Assignment #10

Due Friday, 2 December 2016, at the start of class

Please read Chapters 13, 14, and 15 in the textbook, Sutherland, *Introduction to Metric and Topological Spaces*.

Do the following Exercises and Problems.

Chapter 13, pages 136–137, Exercises:

13.4
13.5
13.10
13.15

Chapter 14, pages 147–149, Exercises:

14.1
14.2

Chapter 15, pages 171–172, Exercises:

15.5

Problem P4. The title of this problem is "the one-point compactification of $\mathbb{R}$."

Consider the set formed by adding one new point to $\mathbb{R}$, with the label "∞":

$$X = \mathbb{R} \cup \{\infty\}.$$

(a) Show that if $\mathcal{T}_\mathbb{R}$ gives the usual topology for $\mathbb{R}$ then

$$\mathcal{T} = \mathcal{T}_\mathbb{R} \cup \Big\{ U \cup \{\infty\} \,:\, U \in \mathcal{T}_\mathbb{R} \text{ such that } (-\infty,-a) \cup (a,\infty) \subseteq U \text{ for some } a \geq 0 \Big\}$$

is a topology for X.

(b) Show that

$$\mathcal{B} = \Big\{ (a,b) \,:\, a < b \Big\} \cup \Big\{ (-\infty,-a) \cup (a,\infty) \cup \{\infty\} \,:\, a \geq 0 \Big\}$$

is a basis for the topology $\mathcal{T}$. (*You may use the result of Exercise 8.5.*)

(c) Let S^1 be the usual unit circle in the plane, namely $S^1 = \{(x,y) \,:\, x^2 + y^2 = 1\}$, with the usual topology. (*I.e. S^1 is a subspace of $\mathbb{R}^2$ with the usual topology.*) Show S^1 is compact. (*You may use the Heine-Borel Theorem, but give at least some explanation for why the hypotheses hold.*)

(d) Let

$$\begin{aligned} f : X &\rightarrow S^1 \\ t &\mapsto \begin{cases} \left(\frac{2t}{t^2+1}, \frac{t^2-1}{t^2+1}\right) & \text{if } t \in \mathbb{R} \\ (0,1) & \text{if } t = \infty \end{cases} \end{aligned}$$

Show f is well-defined (i.e. show $f(t) \in S^1$ for all $t \in X$) and that f is a bijection. Sketch this map; its inverse is called *stereographic projection*.

(e) Show f^{-1} is continuous. (*You may use Proposition 8.12.*)

(f) Show f^{-1} is a homeomorphism. (*You may use Proposition 13.26.*)

(g) Show X is compact. (*This is really easy.*)