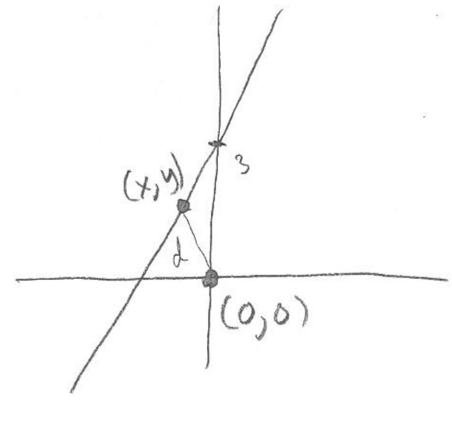
SOLUTIONS

1. Find the point on the line $y = 2x + 3$ which is closest to the origin.

$d^2 = x^2 + (2x+3)^2$

$f(x) = x^2 + (2x+3)^2$ ← minimize this



$f'(x) = 2x + 2(2x+3)\cdot 2 = 10x + 12$

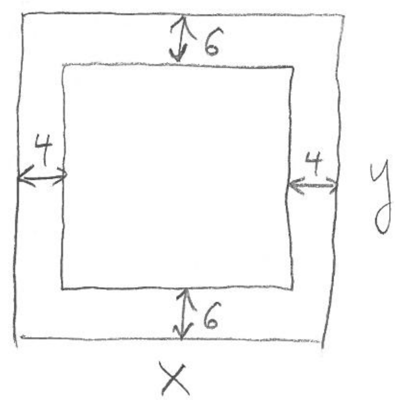
$x = \frac{-12}{10} = -1.2$

$y = 2x+3 = -2.4+3 = 0.6$

point is $(-1.2, 0.6)$

2. The top and bottom margins of a poster are each 6 cm and the side margins are 4 cm. If the area of the printed material on the poster is fixed at 384 cm^2, find the dimensions of the poster with the smallest total area.

$A = xy$



$(x-8)(y-12) = 384$

$\Leftrightarrow \quad y = 12 + \frac{384}{x-8}$

$A(x) = x\left(12 + \frac{384}{x-8}\right), \quad 8 < x < \infty$

$= 12x + 384\frac{x}{x-8}$

$A'(x) = 12 + 384\frac{1\cdot(x-8) - x(1)}{(x-8)^2} = 12 + 384\cdot\frac{-8}{(x-8)^2} = 0$

$(x-8)^2 = \frac{3072}{12} = 256$

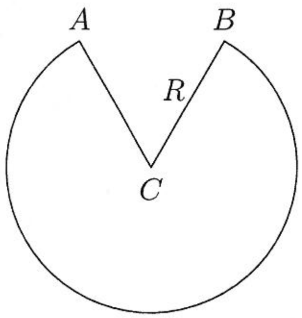
$x - 8 = 16$

$x = 24\text{cm}$

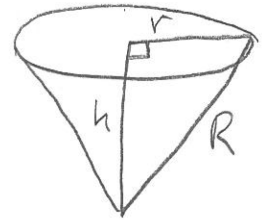
$y - 12 = \frac{384}{16}$

$y = 36\text{ cm}$

3. A cone-shaped drinking cup is made from a circular piece of waxed paper of radius R by cutting out a sector, as shown, and joining the edges CA and CB. Find the maximum capacity of the cup.



$\xrightarrow[\text{and attach } CA \text{ to } CB]{\text{roll up}}$



$R^2 = r^2 + h^2$
$\iff h = \sqrt{R^2 - r^2}$

$$V = \frac{1}{3}\pi r^2 h$$

$$V(r) = \frac{1}{3}\pi r^2 \sqrt{R^2 - r^2}$$

$$V'(r) = \frac{\pi}{3}\left(2r\sqrt{R^2-r^2} + r^2 \frac{1}{\cancel{2}}(R^2-r^2)^{-1/2}(-\cancel{2}r)\right)$$

$$= \frac{\pi}{3} r (R^2-r^2)^{-1/2}\left[2(R^2-r^2) - r^2\right]$$

$$= \frac{\pi}{3} \frac{r(2R^2-3r^2)}{\sqrt{R^2-r^2}} = 0 \iff r = 0 \text{ or}$$

$2R^2 - 3r^2 = 0$

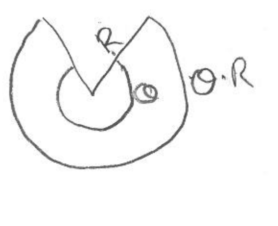
$3r^2 = 2R^2$

$r = \sqrt{\frac{2}{3}}R$

$$V\left(\sqrt{\tfrac{2}{3}}R\right) = \frac{1}{3}\pi\left(\frac{2}{3}R^2\right)\sqrt{R^2 - \frac{2}{3}R^2}$$

$$= \frac{2}{9}\pi R^2 \sqrt{\frac{1}{3}R^2} = \frac{2\pi}{9\sqrt{3}}R^3$$

to cut material:



$$R\theta = 2\pi r \quad \text{so} \quad \theta = \frac{2\pi r}{R} = \frac{2\pi\sqrt{\frac{2}{3}}\cancel{R}}{\cancel{R}}$$

$$= 2\pi\sqrt{\frac{2}{3}} \approx 293^\circ$$