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Analysis for Computer Science
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Exercise Set #7

1. Let $x, y \in \mathbb{R}^n$. Denote by $\alpha = \langle x, y\rangle$, $\beta = ||x||$ and $\gamma = ||y||$.
a) Using the properties of the scalar product and the definition of the Euclidean norm, determine, in terms of α, β and γ, the numbers $\langle x+y, y\rangle$, $\langle x, 2x-3y\rangle$ and $||x-y||$.
b) If $n = 3$, $x = (-1, 2, 3)$ and $y = (-2, 1, -3)$,
  i) compute α, β and γ,
  ii) find all reals $r > 0$ such that the open ball $B(x, r)$ does not contain the point y,
  iii) find all reals t such that the closed ball $\overline{B}(x, 5)$ contains the vector $(1, -1, t)$.

2. Show that if $x, y \in \mathbb{R}^n$, $x \neq y$, there exist $U \in \mathcal{V}(x)$ and $V \in \mathcal{V}(y)$ such that $U \cap V = \emptyset$.

3. Let $x, y \in \mathbb{R}^n$. Prove that:
a) $\|x+y\|^2 - \|x-y\|^2 = 4\langle x, y\rangle$,
b) $\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$ (the parallelogram identity).

4. Two vectors $x, y \in \mathbb{R}^n$ are said to be *orthogonal* if $\langle x, y\rangle = 0$. Which of the following pairs of vectors are orthogonal?
a) $(1, 1, -1)$ and $(1, -3, -2)$, b) $(1, 2, 3)$ and $(4, -3, 1)$, c) $(e, 3, 0)$ and $(-3, e, -2)$.

5. Let $x, y \in \mathbb{R}^n$. Prove that each of the following statements is equivalent to x and y being orthogonal:
a) $\|x+y\| = \|x-y\|$, b) $\|x+y\|^2 = \|x\|^2 + \|y\|^2$.

6. Let $x, y, z \in \mathbb{R}^n \setminus \{0_n\}$. True or false: If $\langle x, y\rangle = \langle x, z\rangle$, is $y = z$?

7. Given $A \subseteq \mathbb{R}^n$ nonempty, for any $x \in \mathbb{R}^n$, define $\operatorname{dist}(x, A) = \inf\{\|x-a\| \mid a \in A\}$. Prove that the function $f : \mathbb{R}^n \to \mathbb{R}$, $f(x) = \operatorname{dist}(x, A)$ is Lipschitz with Lipschitz constant 1.

8. Let $x \in \mathbb{R}^n$ be a unit vector (that is, $\|x\| = 1$). Find a sequence in the open ball $B(0_n, 1)$ which converges to x.

9. In each of the following cases, determine if the sequence $(x^k)_{k\in\mathbb{N}^*}$ in $\mathbb{R}^n$ is convergent or not. If the sequence is convergent, find also its limit:
a) $n = 2$, $x^k = \left(\dfrac{1}{k}, \dfrac{k^2+4k}{2k^2+1}\right)$, b) $n = 2$, $x^k = \left((-1/2)^k, (-1)^k\right)$,
c) $n = 2$, $x^k = \left(\sin k, \dfrac{1}{k^2}\right)$, d) $n = 2$, $x^k = \left(\left(\dfrac{\sqrt{k}}{1+\sqrt{k}}\right)^k, \dfrac{1^1+2^2+\ldots+k^k}{k^k}\right)$,
e) $n = 3$, $x^k = \left(e^{-k}\cos k, e^{-k}\sin k, k\right)$, f) $n = 3$, $x^k = \left(\dfrac{2^k}{k!}, \dfrac{1-4k^7}{k^7+12k}, \dfrac{\sqrt{k}}{e^{3k}}\right)$,
g) $n = 4$, $x^k = \left(\dfrac{2^{2k}}{\left(2+\frac{1}{k}\right)^{2k}}, \dfrac{1}{\sqrt[k]{k!}}, (e^k+k)^{\frac{1}{k}}, \dfrac{\alpha^k}{k}\right)$, where $\alpha \geq 0$ is fixed.

10. Find the interior, closure and boundary for the following subsets of $\mathbb{R}^2$ (determine also which sets are open and which are closed):
a) $A = [0, 1] \times [1, 2]$, b) $A = [0, 1) \times (1, 2]$, c) $A = \{(x, 0) \mid x < 0\} \cup \{(x, y) \mid y < 0\}$,
d) $A = \mathbb{Q} \times \mathbb{Q}$, e) $A = \{0_2\}$, f) $A = \mathbb{R} \times \{0\}$, g) $A = \mathbb{R}^2$, h) $A = \emptyset$.