Selected Solutions to Assignment #7

Exercise 2 (page 96 of B&C). For any z_1, z_2,

$$\operatorname{Log}(z_1 z_2) = \operatorname{Log}(z_1) + \operatorname{Log}(z_2) + 2N\pi i$$

for N equal to one of $-1, 0, 1$.

Proof. Expanding the definition of Log gives

$$\operatorname{Log}(z_1 z_2) = \ln(|z_1 z_2|) + i \operatorname{Arg}(z_1 z_2) = \ln(|z_1|) + \ln(|z_2|) + i \operatorname{Arg}(z_1 z_2).$$

We need to analyze $\operatorname{Arg}(z_1 z_2)$ exactly as is done back on pages 18 and 19 of the textbook. Generally, if $z_1 = |z_1|e^{i\theta}$ and $z_2 = |z_2|e^{i\phi}$ then $\arg(z_1 z_2)$ is the set $\{\theta + \phi + 2n\pi i\}$ where n is any integer, while $\operatorname{Arg}(z_1 z_2)$ is the element of this set which is in the interval $(-\pi, \pi]$. But if $\theta = \operatorname{Arg}(z_1)$ and $\phi = \operatorname{Arg}(z_2)$ then these angles are each in the interval $(-\pi, \pi]$, so $\theta + \phi$ is in the interval $(-2\pi, 2\pi]$. If $\theta + \phi \in (-2\pi, -\pi]$ then $\operatorname{Arg}(z_1 z_2) = \operatorname{Arg}(z_1) + \operatorname{Arg}(z_2) + 2\pi$. If $\theta + \phi \in (-\pi, \pi]$ then $\operatorname{Arg}(z_1 z_2) = \operatorname{Arg}(z_1) + \operatorname{Arg}(z_2)$. If $\theta + \phi \in (\pi, 2\pi]$ then $\operatorname{Arg}(z_1 z_2) = \operatorname{Arg}(z_1) + \operatorname{Arg}(z_2) - 2\pi$. Always, therefore, $\operatorname{Arg}(z_1 z_2) = \operatorname{Arg}(z_1) + \operatorname{Arg}(z_2) + 2N\pi$ for N in the statement of the problem.

Now we conclude

$$\operatorname{Log}(z_1 z_2) = \ln(|z_1|) + \ln(|z_2|) + i\left(\operatorname{Arg}(z_1) + \operatorname{Arg}(z_2) + 2N\pi\right) = \operatorname{Log}(z_1) + \operatorname{Log}(z_2) + 2N\pi i.$$

□

Exercise 2a (page 99 of B&C). The principal value of i^i ia $\exp(-\pi/2)$, a real, positive number.

Proof. Recall the definition: $z^c = e^{c \log(z)}$, where log is multivalued. Noting $z = i = 1\, e^{i\pi/2}$,

$$i^i = e^{i \log(i)} = e^{i(\ln(1) + i\pi/2 + 2n\pi i)} = e^{-\pi/2 - 2n\pi}.$$

The principal value is the choice $n = 0$. □

The proof shows that $i^i = \{e^{-\pi/2}(e^{-2\pi})^n\}$. Therefore i^i *is a sequence of positive numbers trending toward zero!*

Exercise 6 (page 100 of B&C). Suppose a is real. Then $|z^a| = \exp(a \ln |z|)$.

Proof. Recall the definition: $z^c = e^{c \log(z)}$, where log is multivalued. Here $c = a$ is real. Therefore

$$|z^a| = |e^{a \log(z)}| = |e^{a(\ln|z| + i(\operatorname{Arg} z + 2n\pi))}| = |e^{a \ln|z|}||e^{i(\operatorname{Arg} z + 2n\pi)a}| = e^{a \ln |z|}.$$

The last equality follows because e^x is positive if x is real and because $|e^{i\theta}| = 1$ if θ is real. □

Exercise 7 (page 104 of B&C). (a) Show that $1 + \tan^2 z = \sec^2 z$

Proof. Using the definition (19) of tan, identity (7) the definition (20) of sec :

$$1 + \tan^2 z = 1 + \left(\frac{\sin z}{\cos z}\right)^2 = \frac{\cos^2 z + \sin^2 z}{\cos^2 z} = \frac{1}{\cos^2 z} = \sec^2 z$$

□

[(b): The proof that $1 + \cot^2 z = \csc^2 z$ is very similar.]

