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Year: 2015/2016

2 Sequences of real numbers

Definition 2.1. Let $m \in \mathbb{Z}$. A *sequence* in $\mathbb{R}$ is a function $x : \{n \in \mathbb{Z} \mid n \geq m\} \to \mathbb{R}$. We usually write x_n instead of $x(n)$.

Notation: $(x_n)_{n \geq m}$, $(x_n)_{n=m}^{\infty}$.
In general, we consider $m = 1$ and use the notation $(x_n)_{n \geq 1}$, $(x_n)_{n \in \mathbb{N}^*}$, or (x_n).

Limit of a sequence

Definition 2.2. A sequence (x_n) in $\mathbb{R}$ is said to *converge* to $x \in \mathbb{R}$ if

$$\forall \varepsilon > 0, \exists N = N(\varepsilon) \in \mathbb{N}^* \text{ such that } \forall n \geq N \text{ we have } |x_n - x| < \varepsilon.$$

Remark 2.1. A sequence in $\mathbb{R}$ cannot converge to two distinct real numbers.

Definition 2.3. If a sequence (x_n) in $\mathbb{R}$ converges to some $x \in \mathbb{R}$, we say that (x_n) is *convergent*. The number x is called the *limit* of (x_n) and we write

$$\lim_{n \to \infty} x_n = x \quad \text{or} \quad x_n \to x.$$

If (x_n) does not converge to any real number, we say that (x_n) is *divergent*.

Remark 2.2. A sequence (x_n) in $\mathbb{R}$ is
(i) convergent if and only if $\exists x \in \mathbb{R}, \forall \varepsilon > 0, \exists N \in \mathbb{N}^*, \forall n \geq N, |x_n - x| < \varepsilon$.
(ii) divergent if and only if $\forall x \in \mathbb{R}, \exists \varepsilon > 0, \forall N \in \mathbb{N}^*, \exists n \geq N, |x_n - x| \geq \varepsilon$.

Algorithm for proving that (x_n) converges to $x \in \mathbb{R}$:
Step 1: Let $\varepsilon > 0$.
Step 2: Choose $N \in \mathbb{N}^*$ (obtained by manipulating the inequality that needs to be satisfied for n sufficiently large).
Step 3: Show that $\forall n \geq N, |x_n - x| < \varepsilon$.

Definition 2.4. A sequence (x_n) in $\mathbb{R}$ is said to *properly diverge* to $+\infty$ $(-\infty)$ if

$$\forall a \in \mathbb{R}, \exists N \in \mathbb{N}^* \text{ such that } \forall n \geq N \text{ we have } x_n > a \ (x_n < a).$$

In this case $+\infty$ $(-\infty)$ is called the *limit* of (x_n).
Notation: $\lim\limits_{n \to \infty} x_n = +\infty \quad \text{or} \quad x_n \to +\infty \left(\lim\limits_{n \to \infty} x_n = -\infty \quad \text{or} \quad x_n \to -\infty \right)$.
The sequence (x_n) is *properly divergent* if it properly diverges to either $+\infty$ or $-\infty$.

Remark 2.3. Let (x_n) be a sequence in $\mathbb{R}$. If $\lim\limits_{n \to \infty} x_n = x \in \overline{\mathbb{R}}$, we also sometimes say that (x_n) *tends* to x.

Theorem 2.1. *Let (x_n) be a sequence in $\mathbb{R}$ and let $x \in \overline{\mathbb{R}}$. Then*

$$\lim_{n \to \infty} x_n = x \iff \forall V \in \mathcal{V}(x), \exists N \in \mathbb{N}^* \textit{ such that } \forall n \geq N \textit{ we have } x_n \in V$$

(every neighborhood of x contains all terms of (x_n) except a finite number).

Remark 2.4. The above result follows easily since

$$|x_n - x| < \varepsilon \iff -\varepsilon < x_n - x < \varepsilon \iff x - \varepsilon < x_n < x + \varepsilon \iff x_n \in (x - \varepsilon, x + \varepsilon).$$

and

$$x_n > a \ (x_n < a) \iff x_n \in (a, +\infty) \ (x_n \in (-\infty, a)).$$

The "N" from above will be exactly the "N" from Definitions 2.2, 2.4.

Remark 2.5. For the behavior of a sequence w.r.t. its convergence/divergence, a finite number of terms of the sequence is irrelevant. More precisely, we can change or drop the first m ($m \in \mathbb{N}$) terms of a sequence without altering its behavior, nor its limit if it initially exists.

