Selected Solutions to Assignment #1

Exercise 2 (page 4 of B&C). (a) Show that $\operatorname{Re}(iz) = -\operatorname{Im} z$. (b) Show $\operatorname{Im}(iz) = \operatorname{Re} z$.

Proof. Let $z = x + iy$, then $iz = ix - y$, from this we have that

$$\operatorname{Re}(iz) = -y = -\operatorname{Im} z, \quad \operatorname{Im}(iz) = x = \operatorname{Re} z.$$

□

Exercise 5 (page 4 of B&C). PROOF. Let $z_1 = (x_1, x_2)$, $z_2 = (x_2, y_2)$, where x_i, y_i are real. Then, by definition,

$$z_1 z_2 = (x_1 x_2 - y_1 y_2,\, y_1 x_2 + x_1 y_2)$$

and

$$z_2 z_1 = (x_2 x_1 - y_2 y_1,\, y_2 x_1 + x_2 y_1).$$

By using the commutative law for real multiplication we see that the real parts of $z_1 z_2$ and $z_2 z_1$ are the same. By using the commutative law for real addition and for real multiplication we see that the imaginary parts of $z_1 z_2$ and $z_2 z_1$ are the same. Thus $z_1 z_2 = z_2 z_1$. □

Exercise 4 (page 7 of B&C). Prove that if $z_1 z_2 z_3 = 0$ then at least one of three factors is zero.

Proof. If we denote $Z = z_1 z_2$ then we can rewrite our expression as

$$0 = z_1 z_2 z_3 = (z_1 z_2) z_3 = Z z_3.$$

Using the cancellation property for multiplication of complex numbers we deduce that $Z = 0$ or $z_3 = 0$ or both. In the case $Z = 0$, using this property again we see that at least one of factors z_1, z_2 is zero. □

Exercise 7 (page 7 of B&C). Derive the cancellation law

$$\frac{z_1 z}{z_2 z} = \frac{z_1}{z_2}, \quad z_2 \neq 0,\, z \neq 0.$$

Proof. Since $z_2 \neq 0$, $z \neq 0$, we can use property (8) on page 6:

$$\frac{z_1 z}{z_2 z} = \frac{z_1}{z_2}\frac{z}{z} = \frac{z_1}{z_2}.$$

The second equality follows from property (5) on page 6. □

Exercise C1. Prove by mathematical induction that, for $n \geq 2$,

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2} \qquad (1)$$

Proof. We denote $S_n = 1 + 2 + 3 + \cdots + n$. We can check that

$$S_2 = 3 = \frac{2\,(3)}{2}, \quad S_3 = 6 = \frac{3\,(4)}{2}.$$

Let us assume we proved (1) for some n. Then

$$\begin{aligned} S_{n+1} &= 1 + 2 + 3 + \cdots + n + (n+1) = S_n + (n+1) = \frac{n(n+1)}{2} + (n+1) \\ &= \frac{n^2 + n + 2n + 2}{2} = \frac{(n+1)(n+2)}{2}. \end{aligned}$$

Thus formula (1) applies with n replaced by $n+1$. By the principle of mathematical induction, (1) holds for all n. □

Exercise C2. Let $n \geq 1$ be an integer and let k be an integer in the range $1 \leq k \leq n$. Then

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}.$$

Proof. A direct proof. The first equality is by definition:

$$\begin{aligned} \binom{n}{k} + \binom{n}{k-1} &= \frac{n!}{k!(n-k)!} + \frac{n!}{(k-1)!(n-(k-1))!} = \frac{n!}{k!(n-k)!} + \frac{n!}{(k-1)!(n+1-k)!} \\ &\overset{\circledast}{=} \frac{(n+1-k)\,n!}{k!(n+1-k)!} + \frac{k\,n!}{k!(n+1-k)!} \\ &= \frac{(n+1-k+k)\,n!}{k!(n+1-k)!} = \frac{(n+1)!}{k!(n+1-k)!} = \binom{n+1}{k}. \end{aligned}$$

Note that the equality marked "$\circledast$", though obscure looking, is just the usual step of finding a common denominator. □

Note this is the statement that generates Pascal's triangle! Thinking about Pascal's triangle clarify the ranges on n and on k.