Exercise 10 (page 104 of B&C). If $z = x + iy$ then

$$|\sin z| \geq |\sin x| \qquad \text{and} \qquad |\cos z| \geq |\cos x|.$$

Proof. From (15) and (16) on page 102:

$$|\sin z|^2 = \sin^2 x + \sinh^2 y, \qquad |\cos z|^2 = \cos^2 x + \sinh^2 y.$$

Therefore

$$|\sin z|^2 \geq \sin^2 x = |\sin x|^2,$$

and taking square roots (*noting* $f(x) = \sqrt{x}$ *is an increasing function of* x), $|\sin z| \geq |\sin x|$. The same argument gives $|\cos z| \geq |\cos x|$. □

Exercise 2 (page 107 of B&C). Prove that $\sinh 2z = 2 \sinh z \cosh z$

(a) Using the definition (1) Sec. 34.

$$\sinh 2z = \frac{e^{2z} - e^{-2z}}{2} = 2\frac{(e^z - e^{-z})(e^z + e^{-z})}{2 * 2} = 2 \sinh z \cosh z.$$

(b) Using the identity $\sin 2z = 2 \sin z \cos z$ and relations (3), (4) Sec. 34

$$\sinh 2z = -i \sin 2iz = -2i \sin iz \cos iz = 2 \sinh z \cosh z.$$

Exercise 5 (page 110 of B&C).

$$\arctan z = \tan^{-1} z = \frac{i}{2} \log \frac{i + z}{i - z}.$$

Proof. Let $w = \arctan z$, so (by definition of arctan) $z = \tan w$. But then

$$z = \tan w = \frac{\sin w}{\cos w} = \frac{2(e^{iw} - e^{-iw})}{2i(e^{iw} + e^{-iw})} = \frac{1}{i}\frac{e^{iw} - e^{-iw}}{e^{iw} + e^{-iw}} = \frac{1}{i}\frac{e^{i2w} - 1}{e^{i2w} + 1}.$$

(In the last equality we multiply the fraction by $1 = e^{iw}/e^{iw}$.) Rewriting slightly we have

$$iz(e^{i2w} + 1) = e^{i2w} - 1 \qquad \text{or} \qquad e^{i2w}(iz - 1) = -1 - iz.$$

Now we can solve for w, our goal:

$$w = \frac{1}{2i} \log\left(\frac{-1 - iz}{-1 + iz}\right) = \frac{-i}{2} \log\left(\frac{i - z}{i + z}\right) = \frac{i}{2} \log\left(\frac{i + z}{i - z}\right).$$

In the last equality we use $\log(z^{-1}) = -\log z$, which follows from (4) on page 96. □

Exercise 3 (page 115 of B&C). Show that if m, n are integers,

$$\int_0^{2\pi} e^{im\theta} e^{-in\theta}\, d\theta = \begin{cases} 0 & \text{when } m \neq n, \\ 2\pi & \text{when } m = n. \end{cases}$$

Proof. When $m = n$:

$$\int_0^{2\pi} e^{im\theta} e^{-im\theta}\, d\theta = \int_0^{2\pi} d\theta = 2\pi$$

When $m \neq n$:

$$\int_0^{2\pi} e^{im\theta} e^{-in\theta}\, d\theta = \int_0^{2\pi} e^{i\theta(m-n)} d\theta = \frac{1}{i(m-n)} e^{i\theta(m-n)}\Big]_0^{2\pi} = 0$$

□

Exercise 5 (page 116 of B&C). Let us integrate:

$$\int_0^{2\pi} e^{it}\, dt = -ie^{it}\Big]_0^{2\pi} = -i(e^{i2\pi} - e^{i0}) = 0.$$

But the integrand $w(t) = e^{it}$ was never zero. Thus the mean value theorem for integrals of continuous real-valued functions,[1] $\int_a^b w(t)\, dt = w(c)(b-a)$, applies to real valued $w(t)$ but not to complex-valued $w(t)$.

Exercise 3 (page 129 of B&C). Let $f(z) = \pi e^{\pi \bar{z}}$. Evaluate the integral $\int_C f(z)\, dz$ where C is square, orientation is counterclockwise.

Proof. Note that $\int_C f(z)\, dz = \int_{C_1} f(z)\, dz + \int_{C_2} f(z)\, dz + \int_{C_3} f(z)\, dz + \int_{C_4} f(z)\, dz$, where C_1, C_2, C_3, C_4 are sides of square. On the first side $z(x) = x$, $x \in (0, 1)$, so

$$\int_{C_1} f(z)\, dz = \int_0^1 \pi e^{\pi x}\, dx = e^{\pi} - 1$$

On the second side $z(y) = 1 + iy$, $dz = i$, $y \in (0, 1)$, so

$$\int_{C_2} f(z)\, dz = \int_0^1 \pi e^{\pi(1-iy)} i\, dy = 2e^{\pi}.$$

On the next side $z(x) = x + i$, $dz = 1$, $x \in (1, 0)$, so

$$\int_{C_3} f(z)\, dz = \int_1^0 \pi e^{\pi(x-i)}\, dx = e^{\pi} - 1.$$

And on the last side $z(y) = iy$, $dz = i$, $y \in (1, 0)$, so

$$\int_{C_4} f(z)\, dz = \int_1^0 \pi e^{-\pi(iy)} i\, dy = -2.$$

Adding expressions for $\int_{C_1} f(z)\, dz$, $\int_{C_2} f(z)\, dz$, $\int_{C_3} f(z)\, dz$, $\int_{C_4} f(z)\, dz$ we get

$$\int_C f(z)\, dz = 4(e^{\pi} - 1)$$

□

[1] Which appears on page 443 of the current *Calculus I* textbook for Math 200, for instance.