Selected Solutions to Assignment #10

Exercise 3 (page 181 of B&C). Show that if $\lim_{n\to\infty} z_n = z$ then $\lim_{n\to\infty} |z_n| = |z|$

Proof. We use the inequality

$$||a| - |b|| \leqslant |a - b|, \quad a, b \in \mathbb{C}$$

with $a = z$, $b = z_n$:

$$||z| - |z_n|| \leqslant |z - z_n|.$$

Now let $\epsilon > 0$. Because $\lim_{n\to\infty} z_n = z$, there is an N so that for all $n > N$ we have $|z - z_n| < \epsilon$. The above inequality shows that $||z| - |z_n|| \leqslant |z - z_n| < \epsilon$. (*We are now done, by definition.*) □

Exercise 4 (page 181 of B&C). Derive formulas

$$\sum_{n=1}^{\infty} r^n \cos n\theta = \frac{r\cos\theta - r^2}{1 - 2r\cos\theta + r^2}, \quad \sum_{n=1}^{\infty} r^n \sin n\theta = \frac{r\sin\theta}{1 - 2r\cos\theta + r^2}$$

Proof. We write $z = re^{i\theta}$ and note that

$$\sum_{n=1}^{\infty} z^n = \frac{1}{1-z} - 1 = \frac{z}{1-z} \tag{1}$$

We can rewrite the left hand side of the above equality

$$\sum_{n=1}^{\infty} z^n = \sum_{n=1}^{\infty} r^n \cos n\theta + i \sum_{n=1}^{\infty} r^n \sin n\theta \tag{2}$$

For the right hand side of (1) we have:

$$\begin{aligned} \frac{z}{1-z} &= \frac{r\cos\theta + ir\sin\theta}{1 - r\cos\theta - ir\sin\theta} = \frac{(r\cos\theta + ir\sin\theta)(1 - r\cos\theta + ir\sin\theta)}{(1 - r\cos\theta)^2 + r^2\sin^2\theta} \\ &= \frac{r\cos\theta - r^2}{1 - 2r\cos\theta + r^2} + i\frac{r\sin\theta}{1 - 2r\cos\theta + r^2} \end{aligned} \tag{3}$$

Equating the real and imaginary parts of (2) and (3) we obtain desired result. □

Exercise 9 (page 182 of B&C). Let $z_n \to z$ as $n \to \infty$. Show that there exists $M > 0$ such that $|z_n| \leqslant M$.

Remark. Here are two reasonable proofs, either of which is fine.

Proof. Since $z_n \to z$, there exist $n_0 > 0$ such that $|z - z_n| < 1$ if $n > n_0$. Then we have

$$|z_n| = |z + (z - z_0)| \leqslant |z| + |z - z_0| \leqslant |z| + 1, \quad n > n_0.$$

So, if we denote $I = \max_{0<n<n_0} |z_n|$ we have that

$$|z_n| \leqslant \max\{|z| + 1, I\}, \quad n \geqslant 0.$$

□

Proof. We know that $z_n = x_n + iy_n$ where $x_n \to x$, $y_n \to y$ as $n \to \infty$. From real analysis we know that there exist $N, K > 0$ such that $|x_n| \leqslant N$, $|y_n| \leqslant K$ for $n \geqslant 0$. Thus we get

$$|z_n| \leqslant \sqrt{x_n^2 + y_n^2} \leqslant \sqrt{N^2 + K^2}, \quad n \geqslant 0.$$

□

Exercise 2 (page 189 of B&C). Obtain the Taylor expansion for $f(z) = e^z$ around point 1.

Remark. Here are two methods.

Method A. Note that $f^{(n)}(z) = e^z$ so $f^{(n)}(1) = e$. Using the general formula $a_n = f^{(n)}(z_0)/n!$ for the Taylor series coefficients, we get

$$e^z = \sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!}(z-1)^n = e\sum_{n=0}^{\infty} \frac{(z-1)^n}{n!}$$

Method B. We can rewrite $e^z = e\, e^{z-1}$ and expand the last factor using the Maclaurin series for e^z:

$$e^z = e\sum_{n=0}^{\infty} \frac{(z-1)^n}{n!}$$

Exercise 8 (page 189 of B&C). Expand $\cos z$ into a Taylor series about the point $z_0 = \frac{\pi}{2}$.

Proof. We use the identity $\cos z = -\sin(z - \frac{\pi}{2})$ and obtain

$$\cos z = -\sin(z - \frac{\pi}{2}) = -\sum_{n=0}^{\infty} \frac{(-1)^n (z - \frac{\pi}{2})^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} (z - \frac{\pi}{2}) z^{2n+1}}{(2n+1)!}.$$

□