Limit theorems

Definition 2.5. A sequence (x_n) in $\mathbb{R}$ is said to be *bounded* if there exists $M > 0$ such that $\forall n \in \mathbb{N}^*$, $|x_n| \leq M$.

In other words, a sequence (x_n) is bounded if the set of its values $\{x_n \mid n \in \mathbb{N}^*\}$ is bounded. In a similar way one can consider the notions *bounded above* and *bounded below.* A bounded sequence is not necessarily convergent.

Theorem 2.2. *Any convergent sequence is bounded.*

Theorem 2.3. *Let $(x_n), (y_n)$ be sequences in $\mathbb{R}$ that converge to x and y, respectively. Let $c \in \mathbb{R}$. Then the sequences $(x_n + y_n)$, $(x_n - y_n)$, $(x_n \cdot y_n)$ and $(c \cdot x_n)$ converge to $x + y$, $x - y$, $x \cdot y$ and $c \cdot x$, respectively. Moreover, if $y \neq 0$ and $\forall n \in \mathbb{N}^*$, $y_n \neq 0$, then $\left(\dfrac{x_n}{y_n}\right)$ converges to $\dfrac{x}{y}$.*

Theorem 2.4. *Let (x_n) be a sequence in $\mathbb{R}$.*

- *(i) If (x_n) is properly divergent, then $\exists N \in \mathbb{N}^*$ such that $\forall n \geq N$, $x_n \neq 0$ and $\lim\limits_{n\to\infty} \dfrac{1}{x_n} = 0$.*
- *(ii) If (x_n) converges to 0 and $\forall n \in \mathbb{N}^*$, $x_n > 0$ ($x_n < 0$), then $\lim\limits_{n\to\infty} \dfrac{1}{x_n} = +\infty$ $\left(\lim\limits_{n\to\infty} \dfrac{1}{x_n} = -\infty\right)$.*

Theorem 2.5. *Let $(x_n), (y_n)$ be sequences in $\mathbb{R}$ such that $\forall n \in \mathbb{N}^*$, $x_n \leq y_n$.*

- *(i) If (x_n) and (y_n) converge, then $\lim\limits_{n\to\infty} x_n \leq \lim\limits_{n\to\infty} y_n$.*
- *(ii) If $\lim\limits_{n\to\infty} x_n = +\infty$, then $\lim\limits_{n\to\infty} y_n = +\infty$.*
- *(iii) If $\lim\limits_{n\to\infty} y_n = -\infty$, then $\lim\limits_{n\to\infty} x_n = -\infty$.*

Theorem 2.6 (Sandwich Theorem or Squeeze Theorem)**.** *Let $(x_n), (y_n), (z_n)$ be sequences in $\mathbb{R}$ such that $\forall n \in \mathbb{N}^*$, $x_n \leq y_n \leq z_n$. Suppose that (x_n) and (z_n) are convergent and $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} z_n = l \in \mathbb{R}$. Then (y_n) is also convergent and $\lim\limits_{n\to\infty} y_n = l$.*

Monotone sequences

Definition 2.6. Let (x_n) be a sequence in $\mathbb{R}$. Then (x_n) is

- *increasing (decreasing)*: if $\forall n \in \mathbb{N}^*$, $x_n \leq x_{n+1}$ ($x_n \geq x_{n+1}$).
- *strictly increasing (strictly decreasing)*: if $\forall n \in \mathbb{N}^*$, $x_n < x_{n+1}$ ($x_n > x_{n+1}$).
- *monotone (strictly monotone)*: if it is either increasing or decreasing (if it is either strictly increasing or strictly decreasing).

Remark 2.6. In the literature, the terminology non-decreasing (non-increasing) is sometimes used for increasing (decreasing) sequences, while the notion increasing (decreasing) refers to strictly increasing (strictly decreasing) sequences.

Remark 2.7. Any increasing (decreasing) sequence is bounded below (above).

The next result shows that a monotone sequence is either convergent or properly divergent.

Theorem 2.7. *Let (x_n) be a monotone sequence in $\mathbb{R}$ and let $X = \{x_n \mid n \in \mathbb{N}^*\}$. Then (x_n) is convergent (properly divergent) if and only if it is bounded (unbounded). Moreover,*

(i) if (x_n) is increasing, then $\lim\limits_{n\to\infty} x_n = \sup X$.

(ii) if (x_n) is decreasing, then $\lim\limits_{n\to\infty} x_n = \inf X$.

