A TEMPERATURE ONLY FORMULATION FOR ICE SHEETS

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A

THESIS

Presented to the Faculty

of the University of Alaska Fairbanks

in Partial Fulfillment of the Requirements

for the Degree of

MASTER OF SCIENCE

By

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Fairbanks, Alaska

March 2014

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Chapter 1
Introduction

1.1 Background

In the dynamics of large flowing ice masses, i.e. glaciers and ice sheets, temperature has an important role. Ice's deformation rate is sensitive to temperature, which in turn influences ice flow. Similarly, the liquid water content of glacial ice is part of its thermal (i.e. latent) energy, and is a factor in basal sliding; it also may play a role in transporting heat throughout the ice mass. For the field investigator, temperature has practical implications; some methods for determining ice thickness depend on the DC or RF resistivity of the ice, which is sensitive to temperature. Similarly, liquid water present in the ice scatters radio waves, which complicates radar remote sensing.

A common distinction is made in glaciology literature between ice below the pressure melting temperature, *cold ice*, and ice at the pressure-melting temperature, *temperate ice*. It is important to note that liquid water may be present in temperate ice, whereas cold ice consists entirely of frozen water [8, 13]. As noted above, the existence of liquid water is a factor in the internal energy content of ice, in the form of the latent heat of fusion. In practice this water content is expressed as the *liquid water fraction*, $\omega = \rho_{\text{liquid}}/\rho$, where ρ_{liquid} is the partial density (mass of water per volume of the mixture) of liquid water and ρ is the density of the mixture. This liquid water fraction $0 \leq \omega \leq 1$ is typically small, $\omega \sim 0.01$ [8, see section 9.2].

We can classify glaciers and ice sheets depending on the thermal types of ice present. We say glaciers and ice sheets are *cold*, *temperate*, or *polythermal*, if they consist, respectively, entirely of cold ice, entirely of temperate ice, or if both types of ice are present in the glacier. Because of its role in ice dynamics, many models for glaciers and ice sheets model temperature in some form. One type of model for polythermal glaciers explicitly separates the ice into distinct cold and temperate regimes, and tracks the interface between them as a surface [7]. Other models track the *enthalpy* (internal energy), in both domains [1], with temperature being a function of enthalpy.

1.1.1 Our problem

Our goal in this thesis is to develop a formulation for the steady-state temperature of cold and polythermal glaciers—because temperate glaciers are entirely at the pressure-melting temperature, little is left to model. We consider a simplified setup: a one-dimensional problem representing a vertical column of ice, following, as a plug and without deformation, the horizontal motion of the ice. At the top of the column snow falls and, through compression under its own weight, forms single-phase (cold) ice; we assume this new ice's temperature is negative (in °C). Ice then flows downward while being heated by the dissipation of the gravitational potential energy in the viscous flow [8, see the *dissipation power* term in equations (3.92) and (5.14)]. If there is sufficient heating, and if the downward-advecting ice is neither too fast nor too cold, the ice reaches the pressure-melting point. This determines a set of points in the ice-column where the ice transitions from cold to temperate, which we call the *cold-temperate transition surface* or *CTS*[7]. As the ice continues to move downward past the CTS, it becomes a mixture of solid temperate ice and liquid water. This temperate ice is at the pressure-melting temperature, so the addition of energy increases the liquid-water fraction without raising the temperature.

1.1.2 Proposed method

We propose and analyze a model for ice temperature only, without (explicit) reference to the CTS. The CTS is determined, but as part of the solution for temperature. To do so, in Chapter 2 we first derive a free-boundary problem associated with a differential equation for our temperature-only model. Following that, in Chapter 3 we remove the constraint which causes the free-boundary. By presenting some exact solutions to this unconstrained problem, we conclude that the strong form is indeed not a differential-equation boundary-value problem. In Chapter 4 we transform the constrained problem into a variational inequality, which frees us of explicit references to the CTS. We develop the theory of this variational inequality in Chapter 5, and prove some sufficient conditions under which the variational problem is well posed. In Chapter 6, we shift gears and develop numerical methods for approximating solutions to our problem. Finally, in Chapter 7 we discuss directions for further work.

Chapter 2
The Strong Form for Polythermal Glaciers

2.1 Preliminaries

We now formulate mathematically the problem for the temperature of a one-dimensional plug of glacial ice. To start, we examine the steady state thermal energy balances of cold and temperate ice, and the phenomena at the CTS, where cold and temperate ice meet.

We consider the temperature (in °C) of a one-dimensional plug of ice: $u : [0, \ell] \to \mathbb{R}$, where $u(z)$ is the ice temperature at a height z above the solid base. To simplify our problem, we ignore the (real, but small) pressure-dependence of the melting temperature, and instead assume that the pressure-melting temperature is 0°C. We fix the temperature at the base to be the melting temperature, $u(0) = 0°$—noting that geothermal heat causes the bases of glaciers to be warm [13]—and allow the temperature at the ice surface to be any temperature at or below the pressure-melting temperature, $u(\ell) \leq 0°$. In the accumulation areas we model, the ice has a downward vertical velocity: $V : [0, \ell] \to \mathbb{R}$, where $V(z) \leq 0$ for every $z \in [0, \ell]$. We have a free boundary (the CTS) at the location $s \in [0, \ell]$ where the cold ice reaches the pressure-melting temperature

$$s = \sup \left\{ z \in [0, \ell] \middle| u(z) = 0 \right\}. \tag{2.1}$$

2.2 Heat in Cold Ice

Fourier's law of heat conduction states that the heat flux—the amount of heat flowing across an area in a given time—due to conduction is proportional to the gradient of temperature [13], i.e.

$$Q_{\text{cond}} = -k \nabla u,$$

where k is the conductivity. When ice is flowing within the glacier with velocity V, this flow also transports heat via *advection*, creating an additional heat flux

$$Q_{\text{adv}} = \rho c V u,$$

in the direction of flow, where ρ and c are the bulk density and heat capacity of ice, respectively [13]. In our one dimensional case, flow is strictly vertical, so the total vertical heat flux in cold ice becomes:

$$Q_{\text{cold}} = Q_{\text{cond}} + Q_{\text{adv}} = -k u_z + \rho c V u. \tag{2.2}$$

Note that in steady state, because the top of the ice is colder than the bottom, the conductive heat flux Q_{cond} will be positive, corresponding to upward flux.

2.3 Heat in Temperate Ice

In temperate ice, where the temperature is constant, there is no temperature gradient, and thus no conduction. Also, since all temperate ice is at the melting temperature, 0° C, the temperature of the advection of temperate ice contributes nothing to the heat flux. The advection of latent heat in the water fraction is all that remains,

$$Q_{\text{temp}} = \rho L V \omega, \tag{2.3}$$

where $\omega(z)$ is the liquid water fraction and L is the latent heat of fusion.

2.4 Balance of Thermal Energy

Throughout the ice column, strain forces deform the ice, and in the process produce heat, the so-called strain heating or dissipation power, S. The details of strain heating are external to our problem, so we say nothing further about it other than that it is nonnegative

$$S \in L^2([0,\ell]) \text{ where } S(z) \geq 0.$$

Energy is conserved within the ice. In steady-state, we have a conservation of energy in the ice. On an interval, $[a,b] \subseteq (0,\ell)$ the energy flux through the endpoints balances with the strain heating throughout the column:

$$0 = Q(b) - Q(a) - \int_a^b S,$$

that is, the difference between the energy flowing into the interval and the energy flowing out is accounted for by the energy input by strain heating. At a point $z \in (0,\ell)$, we can express this in terms of Q_z:

$$\begin{aligned} 0 &= \lim_{\Delta z \to 0} \frac{Q(z+\Delta z) - Q(z)}{\Delta z} - \frac{1}{\Delta z}\int_{z+\Delta z}^{z} S \\ &= Q_z - S. \end{aligned} \tag{2.4}$$

Below, we state the conservation of thermal energy in the ice column in terms of temperature and water fraction. It says that above the CTS we have a balance of conduction, sensible heat advection, and strain heating. Below the CTS we have a balance of strain heating and latent heat advection. Furthermore we require that temperature be continuous.

Variable	Name	Value	Units
c	Heat Capacity of ice	2009	$\mathrm{J\,kg^{-1}\,K^{-1}}$
k	Conductivity of ice	2.1	$\mathrm{J\,m^{-1}\,K^{-1}\,s^{-1}}$
ρ	Bulk density of ice	910	$\mathrm{kg\,m^{-3}}$
L	Latent heat of fusion	3.34×10^5	$\mathrm{J\,kg^{-1}}$

Table 2.1. Typical Values of Model Parameters.

Now refering to our conservation of energy equation (2.4) and our equations for heat flux throughout the ice (2.2) & (2.3), our conservation system is:

$$-(ku_z)_z + \rho c(Vu)_z = S, \qquad s < z < \ell \tag{2.5}$$

$$u(s^-) = u(s^+), \tag{2.6}$$

$$-ku_z(s^+) + \rho cV(s^+)u(s^+) = \rho LV(s^-)\omega(s^-), \tag{2.7}$$

$$u = 0, \qquad 0 < z < s \tag{2.8}$$

$$\rho L(V\omega)_z = S, \qquad 0 < z < s. \tag{2.9}$$

Note, by continuity (2.6) and (2.8) we have $u(s^+) = u(s^-) = 0$. Which reduces (2.7) to

$$-ku_z(s^+) = \rho LV(s^-)\omega(s^-). \tag{2.10}$$

By our assumptions, the left hand side of (2.10) is nonnegative and the right hand side is nonpositive, so the temperature gradient at the CTS vanishes:

$$u_z(s^+) = 0. \tag{2.11}$$

Typical values for the physical parameters used in this model can be found in Table 1 of [1], which we reprint here as Table 2.1.

2.5 Formal Statement of the Strong Form

From the energy balance equations (2.5)–(2.11) we can extract a formulation for steady-state temperature without reference to the liquid fraction below the CTS:

Problem 1 (The Strong Formulation). Let $V \in C^1([0,\ell])$, and $S \in L^2([0,\ell])$. Find $u : [0,\ell] \to$

$\mathbb{R}$ and $s \in [0, \ell]$ so that the following hold:

$$u(\ell) = T_0 \leq 0, \tag{2.12}$$

$$\rho c\,(Vu)_z = (ku_z)_z + S, \qquad s < z < \ell, \tag{2.13}$$

$$u(s^+) = 0, \tag{2.14}$$

$$u_z(s^+) = 0, \tag{2.15}$$

$$u(z) = 0, \qquad 0 \leq z < s, \tag{2.16}$$

$$u(z) \leq 0. \tag{2.17}$$

It is important to note that a solution to Problem 1 consists of two components: we must determine the location, s, of the CTS in the ice column, while simultaneously solving the differential equation boundary-value problem, (2.12)-(2.15). Problems of this form, where part of the boundary—in our case the CTS—is not known a priori but is instead part of the solution, are known colloquially as *free-boundary problems*. Classical examples of free-boundary problems, including the obstacle problem for Poisson's equation and the Stefan problem have become motivating problems for the theory of variational inequalities, a theory which we will later employ to prove statements about our problem.

Chapter 3
The Unconstrained Problem

As we will see below, solving Problem 1 is distinct from solving the linear ODE (2.5) which appears in the strong form using well-posed boundary values. We can solve the ODE itself exactly in some cases, and for some parameter values we can find exact solutions which satisfy, or fail to satisfy, the constraint that $u(z) \leq 0$.

We drop (2.14)-(2.17) from Problem 1 to arrive at an ODE problem with Dirichlet boundary conditions:

Problem 2 (The Cold Ice ODE). Given $k, \rho, c \in \mathbb{R}^+$, $V \in C^1([0,\ell])$, and $S \in L^2([0,\ell])$, find $u : [0,\ell] \to \mathbb{R}$ which satisfies:

$$u(l) = T_0 \leq 0. \tag{3.1}$$

$$-\left(ku_z\right)_z + \rho c \left(Vu\right)_z = S \tag{3.2}$$

$$u(0) = 0 \tag{3.3}$$

Problem 2 differs from Problem 1 in that the constraint $u \leq 0$ is not present, and thus there is no free boundary at which an extra Neumann condition applies.

