

Newton's method solves a problem

$$f(x) = 0$$

by starting with an initial estimate  $x_0$ . It gets a new estimate from the old estimate by finding where the linearization  $L(x) \approx f(x)$  crosses the  $x$ -axis. (Write down the linearization  $L(x)$  at  $a = x_0$ . Solve  $L(x) = 0$  for  $x$ .) So it uses this formula:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

**Example 1.** Suppose you want to solve the equation

$$3x^4 - x^3 + 6x^2 + 4x - 2 = 0.$$

I do not see how to factor this by hand. (*Do you?*) If we name the left side of the equation " $f(x)$ " and if we go looking for a root of  $f(x)$  then we might notice  $f(0) = -2$  and  $f(1) = 10$ . Because  $f$  is continuous, by the intermediate value theorem there is a root on  $(0, 1)$ . Also, the derivative is an easy calculation:  $f'(x) = 12x^3 - 3x^2 + 12 + 4$ .

Newton's method with initial estimate  $x_0 = 1/2$  gives the next value by

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0.354651162790698$$

(*It is nice to immediately use a serious calculator, namely one with a big screen and the ability to run a loop. Thus I used MATLAB on my laptop.*) The next four iterations gave consistent numbers which strongly suggest  $x = 1/3$  is an exact root:

$$x_0 = 0.5000000000000000$$

$$x_1 = 0.354651162790698$$

$$x_2 = 0.333718551391767$$

$$x_3 = 0.33333461355651$$

$$x_4 = 0.333333333333347$$

$$x_5 = 0.333333333333333$$

Notice how the number of correct digits doubles from  $x_1$  to  $x_4$ . It stops doubling only because the computer only stores about 16 decimal digits. In fact we have discovered how to factor the function:  $f(x) = (x - \frac{1}{3})(3x^3 + 6x + 6)$ .

**Example 2.** Suppose we want to find the maximum of  $g(x) = x^2 - 2\sin x$  on the interval  $[0, 1]$ . We take the derivative and set it to zero to find the critical numbers:

$$2x - 2\cos x = 0$$

You can factor and remove the "2," but otherwise this is an equation I do not know how to solve by hand. So I try Newton's method. Let  $f(x) = x - \cos x$  so  $f'(x) = 1 + \sin(x)$ . Newton's method starting with  $x_0 = 0.5$  gives these five iterates, and the last two agree to all digits:

$$x_0 = 0.5000000000000000$$

$$x_1 = 0.755222417105636$$

$$x_2 = 0.739141666149879$$

$$x_3 = 0.739085133920807$$

$$x_4 = 0.739085133215161$$

$$x_5 = 0.739085133215161$$

In fact  $f(x_4) = 0$  to the accuracy of the computer. Again notice the approximate doubling of the number of correct digits at each Newton step.