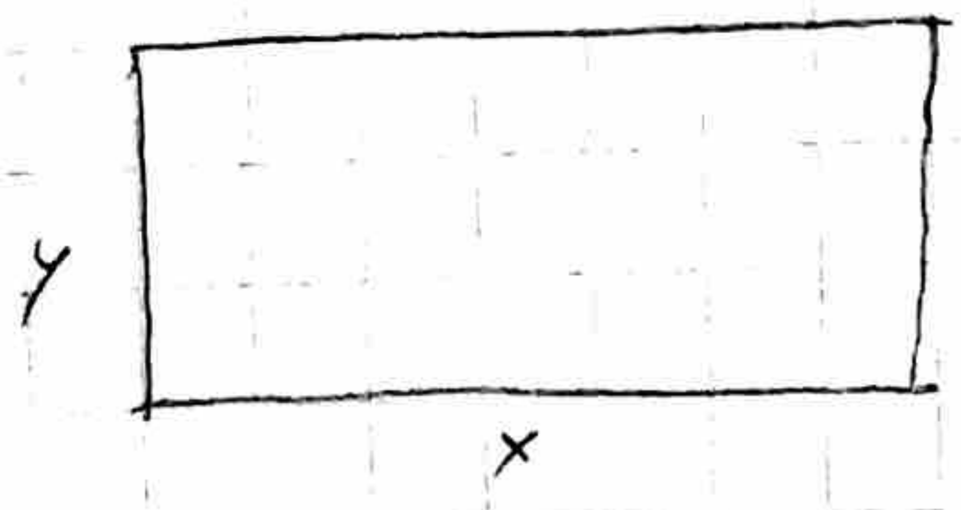
Example 1

A farmer has 3600 ft of fencing and wants to fence off a rectangular area for his cattle. What dimensions should the farmer use to maximize the area for grazing?

maximizing

①



Area = length x width , x = length

y = width

A = area

$$\boxed{A = xy}$$

② 3600 ft of fencing represents the perimeter

$$3600 = 2(x+y)$$

$$\boxed{3600 = 2x + 2y}$$

③ Set $A = xy$ in terms of x

④ Solve for y

$$3600 = 2x + 2y$$
$$-2x \quad -2x$$
$$\frac{3600 - 2x}{2} = \frac{2y}{2}$$

$$\frac{3600 - 2x}{2} = y$$

$$1800 - x = y$$

$$\boxed{y = 1800 - x}$$

⑤ Replace y

$\underline{A}$

$A = xy$

$A = x(1800 - x)$

$A = 1800x - x^2$

$$\boxed{\begin{array}{c} A(x) \\ A(x) = xy \\ A(x) = x(1800 - x) \\ A(x) = 1800x - x^2 \end{array}}$$

⑥ Domain Awareness

The smallest x can be is 0, x cannot be negative since we are working with a rectangular area.

⑦ Set $y = 0$ to get largest x value.

$$3600 = 2x + 2(0)$$

$$=$$

$$\frac{3600}{2} = \frac{2x}{2}$$

$$=$$

$$1800 = x \,,\; x = 1800$$

⑧ Express area A as a function of x

$$A(x) = 1800x - x^2 \,, \quad \text{where} \quad 0 \leq x \leq 1800$$

⑨ Get $A'(x)$

$$A'(x) = \frac{d}{dx}[1800x - x^2]$$

$$=$$

$$\frac{d}{dx}[1800x] - \frac{d}{dx}[x^2]$$

$$=$$

$$1800 \cdot \frac{d}{x}[x] - 2x^{2-1}$$

$$=$$

$$1800 \cdot 1x^{1-1} - 2x$$

$$=$$

$$1800 \cdot 1 \cdot 1 - 2x$$

$$=$$

$$\boxed{A'(x) = 1800 - 2x}$$

⑩ Set $A'(x) = 0$, Get critical Number

$$1800 - 2x = 0$$

$$-1800 \qquad -1800$$

$$\frac{-2x}{-2} = \frac{-1800}{-2}$$

$$=$$

$$\boxed{x = 900}$$

(11) Get Domain Values for $A(x)$

$A(x) = 1800x - x^2$

$x(1800 - x)$

$\boxed{x = 0}$

$$\begin{aligned} 1800 - x &= 0 \\ -1800 \quad & \quad -1800 \\ \hline \frac{-x}{-1} &= \frac{-1800}{-1} \end{aligned}$$

