Assignment #9

Due Friday, 20 November, 2015 at the start of class

Please read Lectures 14, 15, 16, 17, and 20 in Trefethen & Bau.

P18. A *circulant matrix* is one where the diagonals are constant and "wrap around":

$$C = \begin{bmatrix} c_1 & c_m & \dots & c_3 & c_2 \\ c_2 & c_1 & c_m & & c_3 \\ \vdots & c_2 & c_1 & \ddots & \vdots \\ c_{m-1} & & \ddots & \ddots & c_m \\ c_m & c_{m-2} & \dots & c_2 & c_1 \end{bmatrix} \tag{1}$$

In formulas, the entries of $C \in \mathbb{C}^{m\times m}$ are a function of row/column index differences:

$$C_{jk} = \begin{cases} c_{j-k+1}, & j \ge k, \\ c_{m+j-k+1}, & j < k. \end{cases}$$

Here $c_1, \dots, c_m$ are the entries of a column vector, the first column of C. Specifying the first column of a circulant matrix describes it completely.

Here is an extraordinary fact about circulant matrices: Every circulant matrix has a complete set of eigenvectors *that are known in advance*, without knowing the eigenvalues. Specifically, define $f_k \in \mathbb{C}^m$ by

$$(f_k)_j = \exp\left(-i(j-1)(k-1)\frac{2\pi}{m}\right) = e^{-i2\pi(k-1)(j-1)/m}, \tag{2}$$

where, as usual, $i = \sqrt{-1}$. These vectors are *waves*, i.e. combinations of familiar sines and cosines. After some warm-up exercises you will show in part **(d)** that $Cf_k = \lambda_k f_k$.

Also, at some point in doing this problem it may be useful to see the Wikipedia pages for "circulant matrix," "discrete Fourier transform," and "DFT matrix." It may help you organize thoughts.

(a) Define the *periodic convolution* $u * w \in \mathbb{C}^m$ of vectors $u, w \in \mathbb{C}^m$ by

$$(u * w)_j = \sum_{k=1}^m u_{\mu(j,k)} w_k \qquad \text{where} \qquad \mu(j,k) = \begin{cases} j-k+1, & j \ge k, \\ m+j-k+1, & j < k. \end{cases}$$

Show that $u * w = w * u$.

(b) Show that $Cu = v * u$ if C is a circulant matrix and v is the first column of C.

(c) Show that for any m, the eigenvectors $\{f_1, \dots, f_m\}$ are orthogonal.

(d) For the general circulant matrix C in (1) above, confirm the "extraordinary fact" as follows: Give a formula for the eigenvalues λ_k, in terms of the entries $c_1, \dots, c_m$, by showing by direct by-hand calculation that $C f_k = \lambda_k f_k$.

(e) Download this MATLAB function, which builds a circulant matrix with a given first column; notice how it uses the `mod()` function:

http://bueler.github.io/M614F15/matlab/circu.m

Generate the circulant matrix C with first column consisting of 20 random numbers of your choice. Use the result of **(d)** to compute the eigenvalues λ_k, and compare these against the result of `eig()`. Also, generate f_5 from (2) and check numerically that $C f_5 = \lambda_5 f_5$.

Exercise 15.1 in Lecture 15.

Exercise 16.1 in Lecture 16.

Exercise 17.2 in Lecture 17.

Exercise 17.3a in Lecture 17.