Solutions to Quiz # 12

1. **(a)** Apply the ratio test to get

$$L = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{4^{n+1}|x|^{n+1}\sqrt{n}}{\sqrt{n+1}\,4^n|x|^n} = 4|x| \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n+1}} = 4|x|$$

(*The root test also works and gives the same L.*) The series converges if $4|x| < 1$ or $|x| < 1/4$. We test the endpoints:

$$\underline{x = +\frac{1}{4}:} \qquad \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} \text{ converges by the Alternating Series Test}$$

$$\underline{x = -\frac{1}{4}:} \qquad \sum_{n=1}^{\infty} \frac{(-1)^n\,(-1)^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \text{ diverges; } p\text{-series with } p < 1$$

Thus the power series has radius of convergence $R = 1/4$ and interval of convergence $(-1/4, 1/4]$.

(b) The root test is easiest to apply:

$$L = \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left(\frac{|x-2|}{n}\right)^n} = \lim_{n\to\infty} \frac{|x-2|}{n} = 0$$

So the radius of convergence is $R = \infty$ and the interval of convergence is $(-\infty, \infty)$.

Note: *Applying the ratio test would look like*

$$L = \lim_{n\to\infty} \frac{|x-2|^{n+1}\,n^n}{(n+1)^{n+1}\,|x-2|^n} = |x-2| \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n \frac{1}{n+1} = \frac{1}{e}\,0 = 0$$

2. Put $f(x)$ in the form $a/(1-r)$ or $1/(1-r)$. In any case $r = x/3$:

$$f(x) = \frac{2}{3-x} = \frac{2}{3}\frac{1}{1-(x/3)} = \frac{2}{3}\left(1 + \frac{x}{3} + \left(\frac{x}{3}\right)^2 + \left(\frac{x}{3}\right)^3 + \dots\right) = \frac{2}{3}\sum_{n=0}^{\infty} \frac{x^n}{3^n} = \sum_{n=0}^{\infty} \frac{2\,x^n}{3^{n+1}}$$

(*Any of the last three expressions are correct answers.*) Because $|r| < 1$ is required for the convergence of a geometric series, the interval of convergence is $|x/3| < 1$ or $(-3, 3)$.

3. **(a)** The given function $f(x)$ is the derivative of a function $F(x)$ which can be expanded in a geometric series:

$$F(x) = -\frac{1}{1+x} = -\frac{1}{1-(-x)} = -\left(1 + (-x) + (-x)^2 + (-x)^3 + \dots\right)$$

$$= -1 + x - x^2 + x^3 - x^4 + \dots = \sum_{n=0}^{\infty} (-1)^{n+1} x^n$$

Thus, differentiating term-by-term,

$$f(x) = F'(x) = 1 - 2x + 3x^2 - 4x^3 + \dots = \sum_{n=1}^{\infty} (-1)^{n+1} n x^{n-1} = \sum_{k=0}^{\infty} (-1)^k (k+1) x^k$$

(*Any of the last three expressions are correct answers.*)

(b) Since the series for $F(x)$ is a geometric series with radius of convergence $R = 1$, and because the radius does not change when you differentiate a power series, the radius of convergence for the power series for $f(x)$ is $R = 1$.