Example 1

Equation of Sphere

$$V = \frac{4}{3}\pi r^3$$

Rate of balloon increasing in respect to time. $\frac{\Delta \text{ rate of balloon increasing}}{\Delta \text{ time}} = \frac{dr}{dt}$

$$\frac{d}{dt}[V] = \frac{d}{dt}\left[\frac{4}{3}\pi r^3\right]$$

$$\frac{dV}{dt} = \frac{4}{3}\pi \cdot \frac{d}{dt}[r^3]$$

$$\frac{dv}{dt} = \frac{4}{\cancel{3}_1}\pi \cdot \frac{\cancel{3}r^2}{1}\frac{dr}{dt}$$

$$\frac{dv}{dt} = 4\pi r^2 \frac{dr}{dt}$$

Chain It

$f(u) = u^3$ $\quad u = g(x) = r$

u^3 $\quad \frac{d}{dt}[u^3] = 3u^2$

r $\quad \frac{d}{dt}[r] = \frac{dr}{dt}$

$3r^2 \frac{dr}{dt}$, where $r = u$

Solve for $\frac{dr}{dt}$

$$\frac{1}{4\pi r^2} \cdot \frac{dv}{dt} = \frac{dr}{dt} \cdot \cancel{4\pi r^2} \cdot \frac{1}{\cancel{4\pi r^2}}$$

$$\frac{1}{4\pi r^2} \cdot \frac{dv}{dt} = \frac{dr}{dt}$$

$$\frac{dv}{dt} = \frac{60 \text{ cm}^3}{\text{seconds}}$$

$$\frac{1}{4\pi r^2} \cdot (60 \text{ cm}^3/\text{sec})$$

$$\frac{1}{4\pi(10 \text{ cm})^2}(60 \text{ cm}^3/\text{sec})$$

$$\frac{1}{4\pi(100 \text{ cm}^2)}(60 \text{ cm}^3/\text{sec})$$

$$\frac{1}{4\pi} \cdot \frac{60\text{cm}^3/\text{sec}}{100\text{cm}^2} \xrightarrow{\text{Factor}} \boxed{\begin{aligned} &\frac{60\text{cm}^3}{\text{sec}} \div 100\text{cm}^2 \\ &= \frac{60\text{cm}^3}{1} \cdot \frac{1}{100\text{cm}^2} \\ &= \frac{6\cancel{0}\text{cm}^3}{10\cancel{0}\text{cm}^2} \\ &= \boxed{\frac{3\text{ cm}}{5}} \end{aligned}}$$

$$= \frac{1}{4\pi} \cdot \frac{3\text{cm}/\text{sec}}{5}$$

$$= \boxed{\begin{aligned} &\frac{3}{20\pi}\ \text{cm}/\text{sec} \\ &\approx 0.048\ \text{cm}/\text{sec} \end{aligned}}$$

When $r = 10\text{cm}$, the radius of the balloon is increasing at a rate of approximately 0.048 cm/s

Example 2

$y = x^2 + 3x$ and $\frac{dx}{dt} = 4$, find $\frac{dy}{dt}$ when $x = 2$

Differentiate $y = x^2 + 3x$ with respect to time t.

$$\frac{dy}{dt} = \frac{d}{dt}$$

$$\frac{d}{dt}[y] = \frac{d}{dt}[x^2 + 3x]$$

$$\frac{dy}{dt} = \frac{d}{dt}[x^2] + \frac{d}{dt}[3x]$$

$$\boxed{\frac{dy}{dt} = 2x\frac{dx}{dt} + 3\frac{dx}{dt}}$$

Found $\frac{dy}{dt}$

Get $\frac{dy}{dt}$ when $x = 4$

$$\frac{dx}{dt} = 4$$

$$\frac{dy}{dt} = 2(2) \cdot 4 + 3(4)$$

$$= 4 \cdot 4 + 12$$

$$16 + 12$$

$$\boxed{\frac{dy}{dt} = 28}$$

When $dx/dt = 4$ and $x = 2$, $dy/dt = 28$

Chain It

$f(u) = u^2$, $u = g(x) = x$

u^2 $\quad \frac{d}{dt}[u^2] = 2u$

x $\quad \frac{d}{dt}[x] = \frac{dx}{dt}$

$\boxed{2x\frac{dx}{dt}}$, where $x = u$

$f(u) = 3u$, $u = g(x) = x$

$$3u : \frac{d}{dt}[3u] = 3 \cdot \frac{d}{dt}[u] = 3 \cdot 1u^{1-1} = 3 \cdot 1 \cdot u^0 = 3 \cdot 1 \cdot 1 = 3$$

$$x : \frac{d}{dt}[x] = \frac{dx}{dt}$$

$$\boxed{3\frac{dx}{dt}}$$

Example 4

Quanties of Interest

distance x

angle θ

① Get rate quanties

$$\tan(\theta) = \frac{x}{40}$$

② Get Formula

Differentiate Equation with respect to time t.

$$\frac{160\text{ ft}}{\text{sec}} = \frac{dx}{dt} = \frac{d}{dt}\left[\frac{x}{40}\right] = \frac{d}{dt}[\tan\theta]$$

