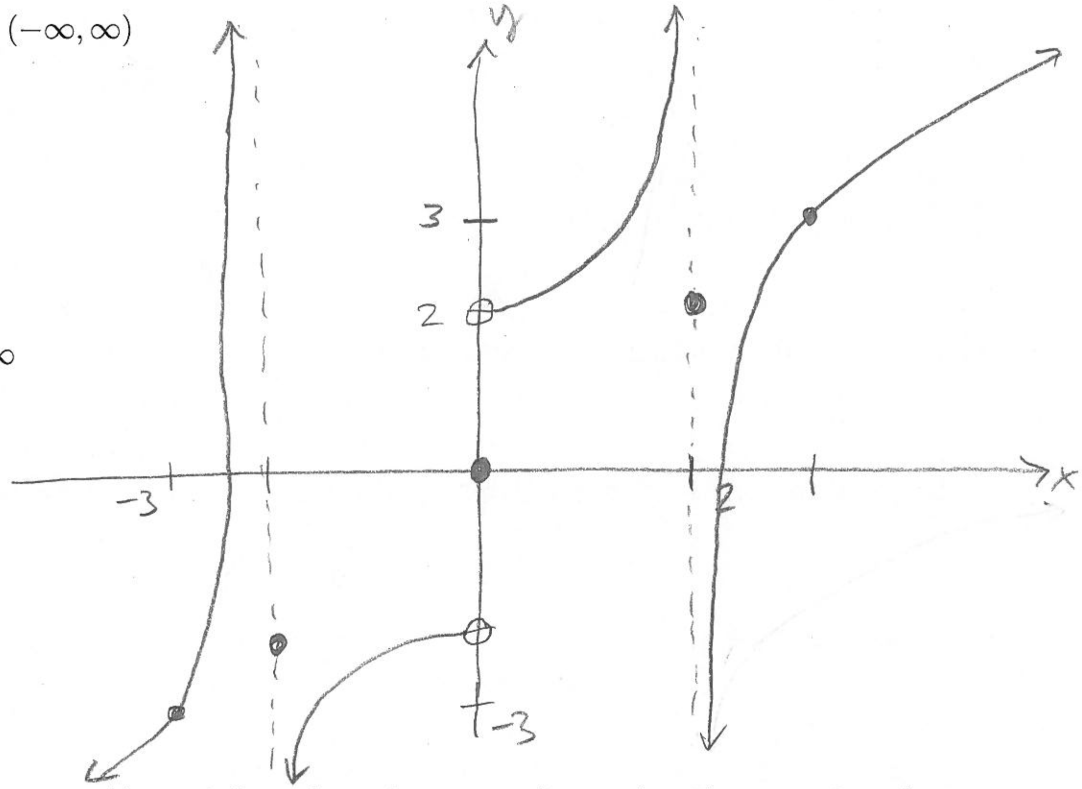
SOLUTIONS

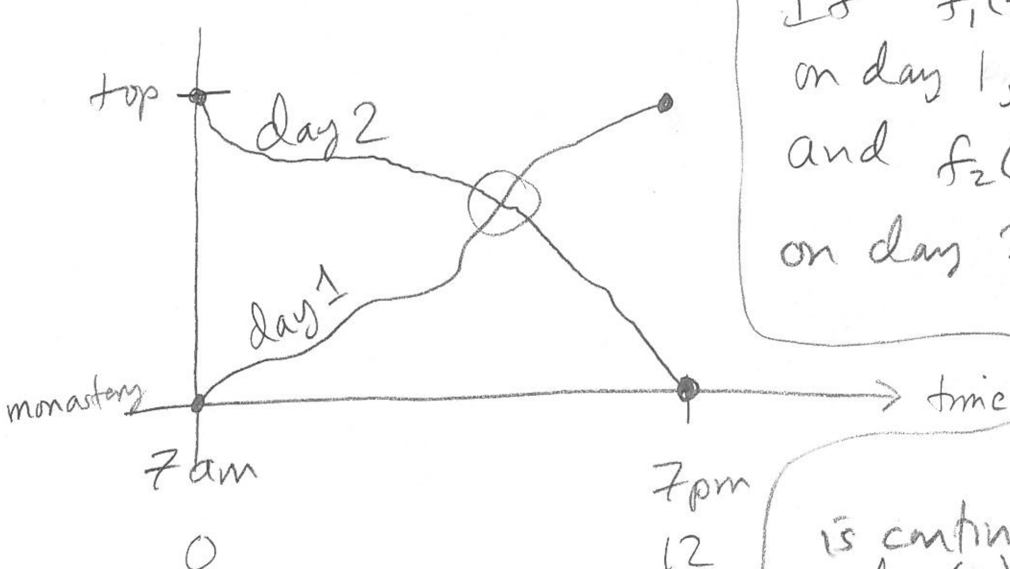
1. Sketch the graph of a function that satisfies all of the given conditions:

- f is continuous except at $x = -2, 0, 2$
- the domain of f is $(-\infty, \infty)$
- f is odd
- $f(3) = 3$
- $\lim_{x\to 0^+} = 2$
- $\lim_{x\to 2^-} f(x) = \infty$
- $\lim_{x\to 2^+} f(x) = -\infty$



2. *A challenge problem, but reasonable. It follows from the Intermediate Value Theorem. Start by sketching elevation versus time for each day, one on top of the other.* ↑ applied to g(t) below

A Tibetan monk leaves the monastery at 7:00 AM and takes his usual path to the top of the mountain, arriving at 7:00 PM. He sleeps the night on top. The next morning he starts at 7:00 AM at the top and takes the same path back, arriving at the monastery at 7:00 PM. Show that there is a point on the path that the monk will cross at exactly the same time of day on both days.



If $f_1(t)$ is his elevation on day 1, for $0 \le t \le 12$, and $f_2(t)$ is his elevation on day 2, then $f_1(t)$ and $f_2(t)$ are continuous, and $g(t) = f_2(t) - f_1(t)$ is continuous, and $g(0) > 0$ and $g(12) < 0$, so there is c so that $g(c) = 0$. So $f_1(c) = f_2(c)$.

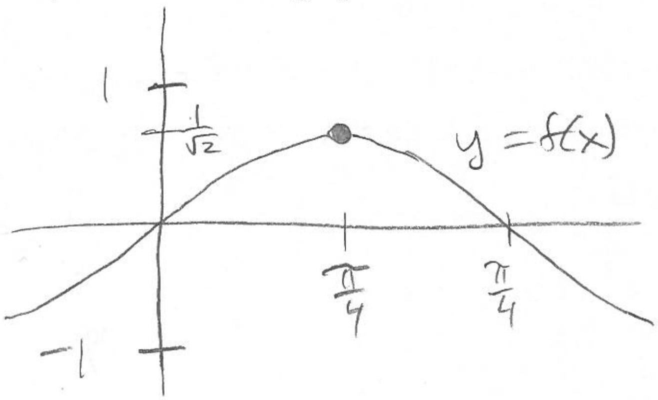
3. Show that f is continuous on $(-\infty, \infty)$, and sketch the graph:

$$f(x) = \begin{cases} \sin x & \text{if } x < \pi/4 \\ \cos x & \text{if } x \geq \pi/4 \end{cases}$$

$$\lim_{x \to \frac{\pi}{4}^-} f(x) = \sin\left(\tfrac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$$

$$\lim_{x \to \frac{\pi}{4}^+} f(x) = \cos\left(\tfrac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$$

$$f\left(\tfrac{\pi}{4}\right) = \cos\left(\tfrac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$$



4. Prove that the equation has at least one real root:

$$\ln x = 3 - 2x$$

(*A calculator can help find an accurate approximation, but this is not required!*)

$f(x) = \ln x - 3 + 2x$ is continuous

$f(1) = \ln 1 - 3 + 2 = 0 - 3 + 2 = -1 < 0$

$f(2) = \ln 2 - 3 + 4 = \ln 2 + 1 > 0$

by I.V.T. there is c in $(1, 2)$ so that $f(c) = 0$:

$$\ln c - 3 + 2c = 0$$

5. For what values of the constant c is the function f continuous on $(-\infty, \infty)$?

$$f(x) = \begin{cases} cx^2 + 2x, & \text{if } x < 2 \\ x^3 - cx, & \text{if } x \geq 2 \end{cases}$$

only issue is at $x = 2$, and we want

$$c \cdot 2^2 + 2 \cdot 2 = 2^3 - c \cdot 2$$

$$4c + 4 = 8 - 2c$$

$$6c = 4$$

$$c = \frac{2}{3}$$