Selected Solutions to Assignment #5

Exercise 9 (page 60 of B&C). *Solution.* Let's write out the details for what the book refers to as "$\Delta w/\Delta z$" at the point $z = 0$:

$$\frac{\Delta w}{\Delta z} = \frac{f(0+\Delta z) - f(0)}{\Delta z} = \frac{\frac{(\overline{\Delta z})^2}{\Delta z} - 0}{\Delta z} = \frac{(\overline{\Delta z})^2}{(\Delta z)^2}.$$

The above expression makes the rest of the question a little clearer. In particular, if $\Delta z = \Delta x + i\,0$ then

$$\frac{\Delta w}{\Delta z} = \frac{(\overline{\Delta x})^2}{(\Delta x)^2} = \frac{(\Delta x)^2}{(\Delta x)^2} = 1,$$

while if $\Delta z = 0 + i\,\Delta y$ then

$$\frac{\Delta w}{\Delta z} = \frac{(\overline{i\Delta y})^2}{(i\Delta y)^2} = \frac{(-i\Delta y)^2}{(i\Delta y)^2} = \frac{-(\Delta y)^2}{-(\Delta y)^2} = 1.$$

On the other hand, if $\Delta z = \Delta x + i\,\Delta x$, a point on the $\Delta y = \Delta x$ line in the $\Delta x, \Delta y$ plane, then

$$\frac{\Delta w}{\Delta z} = \frac{(\overline{\Delta x + i\Delta x})^2}{(\Delta x + i\Delta x)^2} = \frac{(\Delta x)(1-i)^2}{(\Delta x)(1+i)^2} = \frac{(1-i)^2}{(1+i)^2} = \frac{-2i}{2i} = -1.$$

Now we see that there will be no limit as $\Delta z \to 0$. That is,

$$f'(0) = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z}$$

does not exist because as Δz approaches zero along the coordinate axes the difference quotient $\Delta w/\Delta z$ is one while along the 45 degree line in the $\Delta x, \Delta y$ plane the difference quotient $\Delta w/\Delta z$ is *minus* one.

Compare problem # 6 on page 69, done in detail below.

Exercise 1d (page 68 of B&C). *Solution. The theorem says that if the function is complex differentiable at a point then the Cauchy-Riemann equations hold at that point. Therefore if we can show that the C-R equations never hold then the function is nowhere differentiable.*

In this case

$$f(z) = e^x e^{-iy} = e^x(\cos y - i \sin y)$$

so $u = e^x \cos y$, $v = -e^x \sin y$, and

$$u_x = e^x \cos y, \qquad u_y = -e^x \sin y, \qquad v_x = -e^x \sin y, \qquad v_y = -e^x \cos y.$$

Concretely we see that $u_x \neq v_y$ at all points in $\mathbb{C}$ for which $\cos y \neq 0$. However, at the points where $\cos y = 0$ we have $\sin y \neq 0$ so $u_y \neq -v_x$. Therefore $f'(z)$ never exists.

Exercise 2b (page 68 of B&C). *Solution. Here we must show that the C-R equations hold everywhere. Then the theorem on page 63 implies that the complex derivative exists everywhere. We must make this argument for both f and for f'. (Later in the semester we will see that if $f'(z)$ exists in an open set then all higher derivatives exist there, too.)*

For $f(z) = e^{-x}e^{-iy}$ we have $u = e^{-x}\cos y$ and $v = -e^{-x}\sin y$. It is easy to see that the C-R equations apply everywhere. By the theorem we know $f'(z)$ exists everywhere. Note

$$f'(z) = u_x + iv_x = -e^{-x}\cos y + i\,e^{-x}\sin y.$$

The real and imaginary parts of this are $U = -e^{-x}\cos y$ and $V = e^{-x}\sin y$. Again the C-R equations hold. And

$$f''(z) = U_x + iV_x = e^{-x}\cos y - i\,e^{-x}\sin y.$$

Indeed $f''(z) = f(z)$ as stated in the text. Note that we never did write a formula for f directly in terms of the variable z.

Exercise 6 (page 69 of B&C). *Solution.* Let's get formulas for u and v before further thinking. In fact,

$$\frac{(\bar{z})^2}{z} = \frac{x^2 - y^2 - i\,2xy}{x+iy} = \frac{\left[x^3 - 3xy^2\right] + i\,\left[-3xy^2 + y^3\right]}{x^2+y^2},$$

so

$$u = \begin{cases} \frac{x^3-3xy^2}{x^2+y^2}, & (x,y) \neq (0,0), \\ 0, & (x,y) = (0,0), \end{cases}$$

$$v = \begin{cases} \frac{-3xy^2+y^3}{x^2+y^2}, & (x,y) \neq (0,0), \\ 0, & (x,y) = (0,0). \end{cases}$$

Now we must focus on the goal. It is to show that $u_x(0,0) = v_y(0,0)$ and $u_y(0,0) = -v_x(0,0)$. We do not need general formulas; instead we just want certain partial derivatives at the origin.

Therefore we compute, from the definition of the partial derivative,

$$u_x(0,0) = \lim_{h\to 0} \frac{u(0+h,0) - u(0,0)}{h} = \lim_{h\to 0} \frac{\frac{h^3 - 3\,h\,0^2}{h^2+0^2} - 0}{h} = \lim_{h\to 0} \frac{h^3}{h^3} = 1,$$

and also

$$v_y(0,0) = \lim_{h\to 0} \frac{v(0,0+h) - v(0,0)}{h} = \lim_{h\to 0} \frac{\frac{-3\,0\,h^2 + h^3}{0^2+h^2} - 0}{h} = \lim_{h\to 0} \frac{h^3}{h^3} = 1.$$

This shows one of the C-R equations is true at $0 = 0 + i0$. The other C-R equation is also true at 0, by exactly the same kind of argument.

Exercise 7 (page 69 of B&C). *Partial solution.* The desired expressions for v_x and v_y are

$$v_x = v_r\,\cos\theta - v_\theta\,\frac{\sin\theta}{r}, \qquad v_y = v_r\,\sin\theta + v_\theta\,\frac{\cos\theta}{r}.$$

Exercise 8 (page 69 of B&C). *Solution.* We compute, using exercise 7 and the polar form of the Cauchy-Riemann equations

$$\begin{aligned} f'(x_0) = u_x + iv_x &= \left(u_r\cos\theta - u_\theta\frac{\sin\theta}{r}\right) + i\left(v_r\cos\theta - v_\theta\frac{\sin\theta}{r}\right) \\ &= u_r\cos\theta - (-rv_r)\frac{\sin\theta}{r} + i\left(v_r\cos\theta - ru_r\frac{\sin\theta}{r}\right) \\ &= u_r\left(\cos\theta - i\sin\theta\right) + v_r\left(\sin\theta + i\cos\theta\right) \\ &= \left(u_r + iv_r\right)\left(\cos\theta - i\sin\theta\right) = e^{-i\theta}(u_r + iv_r). \end{aligned}$$