Solutions to Quiz # 10

1. (*This can be done by Integral, Comparison, or Limit Comparison tests.*) For the Comparison test we need only notice that $n^2+4 \geq n^2$ so

$$a_n = \frac{1}{n^2+4} \leq \frac{1}{n^2} = b_n.$$

But $\sum b_n$ is a convergent p-series so $\sum a_n$ also converges.

2. To apply the Integral Test we do the following improper integral by the substitution $u = 1+x^2$. It breaks into cases according to the value of p:

$$\begin{aligned}
\int_1^\infty x(1+x^2)^p\,dx &= \frac{1}{2}\int_2^\infty u^p\,du \\
&= \begin{cases} \frac{1}{2(p+1)}\lim_{t\to\infty}\left[t^{p+1}-2^{p+1}\right], & p\neq -1 \\ \frac{1}{2}\lim_{t\to\infty}\left[\ln t - \ln 2\right], & p=-1 \end{cases} \\
&= \begin{cases} -\frac{2^{p+1}}{2(p+1)}, & p<-1 \\ \infty, & p>-1 \\ \infty, & p=-1 \end{cases}
\end{aligned}$$

Thus the series converges for $p<-1$ and diverges for $p \geq -1$.

3. **(a)** $s_1 = (1+2)/(1^3+1) = 3/2$
$s_2 = s_1 + (2+2)/(2^3+2) = 3/2 + 4/10 = 19/10$

(b) The series converges by the Limit Comparison test using $b_n = 1/n^2$, noting $\sum b_n$ is a convergent p-series. I choose to use L'Hopital's rule twice:

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{\frac{n+2}{n^3+n}}{\frac{1}{n^2}} = \lim_{n\to\infty}\frac{(n+2)n^2}{n^3+n} = \lim_{n\to\infty}\frac{n^2+2n}{n^2+1} \overset{\text{L'H}}{=} \lim_{n\to\infty}\frac{2n+2}{2n} \overset{\text{L'H}}{=} \lim_{n\to\infty}\frac{2}{2} = 1$$

Since $c=1$ satisfies $0<c<\infty$, and $\sum b_n$ converges, thus $\sum a_n$ converges.

4. Let $a_n = d_n/10^n$. Since $d_n \leq 9$, let $b_n = 9/10^n$ and notice that

$$a_n = \frac{d_n}{10^n} \leq \frac{9}{10^n} = b_n.$$

But $\sum b_n$ is a convergence geometric series with $r = 1/10$;

$$\sum_{n=1}^\infty b_n = \sum_{n=1}^\infty \frac{9}{10^n} = \frac{9/10}{1-(1/10)} = 1$$

By the Comparison Test we find that $\sum a_n = 0.d_1d_2d_3d_4\ldots$ converges.