Assignment #4

Due Friday, 15 February at the start of class

Please read Chapters 2 and 3 of LeVeque. Problem **P16** below is from Chapter 3.

P14. (*Eigenvalues. Because I assume you have had an undergraduate course in linear algebra, this should be a review topic, but I start with a quick summary.*)

By definition, a vector $v \in \mathbb{R}^m$ is an *eigenvector* of a square matrix $A \in \mathbb{R}^{m\times m}$ if it is nonzero and it has the property that when multiplied by A it merely lengthens or shortens:

$$Av = \lambda v. \tag{1}$$

Here λ is a number which we call the *eigenvalue* corresponding to v. Note "eigen" means something like "property of"; that is, v and λ are in some sense "owned" by A.

Now, if (1) is true then the matrix

$$\lambda I - A$$

has a vector in its *null space*. That is, there is a nonzero vector, namely v, that $\lambda I - A$ sends to zero:

$$(\lambda I - A)v = 0.$$

This equation is true, by the fundamental equivalence in linear algebra, if and only if $\lambda I - A$ is not invertible. In particular, $\det(\lambda I - A) = 0$, which is a polynomial equation with real coefficients because A has real entries:

$$p(\lambda) = \det(\lambda I - A).$$

Finding all the eigenvalues is equivalent to finding all the roots of this polynomial. Some of these roots may be complex:

$$\lambda \in \mathbb{C}.$$

However, because the coefficients of the polynomial are real, if λ is not real then its conjugate $\bar{\lambda}$ is also a root of the polynomial and thus an eigenvalue of A.

Now suppose λ is an eigenvalue of A. Finding a corresponding eigenvector is the task of finding a vector in the null space of a matrix. In particular, Gauss elimination will convert the equation $(\lambda I - A)v = 0$ into an upper triangular equation $Uv = 0$ where U is both upper triangular and has at least one row of zeros. (*We know $\lambda I - A$ is not invertible.*) The matrix equation $Uv = 0$ can be used to generate every eigenvector corresponding to λ, the *eigenspace*, of dimension at least one, corresponding to λ. (*Note that if $Av = \lambda v$ then $A(cv) = \lambda(cv)$, and etc.*)

a) Compute *by hand* the eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

Show all your work. (*Hint: As you expand the determinant, watch for a factor to appear. You may check your work with* MATLAB.)

b) Now consider this matrix-valued function of x:

$$B(x) = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 0 & 1 \\ x & x & 0 \end{bmatrix}.$$

Note that $B(1) = A$ in part **a)**, and if $x \neq 1$ then $B(x)$ is not symmetric. Use MATLAB to generate a single clear figure showing all the eigenvalues of all matrices $B(x)$ for $x \in [-5, 1]$; label this figure reasonably well.

c) Let A be the matrix in part **a)**. Do the following using MATLAB: Choose a vector $u \in \mathbb{R}^3$ at random, for instance `u = randn(3,1)`. Apply A to it 50 times: $w = A^{50}u$. Now compute $\|Aw\|_2/\|w\|_2$. You will get the number 2.5616. Why? Explain in several sentences, using equations if they help. (*By way of a hint, observe that if an $m \times m$ matrix has m distinct eigenvalues then the corresponding eigenvectors form a basis. Also observe that if you only multiply 5 times ($w = A^5u$) then $\|Aw\|_2/\|w\|_2$ is merely close to 2.5616, and it varies as you re-sample u. Note that w is very big in norm; why?*)

P15. (*A general linear ODEBVP, and the advection-dominated case.*) **a)** Recall problem **P10** on Assignment #2; note I handed out a complete solution! Implement the method in that problem in a MATLAB/etc. function with signature

```
[x,U] = genlin(m,xL,xR,p,q,f,alpha,beta)
```

where m is the number of points in the interior, x_L, x_R are the ends of the interval, p, q are coefficients in the ODE, $f(x)$ is the right-hand side of the ODE, and α, β are boundary values. Note that m is an integer and f is a function, but the other inputs are real numbers. Check that, in the appropriate case, your code produces the same numbers as does the verified code

http://bueler.github.io/M615S17/matlab/dave.m

b) Consider the case $x_L = 0, x_R = 1, \alpha = 1, \beta = 0, p = -20, q = 0$, and $f(x) = 0$. Solve this numerically using $m = 3, 5, 10, 20, 50, 200, 1000$. The exact solution to this problem is

$$u(x) = 1 - \frac{1 - e^{20x}}{1 - e^{20}}.$$

Put all these solutions, numerical and exact, on one figure, with decent labeling (e.g. using `legend`).

c) Observe that the solutions for the small values of m are poor but the high m solutions all basically agree. Why do you think that small m values are problematic here, but the same m values are not problematic in `dave.m` for example? Write a few sentences, perhaps based on a bit of research into the textbook or the intertubes. (*Hints: Also try the $p = +20$ case, with other data the same. Observe that in one case the solution does not "feel" the right boundary condition over most of $[0, 1]$, so to speak; in the other case it is the left boundary condition that is not "felt." What is the exact solution to $pu' = 0$ and how does it use boundary conditions?*)

P16. (*Poisson equation on the unit square.*) Based on the ideas in sections 3.1–3.3, write a MATLAB/etc. code that solves Poisson equation (3.5) on the unit square $0 \leq x \leq 1, 0 \leq y \leq 1$, with zero Dirichlet boundary conditions. Find an nonzero exact solution that allows you to verify that the code converges. Show that $\|E\|_\infty \to 0$ as $m \to \infty$ at the rate $O(h^2)$ if $\Delta x = \Delta y = h = 1/(m + 1)$. You may, as usual, use MATLAB's backslash (etc.) to solve the linear system. Also, use `spy` to show the sparsity pattern of the matrix in question for $m = 5$; confirm thereby that the matrix has the the form shown by equation (3.12).