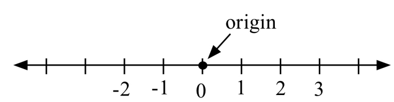
0.2 Lines in the Plane

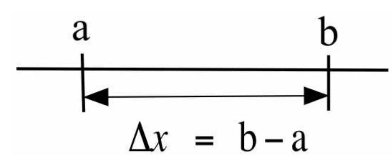
The first graphs and functions you encountered in algebra were straight lines and their equations. These lines were easy to graph, and the equations were easy to evaluate and to solve. They described a variety of physical, biological and financial phenomena such as $d = rt$ relating the distance d traveled to the rate r and time t spent traveling, and $C = \frac{5}{9}(F - 32)$ for converting the temperature in degrees Fahrenheit (F) to degrees Celsius (C).

The first part of calculus—differential calculus—will deal with ideas, techniques and applications of tangent lines to the graphs of functions, so it is important that you understand the graphs, properties and equations of straight lines.

The Real Number Line



The real numbers (consisting of all integers, fractions, rational and irrational numbers) can be represented as a line, called the **real number line**. Once we have selected a starting location, called the **origin**, a positive direction (usually up or to the right), and unit of length, then every number can be located as a point on the number line. If we move from a point $x = a$ to a point $x = b$ on the line, then we will have moved an **increment** of $b - a$. We denote this increment with the symbol Δx (read "delta x").



The capital Greek letter delta (Δ) appears often in calculus to represent the "change" in something.

Caution: Δx does not mean Δ *times* x, but rather the *difference* between two x-coordinates.

- If b is larger than a, then we will have moved in the positive direction, and $\Delta x = b - a$ will be positive.
- If b is smaller than a, then $\Delta x = b - a$ will be negative and we will have moved in the negative direction.
- Finally, if $\Delta x = b - a = 0$, then $a = b$ and we did not move at all.

We can also use the Δ notation and absolute values to express the **distance** that we have moved. On the number line, the distance from $x = a$ to $x = b$ is

$$\text{dist}(a, b) = \begin{cases} b - a & \text{if } b \geq a \\ a - b & \text{if } b < a \end{cases}$$

or:

$$\text{dist}(a, b) = |b - a| = |\Delta x| = \sqrt{(\Delta x)^2}$$

The **midpoint** of the interval from $x = a$ to $x = b$ is the point M such that $\text{dist}(a, M) = \text{dist}(M, b)$, or $|M - a| = |b - M|$. If $a < M < b$,

$$M - a = b - M \Rightarrow 2M = a + b \Rightarrow M = \frac{a + b}{2}$$

It's not difficult to check that this formula also works when $b < M < a$.

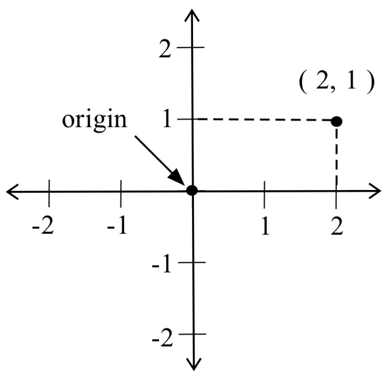
Example 1. Find the length and midpoint of the interval from $x = -3$ to $x = 6$.

Solution. $\text{dist}(-3, 6) = |6 - (-3)| = |9| = 9$; $M = \frac{(-3)+6}{2} = \frac{3}{2}$. ◀

Solutions to Practice problems are at the end of each section.

Practice 1. Find the length and midpoint of the interval from $x = 7$ to $x = 2$.

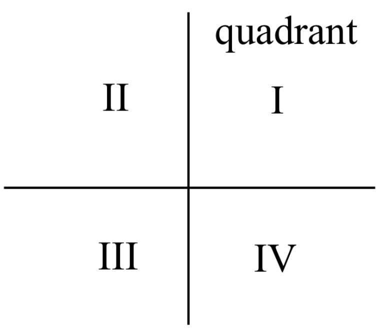
The Cartesian Plane



In this book, a point in the plane will be labeled either with a name, say P, or with an ordered pair, say (x, y), or with both: $P = (x, y)$.

Two perpendicular number lines, called **coordinate axes**, determine a **real number plane**. The axes intersect at a point called the **origin**. Each point P in the plane can be described by an **ordered pair** (x, y) of numbers that specify how far, and in which directions, we must move from the origin to reach the point P. We can locate the point $P = (x, y)$ in the plane by starting at the origin and moving x units horizontally and then y units vertically. Similarly, we can label each point in the plane with the ordered pair (x, y), which directs us how to reach that point from the origin.

This coordinate system is called the **rectangular coordinate system** or the **Cartesian** coordinate system (after René Descartes), and the resulting plane the **Cartesian plane**.

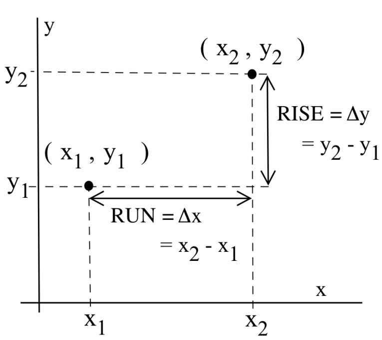


The coordinate axes divide the plane into four **quadrants**, labeled quadrants I, II, III and IV moving counterclockwise from the upper-right quadrant.

