Solutions to Midterm #1

1. **(a)** *Solution.* $u_x = v_y$ and $u_y = -v_x$.

(b) *Solution.* Here $u = e^{-x}\cos y$ and $v = -e^{-x}\sin y$. These two functions are defined in all of $\mathbb{C}$, an open set. All of the first partial derivatives exist and are continuous in all of $\mathbb{C}$. It is easy to check that the Cauchy-Riemann equations hold at every point. By the theorem on sufficiency of the Cauchy-Riemann equations, $f'(z)$ exists at every point in $\mathbb{C}$.

2. Does not exist. (I.e. problem **2** does not exist!)

3. **(a)** *Solution.* The sketch is of an open annulus centered at $-i = (0, -1)$. The inner radius is one and the outer radius 2.

(b) *Solution.* A *domain* is an open connected set.

(c) *Solution.* The set in part **a** is open (and not closed). It is connected and therefore it is a domain.

4. **(a)** *Solution.* $u = x^3 - 3xy^2$, $v = -3yx^2 + y^3$.

(b) *Solution.* $f'(i)$ does not exist because the Cauchy-Riemann equations do not apply at i. In particular, $u_x(0,1) = +3$ but $v_y(0,1) = -3$.

(c) (*Note the typo in the problem; the goal is to show* $f'(0)$ does *exist.*) *Solution.* $f'(0)$ exists by the definition, as follows. (Note that checking Cauchy-Riemann at a point is not sufficient to prove that $f'(z)$ exists). Compute:

$$f'(0) = \lim_{\Delta z \to 0} \frac{\left(\overline{0 + \Delta z}\right)^3}{\Delta z} = \lim_{\Delta z \to 0} \overline{\Delta z}^2 \, \frac{\overline{\Delta z}}{\Delta z}$$

Now the fraction $\overline{\Delta z}/\Delta z$ is a bounded function on the plane, in fact,

$$\left| \frac{\overline{\Delta z}}{\Delta z} \right| = 1.$$

So the limit above is zero because $\lim_{\Delta z \to 0} \overline{\Delta z}^2 = 0$. Thus $f'(0)$ exists and $f'(0) = 0$.

5. $\lim_{z \to z_0} \operatorname{Re} z = \operatorname{Re} z_0$

Proof. Let $\epsilon > 0$. Define δ to be exactly ϵ. If $|z - z_0| < \delta$ then

$$|\operatorname{Re} z - \operatorname{Re} z_0| = |x - x_0| = \sqrt{(x - x_0)^2} \le \sqrt{(x - x_0)^2 + (y - y_0)^2} = |z - z_0| < \delta = \epsilon.$$

□

6. *Solution.* For all $z \ne 0$, $\operatorname{Arg} z$ is the angle $\theta \in (-\pi, \pi]$ so that $z = re^{i\theta}$ for some positive r.

7. *Solution.*

$$\frac{4i}{(1-i)(2-i)(3-i)} = \frac{4i}{(1-3i)(3-i)} = \frac{4i}{-10i} = -\frac{2}{5}.$$