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3 Series of real numbers

Definition 3.1. Let (x_n) be a sequence in $\mathbb{R}$. The *series* with terms x_n is the sequence (s_n) defined by

$$s_n = x_1 + x_2 + \ldots + x_n, \quad n \in \mathbb{N}^*.$$

Notation:

$$\sum_{n\geq 1} x_n \quad \text{or} \quad \sum x_n.$$

For $n \in \mathbb{N}^*$, the number s_n is called the n^{th} *partial sum* of the series. If the sequence (s_n) of partial sums converges, we say that the series $\sum_{n\geq 1} x_n$ is *convergent*. If (s_n) diverges, we say that the series $\sum_{n\geq 1} x_n$ is *divergent*. If the limit $s = \lim\limits_{n\to\infty} s_n$ exists (in $\overline{\mathbb{R}}$), we call s the *sum* of the series and we write

$$s = \sum_{n=1}^{\infty} x_n.$$

Remark 3.1. (i) We denote a series using the symbols $\sum_{n\geq 1} x_n$ or $\sum x_n$, while $\sum_{n=1}^{\infty} x_n$ denotes the sum of a convergent series.
(ii) A series has a sum if and only if the sequence of its partial sums is either convergent (when the sum is finite) or properly divergent (when the sum is infinite).
(iii) We also consider series of the form $\sum_{n\geq m} x_n$ generated by a sequence $(x_n)_{n\geq m}$. Moreover, for any $k \in \mathbb{N}$, $\sum_{n\geq m} x_n$ is convergent if and only if $\sum_{n\geq m+k} x_n$ is convergent. In this case we have

$$\sum_{n=m}^{\infty} x_n = x_m + x_{m+1} + \ldots x_{m+k-1} + \sum_{n=m+k}^{\infty} x_n.$$

Proposition 3.1. *Let $\sum_{n\geq 1} x_n$ and $\sum_{n\geq 1} y_n$ be convergent series and let $c \in \mathbb{R}$. Then*

(i) $\sum_{n\geq 1}(x_n + y_n)$ is convergent and

$$\sum_{n=1}^{\infty}(x_n + y_n) = \sum_{n=1}^{\infty} x_n + \sum_{n=1}^{\infty} y_n.$$

(ii) $\sum_{n\geq 1}(c\, x_n)$ is convergent and

$$\sum_{n=1}^{\infty}(c\, x_n) = c\sum_{n=1}^{\infty} x_n.$$

Theorem 3.1 (The n^{th} Term Test)**.** *If the series $\sum_{n\geq 1} x_n$ converges, then $\lim\limits_{n\to\infty} x_n = 0$.*

Remark 3.2. Thus, if (x_n) is divergent or $\lim\limits_{n\to\infty} x_n \neq 0$, then $\sum_{n\geq 1} x_n$ is divergent. However, the condition $\lim\limits_{n\to\infty} x_n = 0$ is not sufficient for the convergence of $\sum_{n\geq 1} x_n$ (e.g., the harmonic series).

Series with nonnegative terms

Let (x_n) be a sequence in $\mathbb{R}$. If the series $\sum_{n\geq 1} x_n$ is convergent, then, by Theorem 2.2, the sequence (s_n) of partial sums must be bounded, but this condition is not equivalent to the convergence of $\sum_{n\geq 1} x_n$. To see this, consider, for instance, $\sum_{n\geq 1}(-1)^n$. Then

$$s_n = \begin{cases} -1, & \text{if } n \text{ is odd,} \\ 0, & \text{if } n \text{ is even.} \end{cases}$$

This sequence is bounded, but does not converge, so $\sum_{n\geq 1} x_n$ is divergent. However, for series with nonnegative terms, this equivalence holds true.
A series $\sum_{n\geq 1} x_n$ is with *nonnegative terms* if

$$\forall n \in \mathbb{N}^*, x_n \geq 0$$

and is with *positive terms* if

$$\forall n \in \mathbb{N}^*, x_n > 0.$$

Assume that $\sum_{n\geq 1} x_n$ is a series with nonnegative terms. Then $\sum_{n\geq 1} x_n$ converges if and only if the sequence (s_n) of its partial sums is bounded. Indeed, because $\forall n \in \mathbb{N}^*, s_{n+1} = s_n + x_{n+1} \geq s_n$, we know that (s_n) is increasing and so one can apply Theorem 2.7. If (s_n) is unbounded, we have that $\lim\limits_{n\to\infty} s_n = +\infty$. Hence, series with nonnegative terms always have a sum which can be finite or $+\infty$:

$$\sum_{n=1}^{\infty} x_n = \lim_{n\to\infty} s_n = \sup\{s_n \mid n \in \mathbb{N}^*\}.$$

Theorem 3.2 (First Comparison Test)**.** *Let $\sum_{n\geq 1} x_n$ and $\sum_{n\geq 1} y_n$ be series with nonnegative terms satisfying*

$$\exists c > 0, \exists N \in \mathbb{N}^* \text{ s.t. } \forall n \geq N, x_n \leq c\, y_n.$$

(i) If $\sum_{n\geq 1} y_n$ is convergent, then $\sum_{n\geq 1} x_n$ is convergent.

(ii) If $\sum_{n\geq 1} x_n$ is divergent, then $\sum_{n\geq 1} y_n$ is divergent.

