Exercises – Resolution proof method

Exercise 5.1

Are the literals from the following pairs unifiable?

If yes, find their most general unifier. $x, y, z \in Var$, $a, b \in Const$, $f, g \in \mathrm{F}_1, h \in \mathrm{F}_2$, $P \in \mathrm{P}_3$.

1. $P(a, x, g(g(y)))$ and $P(y, f(z), f(z))$;
   $P(x, g(f(a)), f(x))$ and $P(f(y), z, y)$;
   $P(a, x, g(g(y)))$ and $P(z, h(z, u), g(u))$;
2. $P(a, x, f(g(y)))$ and $P(y, f(z), f(z))$;
   $P(x, g(f(a)), f(b))$ and $P(f(y), z, z)$;
   $P(a, x, f(g(y)))$ and $P(z, h(z, u), f(b))$;
3. $P(a, f(x), g(h(y)))$ and $P(y, f(z), g(z))$;
   $P(x, g(f(a)), h(x, y))$ and $P(f(z), g(z), y)$;
   $P(g(y), x, f(g(y)))$ and $P(z, h(z, u), f(u))$;
4. $P(a, g(x), f(g(y)))$ and $P(y, z, f(z))$;
   $P(b, g(f(a)), z)$ and $P(f(y), z, g(y))$;
   $P(a, h(x, b), f(g(y)))$ and $P(z, h(z, u), f(u))$;
5. $P(a, x, g(f(y)))$ and $P(f(z), z, g(x))$;
   $P(a, x, g(f(y)))$ and $P(x, y, g(f(b)))$;
   $P(a, h(x, u), g(f(z)))$ and $P(y, h(y, f(z)), g(x))$;
6. $P(a, y, g(f(z)))$ and $P(z, f(z), x)$;
   $P(y, f(x), z)$ and $P(y, f(y), f(y))$;
   $P(h(x, y), x, y)$ and $P(h(y, x), f(z), z)$;
7. $P(a, x, g(f(y)))$ and $P(f(y), z, x)$;
   $P(x, a, g(b))$ and $P(f(y), f(y), g(x))$;
   $P(h(x, a), f(z), z)$ and $P(h(f(y), x), f(x), a)$;
8. $P(a, x, g(f(y)))$ and $P(f(y), f(z), g(z))$;
   $P(x, g(f(a)), x)$ and $P(f(y), z, h(y, f(y)))$;
   $P(a, h(x, u), f(g(y)))$ and $P(z, h(z, u), g(u))$.

Exercise 5.2

Using general resolution prove that the following formulas are theorems.

1. $U_1 = (\neg B \to \neg A) \to ((\neg B \to A) \to B)$;
2. $U_2 = (B \to A) \wedge (C \to A) \to (B \wedge C \to A)$;
3. $U_3 = (B \to A) \wedge (C \to A) \to (B \vee C \to A)$;
4. $U_4 = (A \to C) \to ((\neg A \to B) \to (\neg B \to C))$;
5. $U_5 = A \vee (B \to C) \to (A \vee B) \to (A \vee C)$;
6. $U_6 = (A \to B) \to ((C \to A) \to (C \to B))$;
7. $U_7 = (A \to B) \to ((\neg A \to C) \to (\neg B \to C))$;
8. $U_8 = (A \to B \wedge C) \to (A \to B) \wedge (A \to C)$.

Exercise 5.3

Using lock resolution check the inconsistency of the following sets of clauses.

