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4 Real-valued functions of one real variable

4.1 Limits of functions

Definition 4.1. Let $A \subseteq \mathbb{R}$. A point $c \in \overline{\mathbb{R}}$ is an *accumulation point* (or a *cluster point* or a *limit point*) of A if

$$\forall V \in \mathcal{V}(c), \quad V \cap (A \setminus \{c\}) \neq \emptyset.$$

The set of all accumulation points of A is called the *derived set* of A and is denoted by A'. A point $a \in A$ is called an *isolated point* of A if it is not an accumulation point of A.

Remark 4.1. Let $A \subseteq \mathbb{R}$.
(i) In the literature, accumulation points of A are usually assumed to be real numbers. However, in order to simplify the further exposition, we consider extended real numbers.
(ii) Accumulation points of A may or may not belong to A.
(iii) A point $a \in A$ is an isolated point of A if and only if $\exists V \in \mathcal{V}(a)$ such that $V \cap A = \{a\}$.

Theorem 4.1. *Let $A \subseteq \mathbb{R}$ and $c \in \overline{\mathbb{R}}$. Then $c \in A'$ if and only if there exists a sequence (x_n) in $A \setminus \{c\}$ such that $\lim\limits_{n\to\infty} x_n = c$.*

Definition 4.2. Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$, $c \in A'$ and $L \in \overline{\mathbb{R}}$. We say that f *has limit L at c* if

$$\forall V \in \mathcal{V}(L), \exists U \in \mathcal{V}(c) \text{ such that } \forall x \in U \cap (A \setminus \{c\}) \text{ we have } f(x) \in V.$$

Remark 4.2. Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$, $c \in A'$.
(i) f has at most one limit at c.
(ii) f *has a limit at* c if $\exists L \in \overline{\mathbb{R}}$, $\forall V \in \mathcal{V}(L)$, $\exists U \in \mathcal{V}(c)$, $\forall x \in U \cap (A \setminus \{c\})$, $f(x) \in V$. In this case L is called the *limit of f at c* and we write

$$\lim_{x\to c} f(x) = L \quad \text{or} \quad f(x) \to L \text{ as } x \to c.$$

We also sometimes say that $f(x)$ *approaches L as x approaches c.*

(iii) f *has no limit at* c if $\forall L \in \overline{\mathbb{R}}$, $\exists V \in \mathcal{V}(L)$, $\forall U \in \mathcal{V}(c)$, $\exists x \in U \cap (A \setminus \{c\})$, $f(x) \notin V$.

Remark 4.3. (i) Limits of functions are not considered at isolated points of the domain, only at accumulation points (which may or may not belong to the domain).
(ii) The limit of a function $f : A \to \mathbb{R}$ at a given $c \in A'$ depends only on the values of f "near" c, while the values of the f "away" from c are irrelevant. Thus, when evaluating the limit of f at c, one can only look at $f|_{W\cap A}$, where $W \in \mathcal{V}(c)$ (recall that $f|_{W\cap A}$ denotes the restriction of f to $W \cap A$).

Theorem 4.2 (ε-δ characterization of limits)**.** *Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$, $c \in \mathbb{R}$, $c \in A'$, $L \in \mathbb{R}$. Then*

(i) $\lim\limits_{x\to c} f(x) = L \iff \forall \varepsilon > 0,\ \exists \delta = \delta(\varepsilon) > 0,\ \forall x \in A,\ 0 < |x - c| < \delta,\ |f(x) - L| < \varepsilon.$

(ii) $\lim\limits_{x\to c} f(x) = +\infty\ (-\infty) \iff \forall \alpha \in \mathbb{R},\ \exists \delta = \delta(\alpha) > 0,\ \forall x \in A,\ 0 < |x - c| < \delta,\ f(x) > \alpha$ $(f(x) < \alpha)$.

