Babeş-Bolyai University, Faculty of Mathematics and Computer Science
Analysis for Computer Science
Year: 2015/2016

6 Integration

6.1 The Riemann integral

Definition 6.1. Let $a, b \in \mathbb{R}$, $a < b$. A *partition* of the interval $[a, b]$ is a finite ordered set $P = (x_0, x_1, \dots, x_n)$ of points in $[a, b]$ such that $a = x_0 < x_1 < \dots < x_{n-1} < x_n = b$. The intervals $[x_{i-1}, x_i]$ $(i = 1, \dots, n)$ are called subintervals of the partition P.
The *norm* of P is $\|P\| = \max\{x_1 - x_0, x_1 - x_2, \dots, x_n - x_{n-1}\}$ (that is, the length of the largest subinterval of the partition P).
Suppose that, for each $i = 1, \dots, n$, ξ_i has been chosen in each subinterval $[x_{i-1}, x_i]$ and denote $\xi = (\xi_1, \dots, \xi_n)$. Then (P, ξ) is called a *tagged partition* of $[a, b]$.

Definition 6.2. Let $a, b \in \mathbb{R}$, $a < b$, $f : [a, b] \to \mathbb{R}$ and (P, ξ) a tagged partition of $[a, b]$ (as in Definition 6.1). Then the sum

$$\sigma(f; P, \xi) = \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1})$$

is called the *Riemann sum* of f corresponding to the tagged partition (P, ξ).

Definition 6.3. Let $a, b \in \mathbb{R}$, $a < b$ and $f : [a, b] \to \mathbb{R}$. We say that f is *Riemann integrable* on $[a, b]$ if there exists $I \in \mathbb{R}$ such that

$$\forall \varepsilon > 0,\ \exists \delta_\varepsilon > 0 \text{ s.t. } \forall (P, \xi) \text{ tagged partition of } [a, b] \text{ with } \|P\| < \delta_\varepsilon,\ |\sigma(f; P, \xi) - I| < \varepsilon.$$

The family of all Riemann integrable functions on $[a, b]$ is denoted by $\mathcal{R}[a, b]$.
If $f \in \mathcal{R}[a, b]$, then $I \in \mathbb{R}$ (given above) is uniquely determined and called the *Riemann integral* (or *definite integral*) of f on $[a, b]$, and we write

$$I = \int_a^b f(x)dx = \int_a^b f.$$

Theorem 6.1 (First Fundamental Theorem of Calculus)**.** *Let $a, b \in \mathbb{R}$, $a < b$ and $f \in \mathcal{R}[a, b]$. Define $F : [a, b] \to \mathbb{R}$,*

$$F(t) = \int_a^t f.$$

Then F is continuous. Moreover, if f is continuous at $c \in [a, b]$, then F is differentiable at c and $F'(c) = f(c)$.

Theorem 6.2 (Second Fundamental Theorem of Calculus, Leibniz-Newton Formula)**.** *Let $a, b \in \mathbb{R}$, $a < b$ and $f \in \mathcal{R}[a, b]$. If $F : [a, b] \to \mathbb{R}$ is an antiderivative of f, then*

$$\int_a^b f = F(b) - F(a).$$

6.2 Improper integrals

Definition 6.4. Let $J \subseteq \mathbb{R}$ be an interval and $f : J \to \mathbb{R}$. We say that f is *locally Riemann integrable* on J if for any $a, b \in J$ with $a < b$ the function f is Riemann integrable on $[a, b]$.

Definition 6.5. Let $a \in \mathbb{R}$, $b \in \mathbb{R} \cup \{+\infty\}$ with $a < b$ and $f : [a, b) \to \mathbb{R}$ a locally Riemann integrable function on $[a, b)$. We say that f is *improperly integrable on* $[a, b)$ if the limit $\lim\limits_{\substack{t \to b \\ t < b}} \displaystyle\int_a^t f(x)dx$ exists in $\mathbb{R}$. In this case this limit is called the *improper integral of* f *on* $[a, b)$ and is denoted, if $b \in \mathbb{R}$, by $\displaystyle\int_a^{b-0} f(x)dx$ and, if $b = +\infty$, by $\displaystyle\int_a^{+\infty} f(x)dx$.

