Selected Solutions to Assignment #8

Exercise 2 (page 133 of B&C). Show that

$$\left|\int_C \frac{dz}{z^4}\right| \leqslant 4\sqrt{2}, \quad C = [i, 1]$$

Proof.

$$\left|\int_C \frac{dz}{z^4}\right| \leqslant \int_C \left|\frac{1}{z^4}\right| dz \leqslant |C| \max_{z\in C}\left\{\frac{1}{|z|^4}\right\}$$

Where $|C|$ is the length of the line segment C: $|C| = \sqrt{2}$. Since the maximum of the $\max_{z\in C}\left\{\frac{1}{|z|^4}\right\}$ is attained when $|z| = \frac{1}{\sqrt{2}}$, we conclude

$$\left|\int_C \frac{dz}{z^4}\right| \leqslant \sqrt{2}\frac{1}{(1/\sqrt{2})^4} = 4\sqrt{2}.$$

□

Exercise 5 (page 135 of B&C). Show that

$$\left|\int_{C_R} \frac{\operatorname{Log} z}{z^2}\, dz\right| < 2\pi\left(\frac{\pi + \ln R}{R}\right), \quad C_R = \{|z| = R\}$$

Proof. We can estimate

$$\left|\int_{C_R} \frac{\operatorname{Log} z}{z^2}\, dz\right| \leqslant \int_{C_R} \left|\frac{\operatorname{Log} z}{z^2}\right| dz$$

Note that $\operatorname{Log} z = \ln|z| + i \operatorname{Arg} z$, so when $z \in C_R$, we have

$$|\operatorname{Log} z| = |\ln|z| + i\operatorname{Arg} z| < \ln R + \pi$$

Thus we can estimate

$$\left|\int_{C_R} \frac{\operatorname{Log} z}{z^2}\, dz\right| < |C_R|\frac{\ln R + \pi}{R^2} = 2\pi R\frac{\ln R + \pi}{R^2} = 2\pi\frac{\ln R + \pi}{R}$$

Let us consider limit

$$\lim_{R\to\infty} \frac{\ln R + \pi}{R} = \left[r = \frac{1}{R}\right] = \lim_{r\to 0} \frac{\ln 1/r + \pi}{1/r} = [l'Hospital's] = \lim_{r\to 0} \frac{-1/r}{-(1/r)^2} = 0.$$

□

Exercise 2 (page 141 of B&C). Evaluate

(a)

$$\int_i^{\frac{i}{2}} e^{\pi z}\, dz = \frac{1}{\pi}e^{\pi z}\Big]_i^{\frac{i}{2}} = \frac{1+i}{\pi}$$

(b)

$$\int_0^{\pi+2i} \cos\frac{z}{2}\, dz = 2\sin\frac{z}{2}\Big]_0^{\pi+2i} = 2\sin\frac{\pi+2i}{2} = \frac{2}{2i}(e^{i\frac{\pi+2i}{2}} - e^{-\frac{\pi+2i}{2}}) =$$

$$-i(e^{-1}e^{i\frac{\pi}{2}} - ee^{-i\frac{\pi}{2}}) = e + e^{-1}.$$

Exercise 5 (page 142 of B&C). Show that

$$\int_{-1}^{1} z^i\, dz = \frac{1+e^{-\pi}}{2}(1-i)$$

Proof. We choose the branch of z^i: $z^i = e^{i\log z}$, $-\pi/2 < \operatorname{Arg} z < 3\pi/2$, then using anti-derivative:

$$\int_{-1}^{1} z^i\, dz = \frac{z^{i+1}}{i+1}\Big]_{-1}^{1} = \frac{e^{(i+1)(\ln 1+i0)} - e^{(i+1)(\ln 1+i\pi)}}{i+1} = \frac{(1-i)(1-e^{-\pi+i\pi})}{(i+1)(1-i)} = \frac{1+e^{-\pi}}{2}(1-i)$$

□

Exercise 1 (page 153 of B&C). (c) The function

$$f(z) = \frac{1}{z^2+2z+2}$$

is analytic inside the contour $|z|=1$, since

$$z^2+2z+2=0 \quad \Rightarrow \quad z_{1,2} = -1 \pm i,$$

and $z_{1,2} \notin \{|z| \leqslant 1\}$. So we can use C-G theorem that implies

$$\int_{|z|=1} \frac{dz}{z^2+2z+2} = 0.$$

(e) The function

$$f(z) = \tan z$$

is analytic inside the contour $|z|=1$, since

$$\cos z = 0 \quad \Rightarrow \quad z_n = \pi/2 + n\pi, \quad n \in \mathbb{Z}$$

and $z_n \notin \{|z| \leqslant 1\}$ for all n. So we can use C-G theorem that implies

$$\int_{|z|=1} \tan z\, dz = 0.$$

Exercise 4 (page 154 of B&C). Show that

$$\int_0^\infty e^{-x^2} \cos 2bx\, dx = \frac{\sqrt{\pi}}{2} e^{-b^2}$$

Proof. We consider function $f(z) = e^{-z^2}$ and integrate it over the rectangle $C = \{a, a+bi, -a+bi, -a\}$. According to C-G theorem,

$$\int_C f(z)\, dz = 0$$

We evaluate integrals over edges. For the first one $z = x + i0$:

$$C_1(a) = \int_{-a}^{a} e^{-z^2}\, dz = 2\int_0^a e^{-x^2}\, dx$$

For the second edge $z = a + iy$

$$C_2(a) = \int_a^{a+ib} e^{-z^2}\, dz = \int_0^b e^{-(a+iy)^2}\, i\, dy = i\int_0^b e^{-a^2} e^{y^2} e^{-2aiy}\, dy = ie^{-a^2}\int_0^b e^{y^2} e^{-2aiy}\, dy$$

For the third, $z = x + ib$,

$$C_3(a) = \int_{a+ib}^{-a+ib} e^{-z^2}\,dz = \int_a^{-a} e^{-(x+ib)^2}\,dx = \int_a^{-a} e^{-x^2} e^{b^2} e^{-2xib}\,dx =$$
$$e^{b^2}\int_a^{-a} e^{-x^2}(\cos 2xb - i\sin 2xb)\,dx = -2e^{b^2}\int_0^a e^{-x^2}\cos 2xb\,dx$$

Here we used the fact that $e^{-x^2}\sin 2xb$ is odd, and integral over symmetric interval of this function is zero. For the forth interval we have $z = -a + iy$

$$C_4(a) = \int_{-a+ib}^{-a} e^{-z^2}\,dz = \int_b^0 e^{-(-a+iy)^2} i\,dy = i\int_b^0 e^{-a^2} e^{y^2} e^{2aiy}\,dy = ie^{-a^2}\int_b^0 e^{y^2} e^{2aiy}\,dy$$

We know that for all a,

$$C_1(a) + C_2(a) + C_3(a) + C_4(a) = 0,$$

so

$$\int_0^a e^{-x^2}\cos 2bx\,dx = \frac{C_3(a)}{-2e^{b^2}} = -\frac{C_1(a) + C_2(a) + C_4(a)}{-2e^{b^2}}$$

Note that since $|e^{\pm 2aiy}| = 1$,

$$\lim_{a\to\infty} C_{2,4}(a) = 0, \quad \lim_{a\to\infty} C_1(a) = \sqrt{\pi}$$

and finally

$$\int_0^\infty e^{-x^2}\cos 2bx\,dx = \frac{\sqrt{\pi}}{2e^{b^2}}$$

$\square$