Solutions to SAMPLE Midterm Exam #1

1. Show that $y = \sqrt{x - \frac{1}{2} + \frac{1}{2}e^{-2x}}$ is a solution of the initial value problem

$$\frac{dy}{dx} = \frac{x}{y} - y, \qquad y(0) = 0.$$

Solution. We compute the two sides of the ODE, using the given formula for $y = y(x)$:

$$\frac{dy}{dx} = \frac{1}{2}\left(x - \frac{1}{2} + \frac{1}{2}e^{-2x}\right)^{-1/2}\left(1 - e^{-2x}\right) = \frac{1}{2}\frac{1 - e^{-2x}}{\left(x - \frac{1}{2} + \frac{1}{2}e^{-2x}\right)^{1/2}},$$

$$\frac{x}{y} - y = \frac{x - y^2}{y} = \frac{-\frac{1}{2} + \frac{1}{2}e^{-2x}}{\left(x - \frac{1}{2} + \frac{1}{2}e^{-2x}\right)^{1/2}} = \frac{1}{2}\frac{1 - e^{-2x}}{\left(x - \frac{1}{2} + \frac{1}{2}e^{-2x}\right)^{1/2}}.$$

Since both sides of the ODE simplify to the same function of x, the given $y(x)$ is a solution of the ODE. Also, $y(0) = \sqrt{0 - (1/2) + (1/2)e^0} = 0$.

2. Solve for $\theta(t)$:

$$\frac{d\theta}{dt} - 6\theta = t.$$

Solution. The equation is linear. $\mu(t) = e^{\int -6\,dt} = e^{-6t}$. The equation is equivalent to

$$\frac{d}{dt}\left(e^{-6t}\theta(t)\right) = e^{-6t}\,t.$$

Integrating both sides with respect to t gives

$$e^{-6t}\theta(t) = \int te^{-6t}\,dt = -\frac{1}{6}te^{-6t} - \frac{1}{6}\int e^{-6t}\,dt = -\frac{1}{6}te^{-6t} + \frac{1}{36}e^{-6t} + c.$$

Solve for $\theta(t)$:

$$\theta(t) = -\frac{1}{6}t + \frac{1}{36} + ce^{6t}.$$

3. Check that the following equation is exact, and then solve the initial value problem:

$$(2x + y)\,dx + (x - 2y)\,dy = 0, \qquad y(1) = 2.$$

Solution. Here $M = 2x + y$, $N = x - 2y$. The check is:

$$1 = \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = 1.$$

So we compute $F(x, y)$, because the solutions of the ODE are its level curves:

$$F(x, y) = \int M(x, y)\,dx + g(y) = x^2 + yx + g(y).$$

On the other hand,

$$x - 2y = N = \frac{\partial F}{\partial y} = x + g'(y).$$

This says $g'(y) = -2y$ so $g(y) = -y^2$ suffices. Thus $F(x,y) = x^2 + yx - y^2$. All solutions of the ODE are of this form:

$$x^2 + yx - y^2 = C.$$

But the initial condition $(x, y) = (1, 2)$ gives $1^2 + (1)(2) - 2^2 = -1 = C$ so the solution of the initial value problem is

$$x^2 + yx - y^2 = -1.$$

4. Solve the following equation explicitly for $y = y(x)$.

$$\frac{dy}{dx} = 4 + y^2.$$

Solution. The equation is separable:

$$\int \frac{dy}{4 + y^2} = \int dx.$$

For the left-hand integral, the substitution $u = y/2$ gives a familiar antiderivative problem:

$$\int \frac{dy}{4 + y^2} = \frac{1}{4} \int \frac{dy}{1 + (y/2)^2} = \frac{2}{4} \int \frac{du}{1 + u^2} = \frac{1}{2} \arctan u = \frac{1}{2} \arctan \left(\frac{y}{2}\right).$$

(Dropping the constant is o.k. here only because we keep it on the right side.) Since the right-hand integral is easy we get

$$\frac{1}{2} \arctan \left(\frac{y}{2}\right) = x + C.$$

But we want an explicit solution so we solve for $y(x)$:

$$y(x) = 2 \tan (2x + c).$$

(Note $c = 2C$.)

5. Use Euler's method with steps of size $h = 1.0$ to approximate the solution of the initial value problem

$$y' = x - y^2, \qquad y(1) = 0$$

at $x = 2$ and $x = 3$.

Solution. *This is done without a calculator, of course.*

Here $y_0 = 0$ while $x_0 = 1$, $x_1 = 2$, $x_2 = 3$ (because $h = 1$). The formula for getting a new y from an old is

$$y_{n+1} = y_n + hf(x_n, y_n) = y_n + 1.0(x_n - y_n^2).$$

Thus

$$y_1 = y_0 + x_0 - y_0^2 = 0 + 1 - 0^2 = 1,$$
$$y_2 = y_1 + x_1 - y_1^2 = 1 + 2 - 1^2 = 2.$$

We are done. Our approximation to $y(2)$ is $y_1 = 1$, and our approximation to $y(3)$ is $y_2 = 2$.