Choose two different indexings for the literals:

1. $S_1 = \{p \vee q, p \vee \neg q \vee r, p \vee \neg q \vee \neg r, \neg p \vee r, \neg p \vee \neg r\}$;
2. $S_2 = \{\neg p \vee \neg q, \neg p \vee q \vee \neg r, p \vee \neg r, \neg p \vee r, p \vee r\}$;
3. $S_3 = \{p \vee q, p \vee \neg q \vee \neg r, \neg p \vee \neg r, r, \neg p \vee r\}$;
4. $S_4 = \{p \vee q, \neg p \vee q \vee \neg r, \neg p \vee q \vee r, \neg q \vee \neg r, \neg q \vee r\}$;
5. $S_5 = \{p \vee \neg q, \neg p \vee \neg q \vee r, \neg p \vee q \vee r, p \vee q, \neg r\}$;
6. $S_6 = \{p \vee q, \neg p \vee q \vee \neg r, \neg p \vee \neg q \vee \neg r, p \vee \neg q, r\}$;
7. $S_7 = \{p \vee \neg q, \neg p \vee \neg q \vee r, \neg p \vee \neg q \vee \neg r, r \vee q, \neg r \vee q\}$;
8. $S_8 = \{p \vee r, p \vee q \vee \neg r, \neg p \vee \neg q \vee r, \neg p \vee q \vee r, \neg r\}$.

Exercise 5.4

Build a linear refutation from the following set of clauses:

1. $S_1 = \{p \vee q \vee r, \neg q \vee r, \neg r, \neg p \vee r\}$;
2. $S_2 = \{p \vee \neg r, q \vee r, \neg q \vee r, \neg p \vee \neg r\}$;
3. $S_3 = \{q \vee r, \neg p, \neg q \vee r, p \vee \neg r\}$;
4. $S_4 = \{\neg p \vee q, p \vee \neg q \vee r, \neg r, p \vee q \vee r, \neg p \vee \neg q\}$;
5. $S_5 = \{p \vee r, \neg q, p \vee q \vee \neg r, \neg p \vee \neg r, q \vee r\}$;
6. $S_6 = \{p \vee q, \neg p \vee q, \neg p \vee \neg q, p \vee \neg q\}$;
7. $S_7 = \{p, q \vee r, \neg p \vee q \vee \neg r, \neg p \vee \neg q\}$;
8. $S_8 = \{p \vee \neg q \vee r, q, \neg p \vee \neg q \vee r, \neg p \vee \neg q \vee \neg r, p \vee \neg r\}$.

Exercise 5.5

Prove the consistency of the following sets of clauses using the level saturation strategy.

1. $S_1 = \{p \vee q \vee r, \neg q \vee r, \neg r, \neg p \vee r\}$;
2. $S_2 = \{p \vee \neg r, q \vee r, \neg q \vee r, \neg p \vee \neg r\}$;
3. $S_3 = \{q \vee r, \neg p, \neg q \vee r, p \vee \neg r\}$;
4. $S_4 = \{\neg p \vee q, p \vee \neg q \vee r, \neg r, p \vee q \vee r, \neg p \vee \neg q\}$;
5. $S_5 = \{p \vee r, \neg q, p \vee q \vee \neg r, \neg p \vee \neg r, q \vee r\}$;
6. $S_6 = \{p \vee q, \neg p \vee q, \neg p \vee \neg q, p \vee \neg q\}$;
7. $S_7 = \{p, q \vee r, \neg p \vee q \vee \neg r, \neg p \vee \neg q\}$;
8. $S_8 = \{p \vee \neg q \vee r, q, \neg p \vee \neg q \vee r, \neg p \vee \neg q \vee \neg r, p \vee \neg r\}$.

Exercise 5.6

Using the set-of-support strategy prove the following deductions:

1. $\neg(p \vee q) \to r, \neg p \vee q \vee r, \neg r \vdash q \wedge \neg r$;
2. $p \vee \neg r, \neg q \to r, \neg q \vdash \neg(p \to q)$;
3. $q \wedge r \to p, p \vee q, q \to r \vdash p$;
4. $r \to p \vee q, \neg p \to r, \neg q \vdash p \wedge \neg q$;
5. $\neg p \to q, (q \to r) \wedge \neg r \vdash p \wedge \neg r$;
6. $q \to p, q \vee r, p \to r \vdash r$;
7. $\neg p \to q \vee r, \neg q, p \to q \vdash \neg(p \vee q) \wedge r$;
8. $r \to p, \neg p, q \to p \vee r \vdash \neg(\neg p \to q \vee r)$.

