Solutions to Midterm Exam #2

1 Suppose $f(z) = u(x, y) + iv(x, y)$ is analytic in an open set. Use the Cauchy-Riemann equations to show that the real part u is harmonic, that is, $u_{xx} + u_{yy} = 0$. What assumptions, if any, are necessary for your derivation to be correct?

Proof. We know that since $f(z)$ is analytic, partial derivatives of u, v exist and satisfy CR equations:

$$u_x(x, y) = v_y(x, y), \quad u_y(x, y) = -v_x(x, y).$$

Assuming that there exist second derivatives of u, v, we can differentiate the first equation with respect to x and the second equation with respect to y:

$$u_{xx}(x, y) = v_{yx}(x, y), \quad u_{yy}(x, y) = -v_{xy}(x, y).$$

Since $v_{yx} = v_{xy}$, we derive that $u_{xx} + u_{yy} = 0$.

For now we must assume that the second derivatives are continuous, so that $v_{yx} = v_{xy}$, to make this argument work. (Later will see that this is not necessary to assume; the fact that f is analytic actually implies that it has continuous derivatives of all orders.) □

2 Compute $\log(-i)$.

Proof. $\log(-i) = \log(1 \cdot e^{-\frac{\pi}{2}}) = \ln 1 + i\left(-\frac{\pi}{2} + 2n\pi\right) = i\left(-\frac{\pi}{2} + 2n\pi\right)$, for $n \in \mathbb{Z}$. □

3 Explain why $\int_{-C} f(z)\,dz = -\int_C f(z)\,dz$.

Proof. By definition, when we have parametrization of contour C by the function $z(t)$, $t \in [a, b]$, then the contour $-C$ is parametrized by $z(-t)$, $t \in [-b, -a]$ (or, equivalently, $z(b+a-t)$, $t \in [a, b]$).

On the other hand

$$\int_C f(z)\,dz = \int_a^b f(z(t))z'(t)\,dt.$$

Using the definition of integral and changing variable we have

$$\int_{-C} f(z)\,dz = \int_{-b}^{-a} f(z(-t))\frac{d}{dt}z(-t)\,dt = -\int_{-b}^{-a} f(z(-t))z'(-t)\,dt$$

$$\overset{[\tau=-t]}{=} \int_b^a f(z(\tau))z'(\tau)\,d\tau = -\int_a^b f(z(\tau))z'(\tau)\,d\tau = \int_C f(z)\,dz$$

(*If the formula "$z(b+a-t)$, $t \in [a, b]$" was used for $-C$ then the substitution is $\tau = b + a - t$.*) □

4 Show that $\exp(z)$ is real if and only if $\operatorname{Im} z = n\pi$ for some integer n.

Proof. If $z = x+iy$ then the representation $e^z = e^x(\cos y + i \sin y)$ implies that e^z is real if and only if $\sin y = 0$. This holds if and only if $\operatorname{Im} z = y = n\pi$, $n \in \mathbb{Z}$. □

5 **(a)** Define "$\cos z$" if z is a complex number.
Definition: $\cos z = \frac{e^{iz}+e^{-iz}}{2}$.

(b) Show that $\cos z = \cos x \cosh y - i \sin x \sinh y$ if $z = x + iy$.

Proof. We use the formula $\cos(x+iy) = \cos x \cos iy - \sin x \sin iy$. Note that $\cos iy = \cosh y$ and $\sin iy = i \sinh y$, we get $\cos(x+iy) = \cos x \cosh y - i \sin x \sinh y$. $\square$

6 (a) Show that $f(z) = \exp(\bar{z})$ is not analytic at any point of the complex plane.

Proof. Let us rewrite our function as $f(z) = e^{\bar{z}} = e^x \cos y - ie^x \sin y$. Thus $u(x,y) = e^x \cos y$, $v(x,y) = -e^x \sin y$. The C-R equation $u_x = v_y$ is only true when $e^x \cos y = -e^x \cos y$. This implies $\cos y = 0$, or $y = \frac{\pi}{2} + k\pi$, $k \in \mathbb{Z}$. The other C-R equation $u_y = -v_x$ is only true when $-e^x \sin y = e^x \sin y$. That is, $\sin y = 0$ or $y = k\pi$, $k \in \mathbb{Z}$. The conditions $\cos y = 0$ and $\sin y = 0$ cannot hold simultaneously. Thus, the C-R equations are satisfied nowhere and the function $f(z) = \exp(\bar{z})$ is analytic nowhere. $\square$

(b) Let C be the contour given by the parameterization $z(t) = t + i(1-t)$ for $0 \le t \le 1$. For the function $f(z)$ in part **(a)**, compute $\int_C f(z)\, dz$.

Proof. We have $z(t) = x(t) + iy(t) = t + i(1-t)$, so $x(t) = t$, $y(t) = 1-t$, $z'(t) = 1-i$ and we can evaluate:

$$\begin{aligned} \int_C e^{\bar{z}}\, dz = \int_0^1 e^{x(t)-iy(t)} z'(t)\, dt = \int_0^1 e^{t-i(1-t)}(1-i)\, dt = e^{-i}(1-i) \int_0^1 e^{t(1+i)}\, dt \\ = e^{-i} \frac{1-i}{1+i} e^{t(1+i)} \Big]_0^1 = \frac{1-i}{1+i}(e - e^{-i}). \end{aligned}$$

$\square$

7 (a) State the definition of "z^c" for z and c complex numbers (and $z \neq 0$).
Definition: $z^c = e^{c \log z}$.

(b) Find the principal value of $(1+i)^{4i}$.

Proof. We know that $1+i = \sqrt{2} e^{i\frac{\pi}{4}}$. Then $(1+i)^{4i} = e^{4i \operatorname{Log}(1+i)} = e^{4i \operatorname{Log}(\sqrt{2} e^{i\pi/4})} = e^{4i(\ln \sqrt{2} + i\pi/4)} = e^{-\pi + 2i \ln 2}$. $\square$

(c) Show that $(z^c)^n = z^{cn}$ if $n = 1, 2, 3, \ldots$.

Proof. Using the definition of z^c twice, we get $(z^c)^n = \left(e^{c \log z}\right)^n = e^{cn \log z} = z^{cn}$. $\square$

8 (a) Find an antiderivative of $f(z) = 1/z^2$. Is this antiderivative multi-valued? Can an antiderivative ever be multi-valued according to the definition in the textbook?

Proof. An antiderivative of $f(z) = \frac{1}{z^2}$ in $\mathbb{C} \backslash 0$ is $F(z) = -\frac{1}{z}$. It is analytic except at $z = 0$ and single-valued. Antiderivatives of any function, if they exist, must be single-valued functions by definition. $\square$

(b) Suppose C is the circle $z = e^{i\theta}$, $-\pi \le \theta \le \pi$. Compute $\int_C \frac{dz}{z^2}$.

Proof. Since $f(z) = \frac{1}{z^2}$ has an antiderivative $F(z)$ as above, and because $-1 = e^{i(-\pi)} = e^{i\pi}$ at both ends of the contour C,

$$\int_C \frac{dz}{z^2} = F(-1) - F(-1) = 0.$$

$\square$

Extra Credit See separate note.