We will often call the horizontal axis the ***x*-axis** and the vertical axis the ***y*-axis** and then refer to the plane as the ***xy*-plane**. This choice of x and y as labels for the axes is a common choice, but we will sometimes prefer to use different labels—and even different units of measurement on the two axes.

Increments and Distance Between Points In The Plane

If we move from a point $P = (x_1, y_1)$ in the plane to another point $Q = (x_2, y_2)$, then we will need to consider two **increments** or changes.



- The increment in the x (horizontal) direction is $x_2 - x_1$, denoted by $\Delta x = x_2 - x_1$.
- The increment in the y (vertical) direction is $y_2 - y_1$, denoted by $\Delta y = y_2 - y_1$.

Computing the **distance** between the points $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ involves a simple application of the Pythagorean Theorem:

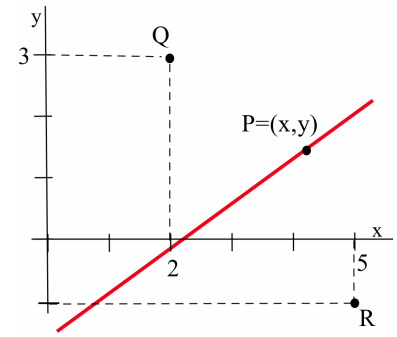
$$\text{dist}(P, Q) = \sqrt{(\Delta x)^2 + (\Delta y)^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

The **midpoint** M of the line segment joining P and Q is:

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$$

where we have just used the one-dimension midpoint formula for each coordinate.

Example 2. Find an equation describing all the points $P = (x, y)$ equidistant from $Q = (2, 3)$ and $R = (5, -1)$.



Solution. The points $P = (x, y)$ must satisfy $\text{dist}(P, Q) = \text{dist}(P, R)$ so:

$$\sqrt{(x-2)^2 + (y-3)^2} = \sqrt{(x-5)^2 + (y-(-1))^2}$$

By squaring each side we get:

$$(x-2)^2 + (y-3)^2 = (x-5)^2 + (y+1)^2$$

Expanding we get:

$$x^2 - 4x + 4 + y^2 - 6y + 9 = x^2 - 10x + 25 + y^2 + 2y + 1$$

and canceling like terms yields:

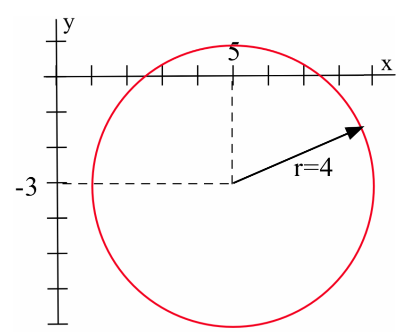
$$-4x - 6y + 13 = -10x + 2y + 26$$

so $y = 0.75x - 1.625$, the equation of a line. Every point on the line $y = 0.75x - 1.625$ is equally distant from both Q and R. ◀

Practice 2. Find an equation describing all points $P = (x, y)$ equidistant from $Q = (1, -4)$ and $R = (0, -3)$.

A circle with radius r and center at the point $C = (a, b)$ consists of all points $P = (x, y)$ at a distance of r from the center C: the points P that satisfy $\text{dist}(P, C) = r$.

Example 3. Find an equation of a circle with radius $r = 4$ and center $C = (5, -3)$.



Solution. A circle consists of the set of points $P = (x, y)$ at a fixed distance r from the center point C, so this circle will be the set of points $P = (x, y)$ at a distance of 4 units from the point $C = (5, -3)$; P will be on this circle if $\text{dist}(P, C) = 4$.

Using the distance formula and rewriting:

$$\sqrt{(x-5)^2 + (y+3)^2} = 4 \quad \Rightarrow \quad (x-5)^2 + (y+3)^2 = 16$$

which we can also express as $x^2 - 10x + 25 + y^2 + 6y + 9 = 16$. ◀

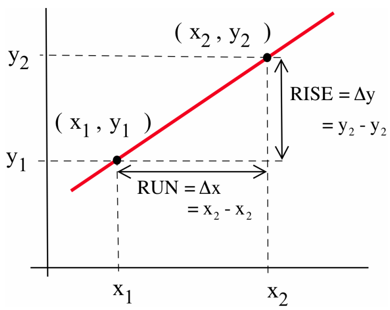
Practice 3. Find an equation of a circle with radius $r = 5$ and center $C = (-2, 6)$.

The Slope Between Points in the Plane

In one dimension (on the number line), our only choice was to move in the positive direction (so the x-values were increasing) or in the negative direction. In two dimensions (in the plane), we can move in infinitely many directions, so we need a precise way to describe direction.

The **slope** of the line segment joining $P = (x_1, y_1)$ to $Q = (x_2, y_2)$ is

$$m = \text{slope from } P \text{ to } Q = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x1} = \frac{\Delta y}{\Delta x}$$



The slope of a line measures how fast we rise or fall as we move from left to right along the line. It measures the rate of change of the y-coordinate with respect to changes in the x-coordinate. Most of our work will occur in two dimensions, and slope will be a very useful concept that will appear often.

