Selected Solutions to Assignment #6

Exercise 2 (page 78 of B&C). Suppose v and V are both harmonic conjugates of u. Then $v = V + const$.

Proof. Using Cauchy-Riemann equation $u_x = v_y = V_y$ we get that

$$v(x, y) = V(x, y) + c(x) + c_1.$$

Using the second Cauchy-Riemann equation $u_y = -v_x = -V_x$ we get that

$$v(x, y) = V(x, y) + d(y) + d_1.$$

Thus $c(x) = 0$, $d(y) = 0$, $c_1 = d_1$ and $v(x, y) = V(x, y) + C$ $\square$

Exercise 5 (page 78 of B&C). Let $f(z) = u(r, \theta) + iv(r, \theta)$ be analytic in a domain D that does not contain the origin. Show that in D the function $u(r, \theta)$ satisfies the partial differential equation

$$r^2 u_{rr}(r, \theta) + r u_r(r, \theta) + u_{\theta\theta}(r, \theta) = 0$$

Proof. Differentiating the Cauchy-Riemann equation $r u_r = v_\theta$ (in polar coordinates) with respect to r we get

$$u_r + r u_{rr} = v_{\theta r} \tag{1}$$

Differentiating the second Cauchy-Riemann equation in the polar coordinates $u_\theta = -r v_r$ with respect to θ we get

$$u_{\theta\theta} = -r v_{\theta r} \tag{2}$$

Equating expressions for $v_{\theta r}$ from (1), (2) we obtain

$$u_r + r u_{rr} = -\frac{u_{\theta\theta}}{r},$$

as desired.

Note that v also satisfies the same equation, which is to say it is also harmonic. $\square$

Exercise 3 (page 89 of B&C). Show that the function $f(z) = e^{\bar{z}}$ is not analytic anywhere.

Proof. Rewrite $f(z) = e^{\bar{z}} = e^x \cos y - ie^x \sin y$. Therefore $u(x, y) = e^x \cos y$ and $v(x, y) = -e^x \sin y$. The Cauchy-Riemann equation $u_x = v_y$ is $e^x \cos y = -e^x \cos y$. This is true only when $\cos y = 0$, or $y = \frac{\pi}{2} + k\pi$, $k \in \mathbb{Z}$. On the other hand, $u_y = -v_x$ says $-e^x \sin y = e^x \sin y$. This equations holds if and only if $\sin y = 0$, or $y = k\pi$, $k \in \mathbb{Z}$. The coordinate y cannot have both these properties.

Therefore the Cauchy-Riemann equations are satisfied nowhere. By the theorem on page 62, or more precisely by its contrapositive, the function is differentiable nowhere. It is analytic in no open set. $\square$

Exercise 10 (page 89 of B&C). (a) Show that if e^z is real, then $\operatorname{Im} z = n\pi$, $n \in \mathbb{Z}$

Proof. The representation
$$e^z = e^x \cos y + ie^x \sin y,$$
implies that e^z is real if and only if $\sin y = 0$, so $\operatorname{Im} z = y = n\pi$, $n \in \mathbb{Z}$. □

(b) Identify restrictions on z such that e^z is pure imaginary.

Proof. The representation
$$e^z = e^x \cos y + ie^x \sin y,$$
implies that e^z is pure imaginary if and only if $\cos y = 0$. This holds if and only if $\operatorname{Im} z = y = \frac{\pi}{2} + n\pi$, $n \in \mathbb{Z}$. □

Exercise 3 (page 94 of B&C). (a) Show that $\operatorname{Log}(1+i)^2 = 2\operatorname{Log}(1+i)$

Proof. We rewrite in polar coordinates:
$$1 + i = \sqrt{2}e^{i\frac{\pi}{4}}, \quad (1+i)^2 = 2e^{i\frac{\pi}{2}}$$
By the definition of Log:
$$\operatorname{Log}(1+i)^2 = \operatorname{Log}(2e^{i\frac{\pi}{2}}) = \ln 2 + i\frac{\pi}{2},$$
$$2\operatorname{Log}(1+i) = 2\operatorname{Log}(\sqrt{2}e^{i\frac{\pi}{4}}) = 2\ln\sqrt{2} + 2i\frac{\pi}{4} = \ln 2 + i\frac{\pi}{2}.$$
□

(b) Show that $\operatorname{Log}(-1+i)^2 \neq 2\operatorname{Log}(-1+i)$

Proof. Polar coordinates:
$$-1 + i = \sqrt{2}e^{i\frac{3\pi}{4}}, \quad (-1+i)^2 = -2i = 2e^{-i\frac{\pi}{2}}$$
By the definition of Log:
$$\operatorname{Log}(-1+i)^2 = \operatorname{Log}(2e^{-i\frac{\pi}{2}}) = \ln 2 - i\frac{\pi}{2},$$
$$2\operatorname{Log}(-1+i) = 2\operatorname{Log}(\sqrt{2}e^{i\frac{3\pi}{4}}) = 2\ln\sqrt{2} + 2i\frac{3\pi}{4} = \ln 2 + i\frac{3\pi}{2}.$$
□

Exercise 8 (page 94 of B&C). Suppose that the point $z = x + iy$ lies in the strip $\alpha < y < \alpha + 2\pi$. Show that when the branch $\log z = \ln r + i\theta$, $r > 0$, $\alpha < \theta < \alpha + 2\pi$ is used, $\log(e^z) = z$.

Proof. For the point z in the strip we have
$$\log e^z = \log e^{x+iy} = \ln e^x + i(\operatorname{Arg} e^{iy} + n\pi)$$
where n is such that $\alpha < \operatorname{Arg} e^{iy} + n\pi < \alpha + 2\pi$. But if it is so, $\operatorname{Arg} e^{iy} + n\pi = y$, and
$$\log e^z = x + iy = z.$$
□