Numeration systems

Numeration systems are used to represent and manipulate numeric values.

We have to distinguish between a numeric value and a numeric representation:

- *a value* (abstract concept) is a measure of an attribute of a collection of objects;
- a *representation* is the means by which we display a value and manipulate it.

A value has different representations.

Ex: The numeric value "5" has more representations:

5 (decimal system) **V** (Roman system) **101**(binary system)
cinci(Romanian language) **five**(English language) **cinq**(French language)

- At the mental level people work with values, but for displaying and manipulating them different numeration systems are used (now the decimal system).
- Computers work with number representations using the binary system.

Numeration system - a set of rules for the representation and manipulation (operations) of numeric values using symbols called **digits**. The total number of digits is called **numeration base** (radix)**.**
!! In the following "value" means the decimal numeric value.

<u>**Classification of numeration systems:**</u>
- positional systems and non-positional systems

1. Non-positional systems: ex: the Roman system

- **digits**: I , V , X , L , C , D , M
  values: 1 5 10 50 100 500 1000
  - each symbol (digit) represents a value
  - !! the value 0 cannot be represented
- it is an additive system: the value of a number representation is obtained as the sum (difference) of the values of its digits according to the following rules.
- **rules**:
  a) the value of two or more identical consecutive digits is the sum of the values of these digits.
  ex: numeric representations: III , CCC, MM
  numeric values: 3 , 300, 2000

  b) the value of a pair of different digits, with the biggest one in front of the other is the sum of these two digits values:
  ex: numeric representations: VI , CL , MD
  numeric values: 5+1=6 , 100+50=150, 1000+500=1500

c) the value of a pair of different digits, with the smallest one in front of the other is the difference of these two digits values:

ex: numeric representations: XC , IX , CM

numeric values: 100-10=90, 10-1=9 , 1000-100=900

d) for big numbers a horizontal line over the digit is used, meaning the multiplication of the digit's value with 1000:

ex: the numeric value of $\overline{\text{V}}$ **is 5000**

!! There are different representations of the same value

Ex: both representations **CDXC** and **XD** have the same value **490**

Example: MCDLIX has the decimal value 1000+500-100+50+10-1= 1459

2. Positional systems: Hindu-Arabic numeration systems

- the position of a digit in a representation implies an association with a "positional value"
- the numeric value is the sum of the positional values of all the digits from the representation

- binary system: base = 2, digits : 0,1 , **ex**: $\mathbf{11001_{(2)}}$
- octal system: base = 8, digits: 0,1,2,3,4,5,6,7, **ex:** $\mathbf{2047_{(8)}}$
- decimal system: base =10, digits: 0,1,2,3,4,5,6,7,8,9, **ex**: **2343**
- hexadecimal system: base =16, digits: 0,1,2,3,4,5,6,7,8,9,A,B,C,D,E,F

  where: $A_{(16)}=10$, $B_{(16)}=11$, $C_{(16)}=12$, $D_{(16)}=13$, $E_{(16)}=14$, $F_{(16)}=15$

  ex: $\mathbf{2AF_{(16)}}$

The calculation of the decimal values of numbers given in different numeration bases represents a conversion method from an arbitrary base into base 10 – a particular case of the *substitution method*.

Let us consider a real number in base ***b*** represented as:

$$(a_m a_{m-1} \dots a_1 a_0 , a_{-1} \dots a_{-n})_{(b)} .$$

The decimal numeric value of this representation is calculated as follows:

N= $a'_0 * b^0 + a'_1 * b^1 + \dots + a'_m * b^m + a'_{-1} * b^{-1} + \dots + a'_{-n} * b^{-n}$, where:

$$a'_0 = (a_0)_{(b)}, \dots, a'_m = (a_m)_{(b)}, a'_{-1} = (a_{-1})_{(b)}, \dots, a'_{-n} = (a_{-n})_{(b)}$$

- the digits from the representation in base ***b*** are converted in decimal and then the operations *,+,/ are performed in base 10.
- if $b < 10$ the digits of the base b are the same in base 10.
- $a'_i * b^i$ is the positional value of the digit of rank (position) ***i*** in the representation
- the decimal numeric value of a number in a base is the sum of the positional values of all the digits from the representation in that base.