Theorem 4.3 (ε-δ characterization of limits)**.** *Let $A \subseteq \mathbb{R}$ such that $+\infty \in A'$, $f : A \to \mathbb{R}$, $L \in \mathbb{R}$. Then* *(i)* $\lim_{x\to\infty} f(x) = L \iff \forall\varepsilon > 0,\ \exists\delta = \delta(\varepsilon) > 0,\ \forall x \in A, x > \delta,\ |f(x) - L| < \varepsilon.$

(ii) $\lim_{x\to\infty} f(x) = +\infty\ (-\infty) \iff \forall\alpha \in \mathbb{R},\ \exists\delta = \delta(\alpha) > 0,\ \forall x \in A, x > \delta,\ f(x) > \alpha\ (f(x) < \alpha).$

In a similar way as above one can state the ε-δ characterization of limits for $x \to -\infty$.

Theorem 4.4 (Sequential characterization of limits, Heine)**.** *Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$, $c \in A'$, $L \in \overline{\mathbb{R}}$. Then*

$$\lim_{x\to c} f(x) = L \iff \forall \textit{ sequence } (x_n) \textit{ in } A \setminus \{c\} \textit{ with } \lim_{n\to\infty} x_n = c \textit{ we have that } \lim_{n\to\infty} f(x_n) = L.$$

Remark 4.4. In fact, one can prove that having $A \subseteq \mathbb{R}$ and $c \in A'$, the function $f : A \to \mathbb{R}$ has a limit at c if and only if

$$\forall \text{ sequence } (x_n) \text{ in } A \setminus \{c\} \text{ with } \lim_{n\to\infty} x_n = c \text{ we have that } (f(x_n)) \text{ has a limit.}$$

Since by Theorem 4.4, limits of functions can be characterized using limits of sequences, limit theorems for functions can be derived from corresponding ones for sequences. We solely include below the following results.

Theorem 4.5. *Let $A \subseteq \mathbb{R}$, $f, g : A \to \mathbb{R}$, $c \in A'$. Suppose $\exists U \in \mathcal{V}(c)$ such that $f(x) \le g(x)$, $\forall x \in U \cap (A \setminus \{c\})$.*

(i) *If $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ exist, then $\lim_{x\to c} f(x) \le \lim_{x\to c} g(x)$.*

(ii) *If $\lim_{x\to c} f(x) = +\infty$, then $\lim_{x\to c} g(x) = +\infty$.*

(iii) *If $\lim_{x\to c} g(x) = -\infty$, then $\lim_{x\to c} f(x) = -\infty$.*

Remark 4.5. Strict inequalities are not preserved: $1 < \dfrac{x+1}{x}, \forall x > 0$, but $\lim_{x\to\infty} \dfrac{x+1}{x} = 1$.

Theorem 4.6 (Sandwich Theorem or Squeeze Theorem for functions)**.** *Let $A \subseteq \mathbb{R}$, $f, g, h : A \to \mathbb{R}$, $c \in A'$. Suppose $\exists U \in \mathcal{V}(c)$ such that $f(x) \le g(x) \le h(x)$, $\forall x \in U \cap (A \setminus \{c\})$. If $L \in \mathbb{R}$ is the limit of both f and h at c, then L is also the limit of g at c.*

One-sided limits

Definition 4.3. Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$, $c \in \mathbb{R}$, $c \in (A_l)'$, where $A_l = A \cap (-\infty, c)$. If $f|_{A_l}$ has a limit at c, then we call this limit the *left-hand limit of f at c* and we write

$$\lim_{\substack{x\to c\\ x<c}} f(x) \quad \text{or} \quad \lim_{x\to c^-} f(x).$$

In a similar way one defines the *right-hand limit of f at c* when considering the set $A \cap (c, +\infty)$. In this case we write

$$\lim_{\substack{x\to c\\ x>c}} f(x) \quad \text{or} \quad \lim_{x\to c^+} f(x).$$

These two limits are called *one-sided limits of f at c.*

Remark 4.6. It may happen that none of the one-sided limit exists, only one of them exists or both or them exist and are different.