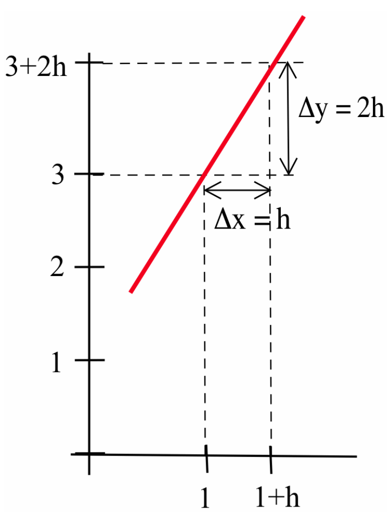
If P and Q have the same x-coordinate, then $x_1 = x_2 \Rightarrow x = 0$. The line from P to Q is thus **vertical** and the slope $m = \frac{\Delta y}{\Delta x}$ is **undefined** because $\Delta x = 0$.

If P and Q have the same y-coordinate, then $y_1 = y_2 \Rightarrow \Delta y = 0$, so the line is **horizontal** and the slope is $m = \frac{\Delta y}{\Delta x} = \frac{0}{\Delta x} = 0$ (assuming $\Delta x \neq 0$).

Practice 4. For $P = (-3, 2)$ and $Q = (5, -14)$, find Δx, Δy, and the slope of the line segment from P to Q.

If the coordinates of P or Q contain variables, then the slope m is still given by $m = \frac{\Delta y}{\Delta x}$, but we will need to use algebra to evaluate and simplify m.

Example 4. Find the slope of the line segment from $P = (1, 3)$ to $Q = (1 + h, 3 + 2h)$.



Solution. $y_1 = 3$ and $y_2 = 3 + 2h$, so $\Delta y = (3 + 2h) - (3) = 2h$; $x_1 = 1$ and $x_2 = 1 + h$, so $\Delta x = (1 + h) - (1) = h$. The slope is:

$$m = \frac{\Delta y}{\Delta x} = \frac{2h}{h} = 2$$

In this example, the value of m is constant (2) and does not depend on the value of h. ◀

Practice 5. Find the slope and midpoint of the line segment from $P = (2, -3)$ to $Q = (2 + h, -3 + 5h)$.

Example 5. Find the slope between the points $P = (x, x^2 + x)$ and $Q = (a, a^2 + a)$ for $a \neq x$.

Solution. $y_1 = x^2 + x$ and $y_2 = a^2 + a \Rightarrow \Delta y = (a^2 + a) - (x^2 + x)$; $x_1 = x$ and $x_2 = a$, so $\Delta x = a - x$ and the slope is:

$$\begin{aligned} m = \frac{\Delta y}{\Delta x} &= \frac{(a^2+a)-(x^2+x)}{a-x} \\ &= \frac{a^2 - x^2 + a - x}{a-x} = \frac{(a-x)(a+x)+(a-x)}{a-x} \\ &= \frac{(a-x)\left((a+x)+1\right)}{a-x} = (a+x)+1 \end{aligned}$$

Here the value of m depends on the values of both a and x. ◄

Practice 6. Find the slope between the points $P = (x, 3x^2 + 5x)$ and $Q = (a, 3a^2 + 5a)$ for $a = x$.

In application problems, it is important to read the information and the questions very carefully—including the units of measurement of the variables can help you avoid "silly" answers.

Example 6. In 1970, the population of Houston was $1,233,535$ and in 1980 it was $1,595,138$. Find the slope of the line through the points $(1970, 1233535)$ and $(1980, 1595138)$.

Solution. $m = \dfrac{\Delta y}{\Delta x} = \dfrac{1595138 - 1233535}{1980 - 1970} = \dfrac{361603}{10} = 36,160.3$ but $36,160.3$ is just a number that may or may not have any meaning to you. If we include the units of measurement along with the numbers we will get a more meaningful result:

$$\frac{1595138 \text{ people} - 1233535 \text{ people}}{\text{year } 1980 - \text{year } 1970} = \frac{361603 \text{ people}}{10 \text{ years}} = 36,160.3 \,\frac{\text{people}}{\text{year}}$$

which says that during the decade from 1970 to 1980 the population of Houston grew at an average rate of $36,160$ people per year. ◄

If the x-unit is time (in hours) and the y-unit is distance (in kilometers), then

$$m = \frac{\Delta y \text{ km}}{\Delta x \text{ hours}}$$

so the units for m are $\frac{\text{km}}{\text{hour}}$ ("kilometers per hour"), a measure of velocity, the rate of change of distance with respect to time.

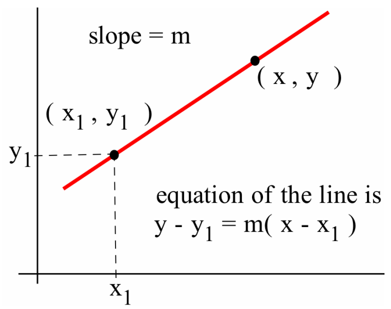
If the x-unit is the number of employees at a bicycle factory and the y-unit is the number of bicycles manufactured, then

$$m = \frac{\Delta y \text{ bicycles}}{\Delta x \text{ employees}}$$

and the units for m are $\frac{\text{bicycles}}{\text{employee}}$ ("bicycles per employee"), a measure of the rate of production per employee.