Examples:

2 3 4 3 = $3*10^0 + 4*10^1 + 3*10^2 + 2*10^3 = 3+40+300+2000$
│ │ │ └─ $3*10^0 = 3$ (positional value)
│ │ └── $4*10^1 = 40$ (positional value)
│ └─── $3*10^2 = 300$ (positional value)
└──── $2*10^3 = 2000$ (positional value)

$4\,3{,}1\,2_{(5)} = 4*5^1 + 3*5^0 + 1*5^{-1} + 2*5^{-2} = 20+3+0{,}2+0{,}08 = 23{,}28$
│ │ │ └─ $2*5^{-2} = 2/25=0{,}08$ (positional value)
│ │ └── $1*5^{-1} = 1/5=0{,}2$ (positional value)
│ └─── $3*5^0 = 3$ (positional value)
└──── $4*5^1 = 20$ (positional value)

$2\,0\,4\,7_{(8)} = 7*8^0 + 4*8^1 + 0*8^2 + 2*8^3 = 7+32+300+1024 = 1363$
│ │ │ └─ $7*8^0 = 7$ (positional value)
│ │ └── $4*8^1 = 32$ (positional value)
│ └─── $0*8^2 = 300$ (positional value)
└──── $2*8^3 = 2*512=1024$ (positional value)

$1\,0\,1\,1_{(2)} = 1*2^0 + 1*2^1 + 0*2^2 + 1*2^3 = 1+2+8 = 11$
│ │ │ └─ $1*2^0 = 1$ (positional value)
│ │ └── $1*2^1 = 2$ (positional value)
│ └─── $0*2^2 = 0$ (positional value)
└──── $1*2^3 = 8$ (positional value)

$2\,A\,F_{(16)} = F_{(16)}*16^0+A_{(16)}*16^1+2_{(16)}*16^2 = 15*1+10*16+2*256 = 15+160+512 = 687$
│ │ └── $F_{(16)}*16^0 = 15$ (positional value)
│ └─── $A_{(16)}*16^1 = 10*16=160$ (positional value)
└──── $2_{(16)}*16^2 = 2*256=512$ (positional value)

Theorem of numeration systems:

Given $b \in N^*, b > 1$ and the natural number N, N≠0, there exist ***n*** and the natural numbers $a_0, a_1, \ldots, a_{m-1}, a_m$, *with* $0 < a_m < b, 0 < a_i < b$ *for* $i = 0,\ldots,m-1$, unique determined, such that N= $a_0 * b^0 + a_1 * b^1 + \ldots + a_m * b^m$.

Remark: The decomposition of N in a sum of powers (with coefficients) of *b* provides a representation of N in base *b:* N= $(a_m a_{m-1} \ldots a_1 a_0)_{(b)}$

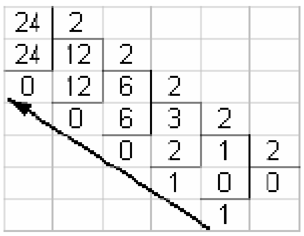
This theorem suggests a **conversion method** of a decimal number into an arbitrary base *b* applying **successive divisions** (performed in base 10) of N by the destination base *b*:

- N is divided by *b* obtaining a quotient and a remainder.
- the quotient is divided by *b* obtaining a new quotient an a new remainder,…

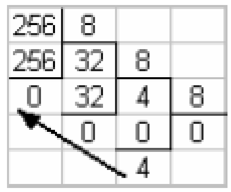
- the process of successive divisions ends when 0 is obtained as quotient.
- the remainders, in the reverse order they are obtained, are the digits of the representation of N in base *b*.