Theorem 4.7 (Characterization of limits using one-sided limits)**.** *Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$, $L \in \overline{\mathbb{R}}$ and let $c \in \mathbb{R}$ be an accumulation point of both the sets $A \cap (-\infty, c)$ and $A \cap (c, +\infty)$. Then*

$$\lim_{x\to c} f(x) = L \iff \lim_{\substack{x\to c\\ x<c}} f(x) = L = \lim_{\substack{x\to c\\ x>c}} f(x).$$

4.2 Continuous functions

Definition 4.4. Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$, $c \in A$. We say that f is *continuous at c* if

$$\forall V \in \mathcal{V}\left(f(c)\right), \exists U \in \mathcal{V}(c) \text{ such that } \forall x \in U \cap A \text{ we have } f(x) \in V.$$

In this case we call c a *continuity point of f*. If f fails to be continuous at c, then we say that f is *discontinuous at c* and that c is a *discontinuity point of f*.
If B is a subset of A, we say that f is *continuous on B* if it is continuous at every point of B.

Remark 4.7. (i) An important difference between the notions of limit and continuity is that the point c is now assumed to belong to A (so that $f(c)$ makes sense).
(ii) If $c \in A \cap A'$, then f is continuous at c if and only if $\lim\limits_{x \to c} f(x) = f(c)$.
(iii) If c is an isolated point of A, then $\exists U \in \mathcal{V}(c)$ such that $U \cap A = \{c\}$. Thus, f is continuous at c.

Theorem 4.8 (Characterizations of continuity)**.** *Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$, $c \in A$. Then f is continuous at c if and only if one of the following conditions are met:*

(i) $\forall \varepsilon > 0$, $\exists \delta = \delta(\varepsilon) > 0$, $\forall x \in A$, $|x - c| < \delta$, $|f(x) - f(c)| < \varepsilon$.

(ii) $\forall$ *sequence* (x_n) *in* A *with* $\lim\limits_{n \to \infty} x_n = c$ *we have that* $\lim\limits_{n \to \infty} f(x_n) = f(c)$.

Remark 4.8. (i) Sums, products and quotients (when defined) of continuous functions are continuous.
(ii) The composition of two continuous functions is continuous.

Classifying discontinuities

Definition 4.5. Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$, let $c \in A$ be a discontinuity point of f. We say that c is a *discontinuity point of the first kind* of f (or that f has a *discontinuity of the first kind* at c) if the one-sided limits of f at c both exist and are finite. The discontinuities that are not of the first kind are called *discontinuities of the second kind.*

Remark 4.9. If c is a discontinuity of the first kind, then it is either a *jump discontinuity* when the one-sided limits do not equal each other or a *removable discontinuity* if the one-sided limits equal each other, but do not equal $f(c)$.

Continuous functions on intervals

Definition 4.6. Let $A \subseteq \mathbb{R}$. A function $f : A \to \mathbb{R}$ is said to be *bounded on A* if there exists $M > 0$ such that $\forall x \in A$, $|f(x)| \leq M$. We say that f *attains its maximum* if there exists $\overline{x} \in A$ such that $\forall x \in A$, $f(x) \leq f(\overline{x})$. Likewise, we say that f *attains its minimum* if there exists $\underline{x} \in A$ such that $\forall x \in A$, $f(\underline{x}) \leq f(x)$. In this case $\overline{x}$ is called a *maximum point* for f and $\underline{x}$ is called a *minimum point* for f.

Theorem 4.9 (Maximum-Minimum Theorem, Weierstrass)**.** *Let $a, b \in \mathbb{R}$ with $a < b$ and let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$. Then f is bounded. Moreover, f attains both its maximum and minimum on $[a, b]$.*

Remark 4.10. (i) The function f can be unbounded if

- the interval is unbounded: $f : [0, +\infty) \to \mathbb{R}$, $f(x) = x$.
- the interval is not closed: $f : (0, 1] \to \mathbb{R}$, $f(x) = 1/x$.
- f is not continuous: $f : [0, 1] \to \mathbb{R}$, $f(x) = \begin{cases} 1/x, & \text{if } x \in (0, 1], \\ 0, & \text{if } x = 0. \end{cases}$

(ii) A maximum (minimum) point is not necessarily unique.

