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Analysis for Computer Science
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Exercise Set #3

1. Prove that the sequence (x_n) defined by

$$x_n = \frac{n-3}{3n^3 - 4n + 5}$$

converges to 0 by using the definition (that is, for every $\varepsilon > 0$, find $N \in \mathbb{N}^*$ s.t. $\forall n \geq N$, $|x_n| < \varepsilon$).

2. Find the limit of the following sequences:

a) $\left(\sin \frac{\pi}{7}\right)^n$, b) $\frac{n + \sin(n^2)}{n + \cos(n)}$, c) $\frac{n^3 + 6n - 6}{-3n^2 + n}$, d) $\frac{1 - 3n^4}{n^4 - 5n^3 + n}$, e) $(n^2+n)^{-\frac{n}{n+1}}$, f) $\frac{2^n - 3^n}{4^n - 5^n}$,
g) $\left(1 + \frac{1}{n}\right)^{\frac{3n}{n+1}}$.

3. Find the limit of the following sequences:

a) $\sqrt{n}\left(\sqrt{n} - \sqrt{n+3}\right)$, b) $\left(\frac{n^2+n+1}{n^2+1}\right)^{\frac{2n^2+n+1}{n+1}}$, c) $\left(1 + \frac{1}{n^3+2n^2}\right)^{n-n^3}$, d) $\frac{2^n}{n!}$,
e) $\frac{n^\alpha}{(1+p)^n}$, where $\alpha \in \mathbb{R}$, $p > 0$, f) $\frac{1 \cdot 1! + 2 \cdot 2! + \ldots + n \cdot n!}{(n+1)!}$, g) $\sqrt[n]{p}$, where $p > 0$,
h) $\sqrt[n]{\sin^2(n^{2015}) + 2\cos^2(n^{2015})}$, i) $\sqrt[n]{1 + 2 + \ldots + n}$, j) $\frac{\sqrt[n]{n!}}{n}$.

4. In each of the following cases, study if the sequence (x_n) is bounded, monotone and convergent. If the sequence is convergent, find also its limit.

a) $x_1 \in (0,1)$, $x_{n+1} = \frac{2x_n + 1}{3}, n \in \mathbb{N}^*$ b) $x_1 = \sqrt{3}$, $x_{n+1} = \sqrt{2x_n + 3}, n \in \mathbb{N}^*$
c) $a > 0$, $x_1 > 0$, $x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right), n \in \mathbb{N}^*$.

5. Define the sequence (x_n) by $x_1 \in (0,1)$ and $x_{n+1} = x_n - x_n^2$, $n \in \mathbb{N}^*$. Check if $(n \cdot x_n)$ converges and, if possible, find also its limit.
Hint: Show first that (x_n) converges and find its limit. To study the convergence of the sequence $(n \cdot x_n)$, use the Stolz-Cesàro Theorem.

6. Check if the following sequence converges and, if possible, find also its limit.

$$x_n = \frac{m_n}{1 + 7n}, \text{ where } m_n^2 \leq n < (m_n + 1)^2 \text{ (see also Ex. 4, Exercise Set \#1).}$$

7. True or false: (x_n) converges if and only if $(|x_n|)$ converges?

8. Give an example of a divergent sequence (x_n) such that the sequence $\left(\frac{1}{n}\sum_{k=1}^{n} x_k\right)$ is convergent.

9. Let (x_n) be a sequence in $\mathbb{Z}$.
True or false: (x_n) is convergent to some x if and only if it is eventually constant equal to x?

10. For $n \in \mathbb{N}^*$, let $a_n, b_n \in \mathbb{R}$ such that $a_n \leq b_n$ and $\lim_{n\to\infty}(b_n - a_n) = 0$. Suppose, in addition, that $\forall n \in \mathbb{N}^*$, $[a_{n+1}, b_{n+1}] \subseteq [a_n, b_n]$. By the Nested Interval Property, $\bigcap_{n=1}^{\infty}[a_n, b_n] \neq \emptyset$.

True or false: can $\bigcap_{n=1}^{\infty}[a_n, b_n]$ contain more than one point?

11. (Koch snowflake) Define a sequence (S_n) of polygons such that S_1 is an equilateral triangle of side length s and S_{n+1} is obtained from S_n by adding to the middle third of each side an equilateral triangle pointing outwards (and removing this middle third). Denote by a_n the area of S_n. Determine the sequence (a_n) and study if it is convergent.

12. Find the limit of the following sequences:

a) $n\left(\left(1+\frac{1}{n}\right)^{1+\frac{1}{n}} - 1\right)$. Hint: Prove first that $\forall n \in \mathbb{N}^*$, $1 \leq \left(1+\frac{1}{n}\right)^{\frac{1}{n}} \leq 1 + \frac{1}{n^2}$.

b) $n\,(\sqrt[n]{e} - 1)$. Hint: $\forall n \in \mathbb{N}^*$, $\left(1+\frac{1}{n}\right)^{n} < e < \left(1+\frac{1}{n}\right)^{n+1}$.

13. Show that $(\sin n)$ does not converge.
True or false: Let $r \in \mathbb{Q}$. Is $(\sin(n!\, r\, \pi))_n$ convergent?

14. Give an example of an unbounded sequence that has a convergent subsequence. Then find an unbounded sequence that has no convergent subsequence. What can you conclude about the Bolzano-Weierstrass Theorem when dropping the boundedness hypothesis for the sequence?

Additional exercises:

15. Let $x \in \mathbb{R}$ and $A \subseteq \mathbb{R}$ nonempty. Show that

a) if A is bounded above, then $x = \sup A$ if and only if x is an upper bound of A and there exists a sequence (x_n) in A which converges to x.

b) if A is bounded below, then $x = \inf A$ if and only if x is a lower bound of A and there exists a sequence (x_n) in A which converges to x.

Apply now these properties for some of the sets given in Ex. 1, Exercise Set #2.

16. Let (x_n) be a sequence in $\mathbb{R}$ and let $x \in \mathbb{R}$. Prove that the following are equivalent:

a) $\lim_{n\to\infty} x_n = x$,

b) $\forall \varepsilon > 0, \exists N \in \mathbb{N}^*, \forall n > N, |x_n - x| < \varepsilon$,

c) $\forall \varepsilon > 0, \exists N \in \mathbb{N}^*, \forall n > N, |x_n - x| \leq \varepsilon$,

d) $\forall \varepsilon > 0, \exists N \in \mathbb{N}^*, \forall n \geq N, |x_n - x| \leq \varepsilon$,

e) $\forall \varepsilon > 0, \exists N \in \mathbb{N}^*, \forall n > N, |x_n - x| < c \cdot \varepsilon$, where $c > 0$ is a constant.

17. Suppose (x_n) is a sequence of real numbers satisfying the following:

$$\exists x \in \mathbb{R}, \exists N \in \mathbb{N}^*, \forall \varepsilon > 0, \forall n \geq N, |x_n - x| < \varepsilon.$$

What remarkable property does (x_n) have?