<u>Examples (F.Boian)</u>:

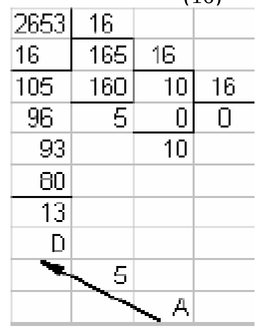
$24 = 11000_{(2)}$



$256 = 400_{(8)}$



$2653 = A5D_{(16)}$



For converting a rational number represented in an arbitrary base with integer part and fractional part we use for the integer part the method of **successive divisions** and for the fractional part we use a complementary method, applying **successive multiplications**:

- the fractional part is multiplied by ***b*** obtaining a number with an integer part and a fractional one;
- we continue with the multiplication of this new fractional part,...
- the process of the successive multiplications continues until one of the following conditions is satisfied:

  a) the fractional part becomes 0;

  b) an established number of digits of the fractional part were calculated;

  c) periodicity is obtained.

- the integer parts, in the order they are obtained during the multiplications process, are the digits of the fractional part in the destination representation.

<u>Example</u>: $543,17 = ?_{(5)}$, with 3 digits on the fractional part

- conversion of the integer part: successive divisions by 5 are applied.

  543: 5 = 108 remainder **3**
  108: 5 = 21 remainder **3**
  21: 5 = 4 remainder **1**
  4:5 = 0 remainder **4**

  $543 = 4133_{(5)}$

- conversion of the fractional part: successive multiplications by 5 are applied.

  0,17*5 = **0**,85
  0,85*5 = **4**,25
  0,25*5 = **1**,25
  0,25*5 = **1**,25 !!! periodicity.

  $0,17 = 0,04(1)_{(5)}$

$543,17 = 4133,04(1)_{(5)}$

base 10	base $2=2^1$	base $4=2^2$	base $8=2^3$	base $16=2^4$
0	0	0	0	0
1	1	1	1	1
2	10	2	2	2
3	11	3	3	3
4	100	10	4	4
5	101	11	5	5
6	110	12	6	6
7	111	13	7	7
8	1000	20	10	8
9	1001	21	11	9
10	1010	22	12	A
11	1011	23	13	B
12	1100	30	14	C
13	1101	31	15	D
14	1110	32	16	E
15	1111	33	17	F

Rapid conversions: conversions between bases which are powers of 2.

1. ***Conversion from the source base $p=2^k$, $p \in \{4=2^2, 8=2^3, 16=2^4\}$ into the destination base 2***

Rule:

Each digit from the source number in base $p=2^k$, the integer part and the fractional one, will be replaced by the corresponding group of k binary digits (adding if it is necessary insignificant zeros on the left) according to the above table .

Example:

$3201_{(4)}$= 11 10 00 01$_{(2)}$

$7205,346_{(8)}$= 111 010 000 101,011 100 110$_{(2)}$

$60FA,B28_{(16)}$= 0110 0000 1111 1010,1011 0010 1000$_{(2)}$

2. ***Conversion from the source base* 2 *into the destination base $q=2^k$***, $q \in \{4=2^2, 8=2^3, 16=2^4\}$

Rules:

- for the **integer part**: from right to left (relative to the decimal point) make groups of k binary digits (eventually we add on the left insignificant zeros to have a complete group); the groups will be replaced by the corresponding digits in base $q=2^k$ according to the above table.
- for the **fractional part**: from left to right (relative to the decimal point) make groups of k binary digits (eventually we add on the right insignificant zeros to have a complete group); the groups will be replaced by the corresponding digits in base $q=2^k$ according to the above table.

<u>**Examples:**</u>

1 001 100 011 010 111$_{(2)}$ = **00**1 001 100 011 010 111$_{(2)}$= $114327_{(8)}$

01 11 00 00 11 00 10 10, 01 10 10$_{(2)}$=$13003022,122_{(4)}$

0011 0001 0101 1001 1100 1010,1010 1000$_{(2)}$=$3159CA,A8_{(16)}$

Conversions between numeration bases for real numbers:

- **the substitution method**
- **the method of successive divisions/multiplications**
- **the method which uses an intermediate base**

1. Substitution method: calculation performed in the destination base

Let $N_{(b)} = (a_m a_{m-1} \dots a_1 a_0, a_{-1} \dots a_{-n})_{(b)}$ be a real number in the source base *b*.

We want to convert the number into the destination base *h*.

