Babeş-Bolyai University, Faculty of Mathematics and Computer Science
Analysis for Computer Science
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Exercise Set #6

1. Find the derivatives of the following functions:
a) $f:\mathbb{R}\to\mathbb{R},\ f(x)=e^{\sin^2 x}$, b) $f:\mathbb{R}\to\mathbb{R},\ f(x)=\sin\left(e^{(x+1)^2}\right)$,
c) $f:\mathbb{R}\to\mathbb{R},\ f(x)=\sin^2 x+\cos^2 x$, d) $f:\mathbb{R}\to\mathbb{R},\ f(x)=\sin^n(x)\cos(nx)\ (n\in\mathbb{N}^*)$,
e) $f:(1,+\infty)\to\mathbb{R},\ f(x)=\ln(\ln x)$, f) $f:(0,+\infty)\to\mathbb{R},\ f(x)=x^x$.

2. Show that $f:[0,\infty)\to\mathbb{R},\ f(x)=\sqrt{x}$ is not differentiable at 0.

3. How many times is the function $f:\mathbb{R}\to\mathbb{R},\ f(x)=\begin{cases}x^2, & \text{if } x\geq 0\\ -x^2, & \text{if } x<0\end{cases}$ differentiable?

4. Find the n^{th} derivative ($n\in\mathbb{N}$) of the following functions:
a) $f:(-1,+\infty)\to\mathbb{R},\ f(x)=\ln(x+1)$, b) $f:\mathbb{R}\to\mathbb{R},\ f(x)=\sin x$,
c) $f:\mathbb{R}\to\mathbb{R},\ f(x)=\cos x$, d) $f:\mathbb{R}\to\mathbb{R},\ f(x)=e^{2x}x^3$, e) $f:\mathbb{R}\to\mathbb{R},\ f(x)=e^x\sin x$.

5. Prove that every differentiable odd function has an even derivative and every differentiable even function has an odd derivative.

6. Let $a,b\in\mathbb{R}$, $a<b$ and $f:[a,b]\to\mathbb{R}$. Suppose that f is continuous on $[a,b]$ and differentiable on (a,b) with bounded derivative. Prove that f is Lipschitz.

7. Let $a,b\in\mathbb{R}$, $a<b$ and $f:[a,b]\to\mathbb{R}$. Suppose that f is continuous on $[a,b]$ and differentiable on (a,b). Prove that $\exists c\in(a,b)$ such that $(c-a)(c-b)f'(c)=a+b-2c$.
Hint: Consider the function $g:[a,b]\to\mathbb{R},\ g(x)=e^{f(x)}(x-a)(x-b)$.

8. Use the Mean Value Theorem to compute $\lim\limits_{n\to\infty} n\left(1-\cos\frac{1}{n}\right)$.

9. Use L'Hôpital's rules to compute $\lim\limits_{x\to 0}\frac{e-(1+x)^{\frac{1}{x}}}{x}$. Then evaluate $\lim\limits_{n\to\infty} n\left(e-\left(1+\frac{1}{n}\right)^n\right)$ (see Ex. 3.(m), Exercise Set #4).

10. Compute the following limits:
a) $\lim\limits_{x\to 0}\frac{e^x+e^{-x}-2}{1-\cos x}$, b) $\lim\limits_{x\to 0}\frac{x^2-\sin^2 x}{x^4}$, c) $\lim\limits_{x\to\infty}\frac{\ln x}{\sqrt{x}}$, d) $\lim\limits_{x\to\infty}\frac{x+\ln x}{x\ln x}$, e) $\lim\limits_{\substack{x\to 0\\ x>0}} x\ln\sin x$,
f) $\lim\limits_{\substack{x\to 0\\ x>0}}(\sin x)^x$.

11. Let $f:\mathbb{R}\to\mathbb{R},\ f(x)=x^3-3x^2+5x+1$. Find the third Taylor polynomial $T_3(x)$ of f at 1.

12. Let $f:[0,\infty)\to\mathbb{R},\ f(x)=\sqrt[3]{x}$. Find the second Taylor polynomial $T_2(x)$ of f at 1 and the remainder term $R_2(x)$ (using Taylor's formula). If $x\in[0.9,1.1]$, find an upper bound for $|R_2(x)|$.

13. Let $f:\mathbb{R}\to\mathbb{R},\ f(x)=\cos x$. Find the second Taylor polynomial $T_2(x)$ of f at 0 and the remainder term $R_2(x)$ (using Taylor's formula). Then show that $\forall x\in\mathbb{R},\ 1-\frac{x^2}{2}\leq\cos x$.

14. For the following functions $f:D\to\mathbb{R}$, find the n^{th} Taylor polynomial $T_n(x)$ of f at 0 and the remainder term $R_n(x)$ (using Taylor's formula). Then show that f can be expanded as a Taylor series around 0 on D and find the corresponding Taylor series expansion:
a) $D=[0,1],\ f(x)=\ln(x+1)$, b) $D=\mathbb{R},\ f(x)=\sin x$, c) $D=\mathbb{R},\ f(x)=\cos x$.