Math F251: Section 3.5 Worksheet                    Wednesday 10 October 2018

1. Find $dy/dx$ by implicit differentiation.

$$y \cos x = x^2 + y^2$$

$$\frac{dy}{dx} \cos x + y(-\sin x) = 2x + 2y \frac{dy}{dx} \Big\} \quad \text{product rule on left,} \atop \text{chain rule on right}$$

$$\frac{dy}{dx}(\cos x - 2y) = 2x + y \sin x$$

$$\boxed{\frac{dy}{dx} = \frac{2x + y \sin x}{\cos x - 2y}}$$

2. Consider the equation
$$\sqrt{x} + \sqrt{y} = 1 \qquad\qquad (*)$$

(a) Find $y'$ by implicit differentiation.

$$\frac{1}{2} x^{-1/2} + \frac{1}{2} y^{-1/2} y' = 0$$

$$\boxed{y' = \frac{-\frac{1}{2} x^{-1/2}}{\frac{1}{2} y^{-1/2}} = -\frac{\sqrt{y}}{\sqrt{x}}}$$

(b) Solve $(*)$ explicitly for $y$ and differentiate to get $y'$ in terms of $x$.

$$\sqrt{y} = 1 - \sqrt{x}$$
$$y = (1 - \sqrt{x})^2$$

$$\boxed{y' = 2(1 - \sqrt{x})\left(-\frac{1}{2} x^{-1/2}\right)}$$
$$= -\frac{1 - \sqrt{x}}{\sqrt{x}} \boxed{= -x^{-1/2} + 1}$$

(c) Check that your solutions in (a) and (b) are consistent.

insert $y = (1 - \sqrt{x})^2$ into (a) result:

$$y' = -\frac{\sqrt{(1 - \sqrt{x})^2}}{\sqrt{x}} = -\frac{1 - \sqrt{x}}{\sqrt{x}}$$

$$= -x^{-1/2} + 1 \quad \checkmark$$

**3.** (*A §3.4 question.*) For what values of $r$ does the function $y = e^{rt}$ satisfy the differential equation $y'' - 4y' + y = 0$?

$$y' = re^{rt}$$
$$y'' = r^2 e^{rt}$$

$$y'' - 4y' + y = r^2 e^{rt} - 4re^{rt} + e^{rt} = 0$$

$$\downarrow \quad (e^{rt} \neq 0)$$

$$r^2 - 4r + 1 = 0$$

$$\boxed{r = \frac{4 \pm \sqrt{16-4}}{2}} = \boxed{2 \pm \sqrt{3}}$$

**4.** For the "cardiod" shown, with the equation and point given, find an equation of the tangent line.

$$x^2 + y^2 = (2x^2 + 2y^2 - x)^2, \qquad (0, \tfrac{1}{2})$$

$$2x + 2yy' = 2(2x^2 + 2y^2 - x)(4x + 4yy' - 1)$$
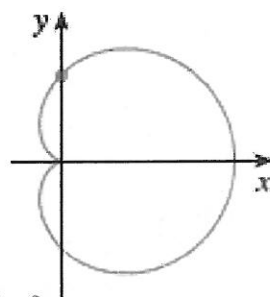
$$= 2(2x^2 + 2y^2 - x)(4x - 1)$$
$$+ 8y(2x^2 + 2y^2 - x)y'$$

$$y'(2y - 8y(2x^2 + 2y^2 - x)) = -2x + 2(2x^2 + 2y^2 - x)(4x - 1)$$

$$y' = \frac{-2x + 2(2x^2 + 2y^2 - x)(4x - 1)}{2y - 8y(2x^2 + 2y^2 - x)}$$

$$m = y'\Big|_{(0, \frac{1}{2})} = \frac{0 + 2(0 + \frac{1}{2} - 0)(-1)}{1 - 4(0 + \frac{1}{2} - 0)} = \frac{-1}{-1} = 1 \qquad \therefore \boxed{y - \tfrac{1}{2} = 1 (x - 0)}$$



**5.** If $xy + e^y = e$, find the value of $y''$ at the point where $x = 0$.

$$1 \cdot y + x \cdot y' + e^y y' = 0$$

$$y' = \frac{-y}{x + e^y}$$

$$y'' = \frac{-y'(x + e^y) + y(1 + e^y y')}{(x + e^y)^2} \quad \downarrow$$

@ $x = 0$: $\quad 0 + e^y = e$

$$\therefore y = 1$$

$$\therefore y' = \frac{-1}{0 + e} = -\frac{1}{e}$$

$$\boxed{y'' = \frac{+\frac{1}{e}(0 + e) + 1(1 + e(\frac{-1}{e}))}{(0 + e)^2} = \frac{1 + 1 - 1}{e^2} = \frac{1}{e^2}}$$