3.1 Exact Solutions to Some Cases

Problem 2 is a non-constant coefficient, linear, second-order, ordinary differential equation, boundary-value problem. Establishing whether such problems are well posed or note is not always straight-forward: a solution may not always exist, and moreover, when one does exist, finding a closed-form for it may be non-trivial. For our purposes, the well-posedness of Problem 2 is not important, and will not be treated seriously—later we will seriously consider conditions under which our constrained variational formulation is well posed. For now, we can find exact solutions for Problem 2 for at least a few cases. These solutions will later serve as test cases to verify that our finite element solvers converge at the rate we expect.

3.1.1 Zero-Velocity and Constant Strain-Heating

Suppose we set $S(z) = S_0 \in \mathbb{R}^+$ and $V(z) = 0$, then (3.2) reduces to:

$$-\left(ku_z\right)_z = S_0$$

Let

$$u(z) = \frac{S_0}{2k}(\ell - z)z + \frac{u_\ell z}{\ell}. \tag{3.4}$$

See Appendix D for verification that u solves Problem 2. In this zero-velocity case, it is straight-forward to derive a critical value for S_0 such that $u(z) > 0$ for some $z \in (0, \ell)$. If $0 \geq u(z)$ for all z, from (3.4), we find:

$$\begin{aligned}
0 &\geq \frac{S_0}{2k}(\ell - z)z + T_0\frac{z}{\ell} \\
0 &\geq S_0\frac{\ell - z}{2k} + \frac{T_0}{\ell} \\
-T_0\frac{2k}{\ell(\ell - z)} &\geq S_0 \\
-\frac{2kT_0}{\ell^2} &\geq S_0. \qquad (3.5)
\end{aligned}$$

Let $S_c = \frac{-2kT_0}{\ell^2}$. It follows that if $S_0 > S_c$, then there is some $z \in (0, \ell)$ so that $u(z) > 0$. In Figure 3.1 (a)-(c) we plot cases where $S_0 < S_c$, $S_0 = S_c$, and $S_0 > S_c$.

3.1.2 Constant-Velocity, Zero Strain-Heating

Consider the case where we set $S(z) = 0$ and $V(z) = V_0$. Then (3.2) reduces to:

$$-ku_{zz} + \rho c V_0 u_z = 0.$$

Then

$$u(z) = T_0\frac{e^{\gamma z} - 1}{e^{\gamma \ell} - 1}$$

where $\gamma = \rho c V_0 / k$, solves Problem 2. A derivation is given in Appendix D.

In Figure 3.1 (d)-(f) we plot some cases of this solution.

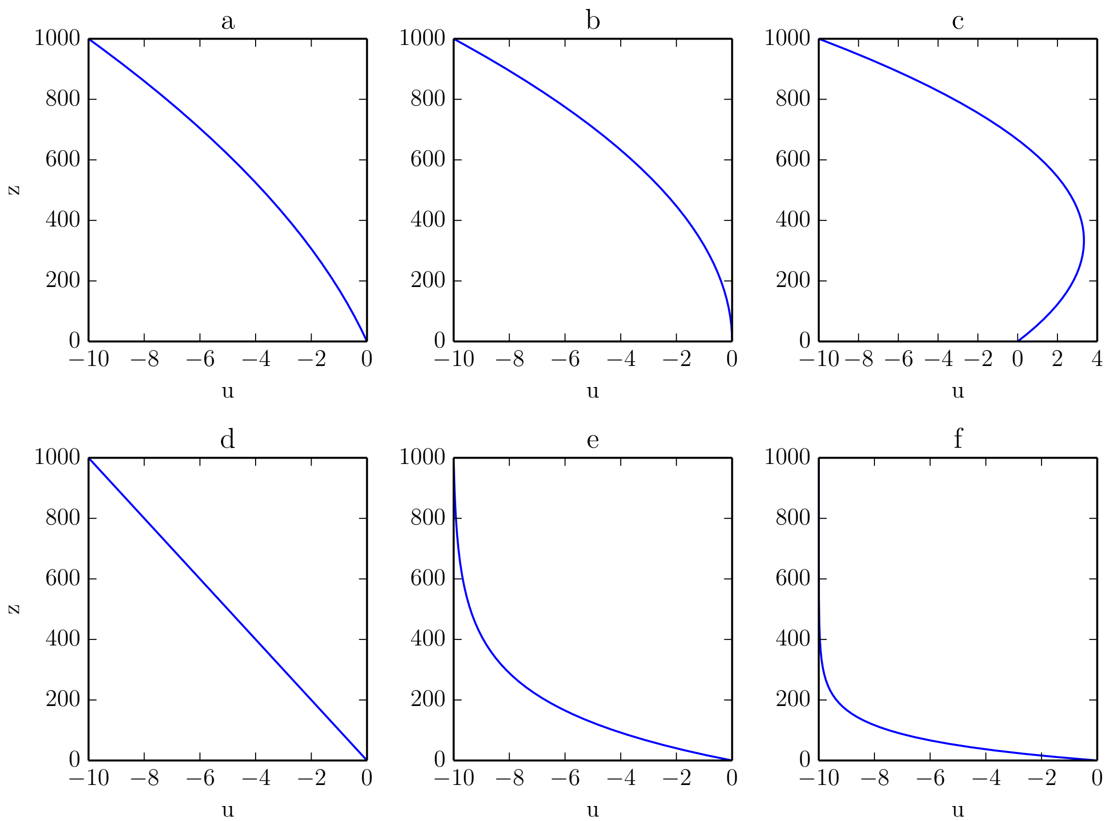

Figure 3.1. Exact solutions to Problem 2. *(a)*: $V = 0$, $S = S_c/2$; *(b)*: $V = 0$, $S = S_c$; *(c)*: $V = 0$, $S = 3\,S_c$; *(d)*: $V = 0$, $S = 0$; *(e)*: $V = -0.2$ m/yr, $S = 0$; *(f)*: $V = -0.5$ m/yr, $S = 0$.

Chapter 4
Variational Formulation

In this chapter we derive the weak form of Problem 1 (page 6). We find that, due to the free boundary, this weak form is a variational inequality. We will see that shifting our problem from one about a differential equation into a variational inequality loosens the regularity requirements for the problem's solution and initial data. This variational formulation benefits by being and is directly approachable theoretically, and will later translate directly to numerical approximation via the finite element method.

4.1 The Weak Form

First, we define the *solution set* $\mathcal{K}$ of admissible functions; solutions to our weak problem will belong to $\mathcal{K}$. In the strong form, solutions to Problem 1 have a square-integrable second derivative; in the weak formulation, for reasons that will be justified below, we will loosen this requirement, and permit solutions to come from a subset of the Sobolev space $H^1((0,\ell))$

$$\mathcal{K} = \left\{ u \in H^1((0,\ell)) | u \leq 0,\ u(0) = 0,\ \text{and } u(\ell) = T_0 \right\}.$$

For a definition of H^1, and related function spaces, see Appendix A.

Now, suppose $u \in \mathcal{K}$ solves Problem 1. Recall from (2.2) that the heat flux in cold ice is

$$Q = -ku_z + \rho c V u. \tag{4.1}$$

This is known as the *sensible heat flux*, the energy flux associated with changes in temperature—in contrast, the *latent heat flux* associated with changes in phase, in the case of ice sheets melting/freezing or the advection of liquid water. We extend Q from the cold ice to the entire ice column $z \in [0,\ell]$. Note that for $z < s$ we have $u(z) = 0$ and $u_z(z) = 0$, so $Q = 0$ below the CTS. Since u solves Problem 1, it is differentiable. Similarly, we know $u_z(s^+) = 0$, so $Q(s^+) = 0$ as well.

In deriving the weak formulation (Problem 3 below) we make the further assumptions that Q has two-sided limits at each point and an integrable derivative. If $\phi \in \mathcal{K}$, then

$$\begin{aligned}\int_0^\ell Q(\phi - u)_z\, dz &= \int_0^s Q(\phi - u)_z\, dz + \int_s^\ell Q(\phi - u)_z\, dz \\ &= [Q(\phi - u)]_0^{s^-} - \int_0^s Q_z(\phi - u)\, dz + [Q(\phi - u)]_{s^+}^\ell - \int_s^\ell Q_z(\phi - u)\, dz \\ &= -\int_0^s Q_z(\phi - u)\, dz - \int_s^\ell Q_z(\phi - u)\, dz. \end{aligned} \tag{4.2}$$

The boundary terms vanish in the last step because $\phi(0) = u(0)$, $\phi(\ell) = u(\ell)$, $Q(z) = 0$ for $0 < z < s$, and $Q(s^+) = 0$.

Since $u(z) = 0$ where $0 < z < s$, and $\phi \in \mathcal{K}$, it follows that $\phi(z) - u(z) = \phi(z) \le 0$ on $0 < z < s$. Note that $Q_z = 0$ on $0 < z < s$ so $Q_z \le S$ on $0 < z < s$. Similarly, by (2.13) we have $Q_z = S$ in the cold ice, $s < z < \ell$. From (4.2), we conclude:

$$\begin{aligned} -\int_0^\ell Q(\phi - u)_z \, dz &= \int_0^s Q_z(\phi - u) \, dz + \int_s^\ell Q_z(\phi - u) \, dz \\ &\ge \int_0^s S(\phi - u) \, dz + \int_s^\ell S(\phi - u) \, dz \\ &= \int_0^\ell S(\phi - u) \, dz. \end{aligned}$$

Expanding Q, we arrive at a *variational inequality* for the temperature u:

$$\int_0^\ell (ku_z - \rho c V u)(\phi - u)_z - S(\phi - u) \, dz \ge 0, \text{ for all } \phi \in \mathcal{K}. \tag{4.3}$$

We will treat (4.3) with an abstract theory later, so we define a bilinear form $a : H^1((0,\ell)) \times H^1((0,\ell)) \to \mathbb{R}$, and linear functional $f : H^1((0,\ell)) \to \mathbb{R}$ via

$$a(u, v) = \int_0^\ell (ku_z - \rho c V u) v_z \, dz \tag{4.4}$$

$$f(v) = \int_0^\ell S v \, dz. \tag{4.5}$$

and rewrite (4.3) as:

$$a(u, \phi - u) \ge f(\phi - u).$$

We now state the weak formulation of the temperature-only formulation for polythermal glaciers:

Problem 3 (The Variational Formulation). Suppose $V \in C^1([0,\ell])$ and $S \in L^2([0,\ell])$. Define the abstract bilinear form $a : H^1((0,\ell)) \times H^1((0,\ell)) \to \mathbb{R}$ by (4.4) and the linear functional $f : H^1((0,\ell)) \to \mathbb{R}$ by (4.5). Find $u \in \mathcal{K}$ so that

$$a(u, \phi - u) \ge f(\phi - u) \tag{4.6}$$

for all $\phi \in \mathcal{K}$.

4.2 An Interior Condition

When comparing our weak (variational) formulation to the strong (ODE) formulation, we should ask under what conditions the variational inequality implies the ODE.

Proposition 1. *If $u \in \mathcal{K}$ solves Problem 3, and if $u \in C^1([0,\ell]) \cap C^2(O)$, $V \in C^1([0,\ell])$ and $S \in C^0([0,\ell])$, then on the set $O = \{z | u(z) < 0\}$ we have equation (2.13), namely*

$$-(ku_z)_z + \rho c(Vu)_z = S.$$

Proof. Let $\psi \in C_c^\infty(O)$. Since ψ is bounded, and u is continuous and negative on O, there exists $\epsilon > 0$ so that $\phi_\pm = u \pm \epsilon\psi \in \mathcal{K}$. Then, by (4.3) with $\phi = \phi_\pm$, we find

$$\pm\epsilon \int_0^\ell (ku_z - \rho cVu)\psi_z - S\psi \, dz \geq 0.$$

So,

$$0 = \int_0^\ell (ku_z - \rho cVu)\psi_z - S\psi \, dz.$$

Integrating by parts, we find:

$$\begin{aligned} 0 &= (ku_z - \rho cVu)\psi|_0^\ell - \int_0^\ell (ku_z - \rho cVu)\psi \, dz - \int_0^\ell S\psi \, dz \\ &= \int_0^\ell \left[-\left(ku_z - \rho cVu\right)_z - S\right]\psi \, dz, \end{aligned}$$

where the boundary terms vanish because ψ has compact support in O. Since ψ has arbitrary support in O, it follows that

$$-\left(ku_z - \rho cVu\right)_z - S = 0$$

almost everywhere on O, i.e. u satisfies the strong-form ODE (2.13) on O. $\square$

Chapter 5
Theory of the Variational Inequality

Now that we have a variational problem, we use the theory of variational inequalities to reach some conclusions about Problem 3 (page 11). In particular we determine two conditions under which Problem 3 is well-posed.