Equations of Lines

Every (non-vertical) line has the property that the slope of the segment between any two points on the line is the same, and this constant slope property of straight lines leads to ways of finding equations to represent non-vertical lines.

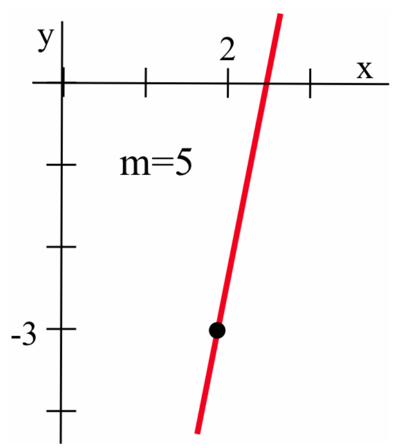


Point-Slope Form

In calculus, we will usually know a point on a line and the slope of that line, so the point-slope form will be the easiest to apply. Other forms of equations for lines can be derived from the point-slope form.

If L is a non-vertical line through a known point $P = (x_1, y_1)$ with a known slope m, then the equation of the line L is:

Point-Slope: $y - y_1 = m(x - x_1)$



Example 7. Find an equation of the line through $(2, -3)$ with slope 5.

Solution. We can simply use the point-slope formula: $m = 5$, $y_1 = -3$ and $x_1 = 2$, so $y - (-3) = 5(x - 2)$, which simplifies to $y = 5x - 13$ ◀

An equation for a **vertical line** through a point $P = (a, b)$ is $x = a$. All points $Q = (x, y)$ on the vertical line through the point P have the same x-coordinate as P.

Two-Point Form

If two points $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ are on the line L, then we can calculate the slope between them and use the first point and the point-slope equation to find an equation for L:

Two-Point: $y - y_1 = m(x - x_1)$ where $m = \dfrac{y_2 - y_1}{x_2 - x1}$

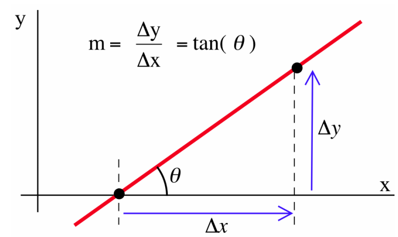
Once we have the slope, m, it does not matter whether we use P or Q as the point. Either choice will result in the same equation for the line once we simplify it.

Slope-Intercept Form

It is common practice to rewrite an equation of a line into the form $y = mx + b$, the **slope-intercept form** of the line. The line $y = mx + b$ has slope m and crosses the y-axis at the point $(0, b)$.

Practice 7. Use the $\frac{\Delta y}{\Delta x}$ definition of slope to calculate the slope of the line $y = mx + b$.

The point-slope and the two-point forms are usually more useful for *finding* an equation of a line, but the slope-intercept form is usually the most useful form for an *answer* because it allows us to easily picture the graph of the line and to quickly calculate y-values given x-values.

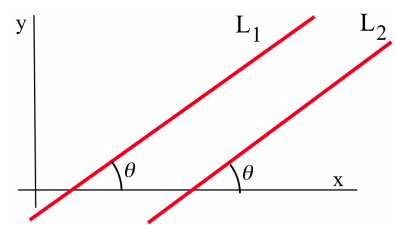


Angles Between Lines

The **angle of inclination** of a line with the x-axis is the smallest angle θ that the line makes with the positive x-axis as measured from the x-axis counterclockwise to the line. Because the slope $m = \frac{\Delta y}{\Delta x}$ and because $\tan(\theta) = \frac{\text{opposite}}{\text{adjacent}}$ in a right triangle, $m = \tan(\theta)$.

The slope of a line is the tangent of its angle of inclination.

Parallel Lines



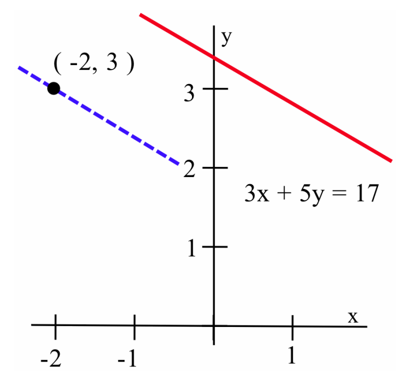
Two parallel lines L_1 and L_2 make equal angles with the x-axis, so their angles of inclination will be equal and hence so will their slopes.

Similarly, if the slopes, m_1 and m_2, of two lines are equal, then the equations of the lines (in slope-intercept form) will always differ by a constant:

$$y_1 - y_2 = (m_1x + b_1) - (m_2x + b_2) = (m_1 - m_2)x + (b_1 - b_2) = b_1 - b_2$$

which is a constant, so the lines will be parallel.

The two preceding ideas can be combined into a single statement:

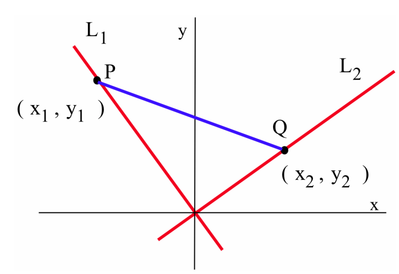


Two non-vertical lines L_1 and L_2 with slopes m_1 and m_2 are parallel if and only if $m_1 = m_2$.