$\boxed{x = 1800}$

(12) Compare Critical Numbers

$x = 0, \quad x = 900, \quad x = 1800$

(13) Get Maximum

$A(x) = 1800x - x^2$

$A(0) = 1800(0) - (0)^2$

$\boxed{A(0) = 0}$

$A(900) = 1800(900) - (900)^2$

$\boxed{A(900) = 810000}$

$A(1800) = 1800(1800) - (1800)^2$

$3.24 \times 10^6 - 3.24 \times 10^6$

$\boxed{A(1800) = 0}$

$\boxed{\text{Maximum Value for } A(x) = 1800x - x^2 \text{ is } 810000, \text{ where } x = 900}$

(14) Get y value

$y = 1800 - x, \quad x = 900$

$y = 1800 - 900$

$\boxed{y = 900}$

(15) Summary

$x = 900, \quad y = 900$

Dimensions of the rectangle that maximize grazing area is 900 ft wide by 900 ft long.

Example 2

Find two numbers who difference is 40 and whose product is a minimum.

minimizing

① Set up equations

$40 = x - y$, where x and y are the two unknown numbers

$P = xy$, where P is the product and x and y are the two unknown numbers

② Set $P = xy$ in terms of x

③ Solve for y

$$40 = x - y$$
$$-x \quad -x$$

$$\frac{40 - x}{-1} = \frac{-y}{-1}$$

$$-40 + x = y, \quad y = x - 40$$

④ Replace y

$$P(x) = xy$$
$$P(x) = x(x - 40)$$
$$P(x) = x^2 - 40x$$

⑤ Domain Awareness

Difference of 40, no restrictions on domain.

Domain of x $(-\infty, \infty)$

⑥ Get $P'(x) = \frac{d}{dx}[x^2 - 40x]$

$$\frac{d}{dx}[x^2] - \frac{d}{dx}[40x]$$

$$2x^{2-1} - 40 \cdot \frac{d}{dx}[x]$$

$$2x - 40 \cdot 1x^{1-1}$$

$$2x - 40 \cdot 1$$

$$\boxed{P'(x) = 2x - 40}$$

⑦ Get Critical Number.

$P'(x) = 0$

$2x - 40 = 0$

$\underline{\quad +40 \quad +40}$

$\frac{2x}{2} = \frac{40}{2}$

$\boxed{x = 20}$ ← 1st Number

⑧ Get $P''(x)$ Second Derivative

$P''(x) = \frac{d}{dx}[2x - 40]$

$=$

$\frac{d}{dx}[2x] - \frac{d}{dx}[40]$

$=$

$2 \cdot \frac{d}{dx}[x] - 0$

$=$

$2 \cdot 1x^{1-1}$

$=$

$2 \cdot 1 \cdot 1$

$=$

$\boxed{P''(x) = 2}$ →

⑨ Apply Second Derivative Test

$P'(x) = 2x - 40$

$P'(20) = 2(20) - 40$

$40 - 40$

$\boxed{P'(20) = 0}$

$P''(x) = 2$

$\boxed{2 > 0}$

$P'(x) = 0$ and $P''(x) > 0$, $P(x) = x^2 - 40x$ have a relative minimum at $x = 20$

(10) Get y value

$y = x - 40$, $x = 20$

$y = 20 - 40$

$\boxed{y = -20}$ ← Second Number

(11) Summary

First Number

$x = 20$

Second Number

$y = -20$

20 and -20 have a difference of 40.

