Selected Solutions to Assignment #9

Exercise 1d (page 162 of B&C). Evaluate the integral $\int_C \frac{\cosh z}{z^4}\,dz$.

SOLUTION. Let $f(z) = \cosh z$. Then

$$\int_C \frac{\cosh z}{z^4}\,dz = \int_C \frac{\cosh z}{(0-z)^4}\,dz = \frac{2\pi i}{3!} f^{(3)}(0) = \frac{2\pi i}{3!}\sinh 0 = 0.$$

Exercise 4 (page 163 of B&C). Let C be a simple closed contour, show that

$$g(w) = \int_C \frac{z^3+2z}{(z-w)^3}\,dz = \begin{cases} 6\pi i w, & \text{if } w \text{ is inside } C, \\ 0, & \text{if } w \text{ is outside } C \end{cases}$$

SOLUTION. If w is outside C then the function $F(z) = \frac{z^3+2z}{(z-w)^3}$ is analytic inside the contour C, and according to C-G theorem, integral is zero. If w is inside C, and if $f(z) = z^3 + 2z$, then we can evaluate:

$$\int_C \frac{z^3+2z}{(z-w)^3}\,dz = \frac{2\pi i}{2!} f''(w) = 6\pi i w.$$

Exercise 7 (page 163 of B&C). Let $C = \{z = e^{i\theta} | -\pi \leqslant \theta \leqslant \pi\}$

a) Show that for any $a \in \mathbb{R}$, $\int_C \frac{e^{az}}{z}\,dz = 2\pi i$.

b) Derive $\int_0^\pi e^{a\cos\theta}\cos(a\sin\theta)\,d\theta = \pi$.

SOLUTION. First,

$$\int_C \frac{e^{az}}{z}\,dz = \int_C \frac{e^{az}}{(z-0)}\,dz = 2\pi i e^{az}|_{z=0} = 2\pi i.$$

On the other hand, using parameterization $z = e^{i\theta}$,

$$\int_C \frac{e^{az}}{z}\,dz = \int_{-\pi}^{\pi} \frac{e^{ae^{i\theta}}}{e^{i\theta}} i e^{i\theta}\,d\theta = i\int_{-\pi}^{\pi} e^{a(\cos\theta + i\sin\theta)}\,d\theta = i\int_{-\pi}^{\pi} e^{a\cos\theta}\left(\cos(a\sin\theta) + i\sin(a\sin\theta)\right)\,d\theta$$
$$= i\int_{-\pi}^{\pi} e^{a\cos\theta}\cos(a\sin\theta)\,d\theta - \int_{-\pi}^{\pi} e^{a\cos\theta}\sin(a\sin\theta)\,d\theta.$$

Equating the imaginary parts we have that

$$\int_{-\pi}^{\pi} e^{a\cos\theta}\cos(a\sin\theta)\,d\theta = 2\int_0^{\pi} e^{a\cos\theta}\cos(a\sin\theta)\,d\theta = 2\pi$$

as desired. (Equating real parts gives $\int_{-\pi}^{\pi} e^{a\cos\theta}\sin(a\sin\theta)\,d\theta = 0$.)

Exercise 1 (page 171 of B&C). Let f be an entire function such that $|f(z)| \leqslant A|z|$, $A > 0$ for all z. Show that $f(z) = a_1 z$, $a_1 \in \mathbb{C}$.

Proof. Let us consider the function $f(z)$ in the disk $D_R = \{|z| \leqslant R\}$. Then it is analytic inside the boundary of the disk C_R and $\max_{z \in C_R} |f(z)| \leqslant AR$. By Cauchy's inequality,

$$|f''(z)| \leqslant \frac{2AR}{R^2}, \quad \text{for all } z \in D_R$$

Since the constant R in the above inequality can be chosen arbitrary large, we conclude that $f''(z) = 0$. Therefore $f(z) = az + b$ for some $a, b \in \mathbb{C}$. Applying the inequality $|f(z)| \leqslant A|z|$ with $z = 0$, we obtain that $b = 0$. So $f(z) = az$. $\square$

Exercise 6 (page 172 of B&C). Let $f(z) = (z + 1)^2$ and R is a triangle with vertices $\{0, 2, i\}$. Find points where $|f(z)|$ has its maximum and minimum values.

Proof. We observe that $|z|^2 = |z^2|$, and using this, we can rewrite

$$|f(z)| = |(z + 1)^2| = |z + 1|^2.$$

So, the function $|f(z)|$ can be interpreted as the square of the distance between points -1 and z. From a picture it is clear that for z in R, the function $|f(z)|$, has its minimum at $z = 0$ and maximum at $z = 2$. $\square$

Exercise 10 (page 172 of B&C). Let z_0 be a zero of the polynomial $P(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n$, $a_n \neq 0$. Show that $P(z) = (z - z_0)Q(z)$ for some polynomial $Q(z)$ of degree $n - 1$.

Proof. Direct calculations show that for all $k \in \mathbb{N}$

$$z^k - z_0^k = (z - z_0)(z^{k-1} + z^{k-2} z_0 + \ldots + z z_0^{k-2} + z_0^{k-1}) = (z - z_0) Q_k(z) \tag{1}$$

Indeed, evaluating the right hand side of the above equality, we get:

$$\begin{aligned}
&(z - z_0)(z^{k-1} + z^{k-2} z_0 + \ldots + z z_0^{k-2} + z_0^{k-1}) \\
&\quad = z^k + z^{k-1} z_0 + z^{k-2} z_0^2 + \ldots + z^2 z_0^{k-1} + z z_0^{k-1} - -z_{k-1} z_0 - z^{k-2} z_0^2 - \ldots - z z_0^{k-1} - z_0^k \\
&\quad = z^k - z_0^k
\end{aligned}$$

Taking arbitrary point z_0 (not necessarily root of polynomial), we have

$$P(z) - P(z_0) = a_1(z - z_0) + a_2(z^2 - z_0^2) + a_3(z^3 - z_0^3) + \ldots + a_n(z^n - z_0^n)$$

To each item in the above expression we apply formula (1) and have that

$$P(z) - P(z_0) = a_1(z - z_0) + a_2(z - z_0)Q_2(z) + a_3(z - z_0)Q_3(z) + \ldots + a_n(z - z_0)Q_n(z) \tag{2}$$

where $Q_k(z)$ is a polynomial of degree $k - 1$, given by the second factor in (1). Thus in (2) we can factor out $(z - z_0)$ and get

$$P(z) - P(z_0) = (z - z_0)(a_1 + a_2 Q_2(z) + a_3 Q_3(z) + \ldots + a_n Q_n(z)) = (z - z_0)Q(z).$$

We see that by construction the degree of $Q(z)$ is at most $n - 1$, and choosing z_0 to be the root of $P(z)$, we get the desired factorization formula.

$\square$