Practice 8. Find an equation of the line that contains the point $(-2, 3)$ and is parallel to the line $3x + 5y = 17$.

Perpendicular Lines

If two lines are perpendicular, the situation is a bit more complicated.



Assume L_1 and L_2 are two non-vertical lines that intersect at the origin (for simplicity), with $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ points away from the origin on L_1 and L_2, respectively. Then the slopes of L_1 and L_2 will be $m_1 = \frac{y_1}{x_1}$ and $m_2 = \frac{y_2}{x_2}$. The line connecting P and Q forms the third side of triangle OPQ, which will be a right triangle if and only if L_1 and L_2 are perpendicular. In particular, L_1 and L_2 are perpendicular if and only if the triangle OPQ satisfies the Pythagorean Theorem:

$$(\text{dist}(O, P))^2 + (\text{dist}(O, Q))^2 = (\text{dist}(P, Q))^2$$

or:

$$(x_1-0)^2+(y_1-0)^2+(x_2-0)^2+(y_2-0)^2 = (x_1-x_2)^2+(y_1-y_2)^2$$

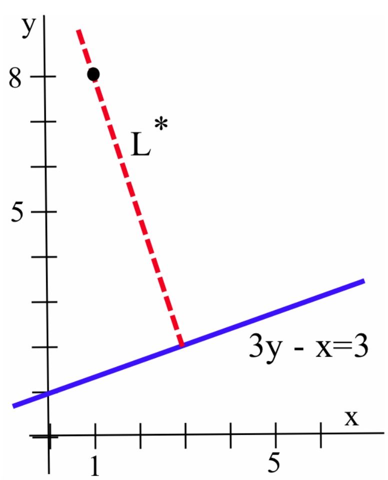
Squaring and simplifying, this reduces to $0=-2x_1x_2-2y_1y_2$, so:

$$\frac{y_2}{x_2}=-\frac{x_1}{y_1}\Rightarrow m_2=\frac{y_2}{x_2}=-\frac{x_1}{y_1}=-\frac{1}{\frac{y_1}{x_1}}=-\frac{1}{m_1}$$

We have just proved the following result:

Two non-vertical lines L_1 and L_2 with slopes m_1 and m_2 are perpendicular if and only if their slopes are negative reciprocals of each other: $m_2=-\frac{1}{m_1}$.

Practice 9. Find an equation of the line that goes through the point $(2,-5)$ and is perpendicular to the line $3y-7x=2$.



Example 8. Find the distance (that is, the shortest distance) from the point $(1,8)$ to the line $L: 3y-x=3$.

Solution. This is a sophisticated problem that requires several steps to solve: First we need a picture of the problem. We will find an equation for the line L^* through the point $(1,8)$ and perpendicular to L. Then we will find the point P where L and L^* intersect. Finally, we will find the distance from P to $(1,8)$.

Step 1: L has slope $\frac{1}{3}$ so L^* has slope $m=-\frac{1}{\frac{1}{3}}=-3$, and L^* has equation $y-8=-3(x-1)$, which simplifies to $y=-3x+11$.

Step 2: We can find the point where L intersects L^* by replacing the y-value in the equation for L with the y-value from our equation for L^*:

$$3(-3x+11)-x=3\Rightarrow x=3\Rightarrow y=-3x+11=-3(3)+11=2$$

which tells us that L and L^* intersect at $P=(3,2)$.

Step 3: Finally, the distance from L to $(1,8)$ is just the distance from the point $(1,8)$ to the point $P=(3,2)$, which is

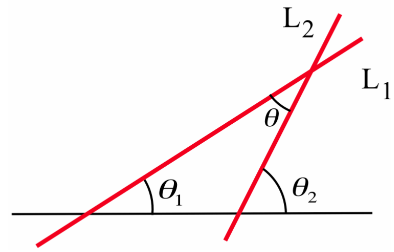
$$\sqrt{(1-3)^2+(8-2)^2}=\sqrt{40}\approx 6.325$$

The distance is (exactly) $\sqrt{40}$, or (approximately) 6.325. ◀

Angle Formed by Intersecting Lines

If two lines that are not perpendicular intersect at a point (and neither line is vertical), then we can use some geometry and trigonometry to determine the angles formed by the intersection of those lines.

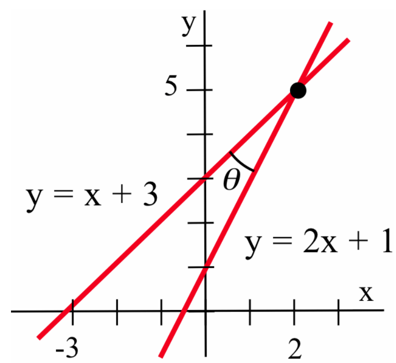
Because θ_2 (see figure at right) is an exterior angle of the triangle ABC, θ_2 is equal to the sum of the two opposite interior angles, so $\theta_2 = \theta_1 + \theta \Rightarrow \theta = \theta_2 - \theta_1$. From trigonometry, we then know that:



$$\tan(\theta) = \tan(\theta_2 - \theta_1) = \frac{\tan(\theta_2) - \tan(\theta_1)}{1 + \tan(\theta_2)\tan(\theta_1)} = \frac{m_2 - m_1}{1 + m_2 m_1}$$

The range of the arctan function is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, so $\theta = \arctan\left(\frac{m_2 - m_1}{1 + m_2 m_1}\right)$ always gives the smaller of the angles. The larger angle is $\pi - \theta$ (or $180^\circ - \theta^\circ$ if we measure the angles in degrees).