Theorem 4.10 (Intermediate Value Theorem, Bolzano-Darboux)**.** *Let $a, b \in \mathbb{R}$ with $a < b$ and let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$. If $\gamma \in \mathbb{R}$ satisfies $f(a) < \gamma < f(b)$ or $f(b) < \gamma < f(a)$, then there exists a point $c \in (a, b)$ such that $f(c) = \gamma$.*

Remark 4.11. (i) Location of Roots: If $f : [a, b] \to \mathbb{R}$ is continuous and $f(a) \cdot f(b) < 0$, then $\exists c \in (a, b)$ such that $f(c) = 0$.
(ii) If $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$, then $f([a, b])$ is a closed bounded interval.
(iii) If I is an interval and $f : I \to \mathbb{R}$ is continuous on I, then $f(I)$ is an interval.

4.3 Differentiation of functions

The concept of derivative

Definition 4.7. Let $A \subseteq \mathbb{R}$ and $c \in A \cap A'$. A function $f : A \to \mathbb{R}$ *has a derivative at* c if the limit

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

exists (in $\overline{\mathbb{R}}$). In this case, the above limit is called the *derivative of* f *at* c and is denoted by $f'(c)$. If f has a finite derivative at c, then f is said to be *differentiable at* c.
If B is a subset of A, we say that f is *differentiable on* B if it is differentiable at every point of B. In this case, the function $f' : B \to \mathbb{R}$, $x \in B \mapsto f'(x) \in \mathbb{R}$ is called the *derivative of* f *on* B. If f is differentiable on A, then f is simply called differentiable.

Remark 4.12. (i) Alternative notation: $\dfrac{df}{dx}$, $\dot{f}$, Df.
(ii) Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$ and $c \in A \cap A'$. If f is differentiable at c, then f is also continuous at c. This follows since $\forall x \in A$, $x \neq c$,

$$f(x) - f(c) = \frac{f(x) - f(c)}{x - c}(x - c).$$

(iii) A function can have a derivative at some point without being continuous at that point: the function $\operatorname{sgn}(x)$ has a derivative at 0, $\operatorname{sgn}'(0) = +\infty$, but it is not continuous at 0.
(iv) A function can be continuous at some point without being differentiable at that point: the function $f(x) = |x|$ is continuous at every $x \in \mathbb{R}$, but has no derivative at 0.
(v) There exist functions that are continuous at every point, but nowhere-differentiable. Such an example was given by K. Weierstrass in the 19^{th} century. In fact, "most" continuous functions are nowhere-differentiable.

Definition 4.8. Let $A \subseteq \mathbb{R}$ and $c \in A \cap (A_l)'$, where $A_l = A \cap (-\infty, c)$. A function $f : A \to \mathbb{R}$ *has a left-hand derivative at* c if the left-hand limit

$$\lim_{\substack{x \to c \\ x < c}} \frac{f(x) - f(c)}{x - c}$$

exists (in $\overline{\mathbb{R}}$). In this case, the above left-hand limit is called the *left-hand derivative of* f *at* c and is denoted by $f_l'(c)$. If f has a finite left-hand derivative at c, then f is said to be *left-hand differentiable at* c.
In a similar way one defines the *right-hand derivative of* f *at* c, denoted by $f_r'(c)$, and the *right-hand differentiability of* f *at* c.

Remark 4.13. If $A = [a, b]$, where $a, b \in \mathbb{R}$ with $a < b$, then the differentiability of $f : A \to \mathbb{R}$ at a is actually the left-hand differentiability of f at a and the differentiability of f at b is actually the right-hand differentiability of f at b.

Theorem 4.11. *Let $A \subseteq \mathbb{R}$, $c \in A \cap A'$ and $f, g : A \to \mathbb{R}$ be functions that are differentiable at c.*

(i) If $\alpha \in \mathbb{R}$, then the function αf is differentiable at c and $(\alpha f)'(c) = \alpha f'(c)$.

(ii) (Sum rule) The function $f + g$ is differentiable at c and $(f+g)'(c) = f'(c) + g'(c)$.

(iii) (Product rule) The function fg is differentiable at c and $(fg)'(c) = f'(c)g(c) + f(c)g'(c)$.

