Stability and Delay

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Outline

This is an applied math talk.

(I.e. abstract a genuine, meaningful problem till you can't remember the original problem. Then state theorems.)

1. Turning model and stability chart
2. DDEs = (difference eqns in ∞ dimensions)
3. Compact operators = (operators you can approximate)
4. Chebyshev collocation = (perfect method for the right problem)
5. A Monte Carlo approach
6. When is a matrix "stable," anyway?

Turning model, a DDE

$x(t)$ is deviation of cutting tool from intended position and $x_{-\tau} = x(t-\tau)$.

Equation of motion ("turning eqn"):

$$m\ddot{x} + c\dot{x} + kx = wK_0(x_{-\tau} - x)$$

m, c, k are mass, damping, and spring constant of *tool*

Cutting process described by a material parameter K_0 and w = (cutting depth) = (nominal chip thickness) = (cutting rate times rotation time).

One rotation occurs in time τ.

Mass, damper, spring

Recall mass-spring system with imposed force F:

$$m\ddot{x} + c\dot{x} + kx = F$$

One can solve it if one solves $m\ddot{z} + c\dot{z} + kz = 0$
(i.e. homogeneous version) and if one can integrate
(i.e. variation-of-parameters)

But our force F depends on position in the *past*.

Stability of turning

Goal is *stability* of turning process, i.e. avoidance of vibrations of tool and workpiece.

Problems with vibrations:

- (potentially) damage to the tool or workpiece
- rough surface

"Stability" means (precisely) that solution $x(t)$ dies away over time. (I.e. asymptotic stability of the zero solution, for the linear problems of interest.)

Practical Goal: A stability chart

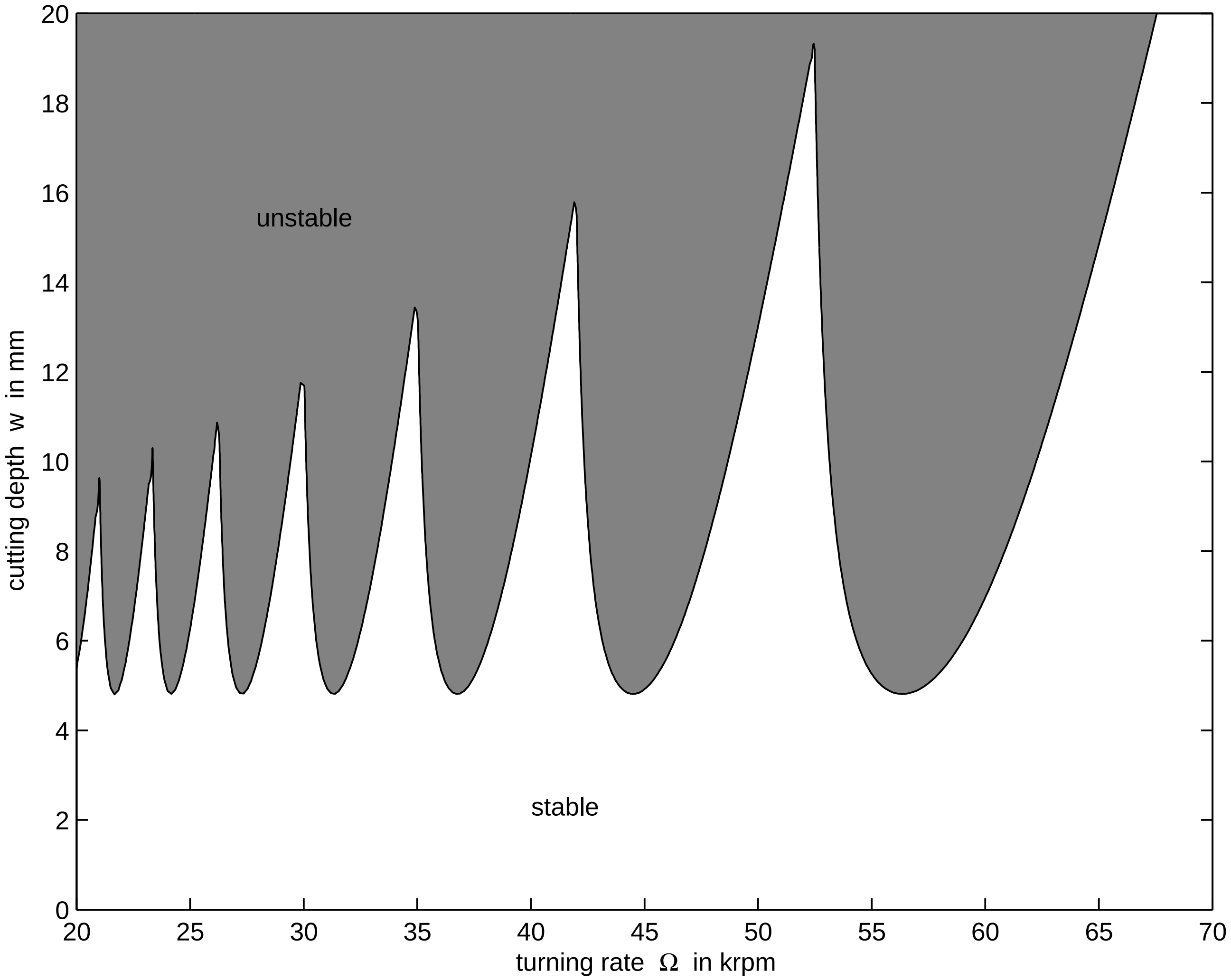
Suppose m, c, k, K_0 are all predetermined.

