SOLUTIONS

Worksheet: Ratio and Root Test problems

Use the ratio and root tests, or other tests as needed, to determine if the series converges or diverges.

A. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+1}{2^n}$ [choose either ratio or root]

ratio test: $\displaystyle \rho = \lim_{n\to\infty} \frac{\frac{(n+1)^2+1}{2^{n+1}}}{\frac{n^2+1}{2^n}} = \lim_{n\to\infty} \frac{(n^2+2n+2)\,\cancel{2^n}}{2^{n+1}(n^2+1)}$

$\displaystyle = \frac{1}{2}\lim_{n\to\infty} \frac{n^2+2n+2}{n^2+1} \overset{L'H\times 2}{=} \frac{1}{2}\cdot 1 = \frac{1}{2} < 1$

$\therefore$ converges

B. $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{n!}$ [factorial ... use ratio]

ratio test: $\displaystyle \rho = \underbrace{\lim_{n\to\infty} \frac{\frac{3^{n+1}}{(n+1)!}}{\frac{3^n}{n!}} = \lim_{n\to\infty} \frac{3^{\cancel{n+1}}\,\cancel{n!}}{(n+1)\cancel{n!}\,\cancel{3^n}}}$

$\displaystyle = \lim_{n\to\infty} \frac{3}{n+1} = 0 < 1$

$\therefore$ converges

C. $\displaystyle\sum_{n=1}^{\infty} \frac{(n-1)^n}{n^n}$ [root test easier ... but inconclusive ... ratio will be also]

root test: $\displaystyle \rho = \lim_{n\to\infty} \sqrt[n]{\frac{(n-1)^n}{n^n}} = \lim_{n\to\infty} \frac{n-1}{n} = 1$

no information

divergence test: $\displaystyle \lim_{n\to\infty} \left(\frac{n-1}{n}\right)^n = \lim_{n\to\infty}\left(1-\frac{1}{n}\right)^n$ recall: $\displaystyle \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x$

$\displaystyle = e^{-1} = \frac{1}{e} \neq 0$

$\therefore$ diverges

D. $\sum_{k=1}^{\infty} \frac{e^k}{k^e}$ [either ratio or root]

ratio test: $\rho = \lim_{k\to\infty} \frac{\frac{e^{k+1}}{(k+1)^e}}{\frac{e^k}{k^e}} = \lim_{k\to\infty} \frac{e^{\cancel{k}+1}\, k^e}{(k+1)^e\, \cancel{e^k}}$

$= e \lim_{k\to\infty} \left(\frac{k}{k+1}\right)^e = e\left(\lim_{k\to\infty} \frac{k}{k+1}\right)^e = e\cdot 1^e = e > 1$

$\therefore$ diverges

E. $\sum_{n=1}^{\infty} \frac{1}{(1+\ln n)^n}$ [root test much easier]

root test: $\rho = \lim_{n\to\infty} \sqrt[n]{\frac{1}{(1+\ln n)^n}} = \lim_{n\to\infty} \frac{1}{1+\ln n} \overset{\text{“}\frac{1}{\infty}\text{”}}{=} 0$

$\therefore$ converges

F. $\sum_{n=1}^{\infty} \frac{(2n)!}{n^{2n}}$ [factorial ... use ratio ... tough limit]

ratio test: $\rho = \lim_{n\to\infty} \frac{\frac{(2n+2)!}{(n+1)^{(2n+2)}}}{\frac{(2n)!}{n^{2n}}} = \lim_{n\to\infty} \frac{(2n+2)(2n+1)\cancel{(2n)!}\, n^{2n}}{(n+1)^{2n+2}\, \cancel{(2n)!}}$

$= \lim_{n\to\infty} \frac{2\cancel{(n+1)}(2n+1)\; n^{2n}}{(n+1)(n+1)^{2n\cancel{+2}}} = \lim_{n\to\infty} 2\,\frac{2n+1}{n+1}\left(\left(\frac{n}{n+1}\right)^n\right)^2$

$= 2\cdot 2\cdot \lim_{n\to\infty} \left(\left(1-\frac{1}{n}\right)^n\right)^2 = 2\cdot 2\cdot(e^{-1})^2 = \frac{4}{e^2} < 1 \quad \therefore$ converges

G. $\sum_{n=1}^{\infty} \frac{n!}{(n+2)!}$ [factorial ... ratio inconclusive ... limit comparison]

ratio test: $\rho = \lim_{n\to\infty} \frac{(n+1)!\,(n+2)!}{(n+3)!\;\; n!} = \lim_{n\to\infty} \frac{(n+1)}{(n+3)} = 1$ no info

but: $a_n = \frac{1}{(n+2)(n+1)}$ so limit compare to $\sum_{n=1}^{\infty} \frac{1}{n^2}$

($p=2$, converges) : $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2}{(n+2)(n+1)} \overset{L'H\times 2}{=} 1 \quad \therefore$ both same

converges