Definition 6.6. Let $a \in \mathbb{R} \cup \{-\infty\}$, $b \in \mathbb{R}$ with $a < b$ and $f : (a, b] \to \mathbb{R}$ a locally Riemann integrable function on $(a, b]$. We say that f is *improperly integrable on* $(a, b]$ if the limit $\lim\limits_{\substack{t \to a \\ t > a}} \displaystyle\int_t^b f(x)dx$ exists in $\mathbb{R}$. In this case this limit is called the *improper integral of* f *on* $(a, b]$ and is denoted, if $a \in \mathbb{R}$, by $\displaystyle\int_{a+0}^{b} f(x)dx$ and, if $a = -\infty$, by $\displaystyle\int_{-\infty}^{b} f(x)dx$.

Definition 6.7. Let $a, b \in \overline{\mathbb{R}}$ with $a < b$ and $f : (a, b) \to \mathbb{R}$ a locally Riemann integrable function on (a, b). We say that f is *improperly integrable on* (a, b) if there exists $c \in (a, b)$ such that f is improperly integrable both on $(a, c]$ and on $[c, b)$. In this case the *improper integral of* f *on* (a, b) is defined in the following way:

$$\begin{aligned}
\text{if } a, b \in \mathbb{R} : \int_{a+0}^{b-0} f(x)dx &= \int_{a+0}^{c} f(x)dx + \int_{c}^{b-0} f(x)dx, \\
\text{if } a \in \mathbb{R}, b = +\infty : \int_{a+0}^{+\infty} f(x)dx &= \int_{a+0}^{c} f(x)dx + \int_{c}^{+\infty} f(x)dx, \\
\text{if } a = -\infty, b \in \mathbb{R} : \int_{-\infty}^{b-0} f(x)dx &= \int_{-\infty}^{c} f(x)dx + \int_{c}^{b-0} f(x)dx, \\
\text{if } a = -\infty, b = +\infty : \int_{-\infty}^{+\infty} f(x)dx &= \int_{-\infty}^{c} f(x)dx + \int_{c}^{+\infty} f(x)dx,
\end{aligned}$$

Remark 6.1. Let $a, b \in \mathbb{R}$ with $a < b$ and $f : [a, b) \to \mathbb{R}$. The expression $\displaystyle\int_a^{b-0} f(x)dx$ is also called an improper integral which is said to *converge* if f is improperly integrable on $[a, b)$ and to *diverge* otherwise. In a similar way one defines convergence and divergence for improper integrals in the cases considered in Definitions 6.6, 6.7.

Theorem 6.3 (Comparison Test for Improper Integrals)**.** *Let $a \in \mathbb{R}$, $b \in \mathbb{R} \cup \{+\infty\}$ with $a < b$ and $f, g : [a, b) \to \mathbb{R}$ be locally Riemann integrable functions on $[a, b)$ satisfying*

$$\exists c \in [a, b) \text{ s.t. } \forall x \in [c, b),\ 0 \leq f(x) \leq g(x). \tag{1}$$

(i) If g is improperly integrable on $[a, b)$, then f is improperly integrable on $[a, b)$.

(ii) If f is not improperly integrable on $[a, b)$, then g is not improperly integrable on $[a, b)$.

Remark 6.2. If in Theorem 6.3, f and g are additionally nonnegative and satisfy instead of (1) the following condition

$$\exists \alpha, \beta > 0, \exists c \in [a, b) \text{ s.t. } \forall x \in [c, b),\ \alpha g(x) \leq f(x) \leq \beta g(x),$$

then f is improperly integrable on $[a, b)$ if and only if g is improperly integrable on $[a, b)$.

Corollary 6.1. *Let $a, b \in \mathbb{R}$ with $a < b$, $f : [a, b) \to [0, +\infty)$ be a locally Riemann integrable function on $[a, b)$ and $p \in \mathbb{R}$ such that the limit $L = \lim\limits_{\substack{x \to b \\ x < b}} (b - x)^p f(x)$ exists.*

(i) If $p < 1$ and $L < +\infty$, then f is improperly integrable on $[a, b)$.

(ii) If $p \geq 1$ and $L > 0$, then f is not improperly integrable on $[a, b)$.

