FIXME Starting Hilbert spaces the right way

Ed Bueler

March 22, 2020

I believe this is the right sequence, better than Muscat and close to Reed & Simon:

1. define $\mathbb{C}$-*inner product space* $(X, \langle \cdot, \cdot \rangle)$; sequilinear and positive-definite

2. $\|x\| = \sqrt{\langle x, x \rangle}$ ...but we don't have triangle inequality yet

3. define *orthogonal* and *ON set*

4. Pythagorean Theorem. if $u, v \in X$ are orthogonal then $\|u + v\|^2 = \|u\|^2 + \|v\|^2$

5. Corollary. if $\{u_i\}_{i=1}^n$ is a finite ON set and $x \in X$ then

$$\|x\|^2 = \sum_{i=1}^n |\langle u_i, x \rangle|^2 + \left\| x - \sum_{i=1}^n \langle u_i, x \rangle u_i \right\|^2$$

   proof. $x = \sum_{i=1}^n \langle u_i, x \rangle u_i - (x - \sum_{i=1}^n \langle u_i, x \rangle u_i)$ is $x = u + v$. check $\langle u, v \rangle = 0$. result follows

6. Bessel's inequality (another corollary). if $\{u_i\}_{i=1}^n$ is a finite ON set and $x \in X$ then

$$\|x\|^2 \ge \sum_{i=1}^n |\langle u_i, x \rangle|^2$$

7. Cauchy-Schwarz (another corollary). $|\langle x, y \rangle| \le \|x\|\|y\|$

   proof. for $y \ne 0$, $\{y/\|y\|\}$ is an ON set with one element so by Bessel

$$\|x\|^2 \ge |\langle y/\|y\|, x \rangle|^2 = \frac{|\langle y, x \rangle|}{\|y\|^2}$$

8. Triangle inequality (another corollary). $\|x + y\| \le \|x\| + \|y\|$

   proof. by Cauchy-Schwarz,

$$\|x + y\|^2 = \|x\|^2 + 2\,\mathrm{Re}\,\langle x, y \rangle + \|y\|^2 \le \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 = (\|x\| + \|y\|)^2$$

9. Corollary. an inner product space is a normed space

10. Parallelogram law. if $x, y \in X$ where $(X, \langle x, y \rangle)$ is an inner product space, then

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

proof. computation; see Muscat Prop 10.8

side note. that this characterizes inner product spaces among normed vector spaces was proven by P. Jordan & J. von Neumann (1935) . . . but don't get distracted now

11. definition. a $\mathbb{C}$-inner product space $(X, \langle \cdot, \cdot \rangle)$ is a *Hilbert space* if it is complete as a normed vector space

12. definition. for a normed vector space X, $A \subset X$ is *convex* if $0 \leq \lambda \leq 1$ and $u, v \in A$ imply $\lambda u + (1-\lambda)v \in A$

13. Fundamental Theorem of Optimization. if $A \subset H$ is a closed, convex subset of a Hilbert space H and if $x \in H$ then there is a unique $y_* \in A$ such that $\|x - y_*\| \leq \|x - y\|$ for all $y \in A$

proof. see Muscat; uses parallelogram law, completeness of H, closedness of A, convexity of A

14. definition. given a subset $A \subset X$ of an inner product space,

$$A^\perp = \{x \in X \,:\, \langle x, a \rangle = 0 \text{ for all } a \in A\}$$

15. lemma. $A^\perp$ is a closed linear subspace of X

proof. Prop 10.9 in Muscat

16. Theorem. if $M \subset H$ is a closed linear subspace of a Hilbert space, and if $x \in H$, then

$$(y_* \text{ is the closest point in } M \text{ to } x) \iff x - y_* \in M^\perp$$

furthermore, $H = M \oplus M^\perp$ and $P : x \mapsto y_*$ defines $P \in B(H)$, an orthogonal projection onto M

proof. see Theorem 10.12 in Muscat

17. calculation. if $(X, \langle \cdot, \cdot \rangle)$ is an inner product space and $x \in X$ then $\phi(y) = \langle x, y \rangle$ defines a continuous linear functional $\phi \in X^*$ because

$$|\phi(y)| = |\, \langle x, y \rangle \,| \leq \|x\| \|y\|$$

so $\|\phi\| \leq \|x\|$

18. definition:

$$J : H \to H^*, \qquad x \mapsto [y \mapsto \langle x, y \rangle]$$

is the Riesz map

19. Riesz Representation Theorem: the Riesz map is bijective, conjugate-linear, and isometric

    proof. see Muscat Theorem 10.16; uses closedness of $\ker \phi$, definition of $M^\perp$, fact $H = M \oplus M^\perp$

20. definition. given $T \in B(X, Y)$ define $T^* \in B(Y, X)$, the *adjoint of* T, by $T^* y = w$ where w represents functional $\phi(x) = \langle y, Tx \rangle_X$, thus

$$\langle T^* y, x \rangle = \langle y, Tx \rangle$$

21. FIXME: selection of other facts about adjoints

22. Gram-Schmidt process. any sequence of vectors in an inner product space $(X, \langle \cdot, \cdot \rangle)$ can be replaced by an ON set with same span

23. definition. $\{u_i\}_{i \in I} \subset X$, where $(X, \langle \cdot, \cdot \rangle)$ is an inner product space, is an *ON basis* if it is ON set and the span (finite linear combinations) is dense; note index set I is arbitrary, possibly uncountable

24. Theorem. every Hilbert space has an ON basis

    proof. page 202 Muscat; better situation than Banach spaces because not all Banach spaces have Schauder bases

25. Lemma. if $\{e_i\}$ is a countable ON set in a Hilbert space H then

$$\left(\sum_i \alpha_i e_i \text{ converges in } H \text{ for } \alpha_i \in \mathbb{C} \right) \iff (\alpha_i) \in \ell^2$$

    proof. Prop 10.30 in Muscat; uses completeness of H and ℓ^2

26. Parseval's identity. if $\{e_i\}$ is a countable ON basis of a Hilbert space H, and if $x \in H$ then

$$x = \sum_i \langle e_i, x \rangle e_i, \qquad \|x\|^2 = \sum_i |\langle e_i, x \rangle|^2$$

    proof. uses Bessel's inequality and previous lemma