Selected Solutions to Assignment #3

Exercise 5 (page 32 of B&C). Let S be the open set of all points such that $|z| < 1$ or $|z-2| < 1$. State why S is not connected.

Proof. There are no polygonal line that connects points $(0,0)$ and $(2,0)$. Such a line necessarily crosses the line $\{x = 1\}$, which does not belong to S. (It is helpful to draw the picture.) □

Exercise 6 (page 32 of B&C). Show that a set S is open if and only if each point in S is an interior point.

Proof. (*This proof follows the book's definition. It is also possible to use the definition given in class, in which case the proof is just as short.*)

Let S be open. Then by definition, it does not contain boundary points, so all points in S are interior. Let now S be a set consisting of interior points, then by definition, it is open (it does not contain boundary points). □

Exercise 9 (page 32 of B&C). Show that any point z_0 of a domain S is an accumulation point of that domain.

Proof. Consider some deleted neighborhood of z_0: $D(z_0) = \{0 < |z - z_0| < \varepsilon\}$. This deleted neighborhood is not necessarily contained in S. Since S is a domain, it is open, so there exist some $\delta > 0$ such that neighborhood $B(z_0) = \{0 < |z - z_0| < \delta\}$ is contained in S. Then the set $D(z_0) \cap B(z_0)$ is not empty, so the deleted neighborhood of $D(z_0)$ contains points from S. Since ε was arbitrary, we have proven that any deleted ε neighborhood of z_0 contains points of S, so z_0 is an accumulation point of S. □

Exercise C3. Write

$$f(z) = \frac{1}{1+z} + \sqrt{z}$$

in the form $f(z) = u(r,\theta) + iv(r,\theta)$

Solution. Note $z = r(\cos\theta + i\sin\theta)$. and $\sqrt{z} = \sqrt{r}(\cos\theta/2 + i\sin\theta/2)$. We arrive at

$$\frac{1}{1+z} + \sqrt{z} = \frac{1}{(r\cos\theta + 1) + ir\sin\theta} + \sqrt{r}(\cos\theta/2 + ir\sin\theta/2)$$

We multiply the numerator and denominator of fraction by $r\cos\theta + 1 - ir\sin\theta$ and have that

$$\frac{1}{1+z} + \sqrt{z} = \frac{r\cos\theta + 1 - ir\sin\theta}{(r\cos\theta+1)^2 + (r\sin\theta)^2} + \sqrt{r}(\cos\theta/2 + ir\sin\theta/2) =$$

$$\left(\frac{r\cos\theta + 1}{r^2 + 1 + 2r\cos\theta} + \sqrt{r}\cos\theta/2\right) + i\left(\frac{-r\sin\theta}{r^2 + 1 + 2r\cos\theta} + \sqrt{r}\sin\theta/2\right)$$

Thus

$$u(r,\theta) = \frac{r\cos\theta + 1}{r^2 + 1 + 2r\cos\theta} + \sqrt{r}\cos\theta/2 \quad \text{and} \quad v(r,\theta) = \frac{-r\sin\theta}{r^2 + 1 + 2r\cos\theta} + \sqrt{r}\sin\theta/2.$$

Exercise 3 (page 42 of B&C). Sketch the region onto which the sector $r \leqslant 1$, $0 \leqslant \theta \leqslant \pi/4$ is mapped by the transformation

a) $\omega = z^2$: *Solution.* Upper right quarter of circle:

$$D = \{r \leqslant 1,\, 0 \leqslant \theta \leqslant \pi/2\}$$

b) $\omega = z^3$: *Solution.*

$$D = \{r \leqslant 1,\, 0 \leqslant \theta \leqslant 3\pi/4\}$$

c) $\omega = z^4$: *Solution.* Upper half of circle:

$$D = \{r \leqslant 1,\, 0 \leqslant \theta \leqslant \pi\}$$

Exercise 5 (page 42 of B&C). Verify that the image of the region $a \leqslant x \leqslant b$, $c \leqslant y \leqslant d$ under the transformation $\omega = e^z$ is the region $e^a \leqslant \rho \leqslant e^b$, $c \leqslant \phi \leqslant d$.

Proof. If $z = x + iy$ then

$$w = e^z = e^{x+iy} = e^x e^{iy}.$$

From this we see that the line segment $[(a,c),(a,d)]$ (the left side of the rectangle) gets mapped to the curve $\{r = e^a,\, c \leqslant \theta \leqslant d\}$. Similarly, the right side of the rectangle, the line segment $[(b,c),(b,d)]$, gets mapped to the curve $\{r = e^b,\, c \leqslant \theta \leqslant d\}$. The lower side gets mapped to the curve (in polar coordinates) $[(e^a,c),(e^b,c)]$, and the upper side $[(a,d),(b,d)]$ get mapped to the curve (in polar coordinates) $[(e^a,d),(e^b,d)]$. $\square$

Exercise 8 (page 43 of B&C). Indicate graphically the vector field represented by

a) $\omega = iz$: *Solution.* The picture looks like this: take some point $z \in \mathbb{C}$, $z \neq 0$. Then the vector $\omega(z)$, "attached" to this point is a vector that can be obtained from the vector z by rotating it by $\pi/2$ angle, counterclockwise.

b) $\omega = z/|z|$: *Solution.* For every point z, the vector $\omega(z)$ attached at this point is the unit vector, directed out of the center of coordinates.