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Analysis for Computer Science
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Exercise Set #8

1. Prove that the function $f\colon \mathbb{R}^2 \setminus \{0_2\} \to \mathbb{R}$, $f(x,y) = \dfrac{xy}{x^2+y^2}$ has no limit at 0_2.

2. Prove that the function $f\colon \mathbb{R}^2 \setminus \{0_2\} \to \mathbb{R}$, $f(x,y) = \dfrac{x^3+y^3}{x^2+y^2}$ has a limit at 0_2 and find this limit.

3. Find all first and second order partial derivatives of the following functions:
a) $f : \mathbb{R}^3 \to \mathbb{R}$, $f(x,y,z) = (1+x^2)ye^z$, b) $f : \mathbb{R}^* \times \mathbb{R}^2 \to \mathbb{R}$, $f(x,y,z) = \dfrac{z^2e^y}{x}$,
c) $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = \sin(xy)$, d) $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = \sin(x \sin y)$,
e) $f : (0,+\infty) \times (0,+\infty) \to \mathbb{R}$, $f(x,y) = x^y$.

4. Let $r > 0$, $A = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 < r^2\}$ and $f : A \to \mathbb{R}$, $f(x,y) = 2\ln \dfrac{r\sqrt{8}}{r^2-x^2-y^2}$. Prove that

$$\forall (x,y) \in A, \quad \frac{\partial^2 f}{\partial x^2}(x,y) + \frac{\partial^2 f}{\partial y^2}(x,y) = e^{f(x,y)}.$$

5. Find the gradient of the following functions at the indicated point:
a) $f : \mathbb{R}^3 \to \mathbb{R}$, $f(x,y,z) = e^{xyz}$ at $(1,2,3)$, b) $f : \mathbb{R}^3 \to \mathbb{R}$, $f(x,y,z) = e^{-x}\sin(yz)$ at $(1,\pi,1)$,

c) $f : \mathbb{R}^3 \to \mathbb{R}$, $f(x,y,z) = e^{2x+y}\cos(3z)$ at $(0,0,\pi/6)$.

6. Let $f, g : \mathbb{R}^n \to \mathbb{R}$ be two partially differentiable functions. Prove that

$$\forall c \in \mathbb{R}^n, \quad \nabla(fg)(c) = f(c)\nabla g(c) + g(c)\nabla f(c).$$

Determine $\nabla(fg)(0,\pi,1)$ for $f, g : \mathbb{R}^3 \to \mathbb{R}$, $f(x,y,z) = xz + \cos y$, $g(x,y,z) = x^2y^3 + y\sin x - 2z$.

7. For the following functions $f : \mathbb{R}^2 \to \mathbb{R}$, study the continuity and the partial differentiablilty at 0_2:

a) $f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}}, & \text{if } (x,y) \neq 0_2 \\ 0, & \text{if } (x,y) = 0_2. \end{cases}$ b) $f(x,y) = \begin{cases} \dfrac{x^4-y^4}{2(x^4+y^4)}, & \text{if } (x,y) \neq 0_2 \\ 0, & \text{if } (x,y) = 0_2. \end{cases}$

c) $f(x,y) = \begin{cases} \dfrac{(xy)^2}{(xy)^2+(x-y)^4}, & \text{if } (x,y) \neq 0_2 \\ 0, & \text{if } (x,y) = 0_2. \end{cases}$