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1 The real numbers: some basic concepts

In the following we consider the usual ordering on $\mathbb{R}$.

Definition 1.1. A set $A \subseteq \mathbb{R}$ is said to be

- *bounded above*: if $\exists u \in \mathbb{R}$ s.t. $\forall a \in A$, $a \leq u$. Every such u is called an *upper bound* of A. If, in addition, $u \in A$, then u is called a *maximum* (or *greatest element*) of A.
- *bounded below*: if $\exists v \in \mathbb{R}$ s.t. $\forall a \in A$, $v \leq a$. Every such v is called a *lower bound* of A. If, in addition, $v \in A$, then v is called a *minimum* (or *least element*) of A.
- *bounded*: if it is both bounded above and below.
- *unbounded*: if it is not bounded.

Remark 1.1. (i) Any $A \subseteq \mathbb{R}$ has at most one maximum (minimum). So, if it exists, we may refer to *the* maximum (minimum) of A instead of *a* maximum (minimum) and we denote it by $\max A$ ($\min A$) - sometimes we also write $\max(A)$ ($\min(A)$).
(ii) If a set has one upper (lower) bound, then it has infinitely many upper (lower) bounds.

Example 1.1. (i) $A = \{a \in \mathbb{R} \mid a \geq 2\}$: unbounded (since it is not bounded above), bounded below by any $v \leq 2$, $\min A = 2$.
(ii) $A = \{a \in \mathbb{R} \mid 0 < a < 1\}$: bounded (above by any $u \geq 1$, below by any $v \leq 0$), no minimum, no maximum.
(iii) $A = \left\{\frac{1}{n} \mid n \in \mathbb{N}^*\right\}$: bounded (above by any $u \geq 1$, below by any $v \leq 0$), $\max A = 1$, no minimum.
(iv) Every nonempty finite set has a minimum and a maximum.

Definition 1.2. Let $A \subseteq \mathbb{R}$.

- If $u \in \mathbb{R}$ is an upper bound of A s.t. for every upper bound u' of A, $u \leq u'$, then u is called a *supremum* (or *least upper bound*) of A.
- If $v \in \mathbb{R}$ is a lower bound of A s.t. for every lower bound v' of A, $v \geq v'$, then v is called an *infimum* (or *greatest lower bound*) of A.

Remark 1.2. (i) Any $A \subseteq \mathbb{R}$ has at most one supremum (infimum). So, if it exists, we may refer to *the* supremum (infimum) of A instead of *a* supremum (infimum) and we denote it by $\sup A$ ($\inf A$) - sometimes we also write $\sup(A)$ ($\inf(A)$).
(ii) If the maximum (minimum) of $A \subseteq \mathbb{R}$ exists, then it is also the supremum (infimum). Conversely, if the supremum (infimum) of A exists and is contained in A, then it is also the maximum (minimum) of A.

Example 1.2. (i) $A = \{a \in \mathbb{Z} \mid 2 \leq a \leq 3\}$: $\max A = \sup A = 3$, $\min A = \inf A = 2$.
(ii) $A = \{a \in \mathbb{R} \mid 0 < a \leq 1\}$: $\max A = \sup A = 1$, $\inf A = 0$, no minimum.

Some additional conventions: We attach to $\mathbb{R}$ two new elements $-\infty$ and $+\infty$ (sometimes we write ∞ instead of $+\infty$) s.t. $\forall x \in \mathbb{R}$, $-\infty < x$ and $x < +\infty$. By $\overline{\mathbb{R}}$ we denote the *extended set of real numbers* defined by $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$. If $A \subseteq \mathbb{R}$ is not bounded above (below), then we set $\sup A = +\infty$ ($\inf A = -\infty$). Moreover, we set $\sup \emptyset = -\infty$ and $\inf \emptyset = +\infty$ (any real number is both an upper and a lower bound of $\emptyset$).

Theorem 1.1 (Supremum Property). *Every nonempty subset of* $\mathbb{R}$ *which is bounded above has a supremum in* $\mathbb{R}$*.*

Remark 1.3. Using Theorem 1.1, one can prove that every nonempty subset of $\mathbb{R}$ which is bounded below has an infimum in $\mathbb{R}$. Indeed, let $A \subseteq \mathbb{R}$, $A \neq \emptyset$, bounded below. Then the set $-A = \{-a \mid a \in A\}$ is nonempty and bounded above, so, by Theorem 1.1, it has a supremum in $\mathbb{R}$. Then $\inf A = -\sup(-A)$.