The smaller angle θ formed by two non-perpendicular lines with slopes m_1 and m_2 is: $\theta = \arctan \dfrac{m_2 - m_1}{1 + m_2 m_1}$



Example 9. Find the point of intersection and the angle between the lines $y = x + 3$ and $y = 2x + 1$.

Solution. Solving the first equation for y and then substituting into the second equation:

$$(x + 3) = 2x + 1 \quad \Rightarrow \quad x = 2 \quad \Rightarrow \quad y = 2 + 3 = 5$$

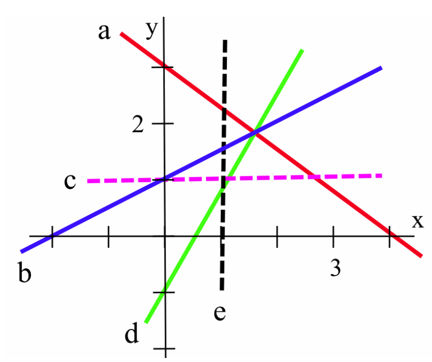
The point of intersection is $(2, 5)$. Because both lines are in slope-intercept form, it is easy to see that $m_1 = 1$ and $m_2 = 2$:

$$\theta = \arctan \frac{m_2 - m_1}{1 + m_2 m_1} = \arctan \frac{2 - 1}{1 + 2 \cdot 1}$$
$$= \arctan \frac{1}{3} \approx 0.322 \text{ radians} = 18.43^\circ$$

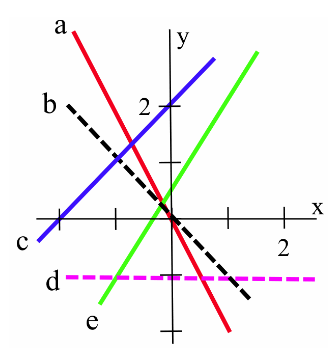
The lines intersect at an angle of (approximately) 18.43°. ◀

0.2 Problems

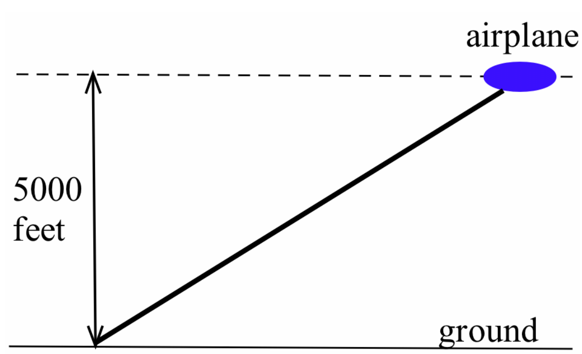
1. Estimate the slope of each line shown below.



2. Estimate the slope of each line shown below.



3. Compute the slope of the line that passes through:
   (a) $(2,4)$ and $(5,8)$
   (b) $(-2,4)$ and $(3,-5)$
   (c) $(2,4)$ and (x,x^2)
   (d) $(2,5)$ and $(2+h, 1+(2+h)^2)$
   (e) (x,x^2+3) and (a,a^2+3)
4. Compute the slope of the line that passes through:
   (a) $(5,-2)$ and $(3,8)$
   (b) $(-2,-4)$ and $(5,-3)$
   (c) $(x,3x+5)$ and $(a,3a+5)$
   (d) $(4,5)$ and $(4+h,5-3h)$
   (e) $(1,2)$ and $(x,1+x^2)$
   (f) $(2,-3)$ and $(2+h,1-(1+h)^2)$
   (g) (x,x^2) and $(x+h,x^2+2xh+h^2)$
   (h) (x,x^2) and $(x-h,x^2-2xh+h^2)$
5. A small airplane at an altitude of 5,000 feet is flying east at 300 feet per second (a bit over 200 miles per hour), and you are watching it with a small telescope as it passes directly overhead.
   (a) What is the **slope** of the telescope 5, 10 and 20 seconds after the plane passes overhead?
   (b) What is the **slope** of the telescope t seconds after the plane passes overhead?
   (c) After the plane passes overhead, is the slope of the telescope increasing, decreasing or staying the same?