5.1 Calculus of Variations

Our guiding result will be a standard one from the theory of the calculus of variations, which gives a sufficient condition under which our variational problem is well posed. We begin with a definition:

Definition 2. *Given a vector space V and bilinear form $a : V \times V \to \mathbb{R}$, we say that a is* coercive *if there exists $\alpha > 0$ so that*

$$|a(u,u)| \geq \alpha \|u\|_V^2 .$$

Coercivity, it turns out, is a sufficient condition for us to establish both existence and uniqueness of solutions to particular variational inequalities, provided our solution spaces are sufficiently nice. We will use the following result to determine a sufficient condition under which Problem 3 has a solution.

Theorem 3 ([10] Theorem 2.1)**.** *Let H be a Hilbert space, and $\mathcal{K} \subseteq H$ be closed and convex. Let $a : H \times H \to \mathbb{R}$ be a coercive bilinear form, and $f \in H'$ be an element of H's dual space Then there exists a unique $u \in \mathcal{K}$ so that*

$$a(u, v-u) \geq f(v-u) \qquad \text{for all } v \in \mathcal{K}. \tag{5.1}$$

Furthermore, the mapping $f \mapsto u$ is Lipshitz. That is, if u_1 and u_2 solve (5.1) for f_1 and f_2 in H', respectively, then

$$\|u_1 - u_2\|_H \leq \frac{1}{\alpha} \|f_1 - f_2\|_{H'} ,$$

where α is a's constant of coercivity. [10]

5.2 Preliminaries

From here out our job is to prove that Problem 3 can be shown, using Theorem 3 to be well posed. To begin, we establish some facts about the function spaces in Problem 3. We begin by stating a technical lemma. Recall the norms [4]

$$\|g\|_{L^p} = \left(\int_0^\ell |g|^p\right)^{\frac{1}{p}}, \qquad \|h\|_{L^\infty} = \operatorname*{ess\,sup}_{x\in(0,\ell)} |h(x)|, \qquad \|f\|_{H^1} = \|f\|_{L^2} + \|f'\|_{L^2}.$$

Lemma 4. *If $g \in H^1((0,\ell))$, then*

$$\|g\|_{L^\infty} \le \max\{\ell^{1/2}, \ell^{-1/2}\} \|g\|_{H^1}.$$

This is an application of Lemma 22 (page 39) to the interval $(0,\ell)$. Using this lemma, we derive a geometric result for our solution space $\mathcal{K}$, defined in Chapter 4.

Proposition 5. *$\mathcal{K}$ is convex and closed.*

Proof. Suppose $f, g \in \mathcal{K}$, and $\tau \in [0,1]$. Consider the convex sum $h = f + (1-\tau)g$. Note that $h \le 0$ because $f, g \le 0$. Moreover, h satisfies the boundary conditions:

$$(\tau f + (1-\tau)g)(0) = \tau \cdot 0 + (1-\tau)\cdot 0 = 0,$$

and

$$(\tau f + (1-\tau)g)(\ell) = \tau T_0 + (1-\tau)T_0 = T_0,$$

so $h \in \mathcal{K}$, and $\mathcal{K}$ is convex.

Now, suppose $\{u_i\}_{i=1}^\infty$ is a sequence in $\mathcal{K}$ so that $u_i \to u$ for some $u \in H^1((0,\ell))$. We need to show $u \in \mathcal{K}$. Let $\epsilon > 0$. Let $C = \max\left\{\ell^{1/2}, \ell^{-1/2}\right\}$. Since $\{u_i\}_{i=1}^\infty$ converges, there exists some $N > 0$ so that $\|u - u_n\|_{H^1} < \epsilon/C$ for all $n \ge N$. Then, for $z \in (0,\ell)$, by Lemma 4,

$$|u(z) - u_n(z)| \le \|u - u_n\|_{L^\infty} \le C\|u - u_n\|_{H^1} < C\frac{\epsilon}{C} = \epsilon.$$

Since $u_n(z) \le 0$ it follows that $u(z) < \epsilon$. Because $\epsilon > 0$ is arbitrary we may conclude that $u(z) \le 0$.

A similar argument shows that

$$\epsilon > |u(0) - u_n(0)| = |u(0) - 0| = |u(0)| \ge 0$$

thus $u(0) = 0$, and

$$\epsilon > |u(\ell) - u_n(\ell)| = |u(\ell) - T_0|$$

thus $u(\ell) = T_0$. So $u \in \mathcal{K}$, and thus $\mathcal{K}$ is closed. $\square$

5.3 Equivalence to a Zero Boundary-Condition Problem

Since $\mathcal{K} \subseteq H^1((0,\ell))$, a standard approach for proving that our variational problem is well-posed requires us to prove that our bilinear form a is coercive on $H^1((0,1))$. These coercivity proofs typically involve integration by parts, and can be complicated by the non-zero boundary terms found in integration by parts for $H^1((0,1))$. We can make things easier on ourselves—both in this continuum problem, and later when we solve Problem 3 numerically—by showing that finding a solution $u \in \mathcal{K}$ to Problem 3 is equivalent to finding an solution to a related variational inequality in a solution space with zero boundary values.

First, define $g : [0,\ell] \to \mathbb{R}$, by $g(z) = (T_0/\ell)z$. Note that $g \in \mathcal{K}$. We define the function space $\mathcal{K}_g \subseteq H_0^1((0,\ell))$:

$$\mathcal{K}_g = \{w \in H_0^1((0,\ell)) : w \leq -g\}.$$

By the same argument as made in Proposition 5, $\mathcal{K}_g$ is closed and convex. Note that if $w \in \mathcal{K}_g$, then $w + g \leq -g + g = 0$, Moreover $(w+g)(0) = 0$ and $(w+g)(\ell) = 0 + T_0 = T_0$, so $w + g \in \mathcal{K}$. Conversely, by similar arguments, if $u \in \mathcal{K}$, then $u - g \in \mathcal{K}_g$.

Now, since a is a bilinear form, if $u \in \mathcal{K}$ and $v \in H_0^1((0,\ell))$, then there exists some $w \in \mathcal{K}_g$ so that $u = w + g$ and

$$\begin{aligned} a(u,v) &= a(w+g,v) \\ &= a(w,v) + a(g,v). \end{aligned}$$

We rewrite (4.6) as:

$$\begin{aligned} a(w,v) &\geq \langle S,v\rangle_{L^2} - a(g,v) \\ &= F(v), \end{aligned}$$

where $\langle \cdot,\cdot\rangle_{L^2}$ is the inner product on $L^2([0,\ell])$ and $F : H_0^1 \to \mathbb{R}$ is defined by:

$$\begin{aligned} F(v) &= \int_0^\ell Sv - kg_z v_z + \rho c V g v_z \, dz \\ &= \int_0^\ell Sv \, dz - k\left[g_z v\Big|_0^\ell - \int_0^\ell g_{zz} v \, dz \right] + \int_0^\ell \rho c V g v_z \, dz \\ &= \int_0^\ell Sv + \rho c V g v_z \, dz. \end{aligned} \tag{5.2}$$

Compare the above with (4.5) which defined $f(v)$. It is now straight-forward to see that finding a solution $u \in \mathcal{K}$ to Problem 3 is equivalent to solving the following:

Problem 4. Let $g = (t_0/\ell)z$. Find $w \in \mathcal{K}_g$ so that

$$a(w, v - w) \geq F(v - w), \qquad \text{for all } v \in \mathcal{K}_g,$$

where $F : H_0^1 \to \mathbb{R}$ and $\mathcal{K}_g$ are defined as above.

We can now leverage what we know about the space $H_0^1((0,\ell))$ to solve Problem 4—and by extension Problem 3—in both the continuum setting, and later when we approximate the problem numerically.

5.4 Well-posedness

Here we establish coercivity of $a(u,v)$, defined in (4.6), thus existence and uniqueness of a solution to the variational inequality, based upon bounds on V. We begin by proving a version of Poincaré's inequality specific to our situation.

Lemma 6 (Poincare's Inequality)**.** *If $u \in H_0^1((0,\ell))$ then*

$$\int_0^\ell u^2 \, dz \leq \ell^2 \int_0^\ell u_z^2 \, dz,$$

or equivalently, $\|u\|_{L^2} \leq \ell \, \|u_z\|_{L^2}$.

Proof. By , there exists a sequence $\{u_i\} \subseteq C^\infty((0,\ell)) \cap H_0^1((0,\ell)$ so that $u_i \to u$ in H_0^1. By the Fundamental Theorem of Calculus

$$u_i(t) = \int_0^t (u_i)_z(z) \, dz.$$

So, by the Cauchy-Schwarz inequality,

$$\begin{aligned}
|u_i(t)| &= \left| \int_0^t (u_i)_z \, dz \right| \\
&\leq \left(\int_0^t 1 \right)^{\frac{1}{2}} \left(\int_0^t (u_i)_z^2 \, dz \right)^{\frac{1}{2}} \\
&= \sqrt{t} \cdot \left(\int_0^t (u_i)_z^2 \, dz \right)^{\frac{1}{2}} \\
&\leq \ell^{\frac{1}{2}} \left(\int_0^\ell (u_i)_z^2 \, dz \right)^{\frac{1}{2}}.
\end{aligned}$$

We square both sides, and integrate over $(0,\ell)$ to find:

$$\begin{aligned}
\int_0^\ell |u_i(t)|^2 \, dt &\leq \int_0^\ell \ell \left(\int_0^\ell (u_i)_z^2 \, dz \right) \, dt \\
&= \ell^2 \int_0^\ell (u_i)_z^2 \, dz,
\end{aligned}$$

or

$$\|u_i\|_{L^2} \leq \ell \, \|(u_i)_z\|_{L^2} .$$

Since convergence in H^1 implies convergence in L^2, we find by continuity:

$$\begin{aligned} \|u\|_{L^2} &= \lim_{i\to\infty} \|u_i\|_{L^2} \\ &\leq \ell \lim_{i\to\infty} \|(u_i)_z\|_{L^2} \\ &= \|u_z\|_{L^2} , \end{aligned}$$

as needed. □

The benefit gained, over the more general result due to Poincaré, is the explicit constant in the inequality. For more general bounded, convex domains, similar estimates can be calculated [14] [2], but for our one-dimensional case, this is sufficient.

Proposition 7. *If $k > \rho c \ell V_0$, where $V_0 = \sup_{z\in[0,\ell]} |V(z)|$, there exists a unique solution to Problem 4, and thus to Problem 3.*

Proof. We show that, under the given assumptions, $a : H_0^1 \times H_0^1 \to \mathbb{R}$, as defined above, is coercive. Suppose $\phi \in H_0^1((0,\ell))$. Then

$$a(\phi,\phi) = \int_0^\ell k\phi_z^2 - \rho c V \phi\phi_z \, dz.$$

So, by the triangle and Cauchy-Schwarz inequalities, we find:

$$\begin{aligned} \left| \int_0^\ell k\phi_z^2 \, dz \right| &= \left| a(\phi,\phi) + \int_0^\ell \rho c V \phi\phi_z \, dz \right| \\ &\leq |a(\phi,\phi)| + \left| \int_0^\ell \rho c V \phi\phi_z \, dz \right| \\ &\leq |a(\phi,\phi)| + \rho c \, \|V\phi\|_{L^2} \, \|\phi_z\|_{L^2} \\ &\leq |a(\phi,\phi)| + \rho c V_0 \, \|\phi\|_{L^2} \, \|\phi_z\|_{L^2} , \end{aligned}$$

By Lemma 6,

$$k \, \|\phi_z\|_{L^2}^2 = \left| \int_0^\ell k\phi_z^2 \, dz \right| \leq |a(\phi,\phi)| + \rho c V_0 \ell \, \|\phi_z\|_{L^2}^2 .$$

Recall that the L^2 and H^1 norms are equivalent on H_0^1 (see Lemma 17). Rearranging then yields:

$$\begin{aligned} |a(\phi,\phi)| &\geq (k - \rho c V_0 \ell) \, \|\phi_z\|_{L^2} \\ &\geq (k - \rho c V_0 \ell) \, \alpha \, \|\phi\|_{H_0^1} , \end{aligned}$$

where $\alpha > 0$ is some real number. Thus, when $k > \rho c \ell V_0$, a is coercive on $H_0^1((0,\ell))$.