Corollary 6.2. *Let $a, b \in \mathbb{R}$ with $a < b$, $f : (a, b] \to [0, +\infty)$ be a locally Riemann integrable function on $[a, b)$ and $p \in \mathbb{R}$ such that the limit $L = \lim\limits_{\substack{x \to a \\ x > a}} (x - a)^p f(x)$ exists.*

(i) If $p < 1$ and $L < +\infty$, then f is improperly integrable on $(a, b]$.

(ii) If $p \geq 1$ and $L > 0$, then f is not improperly integrable on $(a, b]$.

Corollary 6.3. *Let $a \in \mathbb{R}$, $f : [a, +\infty) \to [0, +\infty)$ be a locally Riemann integrable function on $[a, +\infty)$ and $p \in \mathbb{R}$ such that the limit $L = \lim\limits_{x \to \infty} x^p f(x)$ exists.*

(i) If $p > 1$ and $L < +\infty$, then f is improperly integrable on $[a, +\infty)$.

(ii) If $p \leq 1$ and $L > 0$, then f is not improperly integrable on $[a, +\infty]$.

Theorem 6.4 (Integral Test for Convergence of Series)**.** *Let $m \in \mathbb{N}$ and $f : [m, +\infty) \to [0, +\infty)$ be a decreasing function. Then f is improperly integrable on $[m, +\infty)$ if and only if the series $\sum\limits_{n \geq m} f(n)$ is convergent.*

6.3 Multiple integrals

Definition 6.8. A set $A \subseteq \mathbb{R}^n$ is called a *nondegenerate compact interval* in $\mathbb{R}^n$ if there exist $a_i, b_i \in \mathbb{R}$, $a_i < b_i$, $i = 1, \ldots, n$ such that $A = [a_1, b_1] \times [a_2, b_2] \times \ldots \times [a_n, b_n]$.

Definition 6.9. Let $A = [a_1, b_1] \times [a_2, b_2]$ be a nondegenerate compact interval in $\mathbb{R}^2$. If $(x_0, x_1, \ldots, x_p)$ and $(y_0, y_1, \ldots, y_q)$ are partitions of $[a_1, b_1]$ and $[a_2, b_2]$, respectively, then

$$P = \{A_{ij} \mid i = 1, \ldots, p,\ j = 1, \ldots, q\},$$

where $A_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$, is a *partition of* A.
The *norm* of P, denoted by $\|P\|$, is the length of the largest diagonal of any $A_{i,j}$, $i = 1, \ldots, p$, $j = 1, \ldots, q$.
Suppose that, for each $i = 1, \ldots, p$, $j = 1, \ldots, q$, (x^*_{ij}, y^*_{ij}) has been chosen in each A_{ij} and denote $\xi = \big((x^*_{ij}, y^*_{ij})\big)_{\substack{1 \leq i \leq p \\ 1 \leq j \leq q}}$. Then (P, ξ) is called a *tagged partition* of A.

Definition 6.10. Let A be a nondegenerate compact interval in $\mathbb{R}^2$, $f : A \to \mathbb{R}$ and (P, ξ) a tagged partition of A (as in Definition 6.9). Then the sum

$$\sigma(f; P, \xi) = \sum_{i=1}^{p} \sum_{j=1}^{q} f(x^*_{ij}, y^*_{ij})(x_i - x_{i-1})(y_j - y_{j-1})$$

is called the *Riemann sum* of f corresponding to the tagged partition (P, ξ).

Definition 6.11. Let A be a nondegenerate compact interval in $\mathbb{R}^2$ and $f : A \to \mathbb{R}$. We say that f is *Riemann integrable* on A if there exists $I \in \mathbb{R}$ such that

$$\forall \varepsilon > 0,\ \exists \delta_\varepsilon > 0 \text{ s.t. } \forall (P, \xi) \text{ tagged partition of } A \text{ with } \|P\| < \delta_\varepsilon,\ |\sigma(f; P, \xi) - I| < \varepsilon.$$

The family of all Riemann integrable functions on A is denoted by $\mathcal{R}(A)$.
If $f \in \mathcal{R}(A)$, then $I \in \mathbb{R}$ (given above) is uniquely determined and called the *Riemann integral* (or *double integral*) of f on A, and we write

$$I = \iint_A f(x, y)\, dx\, dy.$$

Remark 6.3. (i) Let A be a nondegenerate compact interval in $\mathbb{R}^n$ and $f : A \to \mathbb{R}$. In a similar way as above, one can define the *Riemann integral* (or *multiple integral*) of f on A, denoted by $\displaystyle\int \cdots \int_A f(x_1, \ldots, x_n)\, dx_1 \ldots dx_n$.