(iv) (Quotient rule) If $g(c) \neq 0$, then the function f/g (defined on some neighborhood of c) is differentiable at c and

$$\left(\frac{f}{g}\right)'(c) = \frac{f'(c)g(c) - f(c)g'(c)}{(g(c))^2}.$$

Theorem 4.12 (Chain rule)**.** *Let $I, J \subseteq \mathbb{R}$ be intervals, $c \in I$, $f : I \to J$ and $g : J \to \mathbb{R}$. If f is differentiable at c and g is differentiable at $f(c)$, then $g \circ f : I \to \mathbb{R}$ is differentiable at c and $(g \circ f)'(c) = g'(f(c))f'(c)$.*

Theorem 4.13 (Inverse Function Theorem)**.** *Let $I, J \subseteq \mathbb{R}$ be intervals, $c \in I$ and let $f : I \to J$ be invertible. If f is differentiable at c, $f'(c) \neq 0$ and $f^{-1} : J \to I$ is continuous at $f(c)$, then f^{-1} is differentiable at $f(c)$ and*

$$\left(f^{-1}\right)'(f(c)) = \frac{1}{f'(c)}.$$

Local extrema and derivatives

Definition 4.9. Let $A \subseteq \mathbb{R}$ and $f : A \to \mathbb{R}$. We say that f *attains a local maximum (local minimum)* at $c \in A$ if there exists $V \in \mathcal{V}(c)$ such that c is a maximum point (minimum point) for $f|_{A \cap V}$. We say that f *attains a local extremum* at $c \in A$ if it attains either a local maximum or a local minimum at c.

Theorem 4.14 (Fermat)**.** *Let $a, b \in \mathbb{R}$, $a < b$ and $f : (a,b) \to \mathbb{R}$. If $c \in (a,b)$, f has a derivative at c and f attains a local extremum at c, then $f'(c) = 0$.*

Remark 4.14. (i) The above result may fail if one does not assume that f has a derivative at c (take $f : (-1,1) \to \mathbb{R}$, $f(x) = |x|$, $c = 0$) or if the open interval is replaced by a closed one (take $f : [0,1] \to \mathbb{R}$, $f(x) = x$. Then f attains a minimum at $c = 0$, but $f'(0) = 1$).
(ii) If $f'(c) = 0$, it does not follow that f attains a local extremum at c (take $f : (-1,1) \to \mathbb{R}$, $f(x) = x^3$, $c = 0$).

Theorem 4.15 (Darboux)**.** *Let $a, b \in \mathbb{R}$, $a < b$ and let $f : [a,b] \to \mathbb{R}$ be differentiable. If $\gamma \in \mathbb{R}$ satisfies $f'(a) < \gamma < f'(b)$ or $f'(b) < \gamma < f'(a)$, then there exists a point $c \in (a,b)$ such that $f'(c) = \gamma$.*

Remark 4.15. The derivative of a differentiable function is not always continuous. Take $f : \mathbb{R} \to \mathbb{R}$,

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

Then f is differentiable on $\mathbb{R}$, but f' is not continuous at 0. A function is called *continuously differentiable* if it is differentiable and its derivative is continuous.

Theorem 4.16 (Rolle)**.** *Let $a, b \in \mathbb{R}$, $a < b$ and $f : [a,b] \to \mathbb{R}$. If f is continuous on $[a,b]$, differentiable on (a,b) and $f(a) = f(b)$, then there exists $c \in (a,b)$ such that $f'(c) = 0$.*

Theorem 4.17 (Mean Value Theorem, Lagrange)**.** *Let $a, b \in \mathbb{R}$, $a < b$ and $f : [a,b] \to \mathbb{R}$. If f is continuous on $[a,b]$ and differentiable on (a,b), then there exists $c \in (a,b)$ such that*

$$f(b) - f(a) = f'(c)(b-a).$$

Theorem 4.18 (Generalized Mean Value Theorem, Cauchy)**.** *Let $a, b \in \mathbb{R}$, $a < b$ and $f, g : [a,b] \to \mathbb{R}$. If f, g are continuous on $[a,b]$ and differentiable on (a,b), then there exists $c \in (a,b)$ such that*

$$(f(b) - f(a))g'(c) = (g(b) - g(a))f'(c).$$

L'Hôpital's rules

Theorem 4.19 (L'Hôpital's rules)**.** *Let $a, b \in \overline{\mathbb{R}}$ with $a < b$ and let $f, g : (a, b) \to \mathbb{R}$ be differentiable such that*

(i) $\forall x \in (a, b),\ g'(x) \neq 0$.