So, by Theorem 3, there exists a unique $w \in H_0^1((0,\ell))$ which solves Problem 4. 3. $\square$

This is our first real result, but it's unclear for physical reasons how much it buys us. Referring to Table 2.1 for typical values for k, ρ and c, we find that Proposition 7 requires, for a 1000 m thick glacier, that

$$\begin{aligned} V_0 &< \frac{k}{\rho c \ell} \\ &= \frac{2.1\,\mathrm{J\,m^{-1}\,K^{-1}\,s^{-1}}}{(910\,\mathrm{kg\,m^{-3}})\left(2009\,\mathrm{J\,kg^{-1}\,K^{-1}}\right)(1000\,\mathrm{m})} \\ &\approx 1.1 \times 10^{-9}\,\mathrm{m\,s^{-1}} \\ &\approx 0.036\,\mathrm{m\,a^{-1}}. \end{aligned}$$

This is a restrictive condition on the downward velocity.

However, if we assume some regularity in V, we may prove a useful existence and uniqueness theorem with bounds on the first derivative of V:

Proposition 8. *Let $V \in H^1((0,\ell))$. If there exists some $M \geq 0$ so that $-M \leq V_z(z)$ for all z and if $k - M\ell^2\rho c/2 \geq 0$, then there exists a unique solution to Problem 4.*

Proof. Suppose $\phi \in H_0^1((0,\ell))$. Then

$$a(\phi,\phi) = \int_0^\ell k\phi_z^2 - \rho c V \phi \phi_z \, dz.$$

Noting that $(\phi^2)_z = 2\phi\phi_z$, if $V_0 = \sup_{z\in[0,\ell]} V(z)$ we find:

$$\begin{aligned} a(\phi,\phi) &= \int_0^\ell k\phi_z^2 \, dz - \frac{\rho c}{2} \int_0^\ell V \left(\phi^2\right)_z \, dz \\ &= \int_0^\ell k\phi_z^2 \, dz - \frac{\rho c}{2} \left[V\phi^2\Big|_0^\ell - \int_0^\ell V_z \phi^2 \, dz \right] \\ &= \int_0^\ell k\phi_z^2 \, dz + \frac{\rho c}{2} \int_0^\ell V_z \phi^2 \, dz \\ &\geq \int_0^\ell k\phi_z^2 \, dz - \frac{\rho c}{2} M \int_0^\ell \phi^2 \, dz \\ &= k \left\|\phi_z\right\|_{L^2}^2 - M\frac{\rho c}{2} \left\|\phi\right\|_{L^2}^2 \\ &\geq k \left\|\phi_z\right\|_{L^2}^2 - M\frac{\rho c}{2}\ell^2 \left\|\phi_z\right\|_{L^2}^2 \\ &= \left(k - M\ell^2\rho c/2\right) \left\|\phi_z\right\|_{L^2}^2, \end{aligned}$$

where we have used Lemma 6 and the fact that $\phi(0) = \phi(\ell) = 0$. Now, by Lemma 17, the L^2 and H^1 norms are equivalent in $H_0^1((0,\ell))$, so there exists some $\alpha > 0$ so that

$$\begin{aligned} a(\phi,\phi) &\geq \left(k - M\ell^2\rho c/2\right) \|\phi_z\|_{L^2}^2 \\ &= \left(k - M\rho\ell^2 c/2\right) \alpha \|\phi\|_{H^1}^2 \geq 0. \end{aligned}$$

Thus a is a coercive bilinear form. Again, by Theorem 3, there exists a unique $w \in H_0^1((0,\ell))$ which solves Problem 4. □

This yields the immediate corollary:

Corollary 9. *If $V(z) = V_0$ is a constant function, there exists a unique solution to Problem 3.*

Constant velocity aside, we should see how much flexibility this buys us with nonconstant velocities. Notice V_z is a strain rate with units s^{-1}. Referring to Table 2.1, we find that for an 1000 m thick glacier, we require:

$$\begin{aligned} V_z &> -\frac{2k}{\rho c \ell^2} \\ &= -2 \cdot \frac{2.1\,\mathrm{J\,m^{-1}\,K^{-1}\,s^{-1}}}{(910\,\mathrm{kg\,m^{-3}})\left(2009\,\mathrm{J\,kg^{-1}\,K^{-1}}\right)(1000\,\mathrm{m})\,i^2} \\ &\approx -2.3 \times 10^{-12}\,\mathrm{s^{-1}} \\ &\approx -7.2 \times 10^{-5}\,\mathrm{a^{-1}}\,. \end{aligned}$$

This, it turns out, may be an unreasonable bound; Fig 11.8 in [13] shows that strain rates in a borehole though Devon Island's ice cap in Canada were measured at $\sim 10^{-3}\,\mathrm{a}^{-1}$, which is outside our theoretical limit.

Chapter 6
Numerical Analysis

Now that we know that our problem is well-posed in at least some circumstances, we change gears and numerically approximate its solutions. Our primary tool will be the Finite Element Method (FEM)[3], in which our variational problem suggests a method of numerical approximation.

6.1 The Finite Element Method

Before digging into Problem 3, we'll use a more straightforward problem to illustrate the basics of the FEM, and how the variational form of our problem suggests that it may be the right tool.

Our subject for this illustration is Poisson's equation in one-dimension:

$$-u_{xx} = f, \tag{6.1}$$

where u and f are real-valued functions defined on some open set $U \subseteq \mathbb{R}$. We will concern ourselves with approximating solutions to the boundary-value problem, with Dirichlet (though not necessarily homogeneous) boundary values, defined on the closed unit interval:

Problem 5. Given $n, m \in \mathbb{R}$, find $u \in C^2([0,1])$ which satisfies Poisson's equation (6.1) so that $u(0) = m$ and $u(1) = n$.

6.1.1 The Weak Form of Poisson's Equation

The FEM deals with the weak form of differential problems, so we will derive the weak form of Poisson's equation. This should appear similar to the derivation for Problem 3.

Suppose that $u \in C^2((0,1))$ satisfies (6.1) for all $x \in (0,1)$. Suppose that $\phi \in C_c^\infty((0,1))$, then via integration by parts we find:

$$\begin{aligned} -\int_0^1 f\,\phi\,dx &= \int_0^1 u_{xx}\,\phi\,dx \\ &= -\phi\,u_x\big|_0^1 + \int_0^1 \phi_x\,u_x\,dx \\ &= \int_0^1 \phi_x\,u_x\,dx. \end{aligned} \tag{6.2}$$

We call (6.2) the weak form of Poisson's equation. Note that the weak form only references the first derivative of u, allowing a wider possible range of solutions. We take the left hand side of (6.2) and define the bilinear form $B : H^1((0,1)) \times H_0^1((0,1)) \to \mathbb{R}$ by

$$B(u, \phi) = \int_0^1 \phi_x \, u_x \, dx,$$

where, u_x is the first derivative of u in a *weak sense*, as outlined in Appendix A. We rewrite (6.2), and say that when $u \in H^1$ satisfies

$$B(u, \phi) = \langle f, \phi \rangle_{L^2} \tag{6.3}$$

for all $\phi \in H_0^1$, then u satisfies the weak form of Poisson's equation. We also note that if $u \in C^2$ satisfies the weak form of Poisson's equation, it in turn satisfies the strong form, (6.1), by the same argument as that made in Proposition 1.

Boundary conditions are imposed on the weak problem by selecting a solution space $\mathcal{U} \subseteq H^1$. This is straightforward for our one-dimensional problem but in higher dimensions boundary conditions are imposed in a *trace sense*, which we will not elaborate on here, but which is developed in [4]. For our Dirichlet boundary-values, we define a solution space,

$$\mathcal{U} = \{u \in H^1((0,1)) : u(0) = m, \text{ and } u(1) = n\}.$$

Now, the weak form of Problem 5 is:

Problem 6 (Poisson's Equation (Weak version)). Find $u \in \mathcal{U}$ so that

$$B(u, \phi) = \langle f, \phi \rangle_{L^2},$$

for all $\phi \in H_0^1((0,1))$.

We should note that since our solution space $\mathcal{U}$ is not closed under addition, it is not a vector space. For reasons that will (hopefully) become clear below, this is not ideal, so we will form an equivalent problem to Problem 6 with a solution space which is a vector space, and whose solution can be trivially converted to a solution to Problem 6.

Note that the function $g : [0,1] \to \mathbb{R}$, defined by $g(x) = m + (n-m)x$ satisfies $g(0) = m$ and $g(1) = n$, and moreover $g_{xx} = 0$. So, if $u \in U$ is a solution to Problem 6 then define

$\bar{u} = u - g$, and note that for all $\phi \in H_0^1([0,1])$:

$$\begin{aligned}\int_0^1 f\,\phi\,dx &= \int_0^1 \phi_x\,u_x\,dx \\ &= \int_0^1 \phi_x\,\left(\bar{u}_x + g_x\right)\,dx \\ &= \int_0^1 \phi_x\,\bar{u}_x\,dx + \phi\,g_x\Big|_0^1 - \int_0^1 \phi\,g_{xx}\,dx \\ &= \int_0^1 \phi_x\,\bar{u}_x\,dx.\end{aligned}$$

Then $\bar{u}$ satisfies the weak form of Poisson's equation. By the discussion in Appendix A.4, $\bar{u} \in H_0^1([0,1])$, so we define

Problem 7. Find $\bar{u} \in H_0^1([0,1])$ so that

$$B(\bar{u}, \phi) = \langle f, \phi \rangle_{L^2},$$

for all $\phi \in H_0^1\left((0,1)\right)$.

By the calculation above, the function $u = \bar{u} + g$ then solves Problem 6. We now have a problem whose solution space is a vector space, the structure of which we exploit below.

6.1.2 The Discrete Formulation

The weak formulation couches an ODE problem in terms of infinite-dimensional linear function spaces. The FEM approximates these infinite dimensional spaces with finite-dimensional subspaces, and solves the matrix problem which results.

Suppose we have a partition of $[0,1]$: $0 = x_0 < x_1 < \cdots < x_{J-1} < x_J = 1$. We approximate $H_0^1((0,1))$ by the space $V = \text{span}\{\psi_1, \cdots, \psi_{J-1}\}$, where ψ_i is the continuous, piece-wise linear function where

$$\begin{aligned}\psi_i(x_i) &= 1 \\ \psi_i(x) &= 0 \text{ if } x \notin (x_{i-1}, x_{i+1}).\end{aligned}$$

Note that each $\psi_i \in H_0^1((0,1))$, so V is a finite-dimensional subspace of $H_0^1((0,1))$. We can restrict B to V to form the discrete version of Problem 7:

Problem 8. Find $u \in V \subseteq H_0^1((0,1))$ so that

$$B(u, v) = \langle f, v \rangle_{L^2}, \tag{6.4}$$

for all $v \in V$.

