Propositional logic - exercises

Exercise 1.1.

Check the following properties for $\downarrow$ (’*nor*’), $\uparrow$ (’*nand*’) and $\oplus$ (’*xor*’) connectives using the truth table method.

1. associativity of ’ $\uparrow$ ’ connective:
   $p \uparrow (q \uparrow r) \equiv (p \uparrow q) \uparrow r$;
2. associativity of ’ $\downarrow$ ’ connective:
   $p \downarrow (q \downarrow r) \equiv (p \downarrow q) \downarrow r$;
3. associativity of ’$\oplus$’ connective:
   $p \oplus (q \oplus r) \equiv (p \oplus q) \oplus r$;
4. distribution of ’ $\uparrow$ ’ connective over ’ $\downarrow$ ’ connective:
   $p \uparrow (q \downarrow r) \equiv (p \uparrow q) \downarrow (p \uparrow r)$;
5. distribution of ’ $\downarrow$ ’ connective over ’ $\uparrow$ ’ connective:
   $p \downarrow (q \uparrow r) \equiv (p \downarrow q) \uparrow (p \downarrow r)$;
6. De Morgan’s laws for ’ $\downarrow$ ’ and ’ $\uparrow$ ’:
   $\neg(p \downarrow q) \equiv \neg p \uparrow \neg q$ and $\neg(p \uparrow q) \equiv \neg p \downarrow \neg q$;
7. $p \uparrow (q \vee r) \equiv (p \uparrow q) \wedge (p \uparrow r)$ and $p \downarrow (q \wedge r) \equiv (p \downarrow q) \vee (p \downarrow r)$.
8. $p \downarrow (q \uparrow p) \equiv F$ and $p \uparrow (q \downarrow p) \equiv T$;

Exercise 1.2.

Using the truth table method decide what kind of formula (consistent, inconsistent, tautology, contingent) is U_j , $j \in \{1,2,\ldots,8\}$. Write all the models and anti-models of U_j , $j \in \{1,2,\ldots,8\}$.

1. $U_1 = q \wedge \neg p \wedge r \to \neg p \vee \neg(q \wedge r)$;
2. $U_2 = \neg p \vee \neg(q \wedge r) \to q \wedge \neg p$;
3. $U_3 = \neg p \wedge (\neg q \vee r) \to q \vee \neg p \vee r$;
4. $U_4 = \neg(\neg p \vee q) \vee r \to \neg p \vee (\neg q \vee r)$;
5. $U_5 = \neg p \vee (\neg q \vee \neg r) \to q \wedge \neg p$;
6. $U_6 = \neg p \vee (\neg q \wedge \neg r) \to q \wedge \neg p \wedge r$;
7. $U_7 = p \to (q \wedge r) \vee q \wedge \neg p$;
8. $U_8 = (p \vee q) \wedge \neg r \to p \wedge q \wedge r$.

Exercise 1.3.

Using the truth table method, check if the following logical consequences hold:

1. $p \to q \models (p \to r) \to (p \to q \wedge r)$;
2. $p \to q \models (q \to r) \to (p \to r)$;
3. $p \to (q \to r) \models (p \to q) \to (p \to r)$;
4. $p \to r \models (q \to r) \to ((p \vee q) \to r)$;
5. $p \to q \models (\neg p \to q) \to q$;
6. $p \to q \models (q \to r) \to (p \to q \wedge r)$;
7. $p \to q \models (q \to r) \to (p \to q \vee r)$;
8. $r \to (q \to p) \models (r \to q) \to (r \to p)$.

Exercise 1.4.

Prove that the following formulas are tautologies using the truth table method.