If $n = 3$, we have a *triple integral* and we write $\displaystyle\iiint_A f(x, y, z)\, dx\, dy\, dz$.

(ii) Let $M \subseteq \mathbb{R}^n$ be nonempty, A be a nondegenerate compact interval in $\mathbb{R}^n$ with $M \subseteq A$ and $f : M \to \mathbb{R}$. Define $f' : A \to \mathbb{R}$ by

$$f'(x) = \begin{cases} f(x), & \text{if } x \in M, \\ 0, & \text{if } x \in A \setminus M. \end{cases}$$

If f' is Riemann integrable on A, then we say that f is Riemann integrable on M and we define the *Riemann integral* (or *multiple integral*) of f on M by

$$\int \cdots \int_M f(x_1, \ldots, x_n)\, dx_1 \ldots dx_n = \int \cdots \int_A f'(x_1, \ldots, x_n)\, dx_1 \ldots dx_n.$$

Note that this definition does not depend on the choice of A.

6.3.1 Computing multiple integrals

Theorem 6.5. *Let $A = [a_1, b_1] \times \ldots \times [a_n, b_n]$ be a nondegenerate compact interval in $\mathbb{R}^n$ and $f : A \to \mathbb{R}$ be continuous. Then*

(i) f is Riemann integrable on A.

(ii) $\displaystyle\int \cdots \int_A f(x_1, \ldots, x_n)\, dx_1 \ldots dx_n = \int_{a_1}^{b_1} \left(\int_{a_2}^{b_2} \cdots \left(\int_{a_n}^{b_n} f(x_1, \ldots, x_n)\, dx_n \right) \ldots dx_2 \right) dx_1,$

where one can choose any of the iterated integrals.

Remark 6.4. Alternative notation for the above iterated integral:

$$\int_{a_1}^{b_1} \int_{a_2}^{b_2} \cdots \int_{a_n}^{b_n} f(x_1, \ldots, x_n)\, dx_n \ldots dx_2\, dx_1, \qquad \int_{a_1}^{b_1} dx_1 \int_{a_2}^{b_2} dx_2 \cdots \int_{a_n}^{b_n} f(x_1, \ldots, x_n)\, dx_n.$$

Definition 6.12. A set $M \subseteq \mathbb{R}^2$ is called

- *simple w.r.t. the y-axis*: if there exist $a, b \in \mathbb{R}$, $a < b$ and $\varphi_1, \varphi_2 : [a, b] \to \mathbb{R}$ continuous functions with $\varphi_1 \le \varphi_2$ such that $M = \{(x, y) \in \mathbb{R}^2 \mid a \le x \le b,\ \varphi_1(x) \le y \le \varphi_2(x)\}$.
- *simple w.r.t. the x-axis*: if there exist $c, d \in \mathbb{R}$, $c < d$ and $\psi_1, \psi_2 : [c, d] \to \mathbb{R}$ continuous functions with $\psi_1 \le \psi_2$ such that $M = \{(x, y) \in \mathbb{R}^2 \mid c \le y \le d,\ \psi_1(y) \le x \le \psi_2(y)\}$.

Theorem 6.6. *Let $M \subseteq \mathbb{R}^2$ and $f : M \to \mathbb{R}$ a continuous function.*
If M is simple w.r.t. the y-axis (as in Definition 6.12), then f is Riemann integrable on M and

$$\iint_M f(x, y)\, dx\, dy = \int_a^b \int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y)\, dy\, dx.$$

If M is simple w.r.t. the x-axis (as in Definition 6.12), then f is Riemann integrable on M and

$$\iint_M f(x, y)\, dx\, dy = \int_c^d \int_{\psi_1(y)}^{\psi_2(y)} f(x, y)\, dx\, dy.$$