How can we choose the cutting depth w and the delay τ to get stability? (One rotation occurs in time τ seconds so rotation rate is $\Omega = \frac{60}{\tau}$ RPM. Machinist uses Ω, not τ.)

For industrial use, we want to cut away (metal) from the workpiece as fast as possible for the given tool and material.

SO: How to choose w and Ω large and still have stability?

Example stability chart



Standard form linear DDEs

First order and *vector* form:

$$\dot{\mathbf{y}} = A(t)\,\mathbf{y} + B(t)\,\mathbf{y}_{-\tau}$$

Example: If $\mathbf{y} = \begin{pmatrix} x \\ v \end{pmatrix}$ with $v = \dot{x}$ then *turning eqn* becomes

$$\dot{\mathbf{y}} = \begin{pmatrix} 0 & 1 \\ -\frac{k+wK_0}{m} & -\frac{c}{m} \end{pmatrix} \mathbf{y} + \begin{pmatrix} 0 & 0 \\ \frac{wK_0}{m} & 0 \end{pmatrix} \mathbf{y}_{-\tau}$$

In *milling* examples $A(t), B(t)$ depend on t.

Diff Eq Review

Nonhomogeneous problem

$$(*) \qquad \dot{y} = a(t)y + u(t), \quad y(0) = y_0.$$

Solution method: Find fundamental solution and integrate. ("Variation-of-parameters.")

Note $z(t) = e^{\int_0^t a(s)\,ds}$ solves $\dot{z} = a(t)z$, $z(0) = 1$.
$z(t)$ is "fundamental solution."

Soln to $(*)$ is

$$y(t) = z(t)\left[y_0 + \int_0^t z(s)^{-1}u(s)\,dx\right]$$

Difficulty for non-numerical calculation

For a *system*, i.e. $\mathbf{z} \in \mathbb{R}^n$ and $n \geq 2$, solution of

$$\dot{\mathbf{z}} = A(t)\mathbf{z}, \qquad \mathbf{z}(0) = I$$

is *not*

$$\mathbf{z}(t) = \exp\left(\int_0^t A(s)\, ds\right)$$

in general.

Nevertheless a fundamental soln exists.[a]

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[a] And exponentiating the right matrix gives it. Attend Tim Carlson's MS defense!

Solution of linear DDE

If $\mathbf{y}(t) = \mathbf{y}_0(t)$ for $t \in [-\tau, 0]$ then the solution of

$$\dot{\mathbf{y}} = A(t)\,\mathbf{y} + B(t)\,\mathbf{y}_{-\tau}$$

for $t > 0$ is

$$\mathbf{y}(t) = \Phi(t)\left[\mathbf{y}_0(0) + \int_0^t \Phi(s)^{-1} B(s)\mathbf{y}_0(s-\tau)\,ds\right]$$

if $\Phi(t)$ is fundamental solution of $\dot{\mathbf{z}} = A(t)\mathbf{z}$.

Applies for $t \in [0, \tau]$ but easily extended “by steps” *if* $A(t), B(t)$ periodic with period τ.

Our class of problems

Linear
periodic-coefficient
delay
differential equations
(with delay equal to period)[a].

[a] Or at least rationally related . . .

Solution map of a DDE

$$(U\,\mathbf{f})(t) = \Phi(t)\left[\mathbf{f}(\tau) + \int_0^t \Phi(s)^{-1} B(s)\mathbf{f}(s)\,ds\right]$$

a map[a]

$$U : C[0,\tau] \longrightarrow C[0,\tau].$$

How to Use: Initial function $\mathbf{y}_0 \in C[0,\tau]$. To solve DDE initial value problem $\dot{\mathbf{y}} = A(t)\,\mathbf{y} + B(t)\,\mathbf{y}_{-\tau}$ and $\mathbf{y}(t) = \mathbf{y}_0(t+\tau)$, $t \in [-\tau, 0]$:

$$\mathbf{y}_1 = U\mathbf{y}_0, \quad \mathbf{y}_2 = U\mathbf{y}_1, \quad \mathbf{y}_3 = U\mathbf{y}_2, \ldots$$

Translate $\mathbf{y}_i$ to appropriate interval to make a solution on $[0,\infty)$.