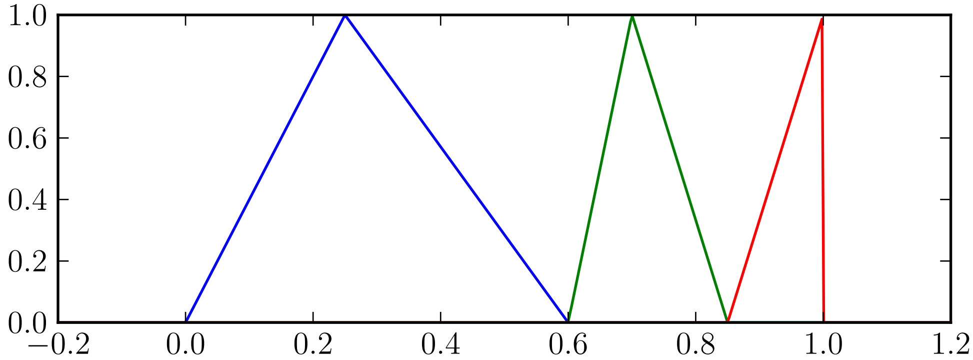

Figure 6.1. Basis functions ψ_i on an irregular grid

Since V is a finite-dimensional subspace and $\{\psi_i\}_{i=1}^{J-1}$ a basis we can now use our tools from linear algebra to solve Problem 8. If $u \in V$, it can be expressed in terms of our basis:

$$u(x) = \sum_{i=1}^{J-1} u_i \psi_i(x).$$

Then, using a basis function, ψ_k as a test function we compute:

$$\begin{aligned} B(u, \psi_k) = \int_0^1 (\psi_k)_x \, u_x dx &= \int_0^1 (\psi_k)_x \sum_{i=1}^{J-1} u_i \, (\psi_i)_x \, dx \\ &= \sum_{i=1}^{J-1} u_i \int_0^1 (\psi_k)_x \, (\psi_i)_x \, dx \\ &= \sum_{i=1}^{J-1} u_i B(\psi_i, \psi_k) \\ &= u_{k-1} B(\psi_{k-1}, \psi_k) + u_k B(\psi_k, \psi_k) + u_{k+1} B(\psi_{k+1}, \psi_k). \end{aligned}$$

This suggests a matrix representation of B on V in terms of $\{\psi_i\}_{i=1}^{J-1}$. In fact, if $v \in V$ and $v = \sum_{k=1}^{J-1} v_k \psi_k$, then

$$B(u, v) = v^T B u = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_{J-1} \end{pmatrix}^T \begin{pmatrix} b_{11} & b_{12} & \cdots & 0 \\ b_{21} & b_{22} & b_{23} & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & & b_{J-1\,J-2} & b_{J-1\,J-1} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{J-1} \end{pmatrix}.$$

We can compute the entries of $B\big|_V$:

$$b_{ij} = B(\psi_i, \psi_j) = \begin{cases} \int_{x_{i-1}}^{i+1} (\psi_i)_x^2 \, dx, & i = j, \\ \int_{x_i}^{x_{i+1}} (\psi_i)_x \, (\psi_{i+1})_x \, dx, & j = i+1, \\ \int_{x_{i-1}}^{x_i} (\psi_i)_x \, (\psi_{i-1})_x \, dx, & j = i-1, \\ 0, & \text{otherwise.} \end{cases}$$

More detail is given in Appendix B And, given v, we can compute $\langle f, v \rangle_{L^2}$:

$$\begin{aligned} \int_0^1 f \, v \, dx &= \int_0^1 f \sum_{i=1}^{J-1} v_i \psi_i \, dx \\ &= \sum_{i=1}^{J-1} v_i \int_0^1 f \, \psi_i \, dx \\ &= \sum_{i=1}^{J-1} v_i \int_{x_{i-1}}^{x_{i+1}} f \, \psi_i \, dx. \end{aligned}$$

Here, we write (6.4) as:

$$[v_i]^T [b_{ij}][u_j] = [v_i]^T [c_i], \tag{6.5}$$

where $c_i = \int_{x_{i-1}}^{x_{i+1}} f \, \psi_i \, dx$. If $B = [b_i j]$ and $c = [c_i]$, linearity tells us that the solution $u \in V$ to

$$Bu = c, \tag{6.6}$$

solves Problem 8.

6.2 Approximation of the Unconstrained Problem

Recall from 4.2, that away from the constraint, the solution of temperature-only variational Problem 3 solves the ODE found in the unconstrained Problem 2. Our first step in developing an approximation of the constrained Problem 3 is to build a FEM approximation of the unconstrained Problem 2.

6.2.1 Weak Formulation

As in our example for Poisson's equation above, we first derive the weak form of (3.2).

Problem 9 (The Weak Form of the Unconstrained Problem). Suppose $V \in C^1([0, \ell])$ and $S \in L^2([0, \ell])$ so that $V(z) \leq 0$ and $S(z) \geq 0$ a.e. Define the set

$$\mathcal{M} = \left\{ u \in H^1((0, \ell)) : u(0) = 0 \text{ and } u(\ell) = T_0 \right\},$$

and recall the bilinear form a from (4.4):

$$a(u, v) = \int_0^\ell (ku_z - \rho cVu)v_z \, dz. \tag{6.7}$$

Find $u \in \mathcal{M}$ so that

$$a(u, \phi - u) = \langle S, \phi - u \rangle_{L^2}, \qquad \text{for all } \phi \in \mathcal{M}. \tag{6.8}$$

Luckily most of the work needed to show that this is indeed the weak form of the unconstrained problem has already been done above in the course of proving Proposition 1. In particular, with very minor adjustments, that argument proves that if some function u satisfies (6.8), and is twice differentiable, then it also satisfies the ODE (2.13).

Since the solution space $\mathcal{M}$ for our weak problem is not a vector space (it's not closed under addition) we again follow our example, and find an equivalent problem with solution space $H_0^1((0, \ell))$. Recall our definition of the function $g : [0, \ell] \to \mathbb{R}$ from section 5.3: $g(z) = (T_0/\ell)z$. Note that $g \in \mathcal{M}$. By the discussion in Appendix A.4, if $u \in \mathcal{M}$, there exists some $w \in H_0^1((0, \ell))$ so that $u = w + g$. Now, if $u, \phi \in \mathcal{M}$, there exists some $w \in H_0^1((0, \ell))$ so that

$$\begin{aligned}
a(u, \phi - u) &= \int_0^\ell (ku_z - \rho cVu)\,(\phi - u)_z \, dz \\
&= \int_0^\ell (k(w + g)_z - \rho cV(w + g))\,\zeta_z \, dz \\
&= \int_0^\ell (kw_z - \rho cVw)\,\zeta_z \, dz + \int_0^\ell (kg_z - \rho cVg)\,\zeta_z \, dz \\
&= a(w, \zeta) + a(g, \zeta),
\end{aligned}$$

where $\zeta = \phi - u =\in H_0^1((0, \ell))$. Variational equality (6.8) in Problem 9 then becomes

$$a(w, \zeta) = \langle S, \zeta \rangle - a(g, \zeta) = F(\zeta),$$

where, by the same calculations as those used above for (5.2),

$$F(\zeta) = \int_0^\ell S\zeta + \rho cVg\zeta_z \, dz. \tag{6.9}$$

We now define the continuum problem, which we will address numerically:

Problem 10. Suppose $V \in C^1([0,\ell])$ and $S \in L^2([0,\ell])$ so that $V(z) \leq 0$ and $S(z) \geq 0$ a.e. Define a by (6.7) and F by (6.9) Find $w \in H_0^1((0,\ell))$ so that

$$a(w,\zeta) = F(\zeta),$$

for all $\zeta \in H_0^1((0,\ell))$.

6.2.2 Finite Element Approximation

Since V, S, and g are known a priori as data, the right-hand side of the variational equality (6.8) can be thought of as a linear functional $F : H_0^1((0,\ell)) \to \mathbb{R}$. Also, since we are in a Hilbert space, a defines a linear map $A : H^1 \to H^1$ so that

$$a(u,v) = \langle Au, v \rangle .$$

Similarly, there exists some $f \in H_0^1$ so that $F(u) = \langle f, u \rangle$ for all $u \in H_0^1$.

Given a finite dimensional linear subspace $U \subseteq H^1((0,\ell))$, we seek a representation of the matrix A, and an approximation of f in U. We can split the bilinear form a into separate terms in the obvious way:

$$\begin{aligned} a(u,v) &= \int_0^\ell k u_z v_z - \rho c V u v_z \, dz \\ &= k \int_0^\ell u_z v_z \, dz - \rho c \int_0^\ell V u v_z \, dz \\ &= k\, b(u,v) - \rho c\, d(u,v). \end{aligned}$$

Following the discussion in Appendix B, we have a finite-dimensional function subspace $V \subseteq H^1$ based on a partition of $[0,\ell]$. We can compute matrices $B, D : V \to V$, corresponding to b and d respectively, so that $\langle B\,u, v\rangle = b(u,v)$ and $\langle D\,u, v\rangle = d(u,v)$. Then, $A = k\,B - \rho c D$, so that $\langle Au, v\rangle = a(u,v)$.

This author has written computer code to do this [5], using the Python packages Numpy and Scipy [9]. We compared the results these schemes calculated to exact solutions calculated in Section 3.1. We calculated the maximum error between the exact solutions u and the approximate solution $\bar{u}$ on the grid points, and found that our scheme converged at a measured rate of $O(\Delta z^2)$ on equally-spaced grids, which can be seen in Figure 6.2. This is the rate one would expect to occur, noting that in this case the finite element scheme outlined in Appendix B is equivalent to the centered-difference finite-difference scheme [12].

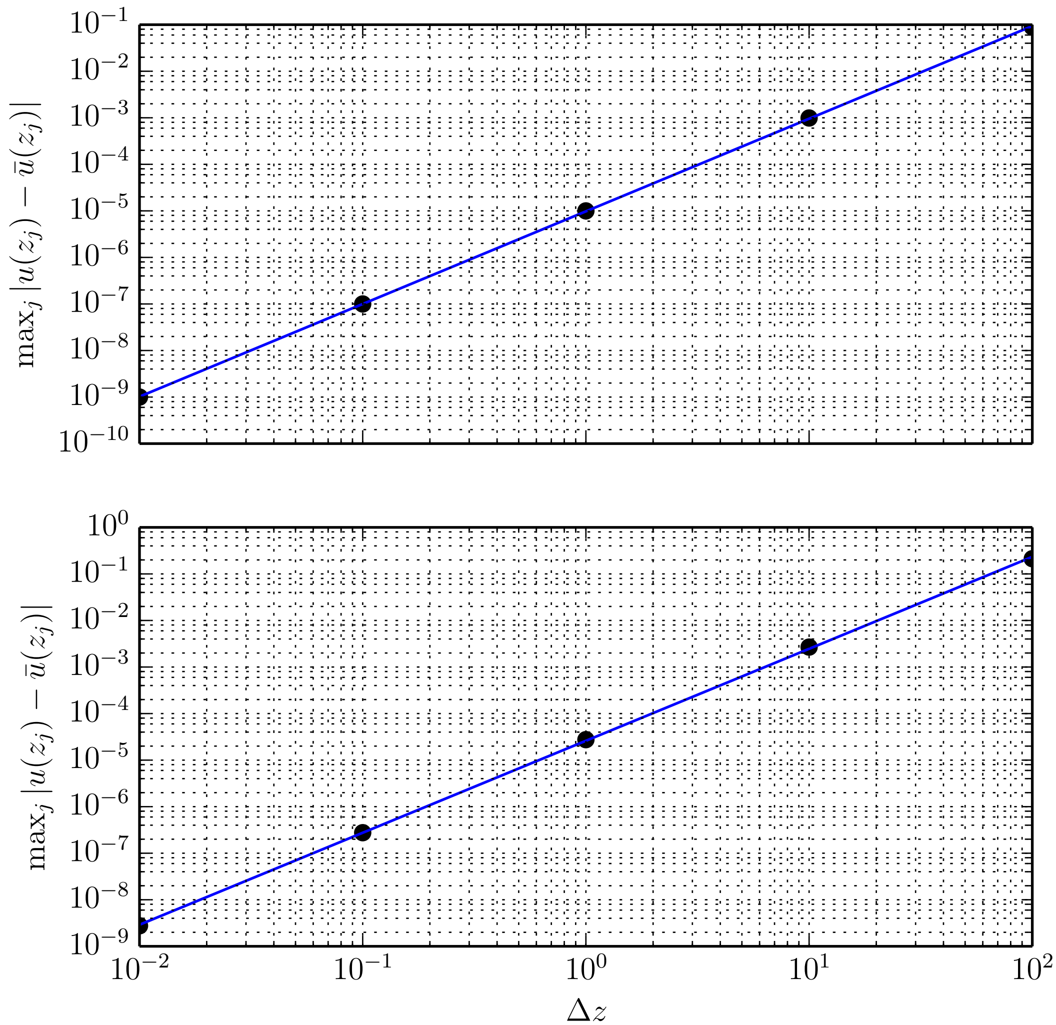

Figure 6.2. Convergence of FEM scheme for the unconstrained problem. Δz is the space between grid points on an equally spaced grid. Here $u(z_i)$ are the vlaues for the exact solution and $\bar{u}(z_i)$ are the values of the FEM approximation, and Δz is the grid spacing. *top* Zero Velocity, Constant Strain: Errors are $O\left(\Delta z^{1.990}\right)$. *bottom* Constant Velocity, Zero Strain: Errors are $O\left(\Delta z^{1.976}\right)$.