<u>*Steps:*</u>

- all the digits from the source representation are converted into the destination base: $(a_i)_{(b)} = (a'_i)_{(h)}, i = -n, \dots, -1, 0, \dots, m-1$
- the base *b* is converted into base *h*: $b = (b')_{(h)}$
- we calculate in base *h* the following sum:

$$(N')_{(h)} = (a'_0)_{(h)} * (b')_{(h)}{}^0 + (a'_1)_{(h)} * (b')_{(h)}{}^1 + \dots + (a'_m)_{(h)} * (b')_{(h)}{}^m +$$
$$+ (a'_{-1})_{(h)} * (b')_{(h)}{}^{-1} + \dots + (a'_{-n})_{(h)} * (b')_{(h)}{}^{-n}$$

Note: The method is recommended for ***b<h***, because:

$(a_i)_{(b)} = (a'_i)_{(h)}, i = -n, \dots, -1, 0, \dots, m-1$, $b = b_{(h)}$, and we have to perform only multiplications/divisions by one digit.

<u>**Example 1**</u>**:** $354_{(6)} = ?_{(8)}$

$354_{(6)} = 4*6^0 + 5*6^1 + 3*6^2$

$3_{(6)}=3_{(8)}$, $5_{(6)}=5_{(8)}$, $4_{(6)}=4_{(8)}$

$6 = 6_{(8)}$

Calculation in base 8:

$4_{(8)}*6_{(8)}{}^0 + 5_{(8)}*6_{(8)}{}^1 + 3_{(8)}*6_{(8)}{}^2 = 4_{(8)}+36_{(8)}+3_{(8)}*44_{(8)} = 42_{(8)}+154_{(8)} = 216_{(8)}$

$\mathbf{354_{(6)} = 216_{(8)}}$

<u>**Example 2**</u>**:** $167_{(8)} = ?_{(5)}$

$167_{(8)} = 1*8^2 + 6*8^1 + 6*8^0$

$1_{(8)}=1_{(5)}$, $6_{(8)}=11_{(5)}$, $7_{(8)}=12_{(5)}$

$8 = 13_{(5)}$

Calculation in base 5:

$1_{(5)}*13_{(5)}{}^2 + 11_{(5)}*13_{(5)}{}^1 + 12_{(5)}*13_{(5)}{}^0 = 224_{(5)}+143_{(5)}+12_{(5)} = 434_{(5)}$

$\mathbf{167_{(8)} = 434_{(5)}}$

<u>**Example 3**</u>: $11011{,}11_{(2)} = ?_{(4)}$

$\mathbf{11011{,}11_{(2)}} = 2^4+2^3+2^1+2^0+2^{-1}+2^{-2} = 2_{(4)}{}^4+2_{(4)}{}^3+2_{(4)}{}^1+2_{(4)}{}^0+2_{(4)}{}^{-1}+2_{(4)}{}^{-2} =$

$= 10_{(4)}{}^2+10_{(4)}*2_{(4)}+2_{(4)}+1_{(4)}+0{,}2_{(4)}+0{,}2_{(4)}/2_{(4)} =$

$= 100_{(4)}+20_{(4)}+2_{(4)}+1_{(4)}+0{,}2_{(4)}+0{,}1_{(4)} = \mathbf{123{,}3_{(4)}}$

2 The method of successive divisions/multiplications: calculation in the source base

b –source base and *h*- destination base.

- it is the generalization of the method of successive divisions/multiplications presented before for h=10.
- in the general case – calculation is in an arbitrary source base *b*.

Note: The method is recommended for $h<b$, because we need to apply only divisions/multiplications by one digit.

<u>Example 1</u>: A5B,ED$_{(16)}$= ?$_{(8)}$ with 3 digits on the fractional part

- conversion of the integer part:

```
 A5B(16)  | 8(16)
 /        | 14B  | 8(16)
 25         /    | 29  | 8(16)
  /        4B       /  | 5   | 8(16)
  5B        /       1   /   | 0
   /        3           5
   3
```

A5B$_{(16)}$= 5133$_{(8)}$

- conversion of the fractional part:

0,ED$_{(16)}$*8$_{(16)}$= 7,68$_{(16)}$

0,68$_{(16)}$*8$_{(16)}$= 3,40$_{(16)}$

0,4$_{(16)}$*8$_{(16)}$= 2,0$_{(16)}$,

0,ED$_{(16)}$= 0,732$_{(8)}$ - **t**he process of the successive multiplications is a finite one.

