## *Exercises - semantic tableaux method*

**Exercise 1.1.**

Using the semantic tableaux method decide what type (consistent, inconsistent, valid) of a formula is $U_j, j \in \{1,2,\ldots,8\}$. If $U_j, j \in \{1,2,\ldots,8\}$ is consistent, find all its models.

1. $U_1 = (p \wedge q) \vee (\neg p \wedge \neg r) \to (q \leftrightarrow r)$;
2. $U_2 = (p \vee q \to r) \to (p \vee r \to q)$;
3. $U_3 = (p \wedge q \to r) \to (p \to r) \wedge q$;
4. $U_4 = (q \vee r \to p) \to (p \to r) \wedge q$;
5. $U_5 = (r \vee q) \vee (p \to \neg r) \to (p \leftrightarrow q)$;
6. $U_6 = (r \wedge q) \vee (\neg p \vee \neg r) \to (p \leftrightarrow q)$;
7. $U_7 = (q \wedge r \to p) \to (p \to r) \wedge q$;
8. $U_8 = (p \wedge r) \vee (\neg p \wedge \neg r) \to (q \leftrightarrow r)$.

**Exercise 1.2.**

Prove that the following formulas are tautologies using the semantic tableaux method:

1. permutation of the premises law: $(p \to (q \to r)) \to (q \to (p \to r))$;
2. separation of the premises law: $(p \wedge q \to r) \to (p \to (q \to r))$;
3. distribution of ' $\to$ ' over ' $\vee$ ': $(p \to q \vee r) \leftrightarrow (p \to q) \vee (p \to r)$;
4. distribution of ' $\vee$ ' over ' $\leftrightarrow$ ': $(p \vee (q \leftrightarrow r)) \leftrightarrow ((p \vee q) \leftrightarrow (p \vee r))$;
5. reunion of the premises law: $(p \to (q \to r)) \to (p \wedge q \to r)$;
6. distribution of implication: $(p \to (q \to r)) \leftrightarrow (p \to q) \to (p \to r)$;
7. distribution of ' $\to$ ' over ' $\wedge$ ': $(p \to q \wedge r) \leftrightarrow (p \to q) \wedge (p \to r)$;
8. distribution of ' $\to$ ' over ' $\leftrightarrow$ ': $(p \to (q \leftrightarrow r)) \leftrightarrow ((p \to q) \leftrightarrow (p \to r))$.

**Exercise 1.3.**

Using the semantic tableaux method, decide if the following logical consequences hold or not. If a logical consequence does not hold find an anti-model of it.

1. $p \to (\neg q \vee r \wedge s),\ p, \neg s \models \neg q$
2. $\neg p \to (\neg q \to r), r \vee q \models (\neg p \to q) \vee r$
3. $p \to (q \vee r \wedge s),\ p, \neg r \models q$
4. $p \to q, r \to t,\ p \wedge r \models q \wedge t$
5. $p \wedge (q \to r), q \vee r \models p \to (q \to r)$
6. $p \to q \models (r \to t) \to (p \wedge r \to q \wedge t)$
7. $p \wedge (q \to r), q \vee r \models p \to (q \to r)$
8. $p \to q \vee r \models (p \to q) \vee (p \to r)$

**Exercise 1.4.**

Write all the anti-models of the propositional formulas $U_1,\ldots,U_8$ using the semantic tableaux method.

1. $U_1 = (p \vee q) \wedge \neg r \to p \wedge q \wedge r$ ;
2. $U_2 = q \wedge \neg p \wedge r \to \neg p \vee \neg(q \wedge r)$ ;
3. $U_3 = p \to (q \wedge r) \vee q \wedge \neg p$ ;
4. $U_4 = \neg p \vee (\neg q \vee r) \to q \vee \neg p \vee r$ ;
5. $U_5 = \neg p \vee (\neg q \vee \neg r) \to q \wedge \neg p$ ;
6. $U_6 = \neg p \vee (\neg q \wedge \neg r) \to q \wedge \neg p \wedge r$ ;
7. $U_7 = \neg p \vee \neg(q \wedge r) \to q \wedge \neg p$ ;
8. $U_8 = \neg(\neg p \vee q) \vee r \to \neg p \vee (\neg q \vee r)$ .