Theorem 3.3 (Second Comparison Test)**.** *Let $\sum_{n\geq 1} x_n$ and $\sum_{n\geq 1} y_n$ be series with positive terms such that $L = \lim\limits_{n\to\infty} \dfrac{x_n}{y_n}$ exists.*

(i) If $L \in (0, +\infty)$, then $\sum_{n\geq 1} x_n$ is convergent if and only if $\sum_{n\geq 1} y_n$ is convergent.

(ii) If $L = 0$ and $\sum_{n\geq 1} y_n$ is convergent, then $\sum_{n\geq 1} x_n$ is convergent.

(iii) If $L = +\infty$ and $\sum_{n\geq 1} y_n$ is divergent, then $\sum_{n\geq 1} x_n$ is divergent.

Theorem 3.4 (Root Test, Cauchy)**.** *Let $\sum_{n\geq 1} x_n$ be a series with nonnegative terms.*

(i) If $\exists q \in [0,1), \exists N \in \mathbb{N}^*$ s.t. $\forall n \geq N$, $\sqrt[n]{x_n} \leq q$, then $\sum_{n\geq 1} x_n$ is convergent.

(ii) If $\exists N \in \mathbb{N}^*$ s.t. $\forall n \geq N$, $\sqrt[n]{x_n} \geq 1$, then $\sum_{n\geq 1} x_n$ is divergent.

(iii) Suppose $L = \lim\limits_{n\to\infty} \sqrt[n]{x_n}$ exists.

1. *If $L < 1$, $\sum_{n\geq 1} x_n$ is convergent.*
2. *If $L > 1$, $\sum_{n\geq 1} x_n$ is divergent.*
3. *If $L = 1$, the test gives no information.*

Theorem 3.5 (Kummer's Test)**.** *Let $\sum_{n\geq 1} x_n$ be a series with positive terms. For $n \in \mathbb{N}^*$, let $c_n > 0$ and define*

$$K_n = c_n \frac{x_n}{x_{n+1}} - c_{n+1}.$$

(i) *If* $\exists r > 0$, $\exists N \in \mathbb{N}^*$ *such that* $\forall n \geq N$, $K_n \geq r$, *then* $\sum_{n\geq 1} x_n$ *is convergent.*

(ii) *If* $\displaystyle\sum_{n=1}^{\infty} \frac{1}{c_n} = +\infty$ *and* $\exists N \in \mathbb{N}^*$ *such that* $\forall n \geq N$, $K_n \leq 0$, *then* $\sum_{n\geq 1} x_n$ *is divergent.*

Corollary 3.1 (Ratio Test, d'Alembert)**.** *Let* $\sum_{n\geq 1} x_n$ *be a series with positive terms.*

(i) *If* $\exists q \in (0,1), \exists N \in \mathbb{N}^*$ *s.t.* $\forall n \geq N$, $\dfrac{x_{n+1}}{x_n} \leq q$, *then* $\sum_{n\geq 1} x_n$ *is convergent.*

(ii) *If* $\exists N \in \mathbb{N}^*$ *s.t.* $\forall n \geq N$, $\dfrac{x_{n+1}}{x_n} \geq 1$, *then* $\sum_{n\geq 1} x_n$ *is divergent.*

(iii) *Suppose* $L = \displaystyle\lim_{n\to\infty} \frac{x_{n+1}}{x_n}$ *exists.*

1. *If* $L < 1$, $\sum_{n\geq 1} x_n$ *is convergent.*
2. *If* $L > 1$, $\sum_{n\geq 1} x_n$ *is divergent.*
3. *If* $L = 1$, *the test gives no information.*

Corollary 3.2 (Raabe's Test)**.** *Let* $\sum_{n\geq 1} x_n$ *be a series with positive terms. For* $n \in \mathbb{N}^*$, *define*

$$R_n = n\left(\frac{x_n}{x_{n+1}} - 1\right).$$

(i) *If* $\exists r > 1, \exists N \in \mathbb{N}^*$ *s.t.* $\forall n \geq N, R_n \geq r$, *then* $\sum_{n\geq 1} x_n$ *is convergent.*

(ii) *If* $\exists N \in \mathbb{N}^*$ *s.t.* $\forall n \geq N, R_n \leq 1$, *then* $\sum_{n\geq 1} x_n$ *is divergent.*

(iii) *Suppose* $L = \displaystyle\lim_{n\to\infty} R_n$ *exists.*

1. *If* $L > 1$, $\sum_{n\geq 1} x_n$ *is convergent.*
2. *If* $L < 1$, $\sum_{n\geq 1} x_n$ *is divergent.*
3. *If* $L = 1$, *the test gives no information.*

Series with arbitrary terms

Definition 3.2. A series $\sum_{n\geq 1} x_n$ is called *absolutely convergent* if the series of absolute values $\sum_{n\geq 1} |x_n|$ is convergent.

Remark 3.3. Clearly, for series with nonnegative terms, absolute convergence is the same as convergence.

Proposition 3.2. *Let* $\sum_{n\geq 1} x_n$ *be an absolutely convergent series. Then* $\sum_{n\geq 1} x_n$ *is convergent.*

Definition 3.3. Let (x_n) be a sequence in $\mathbb{R}$. A series $\sum_{n\geq 1} x_n$ is called *alternating* if

$$\forall n \in \mathbb{N}^*, x_n \cdot x_{n+1} < 0.$$

Theorem 3.6 (Alternating Series Test, Leibniz)**.** *Let* (x_n) *be a decreasing sequence of nonnegative real numbers. Then* $\sum_{n\geq 1}(-1)^{n+1}x_n$ *is convergent if and only if* (x_n) *converges to* 0.