Theorem 2.8 (Stolz-Cesàro)**.** *Let $(x_n), (y_n)$ be sequences in $\mathbb{R}$ such that*

(i) (y_n) is strictly increasing and $\lim\limits_{n\to\infty} y_n = +\infty$,

(ii) $\lim\limits_{n\to\infty} \dfrac{x_{n+1} - x_n}{y_{n+1} - y_n} = L \in \overline{\mathbb{R}}$.

Then $\lim\limits_{n\to\infty} \dfrac{x_n}{y_n} = L$.

Consequences of the Stolz-Cesàro Theorem:

1. If $\lim\limits_{n\to\infty} x_n = x \in \overline{\mathbb{R}}$, then $\lim\limits_{n\to\infty} \dfrac{x_1 + x_2 + \ldots + x_n}{n} = x$.
2. If $\forall n \in \mathbb{N}^*$, $x_n > 0$ and $\lim\limits_{n\to\infty} x_n = x \in \overline{\mathbb{R}}$, then $\lim\limits_{n\to\infty} \sqrt[n]{x_1 \cdot x_2 \cdot \ldots \cdot x_n} = x$.
3. If $\forall n \in \mathbb{N}^*$, $x_n > 0$ and $\lim\limits_{n\to\infty} \dfrac{x_{n+1}}{x_n} = L \in \overline{\mathbb{R}}$, then $\lim\limits_{n\to\infty} \sqrt[n]{x_n} = L$.

Subsequences

Definition 2.7. Let (x_n) be a sequence in $\mathbb{R}$ and let $n_1 < n_2 < \ldots < n_k < \ldots$ be a strictly increasing sequence of natural numbers. Then the sequence $(x_{n_k})_{k\in\mathbb{N}^*} = (x_{n_1}, x_{n_2}, \ldots, x_{n_k}, \ldots)$ is called a *subsequence* of (x_n).

Theorem 2.9. *Let (x_n) be a sequence in $\mathbb{R}$ that converges (properly diverges) and let $x = \lim\limits_{n\to\infty} x_n$. Then any subsequence (x_{n_k}) of (x_n) converges (properly diverges) and $\lim\limits_{k\to\infty} x_{n_k} = x$.*

Theorem 2.10 (Bolzano-Weierstrass)**.** *A bounded sequence in $\mathbb{R}$ has a convergent subsequence.*

Appendix: Rules of calculation for limits

$x + \infty = \infty + x = \infty, \ \forall x \in \mathbb{R},$

$x + (-\infty) = (-\infty) + x = -\infty, \ \forall x \in \mathbb{R},$

$\infty + \infty = \infty, \quad (-\infty) + (-\infty) = -\infty,$

$x \cdot \infty = \infty \cdot x = \begin{cases} \infty, \text{ if } x \in (0, \infty) \\ -\infty, \text{ if } x \in (-\infty, 0), \end{cases}$

$x \cdot (-\infty) = (-\infty) \cdot x = \begin{cases} -\infty, \text{ if } x \in (0, \infty) \\ \infty, \text{ if } x \in (-\infty, 0), \end{cases}$

$\infty \cdot \infty = \infty, \quad (-\infty) \cdot (-\infty) = \infty, \quad \infty \cdot (-\infty) = (-\infty) \cdot \infty = -\infty,$

$\frac{x}{\infty} = \frac{x}{-\infty} = 0, \ \forall x \in \mathbb{R},$

$\frac{1}{0+} = \infty, \quad \frac{1}{0-} = -\infty,$

$x^{\infty} = \begin{cases} \infty, \text{ if } x \in (1, \infty) \\ 0, \text{ if } x \in [0, 1), \end{cases}$

$x^{-\infty} = \begin{cases} 0, \text{ if } x \in (1, \infty) \\ \infty, \text{ if } x \in (0, 1), \end{cases}$

$(\infty)^x = \begin{cases} \infty, \text{ if } x \in (0, \infty) \\ 0, \text{ if } x \in (-\infty, 0), \end{cases}$

$\infty^{\infty} = \infty, \quad \infty^{-\infty} = 0.$

Not defined

$\infty + (-\infty), \quad (-\infty) + \infty,$

$0 \cdot \infty, \quad \infty \cdot 0, \quad 0 \cdot (-\infty), \quad (-\infty) \cdot 0,$

$\frac{\infty}{\infty}, \quad \frac{-\infty}{-\infty}, \quad \frac{\infty}{-\infty}, \quad \frac{-\infty}{\infty},$

$1^{\infty}, \quad 0^0, \quad \infty^0, \quad 1^{-\infty}.$