(ii) $\lim\limits_{\substack{x \to a \\ x > a}} f(x) = L = \lim\limits_{\substack{x \to a \\ x > a}} g(x)$, *where* $L \in \{-\infty, 0, +\infty\}$.

(iii) $\exists \lim\limits_{\substack{x \to a \\ x > a}} \dfrac{f'(x)}{g'(x)} \in \overline{\mathbb{R}}$.

Then $\lim\limits_{\substack{x \to a \\ x > a}} \dfrac{f(x)}{g(x)} = \lim\limits_{\substack{x \to a \\ x > a}} \dfrac{f'(x)}{g'(x)}$.

Remark 4.16. The result for left-hand and two-sided limits is similar.

Higher order derivatives

Definition 4.10. Let $A \subseteq \mathbb{R}$, $c \in A \cap A'$ and $f : A \to \mathbb{R}$. We say that f is *twice differentiable at c* if $\exists V \in \mathcal{V}(c)$ such that f is differentiable on $A \cap V$ and f' is differentiable at c. If f is twice differentiable at c, then we write $(f')'(c) = f''(c) = f^{(2)}(c)$.
In general, for $n \geq 2$, we say that f is *n-times differentiable at c* if $\exists V \in \mathcal{V}(c)$ such that f is $(n-1)$-times differentiable on $A \cap V$ and $f^{(n-1)}$ is differentiable at c. If f is n-times differentiable at c, then we write $\left(f^{(n-1)}\right)'(c) = f^{(n)}(c)$. If B is a subset of A, we say that f is *n-times differentiable on B* if it is n-times differentiable at every point of B. In this case, the function $f^{(n)} : B \to \mathbb{R}$, $x \in B \mapsto f^{(n)}(x) \in \mathbb{R}$ is called the *n^{th} derivative of f on B*.
We say that f is *infinitely differentiable at c* if for every $n \in \mathbb{N}^*$, f is n-times differentiable at c.
Notation: $f = f^{(0)}$, $f' = f^{(1)}$, $f'' = f^{(2)}$, $f''' = f^{(3)}$.

In the following we focus on approximating functions by polynomials. Let $I \subseteq \mathbb{R}$ be an interval, $x_0 \in I$, $f : I \to \mathbb{R}$ and $n \in \mathbb{N}$. Supposing that f is n-times differentiable at x_0, we want to find a polynomial of degree (at most) n, T_n, such that

$$T_n(x_0) = f(x_0), \quad T_n'(x_0) = f'(x_0), \quad T_n''(x_0) = f''(x_0), \quad \ldots, \quad T_n^{(n)}(x_0) = f^{(n)}(x_0). \tag{1}$$

We are looking for T_n of the form

$$T_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \ldots + a_n(x - x_0)^n.$$

Clearly, from (1), we obtain that

$$a_0 = f(x_0), \quad a_1 = f'(x_0), \quad a_2 = \frac{f''(x_0)}{2!}, \quad \ldots, \quad a_n = \frac{f^{(n)}(x_0)}{n!}.$$

The polynomial

$$T_n(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \ldots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n \tag{2}$$

is called the *n^{th} Taylor polynomial of f at the point x_0*.

Remark 4.17. (i) The complete notation for the n^{th} Taylor polynomial of f at the point x_0 would be $T_n(f; x_0)(x)$. However, to simplify the writing we keep the notation $T_n(x)$.
(ii) The polynomial given by (2) is the unique polynomial of degree (at most) n that satisfies (1).
(iii) By definition, the Taylor polynomial approximates well the function at x_0. We are interested to see what happens at points near x_0. To this end, one considers the *remainder function* (or *remainder term*) $R_n(x) = f(x) - T_n(x)$, which is the error between f and T_n and determines the quality of the approximation of f at points in I near x_0. Any formula $f(x) = T_n(x) + R_n(x)$, where R_n is given explicitly, is called *Taylor's formula*.