6. You are at the origin, $(0,0)$, and are watching a small bug at the point $(t,1+t^2)$ at time t seconds.
   (a) What is the **slope** of your line of vision when $t = 5$, 10 and 15 seconds?
   (b) What is the **slope** of your line of vision at an arbitrary time t?
7. The blocks in a city are all perfect squares. A friend gives you directions to a good restaurant: "Go north 3 blocks, turn east and go 5 blocks, turn south and go 7 blocks, turn west and go 3 blocks." How far away (straight-line distance) is it?
8. At the restaurant (see previous problem), a fellow diner gives you directions to a hotel: "Go north 5 blocks, turn right and go 6 blocks, turn right and go 3 blocks, turn left and go 2 blocks." How far away is the hotel from the restuarant?
9. The bottom of a 20-foot ladder is 4 feet from the base of a wall.
   (a) How far up the wall does the ladder reach?
   (b) What is the slope of the ladder?
   (c) What angle does it make with the ground?
10. Let $P = (1,-2)$ and $Q = (5,4)$. Find:
    (a) the midpoint R of the line segment PQ.
    (b) the point T that is $\frac{1}{3}$ of the way from P to Q:
    $$\text{dist}(P,T) = \frac{1}{3}\,\text{dist}(P,Q)$$
    (c) the point S that is $\frac{2}{5}$ of the way from P to Q.
11. If $P = (2,3)$, $Q = (8,11)$ and $R = (x,y)$, where:
    $$x = 2a + 8(1-a),\ y = 3a + 11(1-a),\ 0 \le a \le 1$$
    (a) Verify that R is on the line segment PQ.
    (b) Verify that $\text{dist}(P,R) = (1-a)\cdot\text{dist}(P,Q)$.
12. A rectangular box is 24 inches long, 18 inches wide and 12 inches high.
    (a) Find the length of the longest (straight) stick that will fit into the box.
    (b) What angle (in degrees) does that stick make with the base of the box?
13. The lines $y = x$ and $y = 4 - x$ intersect at $(2,2)$.
    (a) Show that the lines are perpendicular.
    (b) Graph the lines together on your calculator using the "window" $[-10,10] \times [-10,10]$.
    (c) Why do the lines not appear to be perpendicular on the calculator display?
    (d) Find a suitable window so that the lines do appear perpendicular.

14. Two lines both go through the point $(1,2)$, one with slope 3 and one with slope $-\frac{1}{3}$.
    (a) Find equations for the lines.
    (b) Choose a suitable window so that the lines will appear perpendicular, and then graph them together on your calculator..

15. Sketch the line with slope m that goes through the point P, then find an equation for the line.
    (a) $m = 3$, $P = (2,5)$
    (b) $m = -2$, $P = (3,2)$
    (c) $m = -\frac{1}{2}$, $P = (1,4)$

16. Sketch the line with slope m that goes through the point P, then find an equation for the line.
    (a) $m = 5$, $P = (2,1)$
    (b) $m = -\frac{2}{3}$, $P = (1,3)$
    (c) $m = \pi$, $P = (1,-3)$

17. Find an equation for each line.
    (a) L_1 goes through the point $(2,5)$ and is parallel to $3x - 2y = 9$.
    (b) L_2 goes through the point $(-1,2)$ and is perpendicular to $2x = 7 - 3y$.
    (c) L_3 goes through the point $(3,-2)$ and is perpendicular to $y = 1$.

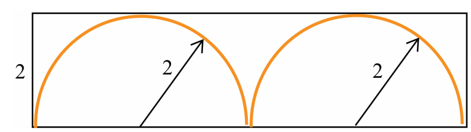
18. Find a value for the constant (A, B or D) so that:
    (a) the line $y = 2x + A$ goes through $(3,10)$.
    (b) the line $y = Bx + 2$ goes through $(3,10)$.
    (c) the line $y = Dx + 7$ crosses the y-axis at the point $(0,4)$.
    (d) the line $Ay = Bx + 1$ goes through the points $(1,3)$ and $(5,13)$.

19. Find the shortest distance between the circles with centers $C_1 = (1,2)$ and $C_2 = (7,10)$ with radii r_1 and r_2 when:
    (a) $r_1 = 2$ and $r_2 = 4$
    (b) $r_1 = 2$ and $r_2 = 7$
    (c) $r_1 = 5$ and $r_2 = 8$
    (d) $r_1 = 3$ and $r_2 = 15$
    (e) $r_1 = 12$ and $r_2 = 1$

20. Find an equation of the circle with center C and radius r when
    (a) $C = (2,7)$ and $r = 4$
    (b) $C = (3,-2)$ and $r = 1$
    (c) $C = (-5,1)$ and $r = 7$
    (d) $C = (-3,-1)$ and $r = 4$

21. Explain how to show, without graphing, whether a point $P = (x,y)$ is inside, on, or outside the circle with center $C = (h,k)$ and radius r.

22. A box with a base of dimensions 2 cm and 8 cm is definitely big enough to hold two semicircular rods with radii of 2 cm (see below).
    (a) Will these same two rods fit in a box 2 cm high and 7.6 cm wide?
    (b) Will they fit in a box 2 cm high and 7.2 cm wide? (Suggestion: Turn one of the rods over.)



