Seminar Nr. 7, Inequalities; Central Limit Theorem; Markov Chains; Point Estimators

Theory Review

Markov's Inequality: $P(|X| \geq a) \leq \frac{1}{a} E(|X|), \forall a > 0.$

Chebyshev's Inequality: $P(|X - E(X)| \geq \varepsilon) \leq \frac{V(X)}{\varepsilon^2}, \forall \varepsilon > 0.$

Central Limit Theorem(CLT) Let $X_1, \dots, X_n$ be independent random variables with the same expectation $\mu = E(X_i)$ and same standard deviation $\sigma = \sigma(X_i)$ and let $S_n = \sum_{i=1}^{n} X_i$. Then, as $n \to \infty$,

$$Z_n = \frac{S_n - E(S_n)}{\sigma(S_n)} = \frac{S_n - n\mu}{\sigma\sqrt{n}} \longrightarrow Z \in N(0,1), \text{ in distribution (in cdf).}$$

Markov Chain with n states $\{X_0, X_1, \dots\}$ discrete random variables with pdf

$$X_i \begin{pmatrix} 1 & 2 & \dots & n \\ P_i(1) & P_i(2) & \dots & P_i(n) \end{pmatrix}, P_i = [P_i(1) \dots P_i(n)], i = 0, 1, \dots$$

- **transition probability matrix** $P = [p_{ij}]_{i,j=\overline{1,n}}$, where $p_{ij} = P(X_{t+1} = j \mid X_t = i)$
- **h-step transition probability matrix** $P^{(h)} = [p_{ij}^{(h)}]_{i,j=\overline{1,n}}$, where $p_{ij}^{(h)} = P(X_{t+h} = j \mid X_t = i)$

$P^{(h)} = P^h$ and $P_i = P_0 P^i$;

steady-state distribution $\pi_x = \lim_{h\to\infty} P_h(x),\ x = 1, \dots,$ found from the system $\pi P = \pi,\ \sum_{x=1}^{n} \pi_x = 1.$

Point Estimators

- method of moments: solve the system $\nu_k = \overline{\nu}_k$, for as many parameters as needed ($k = 1 \dots$ nr. of unknown parameters);
- method of maximum likelihood: solve $\frac{\partial \ln L(x_1, \dots, x_n; \theta)}{\partial \theta_j} = 0$, where $L(x_1, \dots, x_n; \theta) = \prod_{i=1}^{n} f(x_i; \theta)$ is the likelihood function;
- **standard error** of an estimator $\hat{\theta}$: $\sigma_{\hat{\theta}} = \sigma(\hat{\theta}) = \sqrt{V(\hat{\theta})}$;
- an estimator $\hat{\theta}$ is **absolutely correct** for the estimation of the parameter θ, if $E(\hat{\theta}) = \theta$ and $V(\hat{\theta}) \to 0$, as $n \to \infty$;
- **Fisher's information** $I_n(\theta) = -E\left[\frac{\partial^2 \ln L(X_1, \dots, X_n; \theta)}{\partial \theta^2}\right]$; if the range of X does not depend on θ, then $I_n(\theta) = n I_1(\theta)$;
- **efficiency** of an estimator $\hat{\theta}$: $e(\hat{\theta}) = \frac{1}{I_n(\theta) V(\hat{\theta})}$.

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1. (The 3σ Rule). For any random variable X, most of the values of X lie within 3 standard deviations away from the mean.

2. True or False: There is at least a 90% chance of the following happening: when flipping a coin 1000 times, the number of "heads" that appear is between 450 and 550.

3. Installation of some software package requires downloading 82 files. On the average, it takes 15 sec to download a file, with a variance of 16 sec^2. What is the probability that the software is installed in less than 20 minutes?

4. A computer system can operate in two different modes. Every hour, it remains in the same mode or switches to a different mode according to the transition probability matrix

$$P = \begin{bmatrix} 0.4 & 0.6 \\ 0.6 & 0.4 \end{bmatrix}.$$

a) If the system is in Mode I at 5:30 pm, what is the probability that it will be in Mode I at 8:30 pm on the same day?
b) In the long run, in which mode is the system more likely to operate?

5. A sample of 3 observations, $X_1 = 0.4, X_2 = 0.7, X_3 = 0.9$, is collected from a continuous distribution with pdf

$$f(x;\theta) = \begin{cases} \theta x^{\theta-1}, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases},$$

with $\theta > 0$, unknown. Estimate θ by the method of moments and by the method of maximum likelihood.

6. A sample $X_1, \dots, X_n$ is drawn from a distribution with pdf

$$f(x;\theta) = \frac{1}{2\theta} e^{-\dfrac{x}{2\theta}},\ x > 0$$

($\theta > 0$), which has mean $\mu = E(X) = 2\theta$ and variance $\sigma^2 = V(X) = 4\theta^2$. Find
a) the method of moments estimator, $\overline{\theta}$, for θ;
b) the efficiency of $\overline{\theta}$, $e(\overline{\theta})$;
c) an approximation for the standard error of the estimate in a), $\sigma_{\overline{\theta}}$, if the sum of 100 observations is 200.