Babeş-Bolyai University, Faculty of Mathematics and Computer Science
Analysis for Computer Science
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Exercise Set #2

1. Using the following sets, fill in a table with the header:

Set	Set of upper bounds	Set of lower bounds	Min	Max	Inf	Sup

$$A = (-1,1) \cup \{7\},$$
$$B = [-8,\pi) \cap \mathbb{Z},$$
$$C = \{(-1)^n \mid n \in \mathbb{N}\},$$
$$D = \left\{\frac{1}{n} \mid n \in \mathbb{Z} \setminus \{0\}\right\},$$
$$E = \{x \in \mathbb{Z} \mid |2x-3| < 5, |x+1| > 2\},$$
$$F = \{2^m + n! \mid m,n \in \mathbb{N}\},$$
$$G = \left\{\frac{n}{n+1} \mid n \in \mathbb{N}^*\right\},$$
$$H = \left\{\frac{n}{n+m} \mid m,n \in \mathbb{N}^*\right\},$$
$$I = \left\{\frac{3n+7}{n+1} \mid n \in \mathbb{N}\right\},$$
$$J = \left\{\frac{n}{1-n^2} \mid n \in \mathbb{N}, n \geq 2\right\},$$
$$K = \left\{(-1)^n\left(1-\frac{1}{n}\right) \mid n \in \mathbb{N}^*\right\},$$
$$L = \left\{\frac{1}{n+1} + \frac{1+(-1)^n}{2n} \mid n \in \mathbb{N}^*\right\},$$
$$M = \left\{x + \frac{1}{x} \mid x \in \mathbb{R}, x > 0\right\},$$
$$N = \left\{x \in \mathbb{R} \mid x^2 - x + \frac{3}{4} \leq 0\right\},$$
$$O = \left\{x \in \mathbb{R} \mid \frac{x^3+3x}{3x^2+1} < 1\right\},$$
$$P = \left\{x \in \mathbb{Q} \mid x^2 \leq \sqrt{2}\right\}.$$

2. Find two sets A and B such that the following conditions are simultaneously met:

   i) one of the sets is unbounded (but not an interval) and the other is finite,

   ii) $\sup A = \inf B = 2 \in A$,

   iii) for every $a \in A$ and $b \in B$, there exists $c \in \mathbb{R}$ with $a < c < b$.

True or false: is it possible to choose B to be the finite set?

3. True or false: $\displaystyle\bigcap_{n=1}^{\infty}\left(0,\frac{1}{n}\right) = \emptyset$?
What can you say about the Nested Interval Property when replacing the closed intervals with open ones?

4. Decide which of the following sets are neighborhoods of 0. Justify.

$$A = [-1,1],$$
$$B = [-1,1] \cap \mathbb{Z},$$
$$C = (-1,0) \cup (0,1),$$
$$D = (-0.001, +\infty),$$
$$E = \mathbb{R},$$
$$F = \{0\} \cup \left\{\frac{1}{n} \mid n \in \mathbb{Z} \setminus \{0\}\right\},$$
$$G = \mathbb{Q},$$
$$H = \left[1-\frac{3}{2}, 1+\frac{3}{2}\right] \cup (3,4).$$

5. Show that if $x,y \in \mathbb{R}$, $x \neq y$, there exist $U \in \mathcal{V}(x)$ and $V \in \mathcal{V}(y)$ such that $U \cap V = \emptyset$.

Additional exercises:

6. Suppose that B is a bounded subset of $\mathbb{R}$ and let A be a nonempty subset of B. Prove that

$$\inf B \leq \inf A \leq \sup A \leq \sup B.$$

True or false: if, in addition, $\inf A = \inf B$ and $\sup A = \sup B$, does it follow that $A = B$? Justify.

7. Suppose that A and B are nonempty subsets of $\mathbb{R}$ which are bounded above. Show that $A \cup B$ is nonempty and bounded above and

$$\sup(A \cup B) = \max\{\sup A, \sup B\}.$$

8. Suppose that A and B are nonempty subsets of $\mathbb{R}$ such that for every $a \in A$ and $b \in B$, $a \leq b$. Show that $\sup A \leq \inf B$.

9. Suppose that A and B are nonempty subsets of $\mathbb{R}$. Prove that

i) $\sup\{a + b \mid a \in A, b \in B\} = \sup A + \sup B$,

ii) $\sup\{a - b \mid a \in A, b \in B\} = \sup A - \inf B$.

10. Let A be a nonempty subset of $\mathbb{R}$ such that $\inf A > 0$. Prove that

$$\sup\left\{\frac{1}{x} \mid x \in A\right\} = \frac{1}{\inf A}.$$