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Analysis for Computer Science
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Exercise Set #10

1. Study the improper integrability of the following functions on their domains and, in case they are improperly integrable, compute the corresponding improper integrals:

a) $a, b, p \in \mathbb{R}$, $a < b$, $f : [a, b) \to \mathbb{R}$, $f(x) = \dfrac{1}{(b-x)^p}$,

b) $a, b, p \in \mathbb{R}$, $a < b$, $f : (a, b] \to \mathbb{R}$, $f(x) = \dfrac{1}{(x-a)^p}$,

c) $a > 0$, $p \in \mathbb{R}$, $f : [a, \infty) \to \mathbb{R}$, $f(x) = \dfrac{1}{x^p}$, d) $f : [0, \infty) \to \mathbb{R}$, $f(x) = e^{-x}$,

e) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \dfrac{1}{1+x^2}$, f) $f : [1, \infty) \to \mathbb{R}$, $f(x) = \dfrac{1}{x(1+x)}$,

g) $f : (-\infty, 0] \to \mathbb{R}$, $f(x) = xe^{-x^2}$, h) $f : [0, \infty) \to \mathbb{R}$, $f(x) = e^{-x} \sin x$,

i) $f : [0, \infty) \to \mathbb{R}$, $f(x) = \dfrac{\arctan x}{(1+x^2)^{\frac{3}{2}}}$, j) $f : [0, 1) \to \mathbb{R}$, $f(x) = \sqrt{\dfrac{1+x}{1-x}}$,

k) $f : (1, 2] \to \mathbb{R}$, $f(x) = \dfrac{1}{x\sqrt{3x^2 - 2x - 1}}$.

2. Study the improper integrability of the following functions on their domains:

a) $f : \left[0, \dfrac{\pi}{2}\right) \to \mathbb{R}$, $f(x) = \dfrac{1}{\cos x}$, b) $f : [0, 1) \to \mathbb{R}$, $f(x) = \dfrac{1}{\sqrt[4]{1-x^4}}$,

c) $f : (0, 1] \to \mathbb{R}$, $f(x) = \dfrac{1}{\sqrt[3]{x(e^x - e^{-x})}}$, d) $f : (1, \infty) \to \mathbb{R}$, $f(x) = \dfrac{\ln x}{x\sqrt{x^2-1}}$,

e) $f : [0, 1) \to \mathbb{R}$, $f(x) = \dfrac{1}{\sqrt{(1-x^2)(1-k^2x^2)}}$, where $k \in (-1, 1)$,

f) $f : [1, \infty) \to \mathbb{R}$, $f(x) = \dfrac{1}{x^\alpha (1+x^2)^\beta}$, where $\alpha, \beta \in \mathbb{R}$,

g) $f : [1, \infty) \to \mathbb{R}$, $f(x) = \dfrac{x^\alpha \arctan x}{1 + x^\beta}$, where $\alpha, \beta \in \mathbb{R}$.

3. Use the Integral Test to study if the following series are convergent or divergent:

a) $\displaystyle\sum_{n\geq 2} \frac{1}{n(\ln n)^2}$, b) $\displaystyle\sum_{n\geq 1} \frac{\ln n}{n^{3/2}}$, c) $\displaystyle\sum_{n\geq 1} \frac{5n^2}{1+2n^3}$, d) $\displaystyle\sum_{n\geq 1} ne^{-n}$.

4. Compute the following multiple integrals:

a) $\displaystyle\iint_A (\sin x + \sin y)\, dx\, dy$, where $A = [0, \pi/2] \times [0, \pi/4]$,

b) $\displaystyle\iint_A \frac{1}{(x+y)^2}\, dx\, dy$, where $A = [3, 4] \times [1, 2]$,

c) $\displaystyle\iint_A \frac{1}{x^2+y^2}\, dx\, dy$, where $A = [1/a, a] \times [0, 1]$ and $a > 1$,

d) $\displaystyle\iint_A \frac{x}{x^2+y^2}\, dx\, dy$, where $A = [1, 2] \times [0, 1]$,

e) $\displaystyle\iint_A \frac{xy^2}{x^2+1}\, dx\, dy$, where $A = [1, 2] \times [-3, 3]$,

f) $\displaystyle\iint_A \min\{x,y\}\,dx\,dy, \quad \text{where } A=[0,1]\times[0,1],$

g) $\displaystyle\iint_A e^{x+y}\,dx\,dy, \quad \text{where } A=[0,\ln 2]\times[0,\ln 3],$

h) $\displaystyle\iiint_A \frac{x^2z^3}{1+y^2}\,dx\,dy, \quad \text{where } A=[0,1]\times[0,1]\times[0,1],$

i) $\displaystyle\iiint_A \frac{1}{(x+y+z)^3}\,dx\,dy, \quad \text{where } A=[1,2]\times[1,2]\times[1,2].$

5. Let M be the subset of $\mathbb{R}^2$ bounded by the parabola $y=x^2$ and the lines $x=2$ and $y=0$.

a) Express M as a simple set first w.r.t. the y-axis and then w.r.t. the x-axis.

b) Compute $\displaystyle\iint_M xy\,dx\,dy$ in two ways.

c) Compute $\displaystyle\iint_M x\sin\left((4-y)^2\right)\,dx\,dy.$

6. Let M be the subset of $\mathbb{R}^2$ bounded by the parabolas $y=2x^2$ and $y=x^2+1$.

a) Express M as a simple set w.r.t. the y-axis. Is M simple w.r.t. the x-axis?

b) Compute $\displaystyle\iint_M (x+2y)\,dx\,dy.$

7. Let M be the subset of $\mathbb{R}^2$ bounded by the triangle with vertices $(0,0)$, $(1,1)$ and $(-1,1)$. Compute $\displaystyle\iint_M (x^2+y^2)\,dx\,dy.$

8. Let M be the subset of $\mathbb{R}^2$ bounded by the parabola $y^2=2x+6$ and the line $y=x-1$. Compute $\displaystyle\iint_M xy\,dx\,dy.$

9. Let M be the subset of $\mathbb{R}^2$ bounded by the hyperbola $xy=1$ and the lines $x=\sqrt{3}$ and $y=x$. Compute $\displaystyle\iint_M \frac{x}{y^2+1}\,dx\,dy.$

Additional exercises:

1. Consider Dirichlet's function $f:[0,1]\to\mathbb{R}$, $f(x)=\begin{cases}1, & \text{if } x\in\mathbb{Q},\\ 0, & \text{if } x\in\mathbb{R}\setminus\mathbb{Q}.\end{cases}$ Prove that $f\notin\mathcal{R}[0,1]$.

2. Consider Thomae's function $f:[0,1]\to\mathbb{R}$,

$$f(x)=\begin{cases}1/n, & \text{if } x=m/n\in\mathbb{Q} \text{ where } m \text{ and } n>0 \text{ are relatively prime},\\ 0, & \text{if } x\in\mathbb{R}\setminus\mathbb{Q}.\end{cases}$$

Prove that $f\in\mathcal{R}[0,1]$.

3. Consider $f:[0,1]\times[0,1]\to\mathbb{R}$, $f(x,y)=\begin{cases}1, & \text{if } x,y\in\mathbb{Q},\\ 0, & \text{if } x,y\in\mathbb{R}\setminus\mathbb{Q}.\end{cases}$ Prove that $f\notin\mathcal{R}\left([0,1]\times[0,1]\right)$.