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Analysis for Computer Science
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Exercise Set #5

1. Find the set of accumulation points for the following sets:
a) $A = \{1, 2, 3\}$, b) $A = (1, 2) \cup (2, 3)$, c) $A = (-\infty, 1) \cup (2, \infty)$, d) $A = \mathbb{Q}$, e) $A = \mathbb{Z}$,
f) $A = \left\{(-1)^n + \frac{1}{n} \mid n \in \mathbb{N}^*\right\}$.

2. Study the existence of the limit of Dirichlet's function $f : \mathbb{R} \to \mathbb{R}$,

$$f(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q}, \\ 0, & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

at every accumulation point of its domain. Then consider the function $g : \mathbb{R} \to \mathbb{R}$,

$$g(x) = \begin{cases} x, & \text{if } x \in \mathbb{Q}, \\ 0, & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

3. Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x(1 + \sin x)$. Show that f has no limit at $+\infty$.

4. Find:
a) $\lim\limits_{x\to 4} x(5 - x^2)$, b) $\lim\limits_{x\to -\infty} (-x^3 + 2x)$, c) $\lim\limits_{x\to 2} \dfrac{x^3 - 4}{x^2 + 1}$, d) $\lim\limits_{x\to 1} \dfrac{x^2 - 1}{3x - 3}$, e) $\lim\limits_{x\to 1} \dfrac{\sqrt{x} - 1}{x - 1}$,
f) $\lim\limits_{x\to 0} \dfrac{\sqrt{1 + 2x} - \sqrt{1 + x}}{x^2 + 2x}$, g) $\lim\limits_{x\to \infty} \sqrt{x}(\sqrt{x + 1} - \sqrt{x})$, h) $\lim\limits_{x\to 0} \dfrac{\sqrt[3]{1 + x} - 1}{x}$, i) $\lim\limits_{x\to 2} \dfrac{1}{2 - x}$,
j) $\lim\limits_{x\to -3} \dfrac{x^2 - 9}{(x + 3)^2}$, k) $\lim\limits_{\substack{x\to 1\\ x>1}} \left(\dfrac{1}{1 - x} - \dfrac{1}{x^3 - 1}\right)$, l) $\lim\limits_{x\to 0} \dfrac{x^2}{|x|}$, m) $\lim\limits_{x\to \infty} \dfrac{x}{\sqrt{x^2 + 1}}$, n) $\lim\limits_{x\to -\infty} \dfrac{x}{\sqrt{x^2 + 1}}$,
o) $\lim\limits_{x\to -\infty} e^{\frac{|x|+1}{x-1}}$, p) $\lim\limits_{x\to 0} \left(\dfrac{1 + 4x + x^2}{1 + x}\right)^{\frac{1}{x}}$, r) $\lim\limits_{x\to -\infty} \left(\dfrac{x^2 + x + 1}{x^2 - x + 1}\right)^{\sqrt{-x}}$.

5. Find $a, b \in \mathbb{R}$ such that $\lim\limits_{x\to \infty} \left(\sqrt{x^2 - x + 1} + ax + b\right) = 0$.

6. Find the one-sided limits of the function $f : D \to \mathbb{R}$ (with $D \subseteq \mathbb{R}$ the maximal domain of f) at 1, where
a) $f(x) = e^{\frac{1}{x^2-1}}$, b) $f(x) = e^{\frac{x^2-2}{x-1}}$, c) $f(x) = e^{1+\frac{2}{|x-1|}}$, d) $f(x) = \frac{|x|-1}{x-1}$.

7. Study the continuity of the following functions ($n \in \mathbb{N}$) and determine the type of their discontinuities:
a) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \lim\limits_{n\to\infty} \dfrac{e^{nx}}{1 + e^{nx}}$, b) $g \colon \mathbb{R} \setminus \{-1\} \to \mathbb{R}$, $g(x) = \lim\limits_{n\to\infty} \dfrac{x^n + x}{x^{2n} + 1}$,

8. Let $A \subseteq \mathbb{R}$. A function $f : A \to \mathbb{R}$ is called *Lipschitz* if $\exists L \geq 0$ such that

$$|f(x) - f(y)| \leq L|x - y|, \quad \forall x, y \in A.$$

Prove that any Lipschitz function is continuous. Then prove that the function $f : [0, 1] \to \mathbb{R}$, $f(x) = \sqrt{x}$ is not Lipschitz. Thus, there exist continuous functions that are not Lipschitz.

9. Find a function $f : \mathbb{R} \to \mathbb{R}$ that is discontinuous at every point in $\mathbb{R}$ and $|f|$ is continuous on $\mathbb{R}$.

10. Let $f, g : [0, 1] \to \mathbb{R}$ be continuous such that $f(x) = g(x)$, $\forall x \in [0, 1] \cap \mathbb{Q}$. Prove that $f(x) = g(x)$, $\forall x \in [0, 1]$.

True or false: is it enough to assume that f and g are continuous on $[0,1] \setminus \{\alpha\}$ for some $\alpha \in [0,1] \cap (\mathbb{R} \setminus \mathbb{Q})$?

11. A function $f : \mathbb{R} \to \mathbb{R}$ is called *additive* if $f(x+y) = f(x) + f(y)$, $\forall x, y \in \mathbb{R}$. Find all functions f that are continuous and additive on $\mathbb{R}$. Hint: Show first that $f(q) = q \cdot f(1)$, $\forall q \in \mathbb{Q}$.

12. Show that the following functions $f : \mathbb{R} \to \mathbb{R}$ have at least one zero in the set A:
a) $f(x) = (x^2+1)(x-b) + (x^4+1)(x-a)$, $A = (a,b)$ ($a, b \in \mathbb{R}$ with $a < b$),
b) $f(x) = x - \cos x$, $A = \mathbb{R}$, c) $f =$ a polynomial function of odd degree, $A = \mathbb{R}$.

13. (Brouwer's Fixed Point Theorem) Let $a, b \in \mathbb{R}$ with $a < b$ and $f : [a,b] \to [a,b]$ be a continuous function. Prove that f has at least one fixed point (that is, there exists $x_0 \in [a,b]$ such that $f(x_0) = x_0$).

14. Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(\mathbb{R})$ is finite.

15. Find all continuous functions $f : \mathbb{N} \to \mathbb{R}$ and all continuous functions $f : \mathbb{R} \to \mathbb{N}$.

Additional exercises:

16. Study the continuity of Thomae's function $f : \mathbb{R} \to \mathbb{R}$,

$$f(x) = \begin{cases} 1/n, & \text{if } x = m/n \in \mathbb{Q} \text{ where } m \text{ and } n > 0 \text{ are relatively prime,} \\ 0, & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Hint: One can easily show that f is discontinuous at every rational number. Moreover, f is continuous at every irrational number. To show this, note that having an irrational number, denominators of rational numbers that approach it get large.

17. Using the ε-δ characterization of limits, show that

$$\text{(i)} \lim_{x \to 0} x \sin \frac{1}{x} = 0, \quad \text{(ii)} \lim_{x \to 0} \frac{1}{x^2} = +\infty, \quad \text{(iii)} \lim_{x \to \infty} \frac{x^2}{x^2+1} = 1.$$