Theorem 4.20 (Taylor-Lagrange)**.** *Let $a, b \in \mathbb{R}$ with $a < b$, $n \in \mathbb{N}$ and $f : [a, b] \to \mathbb{R}$ be n-times continuously differentiable function which is $(n+1)$-times differentiable on (a, b). Then $\forall x, x_0 \in [a, b]$ with $x \neq x_0$, there exists a point c strictly between x and x_0 such that*

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \ldots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}. \tag{3}$$

Thus, $f(x) = T_n(x) + R_n(x)$, where T_n is the n^{th} Taylor polynomial of f at the point x_0 and

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}. \tag{4}$$

Remark 4.18. (i) The above formula (4) for the remainder R_n is known as the Lagrange form (there are also other expressions of the remainder). In this notes, we will refer to (3) as Taylor's formula.
(ii) If $x = x_0$ and f is $(n+1)$-times differentiable at x_0, then (3) holds with $c = x_0$.
(iii) Note that, for $n = 0$, what we obtain is precisely the Mean Value Theorem. Thus, Theorem 4.20 can be regarded as an extension of the Mean Value Theorem to higher order derivatives since it gives a relation between the values of a function and its higher order derivatives.
(iv) If we can bound $|f^{(n+1)}(c)|$, then we can estimate the error of approximation of $f(x)$.

Definition 4.11. Let $I \subseteq \mathbb{R}$ be an interval and let $f : I \to R$ be infinitely differentiable. For $x_0 \in I$ and $x \in \mathbb{R}$, the series

$$\sum_{n \geq 0} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

is called the *Taylor series of f around x_0*.

Remark 4.19. Problem: At which points x is the above series convergent? If so, is its sum $f(x)$ (when $x \in I$)? Note that $T_n(x)$ is a partial sum of this series. Thus,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n \tag{5}$$

(or, equivalently, $f(x) = \lim_{n\to\infty} T_n(x)$) if and only if $\lim_{n\to\infty} R_n(x) = 0$. If this holds for every $x \in I$, we say that f can be *expanded as a Taylor series around* x_0 on I. Formula (5) is called the *Taylor series expansion of $f(x)$ around x_0*.

II. *Tablou de derivare*

Funcţia	Derivata	Domeniul de derivabilitate
c (constantă)	0	$\mathbb{R}$
x	1	$\mathbb{R}$
x^n, $n \geqslant 1$ întreg	nx^{n-1}	$\mathbb{R}$
x^r, r real	rx^{r-1}	cel puţin $(0, \infty)$
$\sqrt{x}$	$\frac{1}{2\sqrt{x}}$	$(0, \infty)$
$\ln x$	$\frac{1}{x}$	$(0, \infty)$
e^x	e^x	$\mathbb{R}$
a^x, $a > 0$, $a \neq 1$	$a^x \ln a$	$\mathbb{R}$
$\sin x$	$\cos x$	$\mathbb{R}$
$\cos x$	$-\sin x$	$\mathbb{R}$
$\operatorname{tg} x$	$\frac{1}{\cos^2 x}$	$\cos x \neq 0$
$\operatorname{ctg} x$	$-\frac{1}{\sin^2 x}$	$\sin x \neq 0$
$\arcsin x$	$\frac{1}{\sqrt{1-x^2}}$	$(-1, 1)$
$\arccos x$	$-\frac{1}{\sqrt{1-x^2}}$	$(-1, 1)$
$\operatorname{arctg} x$	$\frac{1}{1+x^2}$	$\mathbb{R}$
$\operatorname{arcctg} x$	$-\frac{1}{1+x^2}$	$\mathbb{R}$

arcsin : $[-1, 1] \longrightarrow \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

arccos : $[-1, 1] \longrightarrow [0, \pi]$

arctg : $\mathbb{R} \longrightarrow \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$

arcctg : $\mathbb{R} \longrightarrow]0, \pi[$