1. the left-distribution of ’ $\to$ ’ over ’ $\wedge$ ’: $(p \to (q \wedge r)) \to ((p \to q) \wedge (p \to r))$;
2. the permutation of the premises law: $(p \to (q \to r)) \to (q \to (p \to r))$;
3. the reunion of the premises law: $(p \to (q \to r)) \to (p \wedge q \to r)$;
4. the separation of the premises law: $(p \wedge q \to r) \to (p \to (q \to r))$;
5. the ’cut’ law: $(p \to q) \wedge (p \wedge q \to r) \to (p \to r)$;

6. the left-distribution of ' $\vee$ ' over ' $\rightarrow$ ': $p \vee (q \rightarrow r) \rightarrow ((p \vee q) \rightarrow (p \vee r))$;
7. the syllogism law: $(p \rightarrow q) \wedge (q \rightarrow r) \rightarrow (p \rightarrow r)$;
8. the left-distribution of ' $\rightarrow$ ' over ' $\vee$ ': $(p \rightarrow (q \vee r)) \rightarrow ((p \rightarrow q) \vee (p \rightarrow r))$.

Exercise 1.5.

Transform the formulas U_j, $j \in \{1,2,\ldots,8\}$ into their equivalent conjunctive and disjunctive normal forms. Using one of these forms prove that U_j, $j \in \{1,2,\ldots,8\}$ are valid formulas in propositional logic.

1. $U_1 = (p \rightarrow (q \leftrightarrow r)) \rightarrow ((p \rightarrow q) \leftrightarrow (p \rightarrow r))$;
2. $U_2 = (p \rightarrow q) \wedge (p \wedge q \rightarrow r) \rightarrow (p \rightarrow r)$;
3. $U_3 = (p \wedge q \rightarrow r) \rightarrow (p \rightarrow (q \rightarrow r))$;
4. $U_4 = (p \rightarrow (q \vee r)) \rightarrow ((p \rightarrow q) \vee (p \rightarrow r))$;
5. $U_5 = (p \vee (q \leftrightarrow r)) \rightarrow ((p \vee q) \leftrightarrow (p \vee r))$;
6. $U_6 = (p \rightarrow (q \rightarrow r)) \rightarrow ((p \rightarrow q) \rightarrow (p \rightarrow r))$;
7. $U_7 = (p \rightarrow (q \wedge r)) \rightarrow ((p \rightarrow q) \wedge (p \rightarrow r))$;
8. $U_8 = p \vee (q \rightarrow r) \rightarrow ((p \vee q) \rightarrow (p \vee r))$.

Exercise 1.6.

Using the appropriate normal form write all the models of the following formulas:

1. $U_1 = (p \vee q \rightarrow r) \rightarrow (p \rightarrow r) \wedge q$;
2. $U_2 = \neg(\neg p \vee q) \vee r \rightarrow \neg p \wedge \neg(q \wedge r)$;
3. $U_3 = (p \wedge q \rightarrow r) \rightarrow (p \rightarrow r) \wedge q$;
4. $U_4 = (p \vee q) \wedge \neg r \rightarrow p \wedge q \wedge r$;
5. $U_5 = p \vee \neg(q \wedge \neg r) \rightarrow p \wedge q \wedge \neg r$;
6. $U_6 = (p \vee q \rightarrow r) \rightarrow (q \rightarrow r) \wedge p$;
7. $U_7 = (q \vee r \rightarrow p) \rightarrow (p \rightarrow r) \wedge q$;
8. $U_8 = (q \wedge r \rightarrow p) \rightarrow (p \rightarrow r) \wedge q$.

Exercise 1.7.