The final result: A5B,ED$_{(16)}$= 5133,732$_{(8)}$

<u>Example 2</u>: 123,3$_{(4)}$= ?$_{(2)}$

- conversion of the integer part:

```
 123 (4) | 2(4)
 /       | 31(4)  | 2(4)
  03       /      | 12(4) | 2(4)
  /       11        /     | 3(4)  | 2(4)
  1        /        0       /     | 1    | 2(4)
           1                1            | 0
```

123$_{(4)}$= 11011$_{(2)}$

- conversion of the fractional part:

0,3$_{(4)}$*2$_{(4)}$= 1,2$_{(4)}$

0,2$_{(4)}$*2$_{(4)}$= 1,0$_{(4)}$

0,3$_{(4)}$= 0,11$_{(2)}$ - **t**he process of successive multiplications is a finite one.

The final result: 123,3$_{(4)}$= 11011,11$_{(2)}$

3.The method which uses an intermediate base

$N_{(b)}= N'_{(g)}= N''_{(h)}$

b - the source base

g – the intermediate base

h - the destination base

a) **usually *g* = 10 – calculus in base 10**

- conversion from base *b* into base 10 – using the substitution method,
- conversion from base 10 into base *h* – using the method of successive divisions/multiplications

Example: $365,24_{(7)}= ?_{(3)}$ with 3 digits on the fractional part

b=7, *h*=3 and we choose *g* = 10

- we apply first the substitution method – calculation performed in decimal

$365,24_{(7)} =3*7^2+6*7^1+5*7^0+2*7^{-1}+4*7^{-2}$ = 194+0,2857+0,0816=194, 3673=194,367

- we apply the method of successive divisions/multiplications – calculation in base 10

194:3=64 remainder 2	0,367*3 = 1,10
64:3=21 remainder 1	0,10*3 = 0,30
21:3= 7 remainder 0	0.30*3 = 0.9
7:3= 2 remainder 1	
2:3= 0 remainder 2	

194,366 = $21012,100_{(3)}$

$365,24_{(7)}$ = 3194,366 = $21012,1_{(3)}$

b) **if *b* and *h* are powers of 2, then *g* = 2 and rapid conversions are applied**

Example: $10FA,BC_{(16)}= ?_{(8)}$

$b =16= 2^4$, $h = 8= 2^3$; we choose *g* = 2 and we apply rapid conversions

10FA,BC$_{(16)}$= 0001 0000 1111 1010,1011 1100$_{(2)}$ =

= 001 000 011 111 010,101 111$_{(2)}$ = **10372,57**$_{(8)}$

Example: $2301,123_{(4)}= ?_{(8)}$

$b = 4= 2^2$, $h = 8= 2^3$; we choose *g* = 2 and we apply rapid conversions

2301,123 $_{(4)}$= 10 11 00 01,01 10 11$_{(2)}$ =

= 010 110 001,011 011 $_{(2)}$ = **261,33**$_{(8)}$

c) **if *b* = 10 and *h* = 2 then for efficiency we use *g* = 8 or *g* = 16**

- conversion from base 10 into base *g* - the method of successive divisions/multiplications is applied
- conversion from base *g* into base 2 – rapid conversions are applied

Example: 2345 $=?_{(2)}$

$b = 10$, $h = 2$ and we choose $g = 8$

- conversion from decimal into base 8: the method of successive divisions is applied

2345:8= 293 remainder 1

293:8= 36 remainder 5

36:8= 4 remainder 4

4:8= 0 remainder 4

$2345 = 4451_{(8)}$

- conversion from base 8 into base 2: rapid conversions are applied

$4451_{(8)} = 100\ 100\ 101\ 001_{(2)}$

$2345 = 4451_{(8)} = 100\ 100101001_{(2)}$