Solutions to Assignment # 1

Total of 70 points.

Section 1.1, Exercise 2 (*5 points*) For $f(x) = (x+1)^{1/2}$ we find derivatives $f'(x) = (1/2)(x+1)^{-1/2}$, $f''(x) = -(1/4)(x+1)^{-3/2}$, and $f'''(x) = (3/8)(x+1)^{-5/2}$. With $x_0 = 0$ the third-order polynomial is

$$p_3(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2}(x-x_0)^2 + \frac{f'''(x_0)}{6}(x-x_0)^3 = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3.$$

Section 1.1, Exercise 7 (*5 points*) *This problem can be done with either $f(x) = e^x$ or $f(x) = \sqrt{x}$. But the former is* a lot *easier.*

Let $f(x) = e^x$, $x_0 = 0$, and $x = 1/2$, so that $f(x) = e^{1/2} = \sqrt{e}$. We don't need to take derivatives because the Taylor series at $x_0 = 0$ is already familiar, and on page 3:

$$f(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \dots$$

For any chosen n, $f^{(n+1)}(x) = e^x$ and so the remainder term is

$$R_n(x) = \frac{e^\xi}{(n+1)!}x^{n+1}$$

where $0 = x_0 \le \xi \le x = 1/2$. We want $|R_n(x)| \le 10^{-3}$. A bit of trial-and-error suggests $n = 4$ will do. In fact, because e^x is an increasing function,

$$|R_4(x)| = \frac{e^\xi}{5!}\frac{1}{2^5} \le \frac{e^{1/2}}{120}\frac{1}{32} < \frac{3}{120}\frac{1}{32} = \frac{1}{40 \cdot 32} = \frac{1}{1280} < 10^{-3},$$

because $\xi \le 1/2 < 1$ and $e^1 < 3$. Thus

$$\sqrt{e} = f(1/2) \approx p_4(1/2) = 1 + (1/2) + \frac{(1/2)^2}{2} + \frac{(1/2)^3}{3!} + \frac{(1/2)^4}{4!} = \frac{211}{128} = 1.6484375.$$

(*Octave reports* $\sqrt{e} = 1.6487213$ *so the error is about* $3 \times 10^{-4} < 10^{-3}$.)

Section 1.1, Exercise 11 (*10 points*) **b)** For $f(x) = \ln(1+x)$ the third-order polynomial at $x_0 = 0$ is $p_3(x) = x - \frac{x^2}{2} + \frac{x^3}{3}$. Because $f^{(4)}(x) = -6(1+x)^{-4}$, the remainder term is

$$R_3(x) = \frac{-6(1+\xi)^{-4}}{4!}x^4 = -\frac{x^4}{4(1+\xi)^4}$$

where ξ is between $x_0 = 0$ and x.

Now assume $-1 \le x \le 1$. Then $-1 \le \xi \le 1$ also, so $0 \le 1+\xi \le 2$. Thus

$$|R_3(x)| = \frac{|x|^4}{4(1+\xi)^4} \le \frac{1^4}{4 \cdot 0^4} = +\infty.$$

We cannot find a finite upper bound on the remainder term, given what we know from Taylor's theorem and about ξ. The best bound on the error $|f(x) - p_3(x)|$ on $[-1, +1]$ is infinity.

Because $\lim_{x \to -1} \ln(1+x) = -\infty$, *we cannot do better. In fact $f(-1)$ is not defined so f does not satisfies the hypotheses of Taylor's Theorem.*

c) For $f(x) = \sin(x)$ the third-order polynomial at $x_0 = 0$ is $p_3(x) = x - \frac{x^3}{6}$. Because $f^{(4)}(x) = \sin(x)$, the remainder term is $R_3(x) = \frac{\sin(\xi)}{4!}x^4$ where ξ is between $x_0 = 0$ and x.