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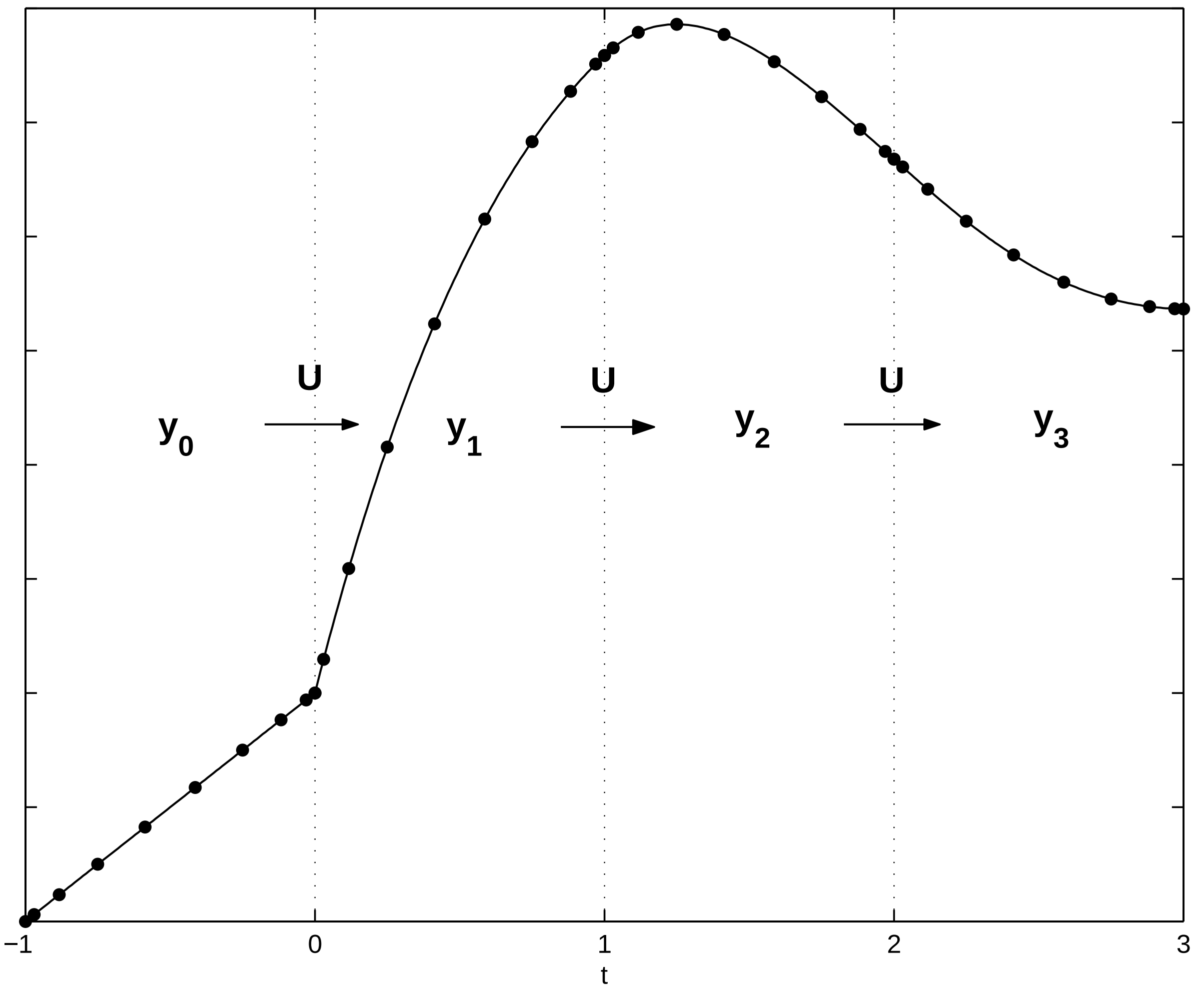
[a] $U : L^p[0,\tau] \longrightarrow L^p[0,\tau]$ for all $p \geq 1$ also.