23. Show that an equation of the circle with center $C = (h,k)$ and radius r is $(x-h)^2 + (y-k)^2 = r^2$.

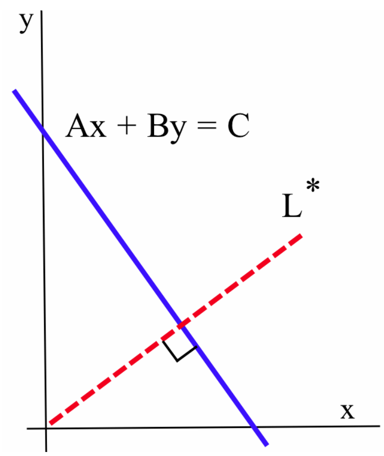
24. Find an equation of the line tangent to the circle $x^2 + y^2 = 25$ at the point P when:
    (a) $P = (3,4)$
    (b) $P = (-4,3)$
    (c) $P = (0,5)$
    (d) $P = (-5,0)$

25. Find an equation of the line tangent to the circle with center $C = (3,1)$ at the point P when:
    (a) $P = (8,13)$
    (b) $P = (-10,1)$
    (c) $P = (-9,6)$
    (d) $P = (3,14)$

26. Find the center $C = (h,k)$ and the radius r of the circle that goes through the three points:
    (a) $(0,1)$, $(1,0)$ and $(0,5)$
    (b) $(1,4)$, $(2,2)$ and $(8,2)$
    (c) $(1,3)$, $(4,12)$ and $(8,4)$

27. How close does
    (a) the line $3x - 2y = 4$ come to the point $(2,5)$?
    (b) the line $y = 5 - 2x$ come to the point $(1,-2)$?
    (c) the circle with radius 3 and center at $(2,3)$ come to the point $(8,3)$?

28. How close does
    (a) the line $2x - 5y = 4$ come to the point $(1,5)$?
    (b) the line $y = 3 - 2x$ come to the point $(5,-2)$?
    (c) the circle with radius 4 and center at $(4,3)$ come to the point $(10,3)$?



29. Follow the steps below (and refer to the figure) to find a formula for the distance from the origin to the line $Ax + By = C$.
    (a) Show that the line L given by $Ax + By = C$ has slope $m = -\frac{A}{B}$.
    (b) Find the equation of the line L^* that goes through $(0,0)$ and is perpendicular to L.
    (c) Show that L and L^* intersect at the point:
    $$(x,y) = \frac{AC}{A^2+B^2}, \frac{BC}{A^2+B^2}$$
    (d) Show that the distance from the origin to the point (x,y) is:
    $$\frac{|C|}{\sqrt{A^2+B^2}}$$

30. Show that a formula for the distance from the point (p,q) to the line $Ax + By = C$ is:
    $$\frac{|Ap+Bq-C|}{\sqrt{A^2+B^2}}$$
    (The steps will be similar to those in the previous problem, but the algebra will be more complicated.)

0.2 Practice Answers

1. Length $= \text{dist}(-7, -2) = |(-7) - (-2)| = |-5| = 5$.

   The midpoint is at $\dfrac{(-7)+(-2)}{2} = \dfrac{-9}{2} = -4.5$.

2. $\text{dist}(P, Q) = \text{dist}(P, R) \Rightarrow (x-1)^2 + (y+4)^2 = (x-0)^2 + (y+3)^2$; squaring each side and simplifying eventually yields $y = x - 4$.

3. The point $P = (x, y)$ is on the circle when it is 5 units from the center $C = (-2, 6)$, so $\text{dist}(P, C) = 5$. Then $\text{dist}\left((x, y), (-2, 6)\right) = 5$, so $(x+2)^2 + (y-6)^2 = 5 \Rightarrow (x+2)^2 + (y-6)^2 = 25$.

4. $\Delta x = 5 - (-3) = 8$ and $\Delta y = -14 - 2 = -16$, so:

$$\text{slope} = \frac{\Delta y}{\Delta x} = \frac{-16}{8} = -2$$

5. slope $= \dfrac{\Delta y}{\Delta x} = \dfrac{(-3+5h)-(-3)}{(2+h)-2} = \dfrac{5h}{h} = 5$. The midpoint is at $\dfrac{(2)+(2+h)}{2}, \dfrac{(-3+5h)+(-3)}{2} \quad = \quad 2 + \dfrac{h}{2}, -3 + \dfrac{5h}{2} \quad .$

6. slope $= \dfrac{\Delta y}{\Delta x} = \dfrac{(3a^2+5a)-(3x^2+5x)}{a-x} = \dfrac{3(a^2-x^2)+5(a-x)}{a-x} = \dfrac{3(a+x)(a-x)+5(a-x)}{a-x} = 3(a+x)+5$

7. Let $y_1 = mx_1 + b$ and $y_2 = mx_2 + b$. Then:

$$\text{slope} = \frac{\Delta y}{\Delta x} = \frac{(mx_2+b)-(mx_1+b)}{x_2-x_1} = \frac{m(x_2-x_1)}{x_2-x_1} = m$$

8. The line $3x + 5y = 17$ has slope $-\frac{3}{5}$, so the slope of the parallel line is $m = -\frac{3}{5}$. Using the form $y = -\frac{3}{5}x + b$ and the point $(-2, 3)$ on the line, we have $3 = -\frac{3}{5}(-2) + b \Rightarrow b = \frac{9}{5} \Rightarrow y = -\frac{3}{5}x + \frac{9}{5}$, or $5y + 3x = 9$.

9. The line $3y - 7x = 2$ has slope $\frac{7}{3}$, so the slope of the perpendicular line is $m = -\frac{3}{7}$. Using the form $y = -\frac{3}{7}x + b$ and the point $(2, -5)$ on the line, we have $-5 = -\frac{3}{7}(2) + b \Rightarrow b = -\frac{29}{7} \Rightarrow y = -\frac{3}{7}x - \frac{29}{7}$, or $7y + 3x = -29$.