Now assume $0 \le x \le \pi$. Then $0 \le \xi \le \pi$ also. However, we use $|\sin(\xi)| \le 1$, so

$$|R_3(x)| = \frac{|\sin(\xi)|}{24} x^4 \le \frac{1}{24}\pi^4 \approx 4.0587.$$

Thus $|f(x) - p_3(x)| \le 4.0587$.

This estimate suggests that $p_3(x)$ is not very useful on $[0, \pi]$. Plotting and comparing $f(x)$ and $p_3(x)$ on this interval confirms this, and suggests the actual maximum error is about 2.02, so our value of about 4.06 is an over-estimate but it gives the right impression anyway.

Section 1.1, Exercise 21 (*5 points*) Suppose $p(x)$ is a polynomial of degree n (or less than n). Then $p^{(n+1)}(x) = 0$ identically. Thus

$$R_n(x) = \frac{p^{(n+1)}(\xi)}{(n+1)!}(x - x_0)^{n+1} = 0.$$

It follows that $p(x) = p_n(x) + 0 = p_n(x)$. The Taylor polynomial of degree n equals $p(x)$ itself.

Section 1.1, Exercise 22 (*5 points*) My initial thinking is that if $|x| \le 1/2$ then t in the integral also satisfies $|t| \le 1/2$. Let $z = \pi t^2/2$. Then $|z| = |\pi t^2/2| \le \pi/8 < 1/2$ if $|t| \le 1/2$. Now considering the remainder term for $\cos(z)$, which is informally like the next term in the Taylor series, we note that if $|z| \le 1/2$ then

$$\frac{1}{6!}|z|^6 \le \frac{1}{720 \cdot 2^6} = \frac{1}{46080} < 10^{-4}.$$

As we will see, this informal version suggests the right power, but now we need to be precise.

Note $p_5(z) = p_4(z) = 1 - z^2/2 + z^4/24$ for $f(z) = \cos(z)$:

$$\cos(z) = p_5(z) + R_5(z) = p_5(z) + \frac{f^{(6)}(\xi(z))}{6!} z^6 = p_5(z) - \frac{\cos(\xi(z))}{720} z^6.$$

Now I do the substitution and integrate:

$$C(x) = \int_0^x \cos(\pi t^2/2)\, dt = \int_0^x p_5(\pi t^2/2)\, dt - \int_0^x \frac{\cos(\xi(\pi t^2/2))}{720} (\pi t^2/2)^6\, dt.$$

A side calculation gets the p_5 integral (see below). We need to bound the remainder term, on the interval $-1/2 \le x \le 1/2$, and we use $|\cos(z)| \le 1$:

$$\left| \int_0^x \frac{\cos(\xi(\pi t^2/2))}{720} (\pi t^2/2)^6\, dt \right| \le \frac{\pi^6}{720 \cdot 2^6} \int_0^{|x|} |\cos(\xi(\pi t^2/2))|\, t^{12}\, dt \le \frac{\pi^6}{46080} \int_0^{|x|} t^{12}\, dt$$
$$= \frac{\pi^6 |x|^{13}}{46080 \cdot 13} \le \frac{\pi^6}{599040 \cdot 2^{13}} = 1.9591 \times 10^{-7}.$$

Thus the error in this approximation is less than 10^{-4}:

$$C(x) \approx x - \frac{\pi^2}{40} x^5 + \frac{\pi^4}{3456} x^9.$$

Section 1.2, Exercise 3 (*5 points*) For $f(x) = \sqrt{1+x}$, $x_0 = 0$, and $n = 1$, Taylor's theorem says

$$f(x) = p_1(x) + R_1(x) = 1 + \frac{1}{2}x - \frac{1}{8(1+\xi)^{3/2}} x^2,$$

because $f'(x) = (1/2)(1+x)^{-1/2}$ and $f''(x) = -(1/4)(1+x)^{-3/2}$, and where ξ is between 0 and x. Suppose $|x| \le 1/2$. Then $|\xi| \le 1/2$ also, so $1 + \xi \ge 1/2$ in particular and therefore

$$\left| f(x) - \left(1 + \frac{1}{2}x\right) \right| = \left| \frac{1}{8(1+\xi)^{3/2}} x^2 \right| \le \frac{1}{8(1/2)^{3/2}} x^2 = C x^2$$

for $C = 2^{-3/2}$. Thus

$$f(x) = 1 + \frac{1}{2}x + \mathcal{O}(x^2),$$

by definition, as $x \to 0$.

