SOLUTIONS

Math 253 Calculus III (Bueler) 16 April 2018

Worksheet: Triple integrals in cartesian and spherical coordinates

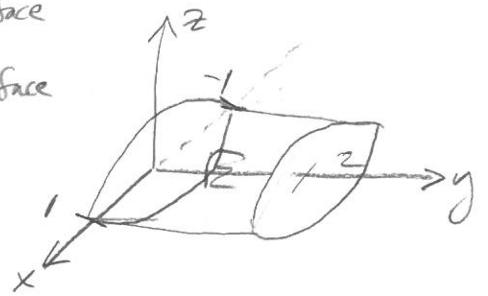
Recall $dV = dx\,dy\,dz$ in cartesian and $dV = \rho^2 \sin\phi \,d\rho\,d\phi\,d\theta$ in spherical coordinates.

A. Suppose E is enclosed by the surfaces $z = x^2 - 1$, $z = 1 - x^2$, $y = 0$, and $y = 2$. Completely set up, but do not evaluate, the triple integral:

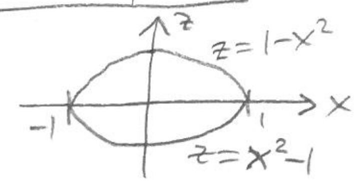
$$\iiint_E (x-y)\,dV \quad = \int_{-1}^{1}\int_0^2\int_{x^2-1}^{1-x^2} x-y \,dz\,dy\,dx$$

$z=x^2-1$ is lower surface

$z=1-x^2$ is uppersurface



in x-z plane:



B. Suppose E is the region between the spheres $x^2 + y^2 + z^2 = 4$ and $x^2 + y^2 + z^2 = 9$. Choose cartesian or spherical coordinates and then evaluate the integral:

$$\iiint_E (x^2 + y^2)\,dV$$

E is much easier to describe in spherical coordinates:

$$E = \{(\rho,\phi,\theta) \mid 2 \le \rho \le 3,\ 0 \le \phi \le \pi,\ 0 \le \theta \le 2\pi\}$$

also:
$$x^2+y^2 = (\rho\sin\phi\cos\theta)^2 + (\rho\sin\phi\sin\theta)^2$$
$$= \rho^2\sin^2\phi\,(\cos^2\theta+\sin^2\theta) = \rho^2\sin^2\phi$$

So:
$$\iiint_E x^2+y^2\,dV = \int_0^{2\pi}\int_0^{\pi}\int_2^3 \rho^2\sin^2\phi\cdot\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$= \left(\int_0^{2\pi} d\theta\right)\left(\int_0^{\pi}\sin^3\phi\,d\phi\right)\left(\int_2^3 \rho^4\,d\rho\right) = 2\pi\left(\int_0^{\pi}(1-\cos^2\phi)\sin\phi\,d\phi\right)\left[\frac{\rho^5}{5}\right]_2^3$$

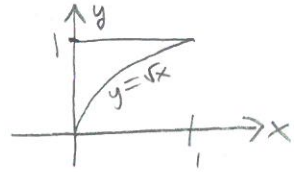
$$\underset{[u=\cos\phi]}{=} 2\pi\left(\int_1^{-1}(1-u^2)(-du)\right)\frac{211}{5} = \frac{2(211)\pi}{5}\int_{-1}^{1} 1-u^2\,du = \frac{1688}{15}\pi$$

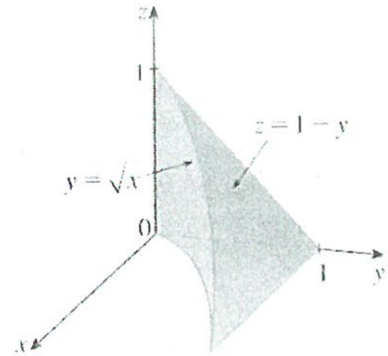
C. Suppose E is the region shown at right. Completely set up, but do not evaluate, the triple integral

$$\iiint_E f(x,y,z)\,dV$$

$$= \int_0^1 \int_{\sqrt{x}}^1 \int_0^{1-y^2} f(x,y,z)\,dz\,dy\,dx$$

projection in $x-y$:





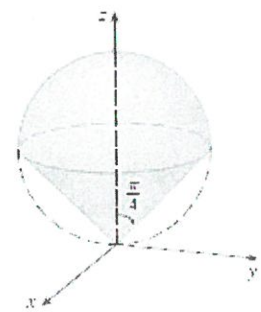
D. The centroid $(\bar{x}, \bar{y}, \bar{z})$ of a three-dimensional object E is computed by integrals for the mass and the three moments assuming constant density K:

$$m = \iiint_E K\,dV,$$

$$M_{yz} = \iiint_E x\,K\,dV, \quad M_{xz} = \iiint_E y\,K\,dV, \quad M_{xy} = \iiint_E z\,K\,dV$$

The centroid is given by these ratios:

$$\bar{x} = \frac{M_{yz}}{m}, \qquad \bar{y} = \frac{M_{xz}}{m}, \qquad \bar{z} = \frac{M_{xy}}{m}.$$



Use spherical coordinates to completely set up, but do not evaluate, the integrals needed to compute the centroid of the region E which is shown. The sphere shown is $x^2 + y^2 + z^2 = z$, which is $\rho = \cos\phi$ in spherical coordinates, and the cone is $z = \sqrt{x^2 + y^2}$.

$$E = \left\{ (\rho, \phi, \theta) \mid 0 \le \theta \le 2\pi, 0 \le \phi \le \tfrac{\pi}{4}, \; 0 \le \rho \le \cos\phi \right\}$$

$$m = \int_0^{2\pi} \int_0^{\pi/4} \int_0^{\cos\phi} K \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$$

$$M_{yz} = \int_0^{2\pi} \int_0^{\pi/4} \int_0^{\cos\phi} K \rho^3 \sin^2\phi \cos\theta \, d\rho\, d\phi\, d\theta$$

$$M_{xz} = \int_0^{2\pi} \int_0^{\pi/4} \int_0^{\cos\phi} K \rho^3 \sin^2\phi \sin\theta \, d\rho\, d\phi\, d\theta$$

$$M_{xy} = \int_0^{2\pi} \int_0^{\pi/4} \int_0^{\cos\phi} K \rho^3 \sin\phi \cos\phi \, d\rho\, d\phi\, d\theta$$