Using the appropriate normal form, prove that the following formulas are inconsistent:

1. $U_1 = (p \rightarrow (q \rightarrow r)) \wedge \neg((p \rightarrow q) \rightarrow (p \rightarrow r))$;
2. $U_2 = (\neg p \vee q) \wedge \neg(\neg q \rightarrow \neg p)$;
3. $U_3 = (p \rightarrow q) \wedge (p \wedge q \rightarrow r) \wedge (p \wedge \neg r)$;
4. $U_4 = (p \rightarrow (q \vee r)) \wedge (\neg(p \rightarrow q) \wedge \neg(p \rightarrow r))$;
5. $U_5 = p \wedge (q \rightarrow r) \wedge ((p \wedge q) \wedge \neg(p \wedge r))$;
6. $U_6 = (p \rightarrow (q \rightarrow r)) \wedge (p \wedge q \wedge \neg r)$;
7. $U_7 = (p \rightarrow (q \rightarrow r)) \wedge \neg(q \rightarrow (p \rightarrow r))$;
8. $U_8 = (p \wedge q \rightarrow r) \wedge \neg(p \rightarrow (q \rightarrow r))$.

Exercise 1.8.

Write all the anti-models of the following formulas using CNF.

1. $U_1 = (q \wedge r \rightarrow p) \rightarrow (p \rightarrow r) \wedge q$;

2. $U_2 = (q \vee r \to p) \to (p \to r) \wedge q$;
3. $U_3 = (p \vee q \to r) \to (q \to r) \wedge p$;
4. $U_4 = p \vee \neg(q \wedge \neg r) \to p \wedge q \wedge \neg r$;
5. $U_5 = p \vee \neg(q \wedge \neg r) \to p \wedge q \wedge \neg r$;
6. $U_6 = (p \wedge q \to r) \to (p \to r) \wedge q$;
7. $U_7 = \neg(\neg p \vee q) \vee r \to \neg p \wedge \neg(q \wedge r)$;
8. $U_8 = (p \vee q \to r) \to (p \to r) \wedge q$.

Exercise 1.9.
Using the definition of deduction, prove the following deductions:

1. $p \to q, r \to t, p \vee r, \neg q \vdash t$;
2. $p \to r, p \vee r \to q, r \vdash q$;
3. $q \to p, t \to r, q \vee t, \neg p \vdash r$;
4. $p \vee (q \to r), p \vee q, \neg p \vdash r$;
5. $\neg p \vee \neg q \vee r, q, p \vdash r$;
6. $p \to \neg q \vee r, p \wedge q, p \vdash r$;
7. $r \vee (q \to p), r \vee q, \neg r \vdash p$;
8. $p \to q, q \to r, r \to t, p \vdash t$.

Exercise 1.10.
Prove the following theorems using the theorem of deduction and its reverse.

1. $\vdash p \vee (q \to r) \to ((p \vee q) \to (p \vee r))$;
2. $\vdash (p \to (\neg r \to q)) \to (r \vee \neg p \vee q)$;
3. $\vdash (p \to (q \to r) \to (p \wedge q \to r)$;
4. $\vdash (p \wedge q \to r) \to (p \to (q \to r))$;
5. $\vdash (p \to (q \to r)) \to (q \to (p \to r))$;
6. $\vdash (p \to (q \to r)) \to ((p \to q) \to (p \to r))$;
7. $\vdash (p \to q) \wedge (p \wedge q \to r) \to (p \to r)$;
8. $\vdash (p \to q) \to ((p \to r) \to (p \to q \wedge r))$.

Exercise 1.11.
Using the theorem of deduction and its reverse prove that:

1. $\vdash (p \to (q \vee r)) \to ((p \to q) \vee (p \to r))$;
2. $\vdash (p \to q) \to ((\neg r \vee p) \to (r \to q))$;
3. $\vdash p \vee (q \to r) \to ((p \vee q) \to (p \vee r))$;
4. $\vdash (p \to r) \to ((q \to r) \to (p \vee q \to r))$;
5. $\vdash (p \to q) \to ((r \to t) \to (p \wedge r \to q \wedge t))$;
6. $\vdash (p \to r) \to ((p \wedge r \to q) \to (p \to q))$;
7. $\vdash (\neg q \vee p) \to ((s \to q) \to (s \to p))$;
8. $\vdash (p \to (q \to r)) \to (p \to (\neg r \to \neg q))$