**Exercise 1.5.**

Using the semantic tableaux method, prove the following properties in predicate logic:

1. ' $\exists$ ' is semi-distributive over ' $\wedge$ ':
   $\models (\exists x)(A(x) \wedge B(x)) \to (\exists x)A(x) \wedge (\exists x)B(x)$ and
   $\not\models (\exists x)A(x) \wedge (\exists x)B(x) \to (\exists x)(A(x) \wedge B(x))$
2. ' $\forall$ ' is semi-distributive over ' $\vee$ ':
   $\models (\forall x)A(x) \vee (\forall x)B(x) \to (\forall x)(A(x) \vee B(x))$ and
   $\not\models (\forall x)(A(x) \vee B(x)) \to (\forall x)A(x) \vee (\forall x)B(x)$
3. ' $\exists$ ' is semi-distributive over ' $\to$ ':
   $\models ((\exists x)A(x) \to (\exists x)B(x)) \to (\exists x)(A(x) \to B(x))$ and
   $\not\models (\exists x)(A(x) \to B(x)) \to ((\exists x)A(x) \to (\exists x)B(x))$
4. ' $\forall$ ' is semi-distributive over ' $\to$ ' :
   $\models (\forall x)(A(x) \to B(x)) \to ((\forall x)A(x) \to (\forall x)B(x))$ and
   $\not\models ((\forall x)A(x) \to (\forall x)B(x)) \to (\forall x)(A(x) \to B(x))$
5. $\models (\exists x)(A(x) \to B(x)) \to ((\forall x)A(x) \to (\exists x)B(x))$ and
   $\not\models ((\exists x)A(x) \to (\exists x)B(x)) \to (\forall x)(A(x) \to B(x))$
6. ' $\exists$ ' is distributive over ' $\vee$ '
   $\models (\exists x)(A(x) \vee B(x)) \leftrightarrow (\exists x)A(x) \vee (\exists x)B(x)$
7. ' $\forall$ ' is distributive over ' $\wedge$ '
   $\models (\forall x)(A(x) \wedge B(x)) \leftrightarrow (\forall x)A(x) \wedge (\forall x)B(x)$
8. $\vdash (\exists x)(P(x) \to Q(x)) \leftrightarrow ((\forall x)P(x) \to (\exists x)Q(x))$

**Exercise 1.6.**

Check the validity of the following first-order formulas using the semantic tableaux method:

1. $U_1 = (\forall x)(\forall y)P(x,y) \leftrightarrow (\exists x)(\forall y)P(x,y)$ ;
2. $U_2 = (\exists x)(\forall y)P(x,y) \leftrightarrow (\forall y)(\exists x)P(x,y)$ ;
3. $U_3 = (\forall y)(\exists x)P(x,y) \leftrightarrow (\exists y)(\exists x)P(x,y)$ ;
4. $U_4 = (\forall x)(\forall y)P(x,y) \leftrightarrow (\forall y)(\forall x)P(x,y)$ ;
5. $U_5 = (\forall y)(\forall x)P(x,y) \leftrightarrow (\forall x)(\exists y)P(x,y)$ ;
6. $U_6 = (\exists y)(\exists x)P(x,y) \leftrightarrow (\exists x)(\forall y)P(x,y)$ ;
7. $U_7 = (\exists y)(\exists x)P(x,y) \leftrightarrow (\forall x)(\exists y)P(x,y)$ ;
8. $U_8 = (\exists y)(\exists x)P(x,y) \leftrightarrow (\exists x)(\exists y)P(x,y)$ .

**Exercise 1.7.**

Using the semantic tableaux method check whether the following logical consequences hold.

1. $(\forall x)(P(x) \to Q(x)), (\forall x)P(x) \models (\forall x)Q(x)$ ;
2. $(\forall x)(\forall y)(Q(x,y) \to P(x,y)), (\forall z)Q(z,z) \models (\forall x)P(x,x)$ ;
3. $P(a), (\forall x)(P(x) \to P(f(x))) \models (\forall x)P(x)$ ;
4. $(\exists x)(\forall y)(P(x,y) \to R(x)), (\forall x)(\forall y)P(x,y) \models (\exists z)R(z)$ ;
5. $(\forall x)(P(x) \to Q(x)), (\exists x)P(x) \models (\exists x)Q(x)$ ;
6. $(\forall x)(\exists y)(P(x,y) \to Q(x,y)), (\exists z)P(z,z) \models (\forall x)Q(x,x)$ ;
7. $(\exists x)(\forall y)(Q(x,y) \to P(x,y)), (\forall z)Q(z,z) \models (\exists x)P(x,x)$ ;
8. $(\forall x)(\forall y)(P(x,y) \to P(y,x)) \models (\forall x)P(x,x)$ .