6.3 Approximation of the Temperature-Only Formulation

We now approximate the temperature-only formulation for glaciers and ice sheets (Problem 3, page 11). As we did for Poisson's equation and the unconstrained problem, the problem we actually solve numerically will be the zero boundary-value value problem (Problem 4, page 16), whose solution is easily transformed into the solution of Problem 3. There are two major differences between solving Problem 3 and the unconstrained Problem 2 in Chapter 3 (page 7):

1. Problem 3 and/or Problem 3 is a variational inequality while Problem 2 is a variational equation.

   Inequality 4.6 still transforms into an inequality with reference to a system of linear equations, but we can no longer drop the test function for free. This turns out not to be an issue, as we will see later.

2. Our solution space is no longer a vector space. Recall that our solutions to Problem 4 are from the set $\mathcal{K}_g \subseteq H_0^1((0,\ell))$, but $\mathcal{K}_g$ is not closed under addition.

Fortunately we have our interior condition, Proposition 1, which tells us that, off the constraint ($u < 0$), the strong form ODE (2.13) holds, whereas on the constraint ($u = 0$), we know the value for u. Using this result, we can modify the Gauss-Seidel technique for solving matrix equations to find numerical solutions to the variational inequality.

6.3.1 The Discrete Problem

Discretizing Problem 4 uses much of the work we have done above in forming the discrete version of the unconstrained problem. Recall that

$$a(u,v) = \int_0^\ell (ku_z - \rho c V u) v_z \, dz$$

and

$$F(v) = \int_0^\ell Sv + \rho c V g v_z \, dz,$$

where k, ρ, and c are positive real numbers, $V \in C^1([0,\ell])$, $S \in L^2([0,\ell])$ and $g(z) = (T_0/\ell)z$. Recall the definition of the solution space

$$\mathcal{K}_g = \{w \in H_0^1((0,\ell)) : w \leq -g\}.$$

Let $\{x_i\}_0^J$ be a partition of $[0,\ell]$ and ψ_i be the piecewise linear functions as above and let $U = \operatorname{span}\{\psi_i\}_{i=1}^{J-1}$. Note $g \in U$. We define the discrete solution space

$$K = \{u \in U : u \leq -g\}.$$

The discrete version of Problem 4 is thus:

Problem 11. Find $\bar{u} \in K$ so that

$$a(\bar{u}, v) \geq F(v),$$

for all $v \in U$.

As above, since a is a bilinear form, and F is a bounded linear functional, there exists a matrix $A : U \to U$ and vector $f \in U$ so that the inequality can be written as

$$\langle Au, v \rangle_{L^2} \geq \langle f, v \rangle_{L^2}.$$

As a result of Proposition 1, we know that for z where the solution to the continuum problem is off of the constraint, i.e. $u(z) < -g(z)$, the variational inequality becomes an equality. This suggests that when solving the discrete problem. We proceed iteratively: we begin by solving the unconstrained problem; if its solution is entirely below the constraint, $-g$, we have also solved the constrained problem, if not—and the unconstrained solution extends past the constraint—we set the points where it is greater than the constraint to the constraint, and use this adjusted "solution" as the starting point guess in an iterative solver.

Iterative matrix solvers are common in numerical linear algebra, especially for the solution of large, sparse systems where Gaussian-Elimination and QR factorization become computationally inefficient [6, 15]. Notable among these are the Krylov subspace methods (which we will mention but not use) and splitting methods such as Jacobi Iteration, Gauss-Seidel, and Successive Overrelaxation. We use a modification of Gauss-Seidel. In this modification, during the back-substitution step for each i, if the value of $u_i^{(k+1)}$ is such that it lies outside of $\mathcal{K}_g$ (i.e. if $u_i^{(k+1)} > -g(x_i)$) then we set $u_i^{(k+1)} = -g(x_i)$. This modification of Gauss-Seidel is known as *Constrained Successive Over-Relaxation* (CSOR).

In the case where our matrix A is symmetric and positive definite, a corollary to Theorem 23 states that Gauss-Seidel, and thus CSOR successfully converges. We should note that, for non-zero velocity, our finite element matrix A is not symmetric, and thus this result does not guarantee that CSOR will converge. Nevertheless, when we use CSOR to attempt to solve the constrained problem, with non-zero velocity, the algorithm *appears* to converge. For a selection of strain heats and ice velocities, we compared our the approximate solutions to both the constrained and unconstrained problems. As can be seen in Figure 6.3, when temperature never crosses the constraint (0°C) the solutions are identical as we expect. When temperature does meet the constraint, as in does when $V = 0$ and $S = 3 \cdot S_c$, it is interesting to note that the effect of the constraint is to push the CTS down in the ice column; the ODE solution has a CTS of ~ 700m, while the constrained variational problem has a CTS of ~ 500m.

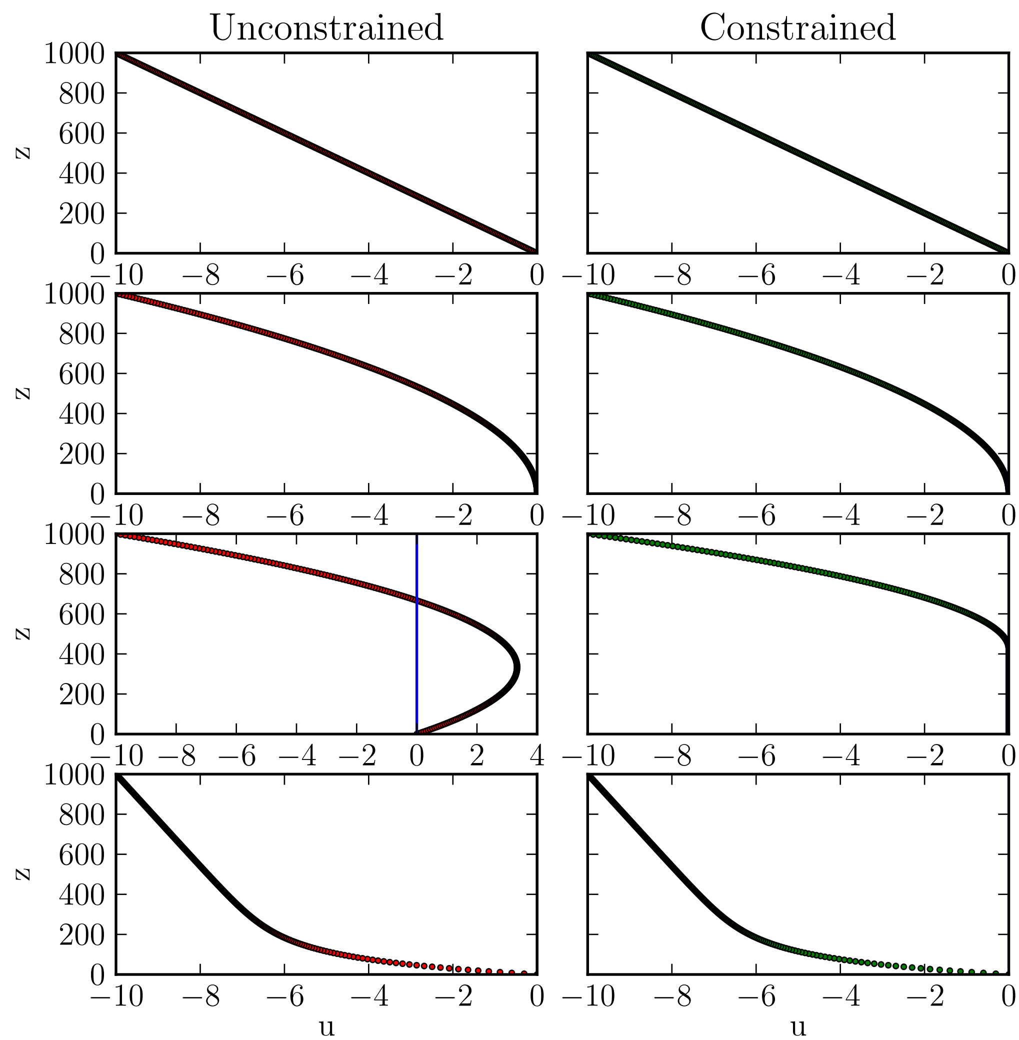

Figure 6.3. Comparison of approximate solutions to the constrained problem with approximate solutions to the unconstrained problem, for a selection of vertical velocities and strain heating terms. From top: (1) $V = 0$, $S = 0$, (2) $V = 0$, $S = S_c$, (3) $V = 0$, $S = 3 \cdot S_c$, (4) $V = -0.5$m/a, $S = 3 \cdot S_c$.

Chapter 7
Discussion

7.1 Theoretical Considerations

Recall that our sufficient conditions for the existence and uniqueness of solutions to Problem 3 require that either V be bounded by $k/\rho c\ell$ (Proposition 7) or that V_z be bounded below by $-2k/\rho c\ell^2$ (Proposition 8). As noted in Chapter 5,these restrictions (the condition on strain-rate V_x, in particular) may be too strict to be unphysical, but looser requirements would be nice to have in hand.

Moreover, throughout this thesis, we have made the assumption that the downward velocity of ice has a continuous derivative. However, the left-hand side of our variational inequality, the bilinear form a, is perfectly well-defined provided V is an integrable function. Since velocity fields of ice in glaciers and ice sheets do not necessarily have a continuous derivative, it could be useful to explore the well-posedness of a version of Problem 3 in which we merely require V to be continuous, but not necessarily continuously differentiable. Towards this, the theory of noncoercive variational inequalities could be a productive line of inquiry.

Finally, while the one-dimensional problem we have presented is interesting, developing an analogous formulation for two and three dimensions would allow us to compare ice-temperature profiles to those arrived at by more complex models which take into account more aspects of ice physics, as well as temperature.

7.2 Numerical and Computational Considerations

This author's software for computing Finite Element matrices [5] has limitations which could be improved upon in future versions to allow for more complete approximation of the problem at hand. In particular it currently accommodates only constant ice velocities, which is notable in that we have a theoretical result for non-constant velocities, and, more importantly, physical ice sheets typically exhibit non-constant velocities. Furthermore, the finite element method which we have outlined, and the software, only addresses approximations to function spaces via piecewise-linear basis functions; considerable theory exists surrounding the use of higher-degree polynomials basis functions to improve error bounds.

Our discussion of the finite element has neglected error analysis. Our analysis was limited to pointing out that in the case of equally spaced grids, the scheme is equivalent to a centered finite-difference scheme, with $\mathcal{O}(\Delta z^2)$ convergence. This analogy to finite differences breaks down on more general grids, and presumably it fails to extend to the more physical two and three-dimensional problems to be seen in the sequel.

In practice, by far the slowest part of the numerical solvers has been our implementation of Gauss-Seidel and CSOR. This is in part because little effort was made to optimize the code, but also because these methods do not have particularly good rates of convergence. Newer FEM techniques, in particular multi-grid methods offer sufficiently better convergence performance (albeit with more theoretical and technical overhead) that switching to them in the future should improve the overall performance of the solver[11].

