=$$
$$\frac{\frac{2n^3}{n} - \frac{3n^2}{n} + \frac{n}{n}}{6n^2} = \frac{2n^2-3n+1}{6n^2}$$
$$=$$
$$\boxed{\frac{(2n-1)(n-1)}{6n^2}}$$

$$\lim_{n\to\infty} L_n = \lim_{n\to\infty} \frac{(n-1)(2n-1)}{6n^2}$$

$$= \lim_{n\to\infty} \frac{2n^2-3n+1}{6n^2}$$

$$= \frac{1}{6} \cdot \lim_{n\to\infty} \left(\frac{2n^2-3n+1}{n^2}\right)$$

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$$= \frac{1}{6} \cdot \lim_{n\to\infty} \left(\frac{2n^2}{n^2} - \frac{3n}{n^2} + \frac{1}{n^2}\right)$$

$$= \frac{1}{6} \cdot \lim_{n\to\infty} \left(2 - \frac{3}{n} + \frac{1}{n^2}\right)$$

$$= \frac{1}{6} \left(\lim_{n\to\infty} 2 - \lim_{n\to\infty} \frac{3}{n} + \lim_{n\to\infty} \frac{1}{n^2}\right)$$

$$= \frac{1}{6} \left(2 - \frac{\lim_{n\to\infty} 3}{\lim_{n\to\infty} n} + \frac{\lim_{n\to\infty} 1}{\lim_{n\to\infty} n^2}\right)$$

$$\frac{3}{\infty} \qquad \frac{1}{\infty}$$

$$= \frac{1}{6}(2) - 0 + 0$$

$$= \frac{2}{6}$$

$$= \frac{1}{3}$$

$$\lim_{n\to\infty} L_n = \lim_{n\to\infty} \frac{(n-1)(2n-1)}{6n^2}$$

$$= \lim_{n\to\infty} \frac{2n^2-3n+1}{6n^2}$$

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$$= \frac{1}{6} \left(2 - \frac{\lim_{n\to\infty} 3}{\lim_{n\to\infty} n} + \frac{\lim_{n\to\infty} 1}{\lim_{n\to\infty} n^2}\right)$$

$$\frac{\lim_{n\to\infty} 3}{\lim_{n\to\infty} n} = \frac{3}{\infty} = 0 \qquad \frac{\lim_{n\to\infty} 1}{\lim_{n\to\infty} n^2} = \frac{1}{\infty} = 0$$

$$\frac{1}{6}(2) - 0 \quad 0$$

$$= \frac{2}{6} = \frac{1}{3}$$