Solution to Midterm Exam #2 *Extra Credit Problem*

Extra Credit By writing $z = re^{i\theta}$, show that $\log(z^{1/n}) = \frac{1}{n}\log(z)$ if $n = 1, 2, 3, \dots$.

Proof. Assume $z \neq 0$ (as nothing is defined if $z = 0$). Note $z^{1/n} = \{r^{1/n}\exp(i\theta/n + i2p\pi/n)\}$, $p = 0, 1, \dots, n-1$, is a set with n elements. Thus

$$\log(z^{1/n}) = \ln\left(r^{\frac{1}{n}}\right) + i\left(\frac{\theta}{n} + \frac{2p\pi}{n} + 2k\pi\right) = \frac{1}{n}(\ln r + i\theta) + i\frac{2(p+nk)\pi}{n}.$$

Here, as noted, $p = 0, 1, \dots, n-1$, while k can be any integer. But then $p + nk$ can be any integer, because every integer m can be divided by k to give remainder $0 \le p < n$, that is, so that $m = p + nk$.

On the other hand,

$$\frac{1}{n}\log z = \frac{1}{n}(\ln r + i\theta + i2j\pi) = \frac{1}{n}(\ln r + i\theta) + i\frac{2j\pi}{n},$$

where j is any integer.

Thus the sets $\log(z^{1/n})$ and $\frac{1}{n}\log(z)$ are the same. □

We necessarily have used the property $\ln(a^b) = b\ln(a)$ for $a > 0$ and b real. This property is equivalent to the fact $(e^A)^B = e^{AB}$ for real numbers A, B. Either way, it is not proven in the text. I can not find a careful proof of it anywhere in my office!

A careful proof would necessarily start from a careful definition of e^x where x is a real number. The two standard candidate definitions are

$$e^x := 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

and

$$e^x := \lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n.$$

Amazingly (to me, anyway) a reasonable starting point for careful definitions is, also,

$$\ln x := \int_1^x \frac{1}{t}\,dt.$$

Follow-on Extra Credit (*5 more exam pts*) Starting from one of these definitions, carefully prove $(e^A)^B = e^{AB}$ for real numbers A, B and, also, $\ln(a^b) = b\ln(a)$ for $a > 0$ and b real.