③ Define $\frac{dx}{dt}$

$$\therefore \frac{dx}{dt} = \sec^2(\theta)$$

④ Get $\frac{dx}{dt}$

Get rate of camera rotating when car is 100 feet from the point on the path closest to the camera. $\frac{d\theta}{dt}$

⑤ Get $\frac{d\theta}{dt}$

$$\frac{dx}{dt} \cdot \frac{1}{40} = \sec^2\theta \cdot \frac{d\theta}{dt}$$

$$\frac{1}{\sec^2\theta} \cdot \frac{1}{40} \cdot \frac{dx}{dt} = \frac{1}{\cancel{\sec^2\theta}} \cdot \cancel{\sec^2\theta} \cdot \frac{d\theta}{dt}$$

$$\frac{1}{40\sec^2\theta} \cdot \frac{dx}{dt} = \frac{d\theta}{dt}$$

When $x = 100$ $\tan\theta = x/40$ → $\tan\theta = \frac{10\cancel{0}}{4\cancel{0}} = \frac{10}{4}$

Use $\sec^2\theta = 1 + \tan^2\theta$

$\tan\theta = 2.5$

$$\frac{1}{40(1+\tan^2\theta)} \cdot \frac{dx}{dt} = \frac{d\theta}{dt}$$

$$\frac{1}{40(1+\tan^2(2.5))} \cdot \frac{dx}{dt} = \frac{d\theta}{dt}$$

$$\frac{1}{40(1+(2.5)^2)} \cdot \frac{dx}{dt} = \frac{d\theta}{dt}$$

$$\frac{1}{40(1+6.25)} \cdot (160) = \frac{d\theta}{dt} \qquad \frac{dx}{dt} = 160 \text{ ft/s}$$

$$\parallel$$

$$\frac{160}{40(7.25)} = \frac{d\theta}{dt}$$

$$\parallel$$

$$\frac{16\cancel{0}}{29\cancel{0}} = \frac{d\theta}{dt}$$

$$\parallel$$

$$.551724$$

Rate at which camera is rotating when car is 100ft from the point on the path closert to the camora is 0.552 rad/s.

Example 3

Rates of Interest

1. Volume of water x
2. Depth of water y

Use Volume of Cone Formula

$$V = \frac{1}{3}\pi r^2 h$$

$$\frac{dV}{dt} = \frac{3\ ft^3}{min}\ ; \quad h = 5ft \text{ deep of water}$$

Solve for r

$$\frac{r}{h} = \frac{4}{8}$$

$$\frac{\cancel{h}}{1} \cdot \frac{r}{\cancel{h}} = \frac{1}{2} \cdot \frac{h}{1}$$

$$\boxed{r = \frac{h}{2}}$$

$$V = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h$$

$$V = \frac{1}{3} \cdot \pi \cdot \frac{h^2}{4} \cdot h$$

$$V = \pi \cdot \frac{1}{3} \cdot \frac{h^3}{4}$$

$$V = \pi \cdot \frac{h^3}{12}$$

$$\boxed{V = \frac{\pi}{12} h^3}$$

$$\frac{d}{dt}[V] = \frac{d}{dt}\left[\frac{\pi}{12} h^3\right]$$

Need to get dh/dt to express V in terms of h.

$$\frac{dv}{dt} = \frac{\pi}{12} \cdot \frac{d}{dt}[h^3]$$

Get $\frac{dh}{dt}$

Chain It

$f(u) = u^3, \quad u = g(x) = h$

$u^3 = \frac{d}{dt}[u^3] = 3u^2$

$h = \frac{d}{dt}[h] = \frac{dh}{dt}$

$$\boxed{3h^2 \frac{dh}{dt}}$$

$$\frac{dv}{dt} = \frac{\pi}{12} \cdot 3h$$

$$\frac{dv}{dt} = \frac{\pi}{\cancel{12}_{4}} \cdot \cancel{3}h^2 \frac{dh}{dt}$$

$$\frac{dv}{dt} = \frac{\pi}{4} h^2 \frac{dh}{dt}$$

$$\frac{4}{\pi h^2} \cdot \frac{dv}{dt} = \frac{dh}{dt} \cdot \frac{\pi h^2}{\cancel{4}}\left(\cancel{\frac{4}{\pi h^2}}\right)$$

$$\frac{dh}{dt} = \frac{4}{\pi h^2} \cdot \frac{dv}{dt} \rightarrow \frac{dv}{dt} = 3\, ft^3/min, \; h = 5$$

$$\frac{dh}{dt} = \frac{4}{\pi h^2}(3\, ft^3/min)$$

$$\frac{dh}{dt} = \frac{4}{\pi (5)^2}(3\, ft^3/min)$$

$$\frac{4}{25\pi} \cdot (3\, ft^3/min)$$

$$\frac{12\, ft^3/min}{\pi (5)^2} \rightarrow \text{Factor} \quad \frac{12\, ft^3}{min} \div \frac{\pi (5)^2}{1}$$

$$\frac{12\, ft^3}{min} \cdot \frac{1}{\pi (5)^2}$$

$$\boxed{\frac{12\, ft^3}{25\pi\, min}}$$

$$\frac{12\, ft^3}{25\pi\, min}$$

$$\boxed{0.153\, ft/min}$$

Rate of water rising is 0.153 ft/min when water is 5 feet deep.