Exercise 5.7

Prove the consistency of the following sets of clauses using linear resolution:

1. $S_1 = \{p \vee r, \neg q, p \vee q \vee \neg r, \neg p \vee \neg r, q \vee r\}$;
2. $S_2 = \{p \vee q, \neg p \vee q, \neg p \vee \neg q, p \vee \neg q\}$;

3. $S_3 = \{p, q \vee r, \neg p \vee q \vee \neg r, \neg p \vee \neg q\}$;

4. $S_4 = \{p \vee \neg q \vee r, q, \neg p \vee \neg q \vee r, \neg p \vee \neg q \vee \neg r, p \vee \neg r\}$.

5. $S_5 = \{p \vee q \vee r, \neg q \vee r, \neg r, \neg p \vee r\}$;

6. $S_6 = \{p \vee \neg r, q \vee r, \neg q \vee r, \neg p \vee \neg r\}$;

7. $S_7 = \{q \vee r, \neg p, \neg q \vee r, p \vee \neg r\}$;

8. $S_8 = \{\neg p \vee q, p \vee \neg q \vee r, \neg r, p \vee q \vee r, \neg p \vee \neg q\}$;

Exercise 5.8

Prove the inconsistency of the following set of clauses using lock resolution.
Try two different indexings for the literals.

1. $S_1 = \{\neg p(x) \vee q(x), p(a), \neg q(x) \vee \neg r(x), \neg w(a), r(y) \vee w(y)\}$;
2. $S_2 = \{p(x) \vee \neg q(x), \neg p(a) \vee r(x), q(x), w(z), \neg r(y) \vee \neg w(y)\}$;
3. $S_3 = \{p(x) \vee q(x) \vee r(x), \neg p(a), \neg q(x), \neg w(a), \neg r(y) \vee w(y)\}$;
4. $S_4 = \{p(x) \vee q(x), \neg p(x) \vee r(x), \neg q(y) \vee r(y), \neg r(x) \vee w(x), \neg w(f(z))\}$
5. $S_5 = \{p(x) \vee q(x), \neg p(a) \vee w(x), \neg q(y) \vee r(y), \neg r(x) \vee w(x), \neg w(a)\}$;
6. $S_6 = \{\neg p(x) \vee \neg q(x), p(z) \vee w(x), q(y) \vee w(y) \vee \neg r(y), \neg r(x) \vee \neg w(x), r(g(a,b))\}$
7. $S_7 = \{p(x) \vee q(x), \neg p(x), \neg q(f(a)) \vee r(z), \neg w(z), \neg r(y) \vee w(y)\}$
8. EMBED Equation.3 $S_8 = \{\neg p(x) \vee q(x) \vee \neg r(x), p(f(b)), \neg q(x), \neg w(y), r(y) \vee w(y)\}$.

Exercise 5.9

Prove the following deductions using linear resolution

1. $(\forall x)(\forall y)(p(y,x) \wedge q(x) \rightarrow q(y)), (\forall x)(\forall y)(r(y,x) \rightarrow q(y)), r(b,a),$
   $r(b,a), p(c,b) \vdash (\exists z)q(z)$;
2. $(\forall x)(p(x) \rightarrow r(x)), (\forall y)(r(y) \rightarrow q(y)), p(a), p(b) \vdash (\exists z)q(z)$;
3. $(\forall x)(\neg p(x) \wedge \neg q(x) \rightarrow r(x)), (\forall y)(r(y) \rightarrow w(y)), (\forall x)(w(x) \rightarrow p(x)),$
   $\neg p(a), \neg p(b), \neg w(c) \vdash (\exists z)q(z)$;
4. $(\forall x)(p(x) \rightarrow r(x)), (\forall y)(r(y) \rightarrow q(y)), r(a), r(b), \neg r(c) \vdash (\exists z)q(z)$;
5. $(\forall x)(\neg p(x) \wedge \neg q(x) \rightarrow r(x)), (\forall y)(r(y) \rightarrow w(y)), (\forall x)(w(x) \rightarrow p(x)),$
   $\neg p(a), \neg w(c) \vdash (\exists z)q(z)$;
6. $(\forall x)(\forall y)(\neg p(y,x) \rightarrow q(y)), (\forall x)(\forall y)(r(y,x) \wedge q(x) \rightarrow q(y)),$
   $r(b,a), \neg p(a,b) \vdash (\exists z)q(z)$;
7. $(\forall x)(p(x) \rightarrow r(x)), (\forall y)(p(y) \rightarrow q(y)), p(a), \neg r(c) \vdash (\exists z)q(z)$;
8. $(\forall x)(p(x) \rightarrow r(x)), (\forall y)(p(y) \rightarrow q(y)), p(a), p(b), \neg p(c) \vdash (\exists z)q(z)$.