“Method of steps” from solution map



($\tau = 1$ here; $\dot{y} = -y + y_{-1}$, $y(t) = t$ for $-1 \leq t \leq 0$)

The view in ∞ dimensions

$$\mathbf{y}_{k+1} = U\mathbf{y}_k, \qquad \mathbf{y}_0 \quad \text{given}$$

is the evolution of the DDE initial value problem

U is a compact linear operator on $C[0,\tau]$

dimension$(C[0,\tau]) = \infty$

our class of DDE

$\subseteq$

linear difference eqn w. compact operator in $C[0,\tau]$

Machining question: is U stable for given parameter values?

On “Stable”

Definition. A matrix/(compact operator) T on a linear space X is *stable* if for every eigenvalue $\lambda \in \mathbb{C}$ of T has $|\lambda| < 1$.

(definition-motivating) Theorem.

T is stable

if and only if

$$\lim_{k\to\infty} T^k v = 0 \text{ for all } v \in X$$

if and only if

$$\lim_{k\to\infty} \|T^k\| = 0.$$

Compact linear operators

X is a vector space with a norm (Banach space)

$T : X \to X$ linear

Definition. T is compact if (equivalently)

1. sequence $\{f_k\} \subset X$ bounded
   (i.e. $\exists M$ s. t. $\|f_k\| \leq M$ for each k)
   $\Longrightarrow \quad \{Tf_k\}$ has a convergent subsequence
2. set $S \subset X$ bounded $\quad \Longrightarrow \quad T(S)$ has compact closure

Relevant quotation

“Mathematicians are like Frenchmen: whatever you say to them they translate into their own language and forthwith it is something entirely different.”

Goethe

Theory of compact operators is very nice; I can’t resist a few facts as in the next slides.

Three compact ideas

A Linear Algebra Theorem. If $\dim X = \infty$ then $\lambda = 0$ is in the spectrum of $T : X \to X$. If T is compact and if $\lambda \neq 0$ is in the spectrum of T then λ is an eigenvalue of T. If $\{\lambda_k\}_{k=1}^{\infty}$ is a sequence of distinct eigenvalues of T then $\lambda_k \to 0$.

An Analysis Theorem. The integral operator $N : f \mapsto \int_a^t f(s)\, ds$ on $C[0, \tau]$ is compact (though it has *no* eigenvalues!).

An Algebra Theorem.

$$\{\text{compact operators}\} \quad \subset \quad \{\text{bounded operators}\}$$

is a two-sided ideal.

$\therefore \quad U$ is compact

Recall $\qquad (U\,\mathbf{f})(t) = \Phi(t)\left[\mathbf{f}(\tau) + \int_0^t \Phi(s)^{-1} B(s)\mathbf{f}(s)\,ds\right]$

So U is a composition (ignoring "$f(\tau)+$")

$$f \overset{\text{multiply}}{\longmapsto} \Phi^{-1}Bf \overset{\text{integrate}}{\longmapsto} \int_0^t \Phi^{-1}Bf\,ds \overset{\text{multiply}}{\longmapsto} \Phi \int_0^t \Phi^{-1}Bf\,ds$$

In fact

$$U = M_\Phi \circ [\delta_\tau + N \circ M_{\Phi^{-1}B}]$$

$\therefore U$ compact by "analysis" and "algebra" theorems on previous page.

The Importance of Being Compact

So what's the big deal about "compact"?

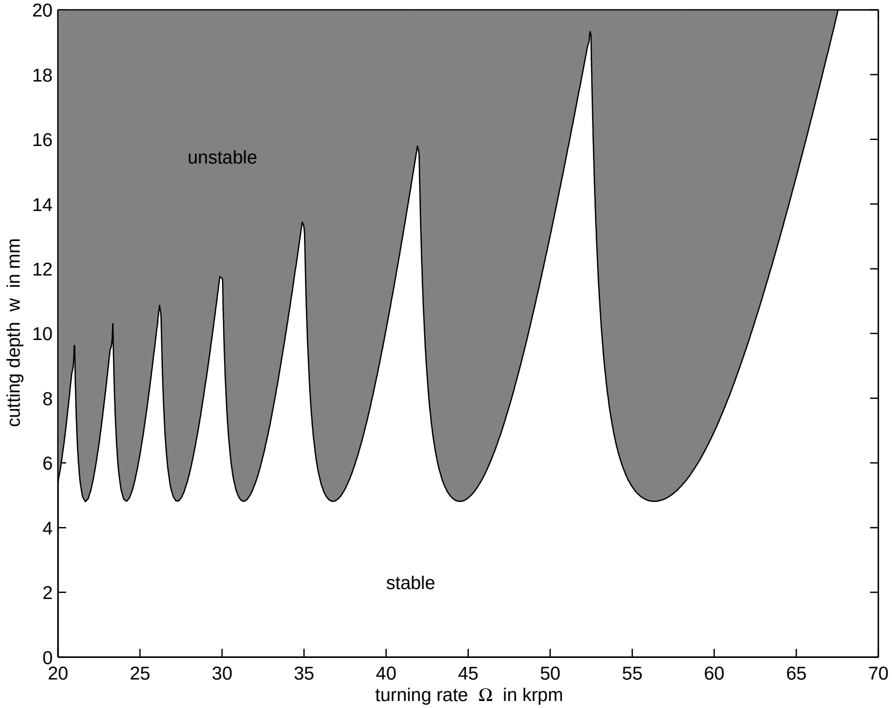
Theorem[a]. T is compact if and only if it is the norm limit of finite rank operators (i.e. matrices).

