Review Topics
for In-Class Midterm Quiz on *Friday 30 October, 2015*

The Midterm Quiz will cover Lectures 1, 2, 3, 4, 5, 6, 7, 8, 10, 11 in Trefethen & Bau. The problems will be of these types: state definitions, state theorems, state algorithms (as pseudocode or Matlab), describe or illustrate basic geometrical ideas, make basic applications of theorems, make basic calculations, prove simple theorems/corollaries.

Definitions. Know how to define:

- matrix-vector product; matrix-matrix product
- adjoint; hermitian; transpose
- inner product; outer product
- unitary
- $\|\cdot\|_p$ of a vector in $\mathbb{C}^m$ for any $1 \le p \le \infty$
- induced matrix norm $\|\cdot\|$
- Frobenius matrix norm $\|\cdot\|_F$
- projector; orthogonal projector
- eigenvalue; eigenvector; singular value

Matrix Factorizations and Constructions. Here $A \in \mathbb{C}^{m\times n}$ unless otherwise stated. Know the properties of the factors. (E.g. know that "$\hat U$ has orthonormal columns and is of same size as A in the $m \ge n$ case of the reduced SVD factorization $A = \hat U \hat\Sigma V^*$.") Be able to use the factorization in simple calculations. Be able to compute the factorization by hand in sufficiently simple cases.

- full SVD: $A = U\Sigma V^*$
- reduced SVD, $m \ge n$: $A = \hat U \hat\Sigma V^*$
- full QR, $m \ge n$: $A = QR$
- reduced QR, $m \ge n$: $A = \hat Q \hat R$
- eigenvalue, $m = n$: $AX = X\Lambda$ and X is invertible
- orthogonal projector, $\hat Q$ has $m \ge n$: $P = \hat Q \hat Q^*$
- orthogonal projector, A full rank with $m \ge n$: $P = A\,(A^*A)^{-1}\,A^*$
- Householder reflector: $F = I - 2vv^*/(v^*v)$ and $Q = \left[\begin{smallmatrix} I & 0 \\ 0 & F \end{smallmatrix}\right]$

Algorithms. Be able to state these algorithms, including the amount of work to leading order:

- matrix-vector and matrix-matrix products (Assignment #1)
- 7.1: classical Gram-Schmidt for QR
- 8.1: modified Gram-Schmidt for QR
- 10.1: Householder triangularization for QR
- 10.2: compute Q^*b after Householder triangularization
- 11.2: QR "modern classical" for least squares on overdetermined "$Ax = b$"
- 11.1: normal equations for least squares on overdetermined "$Ax = b$"

Facts, Formulas, and Inequalities. Know as facts. Be able to prove unless otherwise stated.

- Cauchy-Schwarz: $|x^*y| \leq \|x\|\,\|y\|$ *[proof not required]*
- 1-norm of a matrix is the maximum absolute column sum
- ∞-norm of a matrix is the maximum absolute row sum
- invariance of $\|\cdot\|_2$ and $\|\cdot\|_F$ matrix norms under unitaries
- $\|A\|_F = \sqrt{\sigma_1^2 + \cdots + \sigma_r^2} = \sqrt{\mathrm{tr}(A^*A)}$
- $\|A\|_2 = \sigma_1 = \sqrt{\rho(A^*A)}$
- $\mathrm{rank}(A)$ is number of nonzero singular values (in exact arithmetic)
- for $A \in \mathbb{C}^{m\times n}$, A has full rank if and only if A^*A is nonsingular
- the singular values of A are the square roots of the eigenvalues of A^*A
- if P is a projector then so is $I - P$
- if P is an orthogonal projector then so is $I - P$, and furthermore $I - 2P$ is unitary

Ideas. In this class there are ideas to be comfortable with! In some cases in the list below there are provable theorems, but in other cases there is just a perspective or paradigm to understand:

- L1 and L2: how to think about Ax, $A^{-1}b$, Qx, Q^*b
- L4: the image of the unit sphere under any $m \times n$ matrix is a hyperellipsoid
- L5: sums like this are optimal (in what sense?) approximations of A:

$$A_\nu = \sum_{j=1}^{\nu} \sigma_j u_j v_j^*$$

- construction of orthogonal functions (e.g. orthogonal polynomials) is instance of Gram-Schmidt, and of $A = QR$, in case where columns of A are infinitely long
- the leading-order number of operations in an algorithm can be counted by counting only the number of times the inner-most operations are executed
- the number of operations in an algorithm can be counted by geometric arguments
- Gram-Schmidt is "triangular orthogonalization" while Householder is "orthogonal triangularization"[1]

[1] And Gauss elimination is "triangular triangularization" . . . but do not worry about that on the Midterm Exam!