6. Solve:

$$\frac{dy}{dx} = -3 + \frac{y}{x}.$$

Solution. This is linear, so we put in standard form and see $P(x) = -1/x$, $Q(x) = -3$:

$$\frac{dy}{dx} - \frac{1}{x} y = -3.$$

Then $\mu = e^{\int P} = e^{-\ln|x|} = 1/|x|$. So we multiply by μ and rewrite recognizing the product rule structure:

$$\frac{d}{dx}\left(\frac{1}{|x|}y\right) = -3\frac{1}{|x|}.$$

Multiplying both sides by -1 if necessary allows us to drop the absolute values. Integrating both sides with respect to x gives

$$\frac{y}{x} = -3\ln|x| + c.$$

Thus

$$y(x) = -3x\ln|x| + cx.$$

7. (a). We count burbot (which are really ugly fish that live in the Tanana River among other places) and discover that in 1990 there were 10,000 in the river and that in 1995 there were 12,000. Using the Malthusian model estimate the population in 2000.

Solution. The Malthusian model is "$\frac{dp}{dt} = kp$" for the population $p(t)$ at time t. It has solution

$$p(t) = Ae^{kt}.$$

For convenience we measure time in years from 1990, so we know $p(0) = 10^4$ and $p(5) = 1.2 \times 10^4$. Then $A = 10^4$ and we solve this equation for k:

$$1.2 \times 10^4 = 10^4 e^{5k}.$$

This gives $k = (\ln 1.2)/5$. Thus the estimated 2000 population is

$$p(10) = 10^4 e^{((\ln 1.2)/5)10} = 10^4 e^{2\ln 1.2} = 10^4(1.2)^2 = 14{,}400.$$

(b). Write down the differential equation for part a) (even if you have already!). Then write down a new differential equation for the model: We assume that the population of burbot grows at a rate proportional to the current population minus a constant amount of fishing, which we assume to be 1000 per year.

Solution. The Malthusian model is

$$\frac{dp}{dt} = kp$$

as already stated. The new model is

$$\frac{dp}{dt} = \beta p - 1000.$$

Here β is the proportionality constant, with units of $(\text{year})^{-1}$, just like k in the Malthusian model.

Note that the units of p are (number of fish), while the "1000" is in (number of fish)/year, so "$\frac{dp}{dt} = \beta(p - 1000)$" is not correct. We cannot subtract quantities with different units.

(c). Extra Credit. Solve the model in (b). Then explain how to use the data in (a) to find k and any other unknown constants, and how to estimate the population in 2000.

Solution. The equation from (b) is separable:

$$\int \frac{dp}{\beta p - 1000} = \int dt.$$

Integrating and simplifying gives

$$p(t) = \frac{1}{\beta}\left(Ae^{\beta t} + 1000\right).$$

To use the same data as in (a) to determine constants A and β we would solve these two equations for those variables:

$$10^4 = \frac{1}{\beta}\left(A + 1000\right), \qquad 1.2 \times 10^4 = \frac{1}{\beta}\left(Ae^{5\beta} + 1000\right).$$

Solving these is *not* as easy as in (a). With A, β, $p(10) = \frac{1}{\beta}\left(Ae^{10\beta} + 1000\right)$ is our estimated year 2000 population.

8. The air in a small room 10 ft by 10 ft by 10 ft is 3% carbon monoxide. Starting at $t = 0$, fresh air containing no carbon monoxide is blown into the room at a rate of 100 ft^3/min. The air in the room is kept well-mixed, and air flows out of the room through a vent at the same rate. Determine the time in minutes when the room will have only 0.01% carbon monoxide; your expression for the time may have an elementary function in it.

Solution. *I found it easiest to think in terms of the volume of the CO (= carbon monoxide) part of the gas in the room. There are other ways, but note that we are told nothing related to gas density or mass.*

Let $v(t)$ be the volume of CO in the room, measured in ft^3, at the time t, measured in minutes. Then $v(0) = 0.03\,10^3 = 30\,\text{ft}^3$ because the room has volume $10^3\,\text{ft}^3$ and is 3% CO. Now we say "$\frac{dv}{dt} = (\text{input}) - (\text{output})$":

$$\frac{dv}{dt} = 0 - 100 \cdot \frac{v(t)}{10^3}.$$

This contains the ideas that no CO enters the room while the volume of CO leaving the room is $100\,\text{ft}^3/min$ times the fraction of the gas which is CO. (We assume the gases are well-mixed!) But then $dv/dt = -0.1v$, which has solution $v(t) = v(0)e^{-0.1t} = 30e^{-0.1t}$, so we want to find the time when

$$10 = 30e^{-0.1t}.$$

This time is $t = 10\ln 3$ minutes. (*Which is about 11 minutes because* $\ln 3$ *is just larger than 1.*)