Compact operators are the ones which can be approximated by matrices.

[a] This theorem applies in all Hilbert spaces and most Banach spaces.

Back to stability chart

Every pixel below corresponds to a set of parameter values; pixel is pair (w, Ω). For each pixel there is a DDE and thus a compact operator U. Approximate U by matrix U_m. Make pixel white if U_m stable.



Do we trust it?

Approximate eigenvalues of U

EXAMPLE: Consider scalar DDE

$$\dot{x} = -x + 0.5x_{-1}$$

1. "*By hand*" *method*: Solve characteristic eqn

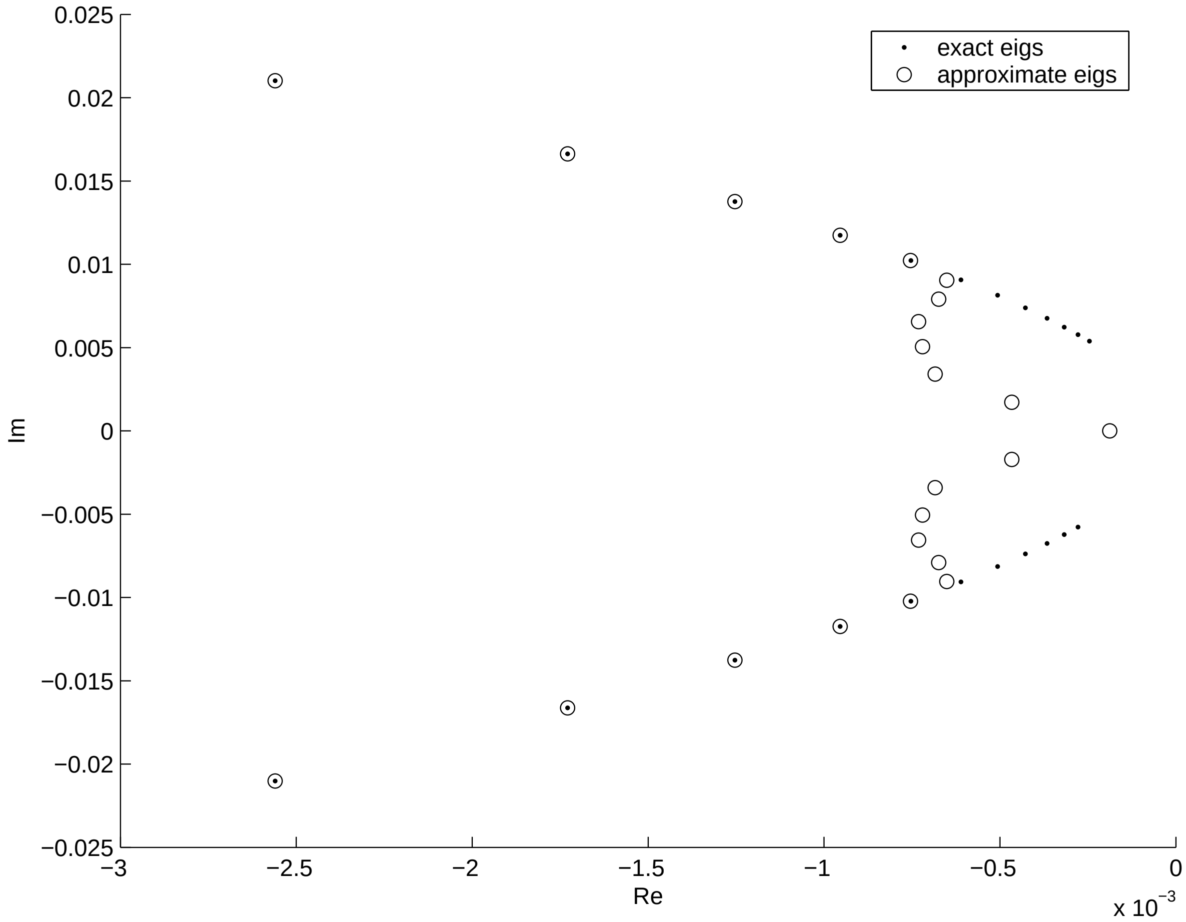
$$\mu = -1 + 0.5e^{-\mu}$$

Each $\mu \in \mathbb{C}$ gives eigenvalue of U.
Solve by bisection, Newton's . . .

