 Name: ____________________

Midterm Quiz on Definitions and Simple Calculations

In class. No notes. No book. No calculator needed.

[60 points total]

1. [5 points] Define the statement "a scheme is (unconditionally) consistent with" a given PDE.

2. (a) [5 points] On Δx (horizontal) versus Δt (vertical) axes, sketch the refinement path

$$\{(\Delta x_i, \Delta t_i)\}_{i=1}^{\infty} = \left\{ \left(\frac{1}{2^i}, \frac{1}{2^i} \right) \right\}_{i=1}^{\infty}.$$

(b) [5 points] Is the explicit scheme for the standard heat equation $u_t = u_{xx}$ stable along the refinement path given in part **(a)**? Explain briefly.

3. (a) [5 points] Consider the standard heat equation $u_t = u_{xx}$ and the implicit scheme

(1)
$$\frac{U_j^{n+1} - U_j^n}{\Delta t} = \frac{U_{j+1}^{n+1} - 2U_j^{n+1} + U_{j-1}^{n+1}}{\Delta x^2}.$$

Draw the stencil for this scheme.

(b) [5 points] Define the truncation error for this scheme.

(c) [5 points] Use Taylor's theorem with remainder to show that

$$\frac{f(t + \Delta t) - f(t)}{\Delta t} = f'(t) + \frac{1}{2}\Delta t f''(\tau)$$

Be sure to show your steps, and explain where τ is. (That is, give bounds on τ.)

(d) [5 points] Using the fact that

$$\frac{f(x+\Delta x)-2f(x)+f(x-\Delta x)}{\Delta x^2}=f''(x)+\frac{1}{12}\Delta x^2 f^{(4)}(\xi),$$

which you do NOT need to prove, and the result of part **(c)**, express the truncation error defined in part **(b)** in a form that shows that the implicit scheme for the heat equation is consistent. Explain in a brief sentence why you know the scheme is consistent from this calculation. (*Hint*. Carefully choose the right location at which to *use* the PDE.)

(e) [5 points] I claim the implicit scheme for the heat equation has order $O(\Delta t^1+\Delta x^2)$. In terms of the result of part **(d)**, justify this statement.

4. [10 points] Again consider the standard heat equation $u_t = u_{xx}$ and the implicit scheme

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} = \frac{U_{j+1}^{n+1} - 2U_j^{n+1} + U_{j-1}^{n+1}}{\Delta x^2}.$$

Using the von Neumann analysis, with substitution $U_j^n = \lambda^n e^{ik(j\Delta x)}$, show that the scheme is unconditionally stable. Show your steps, and be clear at the end why you have shown stability.

5. [5 points] The $\theta = 1/4$ scheme for the standard heat equation is the scheme

$$U_j^{n+1} - U_j^n = \mu \left[\frac{1}{4} \delta_x^2 U_j^{n+1} + \frac{3}{4} \delta_x^2 U_j^n \right]$$

where $\mu = \Delta t / (\Delta x)^2$. The von Neuman substitution gives

$$\lambda(k) = \frac{1 - 3\mu \sin^2(\frac{1}{2} k \Delta x)}{1 + \mu \sin^2(\frac{1}{2} k \Delta x)}$$

For which μ is this scheme stable? That is, show this scheme is conditionally stable and clearly state the condition for stability. Show your steps.

6. [5 points] For the Richardson scheme we can state these equations:

$$\frac{U_j^{n+1} - U_j^{n-1}}{2\Delta t} = \frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{\Delta x^2}$$

$$\frac{u(x_j, t_{n+1}) - u(x_j, t_{n-1})}{2\Delta t} = \frac{u(x_{j+1}, t_n) - 2u(x_j, t_n) + u(x_{j+1}, t_n)}{\Delta x^2} + T(x_j, t_n)$$

Give the equation satisfied by the numerical error e_j^n. That is, *recall* the definition of the numerical error and *state* the equation for it that follows from the above two equations. (Note that you are not asked to *do* anything with the resulting equation.)