When shifting to the more general two and three dimensional setting, the related issues of constructing a mesh on the domain of the ice sheet, and assembling a well-conditioned matrix to represent the matrix A associated with the bilinear form a, on that grid, become important problems from the perspective of numerical linear algebra.

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Appendix A
Function Spaces

Our discussion closely follows that of Chapter 5 in Evans [4]. We include this material primarily to set notation in a self-contained manner.

A.1 Definitions and Notation

Unless otherwise stated, we will assume that $U \subseteq \mathbb{R}^n$ is open and that $n > 0$.

Definition 10. *Function Spaces:*

1. $L^p(U) = \left\{ u : U \to \mathbb{R} \middle| u \text{ is Lebesgue measureable, } \|u\|_{L^p(U)} < \infty \right\}$, *where*
$$\|u\|_{L^p(U)} = \left(\int_U |f|^p \, fd \right)^{\frac{1}{p}}.$$
2. $L^p_{loc}(U) = \left\{ u : U \to \mathbb{R} \middle| u \in L^p(V) \text{ for each } V \subset\subset U \right\}$.
3. $C^k(U) = \left\{ u : U \to \mathbb{R} \middle| u \text{ is } k \text{ times continuously differentiable} \right\}$,
$C^k(\bar{U}) = \left\{ u \in C^k(U) \middle| D^\alpha u \text{ is uniformly continuous for all } \alpha \leq k \right\}$.
4. $C^\infty(U) = \left\{ u : U \to \mathbb{R} \middle| u \text{ is infinitely differentiable} \right\} = \bigcap_{k=0}^\infty C^k(U)$.
5. $C^k_c(U)$, $C^k_c(\bar{U})$, *etc. denote the functions in* $C^k(U)$, $C^k(\bar{U})$ *etc. with compact support.*

A.2 Weak derivatives

We will assume that $U \subset \mathbb{R}$ is open for some $n > 0$.

Definition 11. *Suppose* $u, v \in L^1_{loc}(U)$ *and* $\alpha \in \mathbb{N}$. *We say that* v *is the* αth-weak partial derivative of u,
$$D^\alpha u = v$$
if
$$\int_U u D^\alpha \phi = (-1)^\alpha \int_U v\phi$$
for all $\phi \in C^\infty_c(U)$.

Remark 1. If u has a "strong" derivative, it is also a weak derivative; this is a result of differentiation by parts, and the fact that our "test functions" ϕ are compactly supported, resulting in vanishing boundary terms.

Lemma 12 (Uniqueness of weak derivatives)**.** *Weak derivatives (if they exist) are unique up to a set of measure zero.*

A.2.1 Sobolev Spaces

Fix $1 \leq p \leq \infty$ and let $k \in \mathbb{Z}^+ \cup \{0\}$.

Definition 13. *The* Sobolev Space $W^{k,p}(U)$ *consists of all functions* $u : U \to \mathbb{R}$ *such that* $D^\alpha u$ *exists in the weak sense and belongs to* $L^p(U)$.

Definition 14. *If* $u \in W^{k,p}$, *and* $1 \leq p < \infty$, *we define its* norm *as:*

$$\|u\|_{W^{k,p}(U)} = \left(\sum_{\alpha=0}^{k} \int_U |D^\alpha u|^p \, dx \right)^{1/p}$$

We state without proof some facts about the Sobolev spaces, for proofs refer to Chapter 5 of [4].

Theorem 15. *The Sobolev Spaces are Banach spaces. Moreover* $W^{k,2}(U)$ *is a Hilbert space, and we customarily write* $H^k(U) = W^{k,2}(U)$.

Definition 16. *We denote by*

$$W_0^{k,p}(U)$$

the closure of $C_c^\infty(U)$ *in* $W^{k,p}(U)$. *It's customary to write* $H_0^k(U) = W_0^{k,2}(U)$.

A.3 Some Helpful Results from Functional Analysis

Lemma 17 (An Equivalent Norm on H_0^1)**.** *Given* $U \subset \mathbb{R}^n$, *open. There exist* $a, b \in \mathbb{R}$, $a, b > 0$ *so that if* $u \in H_0^1(U)$,

$$a \|u\|_{H_0^1(U)} \leq \|\nabla u\|_{L^2(U)} \leq b \|u\|_{H_0^1(U)} .$$

That is, the function $u \mapsto \|\nabla u\|_{L^2(U)}$ *is equivalent to the standard norm on* $H_0^1(U)$ *[10].*

A.4 $H^1((a, b))$, Continuity, and Boundary Conditions

For more general (i.e. inhomogeneous) boundary values, our methods for establishing well-posedness of our problems, and later for developing numerical tools to approximate solutions to them, are to "shift" the problem, and show that its solution is equivalent to solving a related homogeneous problem.

Theorem 18 (Evans §5.3.2 Theorem 2)**.** *Assume U is bounded, and suppose as well that $u \in W^{k,p}(U)$ for some $1 \leq p < \infty$. Then there exist functions $u_m \in C^\infty(U) \cap W^{k,p}(U)$ such that*

$$u_m \to u \qquad \text{in } W^{k,p}(U).$$

Lemma 19. *If $v \in H^1((a,b)) \cap C^\infty((a,b))$ then v is Hölder continuous, with exponent $1/2$.*

Proof. Suppose $v \in H^1\left((a,b)\right) \cap C^\infty\left((a,b)\right)$. For any $x, y \in (a,b)$, the Fundamental Theorem of Calculus and the Cauchy-Schwarz inequality tell us that

$$\begin{aligned} |v(x) - v(y)| &= \left| \int_y^x v'(s)\, ds \right| \\ &\leq |x-y|^{1/2} \left(\int_y^x |v'(s)|^2\, ds \right)^{1/2} \\ &\leq |x-y|^{1/2} \left\| v \right\|_{H^1((a,b))}. \end{aligned}$$

That is, v, is Hölder continuous. It is straightforward to show from this inequality that v is bounded as well. □

Lemma 20. *If $v \in H^1((a,b)) \cap C^\infty((a,b))$ then $\|v\|_\infty \leq \max\{(b-a)^{1/2}, (b-a)^{-1/2}\} \|v\|_{H^1}$.*

Proof. Again, letting $x, y \in (a,b)$, the Fundamental Theorem of Calculus tells us that

$$v(x) = v(y) + \int_y^x v'(s)\, ds.$$

So, fixing x and integrating over y:

$$(b-a)\, v(x) = \int_a^b v(x)\, dy = \int_a^b v(y)\, dy + \int_a^b \int_y^x v'(s)\, ds\, dy.$$

We seek norm bounds, so:

$$\begin{aligned} (b-a)|v(x)| &= \left| \int_a^b v(y)\, dy + \int_a^b \int_y^x v'(s)\, ds\, dy \right| \\ &\leq \int_a^b |v(y)|\, dy + \int_a^b \int_y^x |v'(s)|\, ds\, dy. \end{aligned}$$

By the Cauchy-Schwarz inequality, namely $\int_a^b |v(y)|\,dy \le \|1\|_{L^2}\|v\|_{L^2} = \sqrt{b-a}\,\|v\|_{L^2}$:

$$\begin{aligned}(b-a)|v(x)| &\le \sqrt{b-a}\,\|v\|_{L^2} + \int_a^b \int_y^x |v'(s)|\,ds\,dy \\ &\le \sqrt{b-a}\,\|v\|_{L^2} + \int_a^b \int_a^b |v'(s)|\,ds\,dz \\ &\le \sqrt{b-a}\,\|v\|_{L^2} + (b-a)^{3/2}\|v'\|_{L^2}\,.\end{aligned}$$

So, if $M = \max\left\{(b-a)^{1/2}, (b-a)^{-1/2}\right\}$

$$\begin{aligned}|v(x)| &\le (b-a)^{-1/2}\|v\|_{L^2} + (b-a)^{1/2}\|v'\|_{L^2} \\ &\le \max\{(b-a)^{1/2}, (b-1)^{-1/2}\}\,\|v\|_{H^1}\,.\end{aligned}$$

Since this holds for any $x \in [0, \ell]$, it follows that

$$\|v\|_\infty = \sup_{x\in(a,b)} |v(x)| \le \max\{(b-a)^{1/2}, (b-a)^{-1/2}\}\,\|v\|_{H^1}\,.$$

□

Lemma 21. *$H^1((a,b)) \subseteq C^0([a,b])$.*

Proof. Suppose $u \in H^1((a,b))$. By Theorem 18, there exists a sequence $\{u_n\}_{n=0}^\infty \subset H^1((a,b)) \cap C^\infty((a,b))$ which converge to u in $H^1(a,b))$. Since convergence in H^1 implies convergence in L^2, we know that some subsequence converges pointwise a.e., we identify $\{u_n\}$ with this subsequence. Let $\bar{u}$ be this pointwise (a.e) limit.

Given $m, n \in \mathbb{N}$ by Lemma 20:

$$\|u_n - u_m\|_\infty \le \max\{(b-a)^{1/2}, (b-a)^{-1/2}\}\,\|u_n - u_m\|_{H^1}\,,$$

so $\{u_m\}$ is Cauchy in C^0. Thus $u_n \to \bar{u}$ uniformly. As mentioned above, since $u_m \to u$ pointwise almost everywhere, $u = \bar{u}$ in H^1. Thus $u \in C^0([a,b])$. □

Lemma 22. *If $u \in H^1((a,b))$, then $\|u\|_\infty \le \max\{(b-a)^{-1/2}, (b-a)^{1/2}\}\,\|u\|$.*

Proof. By Lemma 21, there exists a sequence $\{u_i\} \subseteq H^1((a,b)) \cap C^\infty((a,b))$ so that $u_i \to u$ in L^∞ and H^1. Now, by continuity:

$$\begin{aligned}\|u\|_\infty &= \lim_{i\to\infty} \|u_i\|_\infty \\ &\le \max\{(b-a)^{-1/2}, (b-a)^{1/2}\} \lim_{i\to\infty} \|u_i\|_{H^1} \\ &= \max\{(b-a)^{-1/2}, (b-a)^{1/2}\}\,\|u\|\,.\end{aligned}$$

□

Strictly speaking, $u \in H^1((a,b))$ is an equivalence class of L^2 functions, for which pointwise values are not well-defined. This presents a problem when we are interested in using Sobolev spaces to solve boundary-value problems, where we've specified $u(a)$ and $u(b)$. In the one-dimensional case, with Dirichlet boundary conditions, what we've noted above gives us a way to do this. We'll abuse the above notation slightly and identify u and its continuous representative $\bar{u}$. For the purposes of boundary values, we can identify $u(a) = \bar{u}(a)$ and $u(b) = \bar{u}(b)$.

Lemma 21 also shows that, given $u \in H_0^1((a,b))$, the continuous representative of u satisfies: $\lim_{x \to a} u(x) = 0 = \lim_{x \to b} u(x)$. We can think of H_0^1 as the set of functions in H^1 with "zero-boundary values". The converse of this, that the functions in $H^1((1,0))$ with zero boundary values, are also members of $H_0^1((0,1))$ requires more work, but may be shown using smoothing operators known as mollifiers. We refer the reader to [4] for such a discussion.

We should note that these results do not extend to higher dimensions as stated, though some of their spirit does.

Appendix B
Some Finite Element Computations

B.1 Some Calculations

In the development given above for the Finite Element Method, we find approximate solutions to the weak form of a PDE:

$$B(u,v) = f(v) \text{ for all } v \in \mathcal{V},$$

where $B : \mathcal{U} \times \mathcal{V} \to \mathbb{R}$ is a bilinear form, and $f : \mathcal{V} \to \mathbb{R}$ is a bounded linear functional, and $\mathcal{U}$ and $\mathcal{V}$ are infinite-dimensional spaces. We approximate the function spaces $\mathcal{U}$ and $\mathcal{V}$ by finite-dimensional subspaces, U and V respectively. Then we calculate the entries of the matrix $B|_{U\times V}$.