2. *Numerical method*: Approximate U by a finite size matrix; find its eigenvalues numerically (by QR). In next slide: U approximated by 30×30 matrix.

Compare eigenvalues.

Accuracy of numerical eigenvalues



(Largest 7 eigenvalues are outside of frame. They are accurate to more than 12 digits. Only smallest eigenvalues ($\leq .01$) are inaccurate.)

Numbers

Number of the eigenvalue, ordered by magnitude	Relative error $\frac{\lvert\lambda_{\text{exact}} - \lambda_{\text{colloc.}}\rvert}{\lvert\lambda_{\text{exact}}\rvert}$
8	1.75×10^{-12}
10	8.01×10^{-10}
12	8.03×10^{-8}
14	3.29×10^{-6}
16	1.13×10^{-4}
20	3.67×10^{-2}
24	0.341
28	2.28
30	28.65

Result: more than 100 digits of correct eigenvalue info from 30×30 matrix approx.

Cost of a stability chart

Most DDE stability problems cannot be done “by hand” as above, for example :

$m\ddot{x} + c\dot{x} + kx = wh(t)(x_{-\tau} - x)$ is “milling equation”; $h(t)$ not constant.

Thus we need numerical method to produce $m \times m$ approximation to U. Then time to produce chart is

$$O((\text{\# of pixels}) \cdot m^3)$$

with standard estimates on QR method for eigenvalues.

So m matters! Small is good.

Approximating a DDE

Consider scalar DDE

$$\dot{y} = a(t)\,y + b(t)\,y_{-\tau}.$$

Approximate $y(t)$ on $[0,\tau]$ by m values $y_0,\ldots,y_{m-1}$ formed into a vector Y.

Approximation of $\dot{y}$ is a matrix acting on Y.

Each approximation method gives a differentiation matrix D:

- equally spaced; finite difference/element differentiation
- equally spaced; Fourier trig differentiation
- irregularly spaced (Chebyshev: $t_j = cos(j\pi/m)$); polynomial differentiation

Three reasons for Chebyshev

- Coeffs in DDEs are (for cases appearing in applications) smooth (in fact, analytic) or piecewise smooth.
  Thus polynomial and Fourier approximation ("spectral approximation") work faster than finite diff, finite elem or wavelets, etc.
- Though the coefficients in our DDE are periodic the solutions are not. Thus Fourier not so good.
- Chebyshev points are nearly optimal interpolation points for minimizing uniform error.

How accurate is Chebyshev collocation?

Theorem. Let p_m be Chebyshev interpolating polynomial for f using m collocation points. If f is analytic in a $\mathbb{C}$-neighborhood R of $[-1, 1]$ then there exists C independent of m such that

$$\|f - p_m\|_\infty \leq C(S + s)^{-m}$$

where S, s are semimajor, semiminor axes of ellipse E such that $[-1, 1] \subset E \subset R$.

In practice, p_m improves by a fixed number of digits per increase by one in m.

Accuracy of DDE collocation soln

Theorem. Consider $\dot{y} = ay + b(t)y_{-\tau}$. Let $q = \Pi_m(by_{-\tau})$ be the interpolating polynomial of the delayed term on the interval $t \in [t_0, t_0 + \tau]$. Find m point collocation solution $p(t)$, an $m-1$ degree polynomial. Then

$$\|y - p\|_\infty \leq \tau e^{a\tau} \left(\|q - by_{-\tau}\|_\infty + |\alpha| \right)$$

for $\alpha = \dot{p}(t_0) - ay(t_0) - b(t_0)y(t_0 - \tau)$.

In practice, if $b, y_{-\tau}$ are analytic then, because $\alpha \to 0$ exponentially in m, this is "spectral convergence".

In any case it is *a posteriori*: it is easy to use to find m so as to get a desired accuracy.

So . . .

1. Numerical stability: As m increases no spurious modes are excited (assuming no rounding error).

2. Accuracy of approximation: Size of nondelayed coefficient a controls how much the error grows from one step to next, after fixed error in delayed term.

3. Practical choice of m for system $\dot{\mathbf{y}} = A(t)\mathbf{y} + B(t)y_{-\tau}$: choose m large enough to approximate $e^{\|A\|t}$, coefficient $B(t)$, and initial function to desired accuracy. Then accuracy of solution is proportional.

