Selected Solutions to Assignment #4

Exercise 3c (page 53 of B&C). Let $P(z)$, $Q(z)$ be polynomials and $Q(z_0) \neq 0$. Show that

$$\lim_{z \to z_0} \frac{P(z)}{Q(z)} = \frac{P(z_0)}{Q(z_0)}.$$

Proof. Since P, Q are polynomials they are continuous functions; see equation (11) on page 48. We know that $\lim_{z \to z_0} P(z) = P(z_0)$ and $\lim_{z \to z_0} Q(z) = Q(z_0) \neq 0$. Thus we can use Theorem 2, Sec. 15, which proves the statement. $\square$

Exercise 4 (page 53 of B&C). Use mathematical induction and properties of limits to show that

$$\lim_{z \to z_0} z^n = z_0^n.$$

Proof. For $n = 1$ we have that $\lim_{z \to z_0} z = z_0$. Suppose that we proved the statement for some integer N, $N > 1$. We need to show that $\lim_{z \to z_0} z^{N+1} = z_0^{N+1}$.

Notice that $z^{N+1} = z^N z$. The induction hypothesis is that $\lim_{z \to z_0} z^N = z_0^N$. Now we can apply property (9) in Sec 15 and deduce

$$\lim_{z \to z_0} z^{N+1} = \lim_{z \to z_0} z^N z = z_0^N z_0 = z_0^{N+1}.$$

. $\square$

Exercise 9 (page 54 of B&C). Let $\lim_{z \to z_0} f(z) = 0$. Suppose there exists $M > 0$ such that $|g(z)| \leqslant M$ in some neighborhood of z_0. Then

$$\lim_{z \to z_0} f(z)g(z) = 0.$$

Proof. Let $\varepsilon > 0$. Since $\lim_{z \to z_0} f(z) = 0$, we can find such a $\delta > 0$ so that $|f(z)| \leqslant \varepsilon/M$ for all z so that $|z - z_0| < \delta$. Make δ smaller, if needed, so that we also have $|g(z)| \leqslant M$ for all z satisfying $|z - z_0| < \delta$.

Let $F(z) = f(z)g(z)$. If z satisfies $|z - z_0| < \delta$ tsatisfieshen

$$|F(z) - 0| = |F(z)| \leqslant |f(z)||g(z)| \leqslant |f(z)|M \leqslant \frac{\varepsilon}{M} M = \varepsilon$$

By definition, $\lim_{z \to z_0} f(z)g(z) = 0$. $\square$

Exercise 2 (page 59 of B&C). Show that a polynomial

$$P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$$

is differentiable everywhere and find its derivative and coefficients.

Proof. Using formulas (1), (2), and (3) in Sec 19 we get that

$$P'(z) = a_1 + 2a_2 z + 3a_3 z^2 + \cdots + n a_n z^{n-1} \tag{1}$$

Plugging $z = 0$ in the last formula gives

$$a_1 = P'(0).$$

Differentiating (1) we obtain

$$P''(z) = 2\,a_2 + 3(2)\,a_3 z + 4(3)\,a_4 z^2 + \cdots + n(n-1)\,a_n z^{n-2} \tag{2}$$

Plugging here $z = 0$ we have $a_2 = P''(0)/2$. Repeating this procedure, we get

$$a_n = \frac{P^{(n)}(0)}{n!}.$$

(A proof by induction is tedious but possible ...) □

Exercise 3 (page 60 of B&C). Apply the definition to get the derivative of $f(z) = 1/z$.