Section 1.2, Exercise 6 (*5 points*) We take the summation formula as given:

$$\sum_{k=0}^{n} r^k = \frac{1 - r^{n+1}}{1 - r}.$$

Note this requires $r \neq 1$. (If $r = 1$ then $\sum_{k=0}^{n} r^k = \sum_{k=0}^{n} 1 = n + 1$.) A slight rearrangement says

$$\sum_{k=0}^{n} r^k - \frac{1}{1-r} = -\frac{r^{n+1}}{1-r}.$$

Suppose $|r| \leq 1/2$ so that $1 - r \geq 1/2$ and $1/(1-r) \leq 2$. Then

$$\left| \sum_{k=0}^{n} r^k - \frac{1}{1-r} \right| = \frac{|r|^{n+1}}{1-r} \leq 2|r|^{n+1}.$$

This shows that

$$\sum_{k=0}^{n} r^k = \frac{1}{1-r} + \mathcal{O}(r^{n+1}),$$

by definition, as $r \to 0$.

Section 2.1, Exercise 1 (*5 points*) **a)** Using the form at the top of page 44,

$$p_a(x) = x^3 + 3x + 2 = 2 + x(3 + x^2) = 2 + x(3 + x(0 + 1 \cdot x))$$

b) $p_b(x) = 1 + x(0 + x(4 + x(0 + x(2 + x(0 + 1 \cdot x)))))$

Section 2.1, Exercise 4 (*10 points*) I wrote the following code, which includes comments (i.e. a "`help`" file) which are intended to be a good model for such things:

horner.m

```
function [p, dp] = horner(a,x)
% HORNER Evaluate a polynomial of degree n, and its derivative.  The input is
% a list (array)  a  of n+1 coefficients:
%   p(x) = a(1) + a(2) x + a(3) x^2 + ... + a(n+1) x^n
% Uses Horner's rule to reduce the number of operations:
%   p(x) = a(1) + x (a(2) + x (a(3) + ... + a(n+1)*x) ... )
% Example:  To evaluate p(x)=x^3+3x+2 and p'(x)=3x^2+3 at x=4 do
%   >> [p,dp] = horner([2 3 0 1],4)
%   p =  78
%   dp =  51
% Note asking for one output gives the first only:
%   >> p = horner([2 3 0 1],4)
%   p =  78
% See also: POLYVAL

n  = length(a) - 1;
p  = a(n+1);
dp = n * a(n+1);
for k = n:-1:1
    p  = a(k) + x * p;
    if k > 1
        dp = (k-1) * a(k) + x * dp;
```

```
    end
end
```

Evaluating the polynomial, and its derivative, from **1(a)** with $x = 4$:

```
>> [p, dp] = horner([2 3 0 1],4)
p =  78
dp =  51
```

Evaluating the polynomial, and its derivative, from **1(b)** with $x = -3$:

```
>> [p, dp] = horner([1 0 4 0 2 0 1],-3)
p =  928
dp = -1698
```

(*To confirm that these values are correct I entered*

`evaluate p(x)=x^6+2x^4+4x^2+1 at x=-3`

and similar, into Wolfram alpha. *While I could also confirm such a calculation by hand, my by-hand and programming errors may be correlated, while that is unlikely for* alpha. *Using* alpha*, and suitably-generic inputs x, in this way represents fairly robust verification. On the other hand we are planning to do tasks which* alpha *can't do!*)

P1 (*5 points*) Let $f(x) = x^{1/4}$ and $x_0 = 625$. Note that $5^4 = 625$ so $f(x_0) = 5$. Since $x = 626$ is so close to x_0, with $x - x_0 = 1$, we optimistically try the $n = 1$ case of Taylor's theorem. Taking derivatives: $f'(x) = (1/4)x^{-3/4}$ and $f''(x) = -(3/16)x^{-7/4}$. Then:

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(\xi)}{2}(x - x_0)^2 = (625)^{1/4} + \frac{1}{4}\frac{1}{(625)^{3/4}}\,1 - \frac{3}{32}\frac{1}{\xi^{7/4}}\,1^2$$
$$= 5 + \frac{1}{4}\frac{1}{5^3} - \frac{3}{32}\frac{1}{\xi^{7/4}} = 5 + \frac{1}{500} - \frac{3}{32}\frac{1}{\xi^{7/4}} = 5.002 - \frac{3}{32}\frac{1}{\xi^{7/4}}.$$

(*Every "=" symbol above really is equality!*) All we know about ξ is that $625 = x_0 \le \xi \le x = 626$.

The remainder term can be bounded by using the fact negative powers are decreasing functions:

$$|R_1(x)| = \left|\frac{3}{32}\frac{1}{\xi^{7/4}}\right| \le \frac{3}{32}\frac{1}{x_0^{7/4}} = \frac{3}{2^5}\frac{1}{5^7} = \frac{3}{10^5 5^2} = \frac{3}{25}10^{-5} = 1.2 \times 10^{-6}.$$

So using just the first two terms, that is, using $p_1(x)$, gives less than 10^{-5} error, as desired:

$$626^{1/4} \approx 5.002.$$

(*And Octave says* $626^{1/4} = 5.0019988\dots$. *The actual error is very-slightly smaller, as predicted.*)

P2 (*10 points*) After thinking about how to do factorials as `for` loops, I wrote the following program which is an m-file (a general term) and a function (a specific term):

combin.m

```
function z = combin(n,k)
% COMBIN  Compute the number of combinations (ways of choosing) k items out
% of n total:
%                   / n \         n!         n (n-1) ... (n-k+1)
%   combin(n,k)  =  |   |  =  ----------  =  -------------------
%                   \ k /      k! (n-k)!        k (k-1) ... 2
```

```
% Does not use the built-in FACTORIAL function.  See also PASCAL10 which uses
% COMBIN to generate part of Pascal's triangle.
% Example:
%   >> combin(6,4)

if (mod(n,1) ~= 0 | mod(k,1) ~= 0)
    error('inputs must be integers')
end
if (n < 0 | k < 0)
    error('inputs must be nonnegative')
end
if (n < k)
    error('n >= k is required')
end

if (n == 0 | k == 0 | k == n)
    z = 1;
else
    z = n / k;
    for j = 1:k-1
        z = z * (n - j) / (k - j);
    end
end
```

For example, to compute "6 choose 4", and then the number of 5-card hands in a regular deck of 52 cards, do

```
>> combin(6,4)
ans =  15
>> combin(52,5)
ans =  2598960
```

(These are correct; look them up, use Wolfram *alpha*, or think it through.)

Also notice the error messages one gets from various nonsensical inputs:

```
>> combin(6,-4)    % BAD
>> combin(6,pi)    % BAD
>> combin(6,8)     % BAD
```

For the first ten rows of Pascal's triangle I wrote an m-file which is a script, so one runs it with no arguments:

pascal10.m

```
% PASCAL  Calls COMBIN to generate 10 rows of Pascal's triangle.

for X = 1:10
    for Y = 0:X
        fprintf('%d ',combin(X,Y))
    end
    fprintf('\n')
end
```

```
>> pascal10
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1
1 10 45 120 210 252 210 120 45 10 1
```

Note the use of `fprintf()`, which is generally a "print formatted text" command. The "`%d`" is a placeholder for an integer, such as the output from `combin()`, and the special character "
`n`" is a newline character. (For more info see the Matlab online help at
`www.mathworks.com/help/matlab/ref/fprintf.html`.)

(*It is interesting to also run* `>> pascal(10).` *A* MATLAB *built-in function also does the job.*)