Back to stability chart

There's a "naive" method for finding stability charts! Recall we want to know if

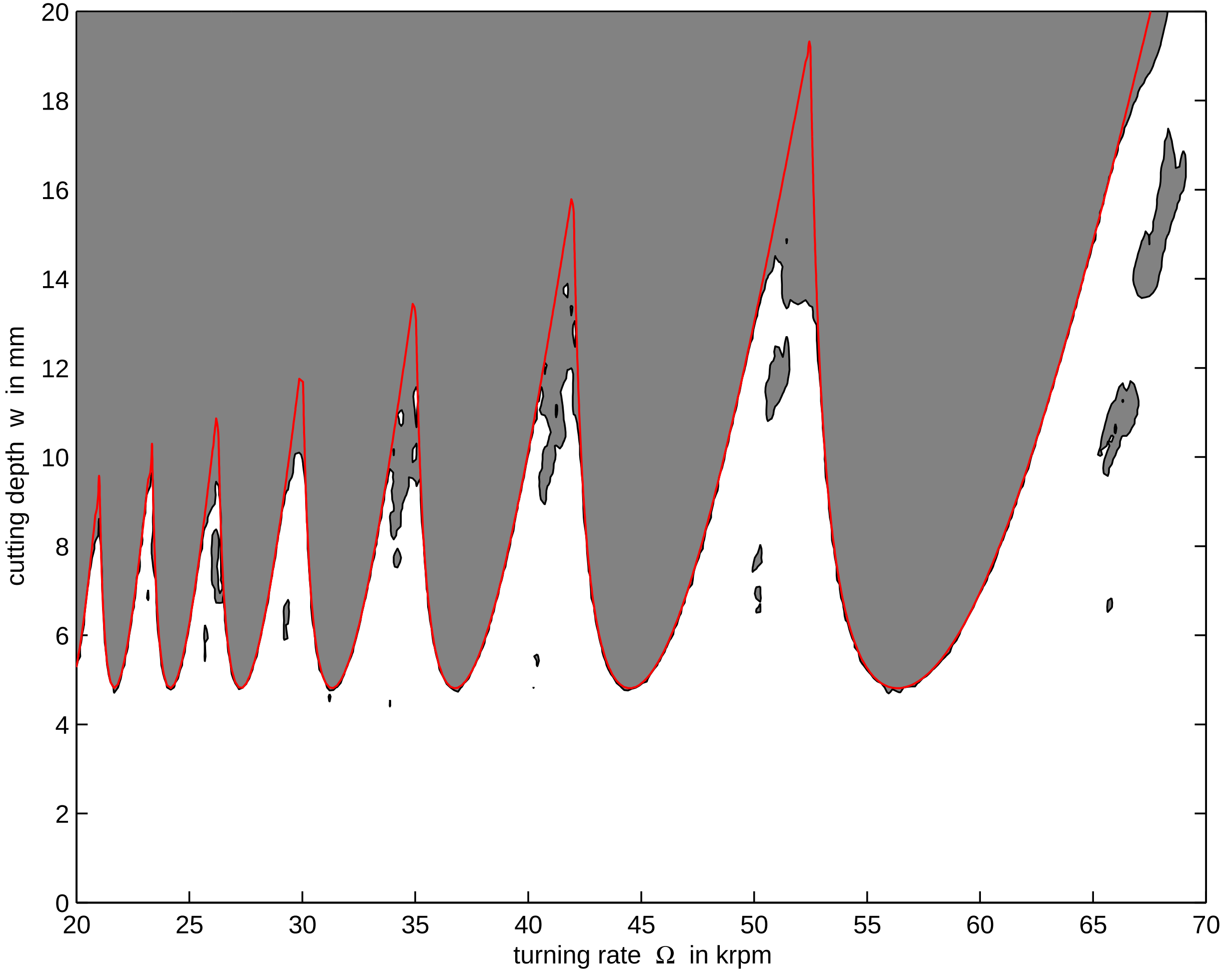
$$\dot{\mathbf{y}} = A(t)\,\mathbf{y} + B(t)\,\mathbf{y}_{-\tau}$$

is stable, for various parameter values (parameters determine $A(t)$, $B(t)$).

naive "Monte Carlo" method: Pick initial function $\mathbf{y}_{-\tau}$ "at random"; iterate forward N delay intervals τ; see if $\mathbf{y}$ is growing or shrinking.