In practice, it can be useful to split the bilinear form B into several terms, whose matrices are more easily calculated, and then add them together to assemble the matrix of B. One advantage of this, is that it allows us to write computer code which can assemble the matrices for a wider variety of weak-forms. Below we will calculate several of these which are pertinent for our temperature formulation for ice sheets.

Given a partition $\{x_k\}_{k=0}^J$ of $[a,b]$, a piece-wise linear basis element looks like:

$$\psi_i(x) = \begin{cases} \frac{1}{x_i - x_{i-1}}(x - x_{i-1}) & x \in (x_{i-1}, x_i) \\ -\frac{1}{x_{i+1} - x_i}(x - x_{i+1}) & x \in (x_i, x_{i+1}) \\ 0 & \text{otherwise} \end{cases}$$

with the notable exceptions being ψ_0 and ψ_J, which have form:

$$\psi_0 = \begin{cases} -\frac{1}{x_1 - x_0}(x - x_1) & , x \in (x_0, x_1) \\ 0 & , \text{otherwise} \end{cases}$$

and

$$\psi_J = \begin{cases} \frac{1}{x_J - x_{J-1}}(x - x_{J-1}) & , x \in (x_{J-1}, x_J) \\ 0 & , \text{ otherwise.} \end{cases}$$

B.1.1 First order terms

We'd like to calculate

$$\int_0^1 (\psi_i)_x \, \psi_j dx.$$

There are three cases to consider: $j = i-1$, $i = j$ and $j = i+1$, all other cases are zero.

1.

$$\int_0^1 \psi_j' \psi_{j+1}\, dx = -\frac{1}{2}$$

2.

$$\begin{aligned}\int_0^1 \psi_j' \psi_{j-1}\, dx &= \psi_j \psi_{j-1}(x)|_0^1 - \int_0^1 \psi_j \psi_{j-1}'\, dx \\ &= 0 - \left(-\frac{1}{2}\right) = \frac{1}{2}.\end{aligned}$$

3.

$$\begin{aligned}\int_0^1 \psi_j' \psi_j\, dx &= \int_{x_{j-1}}^{x_{j+1}} \psi_j' \psi_j\, dx \\ &= \int_{x_{j-1}}^{x_j} \frac{1}{x_j - x_{j-1}} \frac{(x - x_{j-1})}{x_j - x_{j-1}}\, dx + \int_{x_j}^{x_{j+1}} \frac{-1}{x_{j+1} - x_j} \frac{x_{j+1} - x}{x_{j+1} - x_j}\, dx \\ &= \frac{1}{2} - \frac{1}{2} = 0.\end{aligned}$$

B.1.2 Stiffness Matrix Entries

To compute $\int_0^1 (\phi_i)_x (\phi_j)_x\, dx$ is easier than the above calculation. Note that for $0 < k < J$:

$$(\psi_k)_x(x) = \begin{cases} \frac{1}{x_i - x_{i-1}} & , x \in (x_{i-1}, x_i) \\ -\frac{1}{x_{i+1} - x_i} & , x \in (x_i, x_{i+1}) \\ 0 & \text{otherwise.} \end{cases}$$

And at the boundary:

$$(\psi_0)_x(x) = \begin{cases} -\frac{1}{x_1 - x_0} & , x \in (x_0, x_1) \\ 0 & , \text{otherwise} \end{cases}$$

and

$$(\psi_J)_x(x) = \begin{cases} \frac{1}{x_J - x_{J-1}} & , x \in (x_{J-1}, x_J) \\ 0 & , \text{otherwise} \end{cases}.$$

As before, we note that if $1 < |i - j|$ then

$$\int_0^1 (\psi_i)_x (\psi_j)_x dx = 0.$$

By symmetry, we'll just calculate the case where $j = i + 1$. Note:

$$\begin{aligned}\int_0^1 (\psi_i)_x \, (\psi_{i+1})_x \; dx &= \int_{x_i}^{x_{i+1}} -\frac{1}{(x_{i+1} - x_i)^2} \; dx \\ &= \frac{-1}{x_{i+1} - x_i}.\end{aligned}$$

Now, if $0 < i < J$, and $i = j$, we see that

$$\begin{aligned}\int_0^1 (\psi_i)_x^2(x) \; dx &= \int_{x_{i-1}}^{x_i} \frac{1}{(x_i - x_{i-1})^2} \; dx + \int_{x_i}^{x_{i+1}} \frac{1}{(x_{i+1} - x_i)^2} \; dx \\ &= \frac{1}{x_i - x_{i-1}} + \frac{1}{x_{i+1} - x_i}.\end{aligned}$$

Finally, on the boundary:

$$\begin{aligned}\int_0^1 (\psi_0)_x^2 \; dx &= \int_{x_0}^{x_1} \frac{1}{(x_1 - x_0)^2} \; dx \\ &= \frac{1}{x_1 - x_0},\end{aligned}$$

and

$$\begin{aligned}\int_0^1 (\psi_J)_x^2 \; dx &= \int_{x_{J-1}}^{x_J} \frac{1}{(x_J - x_{J-1})^2} \; dx \\ &= \frac{1}{x_J - x_{J-1}}.\end{aligned}$$

Appendix C
Iterative Matrix Methods

Appendix B addresses the assembly of a matrix A

$$A\,x = b. \tag{C.1}$$

where A approximates a differential operator via the finite element element method. This linear system is the discretized approximation of the weak form of our PDE. How do we go about solving this linear system? The most straight-forward approach would be to directly solve the system via Gaussian elimination, but given that A is large $(n \times n)$ and relatively sparse, Gaussian elimination, with algorithmic complexity $\mathcal{O}(2n^3/3)$, and a tendency to spoil the sparse structure of formerly sparse matrices [15], rapidly becomes less useful than other methods which exploit A's sparsity.

Of particular interest to us, in part because of their use in solving the constrained problems we will need to address later when approximately solving variational inequalities, will be iterative matrix-splitting methods, in particular Gauss-Seidel and the related Successive Over-Relaxation.

C.1 Jacobi Iteration

To warm up, we outline the Jacobi Iteration method. Our duscussion closely follows [6]. We start by decomposing A into

$$(U + D + L) = A$$

where U, L and D are the upper, lower and diagonal components of A. We reorganize this into:

$$M_J\,x = b - N_J\,x \tag{C.2}$$

$$x = M_J^{-1}\,(b - N_J\,x) \tag{C.3}$$

where $M_J = D$ and $N_J = L + U$. We require that A have non-zero diagonal for (C.3) to make sense, so M_J is non-singular. Of course, M_J^{-1} is trivial to compute.

We now define the Jacobi Iteration. Given an initial guess for x, $x^{(0)}$, we determine the next term $x^{(k+1)}$ from $x^{(k)}$ by:

$$x^{(k+1)} = M_J^{-1}(b - N_J\,x^{(k)}).$$

Provided D has no zeros on the diagonal, the convergence theorem presented below applies.

C.2 Gauss-Seidel Iteration

A drawback of the Jacobi Iteration is that while we can calculate $x_i^{(k)}$ before $x_{i+1}^{(k)}$, we don't use our new estimate to calculate $x_{i+1}^{(k)}$. Gauss-Seidel iteration fixes this. As with Jacobi iteration, we require that the no entry on the diagonal be zero, then we decompose $A = U + D + L$, and rewrite our problem as

$$M_{GS}\, x = d - N_{GS}\, x \tag{C.4}$$

where $M_{GS} = D + L$ and $N_{GS} = U$ are both triangular. As before, we start with a guess, $x^{(0)}$, and calculate $x^{(k+1)}$ via $M_{GS}\, x^{(k+1)} = d - N_{GS}\, x^{(k)}$, but now, via back substitution, the value of $x_j^{(k+1)}$ is used to calculate $x_{j+1}^{(k+1)}$.

C.3 Convergence

Both the Jacobi algorithm and Gauss-Seidel belong to a family of iterative matrix solvers referred to as *splitting algorithms* [6], which take the matrix equation (C.1) and rewrite it as

$$M\, x^{(k+1)} = d + N\, x^{(k)} \tag{C.5}$$

where $A = M - N$.

We define the *error at step* k, $e^{(k)}$ by:

$$e^{(k)} = x^{(k)} - x$$

and subtract $M\, x = d + N\, x$ from (C.5) to obtain:

$$\begin{aligned} M\, (x^{(k+1)} - x) &= N\, (x^{(k)} - x) \\ M\, e^{(k+1)} &= N\, e^{(k)} \\ e^{(k+1)} &= M^{-1}N\, e^{(k)}. \end{aligned} \tag{C.6}$$

So $e^{(k)} = \left(M^{-1}N\right)^k\ e^{(0)}$. From this we find:

$$\begin{aligned} \left\|e^{(k)}\right\| &= \left\|\left(M^{-1}N\right)^k\ e^{(0)}\right\| \\ &\le \left\|\left(M^{-1}N\right)^{k+1}\right\|\ \left\|e^{(0)}\right\| \\ &\le \left(\left\|M^{-1}N\right\|\right)^k \left\|e^{(0)}\right\|. \end{aligned} \tag{C.7}$$

Evidently, the algorithm's convergence depends on the behavior of $\left(M^{-1}N\right)^k$ as $k \to \infty$. As (C.7) shows us, if $\|M^{-1}N\| < 1$, then the scheme converges, i.e. $e^{(k)} \to 0$.

However, a necessary and sufficient condition for convergence is determined by the spectral radius of $M^{-1}N$. Recall, that the spectral radius ρ of $M^{-1}N$ is $\rho\left(M^{-1}N\right) = \max_{\lambda \in \lambda(M^{-1}N)} |\lambda|$, where $\lambda(M^{-1}N)$ is the set of all eigenvalues of $M^{-1}N$, is proven in [6].

Theorem 23. *Suppose $A = M - N$ is a splitting of a non-singular matrix $A \in \mathbb{R}^{m \times n}$, Assuming M is nonsingular, the iteration (C.5) converges to $x = A^{-1}b$ for all starting vectors $x^{(0)} \in \mathbb{R}^m$ if, and only if, $\rho\left(M^{-1}N\right) < 1$.*

Appendix D

Exact Solutions to the Unconstrained Problem

D.1 Constant-Advection, Zero Strain-Heating

We claim that

$$u(z) = T_0 \frac{e^{\gamma z} - 1}{e^{\gamma \ell} - 1}$$

solves Problem 2 on page 8 when $S(z) = 0$ and $V(z) = V_0$. Note:

$$u_z = \frac{T_0}{e^{\gamma \ell} - 1} \gamma e^{\gamma z}$$

and

$$u_{zz} = \frac{T_0 \gamma^2}{e^{\gamma \ell} - 1} e^{\gamma z}.$$

So,

$$\begin{aligned}
-k u_{zz} + \rho c V_0 u_z &= -k \frac{T_0 \gamma^2}{e^{\gamma \ell} - 1} e^{\gamma z} + \rho c V_0 \frac{T_0}{e^{\gamma \ell} - 1} \gamma e^{\gamma z} \\
&= -\frac{T_0 \rho^2 c^2 V_0^2}{k(e^{\gamma \ell} - 1)} e^{\gamma z} + \frac{T_o \rho^2 c^2 V_0^2}{k(e^{\gamma \ell} - 1)} e^{\gamma z} \\
&= 0,
\end{aligned}$$

so u solves Problem 2. Note that also

$$u(0) = 0$$

and

$$u(\ell) = T_0 \frac{e^{\gamma \ell} - 1}{e^{\gamma \ell} - 1} = T_0,$$

so u satisfies boundary conditions (3.3) and (3.1).

D.2 Zero Advection and Constant Strain-Heating

We claim that

$$u(z) = \frac{S_0}{2k}(\ell - z)z + \frac{u_\ell z}{\ell}$$

solves Problem 2 when $V(z) = 0$ and $S(z) = S_0$. Note:

$$u_z(z) = -\frac{S_0}{k} z + \frac{S_0 \ell}{2k} + \frac{T_0}{\ell},$$

and

$$-k u_{zz} = S_0.$$

and:

$$u(0) = 0$$
$$u(\ell) = T_0,$$

so u solves Problem 2, with boundary conditions (3.3) and (3.1).