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Exercise Set #1

Principle of Induction: Let $n_0 \in \mathbb{N}$ and let $P(n)$ be a property defined for any natural number $n \in \mathbb{N}$, $n \geq n_0$. Suppose that

i) $P(n_0)$ is true,

ii) $\forall k \geq n_0$, if $P(k)$ is true, then $P(k+1)$ is also true.

Then $P(n)$ is true, $\forall n \geq n_0$.

1. Prove that $\forall n \in \mathbb{N}$, $n \geq 4$ we have that $n! \geq 2^n$. Then show that $\forall n \in \mathbb{N}^*$, $n! \geq 2^{n-1}$.

2. Prove that $\forall n \in \mathbb{N}^*$ we have that $4 \sum_{m=1}^{n} m^3 = n^2(n+1)^2$.

3. Prove that $\forall n \in \mathbb{N}$, $n \geq 2$ we have that $\sum_{m=1}^{n} \frac{1}{\sqrt{m}} > \sqrt{n}$.

4. Prove that $\forall n \in \mathbb{N}^*$, $\exists m \in \mathbb{N}^*$ such that $m^2 \leq n < (m+1)^2$.

5. Prove that for any positive real numbers $a_1, a_2, \ldots, a_n > 0$ satisfying $a_1 \cdot a_2 \cdot \ldots \cdot a_n = 1$, we have that $a_1 + a_2 + \ldots + a_n \geq n$.

6. Let $a_1, a_2, \ldots, a_n > 0$. Prove that $H_n \leq G_n \leq A_n$, where

$$
\begin{aligned}
H_n &= \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n}} \quad \text{(the harmonic mean)},\\
G_n &= \sqrt[n]{a_1 \cdot a_2 \cdot \ldots \cdot a_n} \quad \text{(the geometric mean)},\\
A_n &= \frac{a_1 + a_2 + \ldots + a_n}{n} \quad \text{(the arithmetic mean)}.
\end{aligned}
$$

Hint: one solution consists in applying Exercise 5 for $\frac{a_i}{\sqrt[n]{a_1 \cdot a_2 \cdot \ldots \cdot a_n}}$ instead of a_i to get that $G_n \leq A_n$. This inequality considered for $\frac{1}{a_i}$ instead of a_i then yields $H_n \leq G_n$.
Remark: Equality holds in each of the above inequalities if and only if $a_1 = a_2 = \ldots = a_n$.

7. Using the inequality $G_n \leq A_n$, prove that $\forall x > 0$, $\forall n \in \mathbb{N}^*$,

i) $(1+x)^n \geq 1 + nx$.
Remark: Actually, the above inequality holds $\forall x \geq -1$. This inequality is known as Bernoulli's inequality.

ii) $\frac{x^n}{1 + x + \ldots + x^{2n}} \leq \frac{1}{2n+1}$.

8. Let $x, y \in \mathbb{R}$. Prove that

i) $|x+y| \leq |x| + |y|$ (the triangle inequality),

ii) $||x| - |y|| \leq |x - y|$.