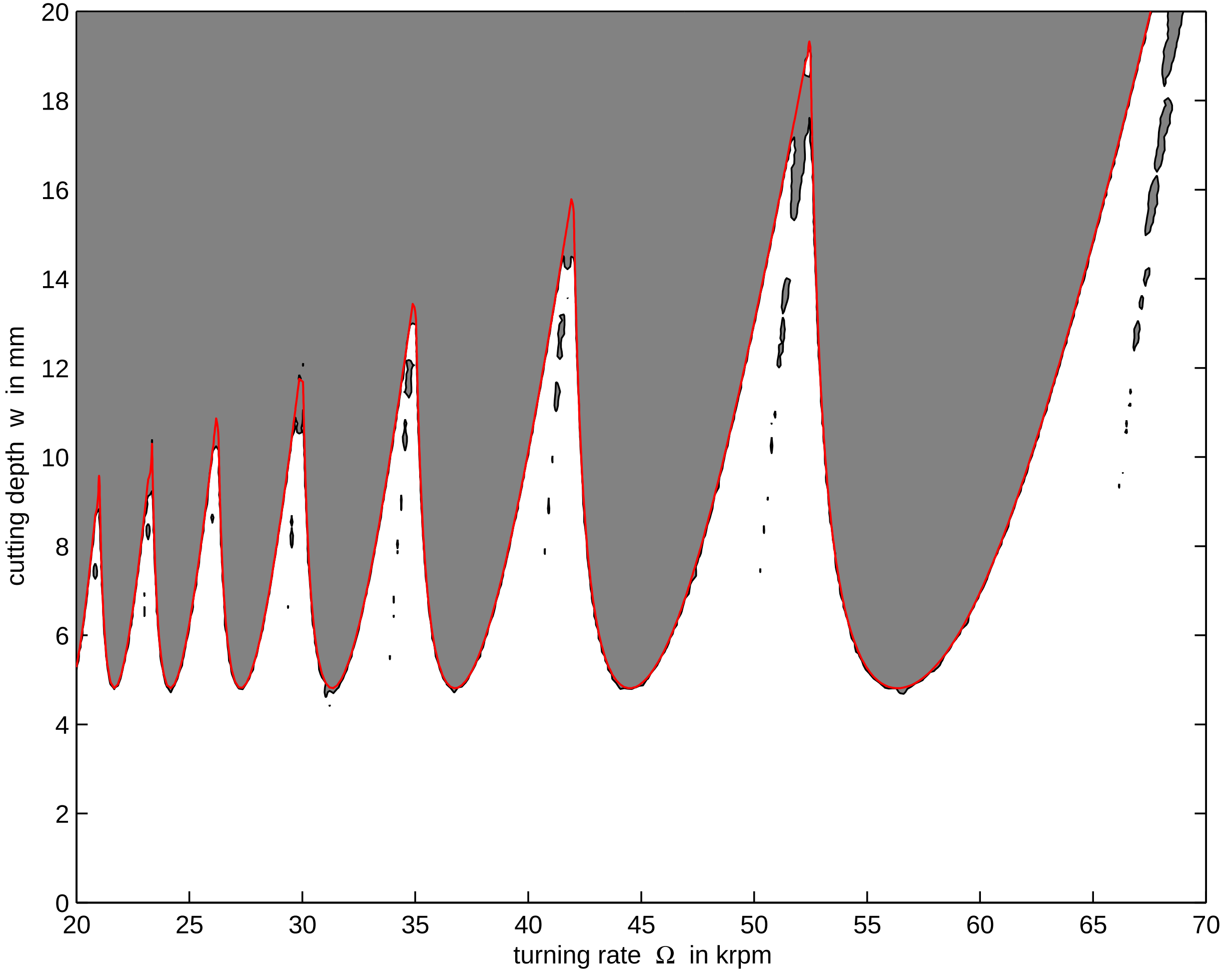
Make stability chart by doing above for each set of parameters.

Monte Carlo version



Iterated $N = 10$ delay intervals. White is stable. Red curve is previous “exact” result.

Again; wait longer



Iterated $N = 40$ delay intervals.

What we think is happening

For “bad” parameter values which produce growth after 10 or 40 intervals, we believe U is “not normal enough”.

Definition. A matrix/operator T is *normal* if $T^*T = TT^*$.

A normal matrix/(compact operator) has orthogonal eigenvectors corresponding for each of its eigenvalues and they form a complete basis. Furthermore

$$\|T^k\| = \rho(T)^k$$

where $\rho(T)$ is max size of eigenvalue. (Nonnormal: “$\geq$”.)

T is stable and normal $\implies$ decay happens immediately
T is stable and *not* normal $\implies$ wait awhile for decay. . .

Why we really care about U

We are interested in the *real* and *nonlinear* dynamical system. The linear system was just an approximation. Questions about nonlinear DDE:

- find fixed points and periodic orbits
- nature of bifurcations?

To study these we need
bases for spaces of stable and unstable directions
and thus it is nice to have a good approximation to U.

But that's another talk . . .
The END. Thanks for your attention!