Exercise 5.10

Using a refinement of predicate resolution prove:

1. the semidistributivity of ' $\forall$ ' over ' $\vee$ ':
   $\vdash (\forall x)p(x) \vee (\forall x)q(x) \rightarrow (\forall x)(p(x) \vee q(x))$ and
   $$\nvdash (\forall x)(p(x) \vee q(x)) \rightarrow (\forall x)p(x) \vee (\forall x)q(x)$$
2. the semidistributivity of ' $\exists$ ' over ' $\wedge$ ':
   $\vdash (\exists x)(p(x) \wedge q(x)) \rightarrow (\exists x)p(x) \wedge (\exists x)q(x)$ and
   $$\nvdash (\exists x)p(x) \wedge (\exists x)q(x) \rightarrow (\exists x)(p(x) \wedge q(x))$$

3. $\vdash (\exists x)(p(x) \to q(x)) \leftrightarrow ((\forall x)p(x) \to (\exists x)q(x))$;

4. the distributivity of ‘ $\exists$ ’ over ‘ $\vee$ ’:
$\vdash (\exists x)(p(x) \vee q(x)) \leftrightarrow (\exists x)p(x) \vee (\exists x)q(x)$;

5. $\vdash (\exists x)p(x) \vee (\exists x)(p(x) \wedge q(x)) \leftrightarrow (\exists x)p(x)$;

6. the semidistributivity of ‘ $\forall$ ’ over ‘ $\to$ ’:
$\vdash (\forall x)(p(x) \to q(x)) \to ((\forall x)p(x) \to (\forall x)q(x))$ and
$\nvdash ((\forall x)p(x) \to (\forall x)q(x)) \to (\forall x)(p(x) \to q(x))$

7. $\vdash (\forall x)p(x) \wedge ((\forall x)p(x) \vee (\forall x)q(x)) \leftrightarrow (\forall x)p(x)$;

8. the distributivity of ‘ $\forall$ ’ over ‘ $\wedge$ ’:
$\vdash (\forall x)p(x) \wedge (\forall x)q(x) \leftrightarrow (\forall x)(p(x) \wedge q(x))$.

Exercise 5.11

Check if the following formulas are theorems using general resolution.

1. $U_1 = (\forall x)(\forall y)p(x,y) \leftrightarrow (\forall y)(\forall x)p(x,y)$;
2. $U_2 = (\exists y)(\exists x)p(x,y) \leftrightarrow (\exists x)(\exists y)p(x,y)$;
3. $U_3 = (\forall x)(\forall y)p(x,y) \leftrightarrow (\exists x)(\forall y)p(x,y)$;
4. $U_4 = (\exists x)(\forall y)p(x,y) \leftrightarrow (\forall y)(\exists x)p(x,y)$;
5. $U_5 = (\exists y)(\exists x)p(x,y) \leftrightarrow (\forall x)(\exists y)p(x,y)$;
6. $U_6 = (\forall y)(\forall x)p(x,y) \leftrightarrow (\forall x)(\exists y)p(x,y)$;
7. $U_7 = (\exists y)(\exists x)p(x,y) \leftrightarrow (\exists x)(\forall y)p(x,y)$;
8. $U_8 = (\forall y)(\exists x)p(x,y) \leftrightarrow (\exists y)(\exists x)p(x,y)$.