Proof. Using the definition we get:

$$f'(z) = \lim_{h\to 0} \frac{f(z+h) - f(z)}{h} = \lim_{h\to 0} \frac{\frac{1}{z+h} - \frac{1}{z}}{h} = \lim_{h\to 0} \frac{z - z - h}{(z+h)zh} = \lim_{h\to 0} \frac{-1}{(z+h)z}$$

Since $z \neq 0$, we can use Theorem 2, Sec 15 and get that

$$f'(z) = \lim_{h\to 0} \frac{-1}{(z+h)z} = -\frac{1}{z^2}.$$

□

Exercise 8 (page 60 of B&C). a) $f(z) = \bar{z}$. We have that

$$\lim_{h\to 0} \frac{f(z+h) - f(z)}{h} = \lim_{h\to 0} \frac{\bar{z} + \bar{h} - \bar{z}}{h} = \lim_{h\to 0} \frac{\bar{h}}{h}.$$

When h converges to zero along the real axis ($h = x$), we have that $\lim_{h\to 0} \frac{\bar{h}}{h} = \lim_{h\to 0} \frac{h}{h} = 1$. When h converges to zero along the imaginary axis ($h = iy$), then $\lim_{h\to 0} \frac{\bar{h}}{h} = \lim_{h\to 0} \frac{-h}{h} = -1$. Thus derivative of f does not exist at any point in $\mathbb{C}$.

b) $f(z) = \operatorname{Re} z$. We have that

$$\lim_{h\to 0} \frac{f(z+h) - f(z)}{h} = \lim_{h\to 0} \frac{\operatorname{Re} z + \operatorname{Re} h - \operatorname{Re} z}{h} = \lim_{h\to 0} \frac{\operatorname{Re} h}{h}.$$

When h converges to zero along the real axis ($h = x$), $\lim_{h\to 0} \frac{\operatorname{Re} h}{h} = \lim_{h\to 0} \frac{h}{h} = 1$. When h converges to zero along the imaginary axis ($h = iy$), then $\lim_{h\to 0} \frac{\operatorname{Re} h}{h} = 0$. Thus derivative of f does not exist at any point in $\mathbb{C}$.

b) $f(z) = \operatorname{Im} z$. We have that

$$\lim_{h\to 0} \frac{f(z+h) - f(z)}{h} = \lim_{h\to 0} \frac{\operatorname{Im} z + \operatorname{Im} h - \operatorname{Im} z}{h} = \lim_{h\to 0} \frac{\operatorname{Im} h}{h}.$$

When h converges to zero along the real axis ($h = x$), $\lim_{h\to 0} \frac{\operatorname{Im} h}{h} = 0$. When h converges to zero along the imaginary axis ($h = iy$), then $\lim_{h\to 0} \frac{\operatorname{Im} h}{h} = -i$. Thus derivative of f does not exist at any point in $\mathbb{C}$.

Exercise C4. Identify a single limit which, once proven, will allow us to show that

$$f(z) = e^z \quad \Rightarrow \quad f'(z) = e^z.$$

What algebraic fact about e^z do we use?

Proof. Using the definition of derivative we get

$$f'(z) = \lim_{h\to 0} \frac{f(z+h) - f(z)}{h} = \lim_{h\to 0} \frac{e^{z+h} - e^z}{h} = e^z \lim_{h\to 0} \frac{e^h - 1}{h}$$

Thus, the limit we need is

$$\lim_{h\to 0} \frac{e^h - 1}{h} = 1$$

and the algebraic fact we used is $e^{a+b} = e^a e^b$, for $a, b \in \mathbb{C}$. □

Exercise C5. Identify limits which, once proven, will allow us to show that

$$f(z) = \sin z \quad \Rightarrow \quad f'(z) = \cos z.$$

What trigonometric fact do we use?

Proof. Using the definition of derivative we get

$$\begin{aligned} f'(z) = \lim_{h\to 0} \frac{f(z+h) - f(z)}{h} &= \lim_{h\to 0} \frac{\sin(z+h) - \sin z}{h} \\ &= \lim_{h\to 0} \frac{\sin z \cos h + \sin h \cos z - \sin z}{h} = \sin z \lim_{h\to 0} \frac{\cos h - 1}{h} + \cos z \lim_{h\to 0} \frac{\sin h}{h} \end{aligned}$$

Thus, two limits we need are

$$\lim_{h\to 0} \frac{\cos h - 1}{h} = 0 \qquad \text{and} \qquad \lim_{h\to 0} \frac{\sin h}{h} = 1.$$

The trigonometric fact we used is $\sin(a+b) = \sin a \cos b + \sin b \cos a$, for $a, b \in \mathbb{C}$. □