$P(20) = (20)^2 - 40(20)$

$400 - 800$

$\boxed{P(20) = -400}$

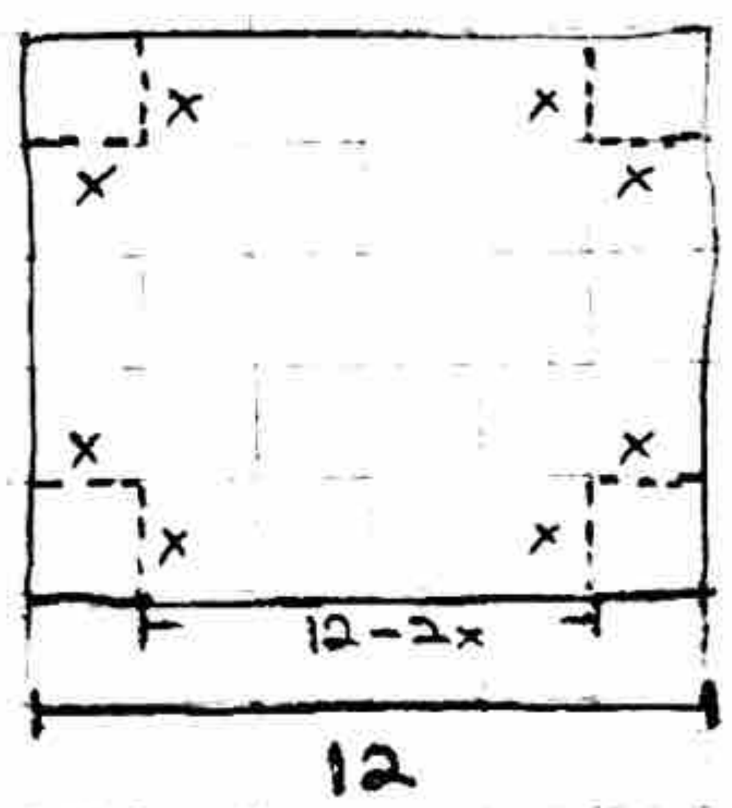
$20 \cdot -20 = -400$

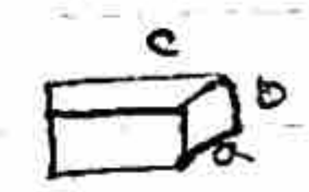
Product of 20 and -20 is the minimum of $P(x) = x^2 - 40x$

Example 3

An open-top box is to be made by cutting small congruent squares from the corners of a 12 inch by 12 inch sheet of cardboard, and bending up the sides. How large should the squares cut from the corners be made to maximize the volume of the box?

① Set equation for what is being maximized, volume.



Volume of rectangular prism: 

Volume = a·b·c

$a = 12-2x$

$b = x$

$c = 12-2x$

Volume $= (12-2x)\cdot x\cdot(12-2x)$

$V(x) = x(12-2x)^2$

Expand

$V(x) = x(12-2x)^2$

$= x(12-2x)(12-2x)$

$= x(144-24x-24x+4x^2)$

$= x(144-48x+4x^2)$

$V(x) = 144x - 48x^2 + 4x^3$

or

$V(x) = 4x^3 - 48x^2 + 144x$

② Express volume as a function of x

$x = 0$

$12-2x = 0$

$+2x \quad +2x$

$\frac{12}{2} = \frac{2x}{2}$

$6 = x$

$x = 6$

0 6

$\boxed{\text{Use } V(x) = 4x^3 - 48x^2 + 144x,\ 0 \le x \le 6}$

There are only two x values where $y=0$ for $V(x)$: $x=0$, $x=6$

③ Get $V'(x)$

$$V'(x) = \frac{d}{dx}\left[4x^3 - 48x^2 + 144x\right]$$

$$= \frac{d}{dx}\left[4x^3\right] - \frac{d}{dx}\left[48x^2\right] + \frac{d}{dx}\left[144x\right]$$

$$= 4 \cdot \frac{d}{dx}\left[x^3\right] - 48 \cdot \frac{d}{dx}\left[x^2\right] + 144 \cdot \frac{d}{dx}\left[x\right]$$

$$= 4 \cdot 3x^{3-1} - 48 \cdot 2x^{2-1} + 144 \cdot 1x^{1-1}$$

$$= 4 \cdot 3x^2 - 48 \cdot 2x + 144 \cdot 1x^0$$

$$= \boxed{V'(x) = 12x^2 - 96x + 144}$$

④ Factor $V'(x)$

$$12x^2 - 96x + 144$$

$$= 12(x^2 - 8x + 12)$$

$1 \cdot 12$

$2 \cdot 6$

$$\boxed{V'(x) = 12(x-2)(x-6)}$$

⑤ Get Critical Values

$$V'(x) = 0$$

$$12(x-2)(x-6) = 0$$

$$\Downarrow$$

$12 = 0$ False

$x - 2 = 0$
$+2 \quad +2$
$\boxed{x = 2}$

$x - 6 = 0$
$+6 \quad +6$
$x = 6$

← 6 is part of the interval, so dismiss

⑥ Compare Values for $V(x) = 4x^3 - 48x^2 + 144x$

$x = 0$ $\quad\quad$ $x = 2$ $\quad\quad$ $x = 6$

$$V(0) = 4(0)^3 - 48(0)^2 + 144(0)$$
$$0 - 0 + 0$$
$$\boxed{V(0) = 0}$$

$$V(2) = 4(2)^3 - 48(2)^2 + 144(2)$$
$$4 \cdot 8 - 48 \cdot 4 + 288$$
$$32 - 192 + 288$$
$$V(2) = 128$$

$$V(6) = 4(6)^3 - 48(6)^2 + 144(6)$$
$$4 \cdot 216 - 48 \cdot 36 + 864$$
$$864 - 1728 + 864$$
$$864 - 864$$
$$\boxed{V(6) = 0}$$

⑦ Summary

Cutting 2 inches by 2 inches from each corner will maximize the volume of the box.