Assignment #9 = Take-home Final Assignment

Due *Tuesday 8 May, 2012 at 5pm* in my office or mailbox.

This assignment is worth a total of 70 points, twice the usual value.

Rules. You *may* use any reference, print or electronic, as long as it is clearly cited. Please refer specifically to equations (or ideas) in the textbook if that promotes clarity. You *may not* search out complete solutions to these particular problems, whether or not they exist. You *may not* talk or communicate about this exam with any person other than me:

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Read sections 6.1, 6.2, and 6.3 of the textbook Morton & Mayers, 2nd ed. Much of this assignment covers earlier sections of the textbook, but problem **4** is from Chapter 6.

1. Consider the explicit method (2.153) for equation (2.150). This is a "staggered grid" method; the diffusivity $p(x,t)$ is evaluated half-way between regular grid points x_j.

(a) *(5 points)* Consider this PDE on $0 < x < 1$, $t > 0$ with an added source term,

$$u_t = (p(x,t)u_x)_x + f(x,t) \tag{1}$$

where $p(x,t) = p(x) = \frac{1}{2}\arctan(x+1)$ is an example diffusivity which is actually independent of t. Suppose there are Dirichlet boundary conditions $u(0,t) = 0$, $u(1,t) = 0$. Choose a source function $f(x,t)$ and an initial condition $u(x,0)$ so that you know the exact solution $u(x,t)$. (Hint: *What should you actually choose first?*) Show that you actually have an exact solution.

(b) *(10 points)* Implement scheme (2.153) in MOP (=Matlab/Octave/pylab) and test its convergence using the exact solution from part **(a)**. Test the program by showing that the maximum error at $t = 0.2$ decreases as $J = 20, 40, 80$. You will need to choose time steps appropriately from a stability condition.

2. In section 2.15 an explicit, centered-difference $O(\Delta t + \Delta x^2)$ method is considered for this problem where $a(x,t)$ and $b(x,t) > 0$ are given:

$$u_t = b(x,t)u_{xx} - a(x,t)u_x. \tag{2}$$

This explicit method converges if there are two conditions, (2.144) and (2.145). A natural question: Can implicitness remove both conditions and give unconditional convergence?

(a) *(5 points)* To answer this question, consider the case where $a(x,t) = a_0$ and $b(x,t) = b_0 > 0$ are constants. State the simplest implicit centered-difference scheme for equation (2), which only involves grid values $U_j^n, U_{j-1}^{n+1}, U_j^{n+1}, U_{j+1}^{n+1}$. Draw its stencil. Compute the truncation error. (Note there is no request to implement the scheme in either part of this problem.)

(b) *(5 points)* Show that this scheme converges only conditionally, and that condition (2.144) on mesh Péclet number remains. What consequences does this condition have on practical computations?

3. **(a)** *(10 points)* Implement in MOP the scheme from problem **1** on the following porous medium equation, a nonlinear "heat-like" equation,

$$u_t = (3\,u^2\,u_x)_x,$$

wherein $p(x,t) = 3u^2$ is a function of the solution itself. This PDE is described in J. Ockendon, et al., *Applied Partial Differential Equations*, Oxford U. Press 2003, section 6.6. In particular, your implementation will need to choose a reasonable way for approximating the staggered grid values $p^n_{j+1/2}$ from the regular-grid values of u.

(b) *(10 points)* Show that this formula

$$u(x,t) = \begin{cases} (t+1)^{-1/4}\left[1 - \frac{1}{12}(t+1)^{-1/2}x^2\right]^{1/2}, & |x| < \sqrt{12}\,(t+1)^{1/4}, \\ 0, & |x| \geq \sqrt{12}\,(t+1)^{1/4}, \end{cases} \tag{3}$$

gives a not-at-all-obvious exact solution. In showing this, consider only the set where $|x| < \sqrt{12}\,(t+1)^{1/4}$. Plot this solution at $t = 0, 5, 20$ on the interval $-10 \leq x \leq 10$.

(c) *(5 points)* Test your code from part **(a)** on this exact solution, on the same interval $-10 \leq x \leq 10$, taking your initial values $u(x,0)$ from the values of the exact solution. Evaluate the numerical error at time $t = 20$ and use $J = 100, 200$ subintervals. You will need to determine a stable time step. This can come from the initial values.

4. *(10 points)* Implement in MOP method (6.29b) for equation (6.20) in MORTON & MAYERS, 2ND ED. In particular, approximate the solution $u(x,y)$ on the square $(x,y) \in [-1,1] \times [-1,1]$ if

$$a(x,y) = 1.2 + \sin(\pi y)$$

and

$$f(x,y) = \begin{cases} \cos\left(\pi(x^2+y^2)^{1/2}\right), & x^2+y^2 < 1/4, \\ 0, & \text{otherwise.} \end{cases}$$

Use boundary conditions $u = 0$ on all edges of the square. Be sure to use sparse matrix storage. Solve the linear system by "`A\b`". Show the result as a surface plot. First use $\Delta x = \Delta y = 0.1$ and then $\Delta x = \Delta y = 0.02$. State the sizes of these two matrix problems and time the solutions using `tic` and `toc`.

(Comment: *A physical interpretation of this PDE problem: $u(x,y)$ is the equilibrium (steady) temperature of a square plate with nonconstant conductivity $a(x,y)$ and heat source $f(x,y)$.*)

5. Consider the advection equation $u_t + au_x = 0$ with $a \geq 0$ constant. Let $\nu = a\Delta t/\Delta x$. Consider the implicit upwind method

$$U_j^{n+1} - U_j^n + \nu(U_j^{n+1} - U_{j-1}^{n+1}) = 0,$$

(a) *(5 points)* Compute the truncation error of this method.

(b) *(5 points)* Show by a maximum principle argument, starting as usual with the definition $e_j^n = U_j^n - u(x_j, t_n)$, that the scheme is unconditionally convergent.

(Comment: *Thus there exists a scheme for advection that does not require CFL for convergence, but it is implicit. Section 4.8 illustrates another unconditionally convergent implicit scheme for advection equations, the box scheme with higher accuracy. Unconditionally convergent advection schemes with fixed stencil do not, however, produce high quality solutions if Δt exceeds the CFL value by a large amount. They still produce the oscillations we have seen from explicit schemes.*)