1.11 Tabel de integrale nedefinite

Peste tot în acest tabel J este un interval $\subset \mathbf{R}$

1.	$f: \mathbf{R} \to \mathbf{R}$ $f(x) = x^n;\ n \in \mathbf{N}$	$\int x^n dx = \frac{x^{n+1}}{n+1} + \mathcal{C}.$
2.	$f: J \to \mathbf{R};\ J \subset (0,\infty)$ $f(x) = x^a;\ a \in \mathbf{R} \setminus \{-1\}$	$\int x^a dx = \frac{x^{a+1}}{a+1} + \mathcal{C}.$
3.	$f: \mathbf{R} \to \mathbf{R}$ $f(x) = a^x;\ a \in \mathbf{R}_+^* \setminus \{1\}$	$\int a^x dx = \frac{a^x}{\ln a} + \mathcal{C}.$
4.	$f: J \to \mathbf{R};\ J \subset \mathbf{R}^*$ $f(x) = \frac{1}{x}$	$\int \frac{1}{x} dx = \ln \lvert x \rvert + \mathcal{C}.$
5.	$f: J \to \mathbf{R};\ J \subset \mathbf{R} \setminus \{-a, a\}$ $f(x) = \frac{1}{x^2 - a^2},\ \{a \neq 0\}$	$\int \frac{1}{x^2 - a^2} dx = \frac{1}{2a} \ln \left\lvert \frac{x-a}{x+a} \right\rvert + \mathcal{C}.$
6.	$f: \mathbf{R} \to \mathbf{R}$ $f(x) = \frac{1}{x^2 + a^2};\ a \neq 0$	$\int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \operatorname{arc\,tg} \frac{x}{a} + \mathcal{C}.$
7.	$f: \mathbf{R} \to \mathbf{R}$ $f(x) = \sin x$	$\int \sin x\, dx = -\cos x + \mathcal{C}.$
8.	$f: \mathbf{R} \to \mathbf{R}$ $f(x) = \cos x$	$\int \cos x\, dx = \sin x + \mathcal{C}.$
9.	$f: J \to \mathbf{R};\ J \subset \mathbf{R} \setminus \left\{ (2k+1)\frac{\pi}{2} \,\middle\vert\, k \in Z \right\}$ $f(x) = \frac{1}{\cos^2 x}$	$\int \frac{1}{\cos^2 x} dx = \operatorname{tg} x + \mathcal{C}.$
10.	$f: J \to \mathbf{R};\ J \subset \mathbf{R} \setminus \{k\pi \mid k \in \mathbf{Z}\}$ $f(x) = \frac{1}{\sin^2 x}$	$\int \frac{1}{\sin^2 x} dx = -\operatorname{ctg} x + \mathcal{C}.$
11.	$f: J \to \mathbf{R};\ J \subset \mathbf{R} \setminus \left\{ (2k+1)\frac{\pi}{2} \,\middle\vert\, k \in \mathbf{Z} \right\}$ $f(x) = \operatorname{tg} x$	$\int \operatorname{tg} x\, dx = -\ln \lvert \cos x \rvert + \mathcal{C}.$
12.	$f: J \to \mathbf{R};\ J \subset \mathbf{R} \setminus \{k\pi \mid k \in \mathbf{Z}\}$ $f(x) = \operatorname{ctg} x$	$\int \operatorname{ctg} x\, dx = \ln \lvert \sin x \rvert + \mathcal{C}.$
13.	$f: \mathbf{R} \to \mathbf{R}$ $f(x) = \frac{1}{\sqrt{x^2 + a^2}};\ a \neq 0$	$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln(x + \sqrt{x^2 + a^2}) + \mathcal{C}.$
14.	$f: J \to \mathbf{R}$, $a > 0$, $\begin{cases} J \subset (-\infty, -a) \\ \text{sau} \\ J \subset (a, \infty) \end{cases}$ $f(x) = \frac{1}{\sqrt{x^2 - a^2}}$	$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln \lvert x + \sqrt{x^2 - a^2} \rvert + \mathcal{C}.$
15.	$f: J \to \mathbf{R};\ J \subset (-a, a), a > 0,$ $f(x) = \frac{1}{\sqrt{a^2 - x^2}}$	$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \arcsin \frac{x}{a} + \mathcal{C}.$