Consequences of the Supremum Property:

1. (Nested Interval Property). *For* $n \in \mathbb{N}^*$*, consider the closed intervals* $I_n = [a_n, b_n]$*, where* $a_n \leq b_n$*, and suppose that* $I_n \supseteq I_{n+1}$*. Then* $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$ *(that is,* $\exists x \in \mathbb{R}$ *s.t.* $\forall n \in \mathbb{N}^*, x \in I_n$*).*

$$I_1 \supseteq I_2 \supseteq \ldots \supseteq I_n \supseteq I_{n+1} \supseteq \ldots \text{ - nested sequence of closed intervals}$$

Proof. Let $A = \{a_k \mid k \in \mathbb{N}^*\}$. Then, $\forall n \in \mathbb{N}^*$, b_n is an upper bound of A. Hence, A is nonempty and bounded above, so, by Theorem 1.1, it has a supremum in $\mathbb{R}$. Thus, $\forall n \in \mathbb{N}^*$, $a_n \leq \sup A \leq b_n$, from where, $\sup A \in \bigcap_{n=1}^{\infty} I_n$. □

2. (Archimedean Property - AP). *Let* $x \in \mathbb{R}$*. Then* $\exists n \in \mathbb{N}$ *s.t.* $n > x$*.*

Proof. Suppose $x \geq n$, $\forall n \in \mathbb{N}$. Then $\mathbb{N}$ is nonempty and bounded above by x, so, by Theorem 1.1, it has a supremum $u \in \mathbb{R}$. Since $u - 1 < u$, $u - 1$ cannot be an upper bound of $\mathbb{N}$. This means that $\exists m \in \mathbb{N}$ s.t. $u - 1 < m$. Thus, $u < m + 1 \in \mathbb{N}$, which is a contradiction to the fact that u is an upper bound of $\mathbb{N}$. □

Applications of (AP):

1. Example 1.1. iii) $A = \left\{ \frac{1}{n} \mid n \in \mathbb{N}^* \right\}$, $\inf A = 0$.

Proof. Since $A \neq \emptyset$ and bounded below by 0, using Remark 1.3, it has an infimum in $\mathbb{R}$. Note that $\inf A \geq 0$. Let $\varepsilon > 0$. By (AP), $\exists n \in \mathbb{N}$ s.t. $n > 1/\varepsilon > 0$. Thus, $n \in \mathbb{N}^*$ and $1/n < \varepsilon$. Hence, $0 \leq \inf A \leq 1/n < \varepsilon$. As $\varepsilon > 0$ is arbitrary, $\inf A = 0$. □

2. (Density Property of $\mathbb{Q}$). *Let* $x, y \in \mathbb{R}$*,* $x < y$*. Then* $\exists q \in \mathbb{Q}$ *s.t.* $x < q < y$*.*

Definition 1.3. The *absolute value* of $x \in \mathbb{R}$ is $|x| = \begin{cases} x, & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$

The *distance* between $x, y \in \mathbb{R}$ is $|x - y|$ (sometimes denoted also by $d(x, y)$).

Definition 1.4. A *neighborhood* of $x \in \mathbb{R}$ is a set $V \subseteq \mathbb{R}$ for which $\exists \varepsilon > 0$ s.t. $(x - \varepsilon, x + \varepsilon) \subseteq V$. A neighborhood of $+\infty$ is a set $V \subseteq \mathbb{R}$ for which $\exists a \in \mathbb{R}$ s.t. $(a, +\infty) \subseteq V$. Likewise, a neighborhood of $-\infty$ is a set $V \subseteq \mathbb{R}$ for which $\exists a \in \mathbb{R}$ s.t. $(-\infty, a) \subseteq V$.

For $x \in \overline{\mathbb{R}}$, we denote by $\mathcal{V}(x)$ the family of all neighborhoods of x.

Properties. Let $x \in \overline{\mathbb{R}}$. Then
(i) if $x \in \mathbb{R}$ and $V \in \mathcal{V}(x)$, then $x \in V$.
(ii) if $V \in \mathcal{V}(x)$ and $U \subseteq \mathbb{R}$ s.t. $V \subseteq U$, then $U \in \mathcal{V}(x)$.
(iii) if $U, V \in \mathcal{V}(x)$, then $U \cap V \in \mathcal{V}(x)$.