Selected Solutions to Assignment #2

Exercise 3 (page 11 of B&C). Verify that $\sqrt{2}|z| \geqslant |\operatorname{Re} z| + |\operatorname{Im} z|$

Solution. We square both sides of inequality and get

$$2|z|^2 = 2(|\operatorname{Re} z|^2 + |\operatorname{Im} z|^2) \geqslant |\operatorname{Re} z|^2 + 2|\operatorname{Re} z||\operatorname{Im} z| + |\operatorname{Im} z|^2$$

Rearranging terms we get

$$|\operatorname{Re} z|^2 - 2|\operatorname{Re} z||\operatorname{Im} z| + |\operatorname{Im} z|^2 = (|\operatorname{Re} z| - |\operatorname{Im} z|)^2 \geqslant 0$$

The validity of the last inequality implies the validity of the initial one.

Exercise 4 (page 11 of B&C). Sketch the set of points determined by the given conditions

a) $|z - 1 + i| = 1$

The set of points is the circle with the center at $(1, -1)$ and radius one.

b) $|z + i| \leqslant 3$

The set of points is the closed disc (ball) with center at $(0, -1)$ and radius three.

c) $|z - 4i| \geqslant 4$

The set of points is the exterior of the open disc (ball) with center at $(0, 4)$ and radius four. The set is closed but unbounded.

Exercise 11 (page 14 of B&C).

a) Prove that z is real if and only if $\bar{z} = z$.

Proof. Let $z = x + iy$. If z is real, then $y = 0$ and $z = \bar{z} = x$. Conversely, suppose that $\bar{z} = z$. Then $x - iy = x + iy$. The last equality implies $y = -y$, so $y = 0$. It means that $z = x$, i.e. z is real. □

b) Prove that z is either real or pure imaginary if and only if $z^2 = \bar{z}^2$.

Proof. Let $z = x + iy$, so $\bar{z} = x - iy$. Consider the two expressions

$$z^2 = x^2 - y^2 + 2ixy \qquad \text{and} \qquad \bar{z}^2 = x^2 - y^2 - 2ixy.$$

Suppose $\bar{z}^2 = z^2$. Then $4ixy = 0$. Thus either $x = 0$ (and z is pure imaginary) or $y = 0$ (and z is pure real).

On the other hand, if z is pure imaginary or pure real then one of $x = 0$ or $y = 0$ is true, so $4ixy = 0$. The above expressions for $\bar{z}^2$ and z^2 show they are equal. □

Exercise 15 (page 14 of B&C). Show that the hyperbola $x^2 - y^2 = 1$ may be written $z^2 + \bar{z}^2 = 2$.

Proof. Plugging $z = x + iy$ and $\bar{z} = x - iy$ into $z^2 + \bar{z}^2 = 2$, we get

$$x^2 - y^2 + 2ixy + x^2 - y^2 - 2ixy = 2.$$

Dividing both parts of the last equation by two, we get $x^2 - y^2 = 1$. □

Exercise 5 (page 21 of B&C). Use the $n = 3$ de Moivre's formula to derive trigonometric identities.

Solution. Let us use de Moivre's formula with $n = 3$:

$$(\cos\theta + i\sin\theta)^3 = (\cos 3\theta + i\sin 3\theta)$$

Evaluating the left hand side of the above equation, we have:

$$\begin{gathered}(\cos\theta + i\sin\theta)^3 = \cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta = \\ \cos^3\theta - 3\cos\theta\sin^2\theta + i(3\cos^2\theta\sin\theta - \sin^3\theta) = \cos 3\theta + i\sin 3\theta.\end{gathered}$$

Equating real and imaginary parts, we get:

$$\begin{aligned}\cos 3\theta &= \cos^3\theta - 3\cos\theta\sin^2\theta,\\ \sin 3\theta &= 3\cos^2\theta\sin\theta - \sin^3\theta.\end{aligned}$$

Exercise 7 (page 21 of B&C). Show that if $\operatorname{Re} z_1 > 0$ and $\operatorname{Re} z_2 > 0$, then

$$\operatorname{Arg}(z_1 z_2) = \operatorname{Arg} z_1 + \operatorname{Arg} z_2.$$

Proof. Let $z_1 = r_1 e^{i\theta_1}$, $z_2 = r_2 e^{i\theta_2}$. Due to the condition $\operatorname{Re} z_{1,2} > 0$, we have that $-\pi/2 < \theta_{1,2} < \pi/2$ and consequently, $-\pi < \theta_1 + \theta_2 < \pi$. Then

$$\operatorname{Arg}(z_1 z_2) = \operatorname{Arg}(r_1 r_2 e^{i(\theta_1+\theta_2)}) = \theta_1 + \theta_2 = \operatorname{Arg} z_1 + \operatorname{Arg} z_2.$$

□

Exercise 4 (page 28 of B&C). Find all cube roots of $z_0 = -4\sqrt{2} + 4\sqrt{2}i$

Solution. We can rewrite z_0 as

$$z_0 = 4\sqrt{2}(-1+i) = 8\left(-\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right) = 8e^{i\frac{3\pi}{4}}$$

Then the principal cube root is

$$c_0 = 2e^{i\frac{\pi}{4}} = \sqrt{2} + \sqrt{2}i.$$

We know that

$$\omega_3 = e^{i\frac{2\pi}{3}} = \frac{-1+\sqrt{3}i}{2}$$

then

$$(\omega_3)^2 = e^{i\frac{4\pi}{3}} = \frac{-1-\sqrt{3}i}{2}$$

Using these equalities, we get

$$\begin{aligned}c_0\omega_3 &= \sqrt{2}(1+i)\frac{-1+\sqrt{3}i}{2} = \frac{-1-\sqrt{3}+i(\sqrt{3}-1)}{\sqrt{2}},\\ c_0(\omega_3)^2 &= \sqrt{2}(1+i)\frac{-1-\sqrt{3}i}{2} = \frac{-1+\sqrt{3}-i(\sqrt{3}+1)}{\sqrt{2}},\end{aligned}$$