Solutions to Quiz # 6

1. Note

$$\frac{dy}{dx} = \frac{1}{\sec x} \sec x \tan x = \tan x$$

so, because the integrand is even,

$$L = \int_{-\pi/4}^{\pi/4} \sqrt{1 + \tan^2 x}\, dx = 2 \int_0^{\pi/4} \sqrt{\sec^2 x}\, dx = 2 \int_0^{\pi/4} \sec x\, dx$$
$$= 2 \ln|\sec x + \tan x|\Big]_0^{\pi/4} = 2\left(\ln(\sqrt{2}+1) - \ln(1+0)\right) = 2\ln(\sqrt{2}+1)$$

2. By substituting $x = 0$ and then $y = 0$ we see that the ellipse crosses the x-axis at $(\pm a, 0)$ and it crosses the y-axis at $(0, \pm b)$. Choosing to solve for y, we get a formula for the upper half of the ellipse:

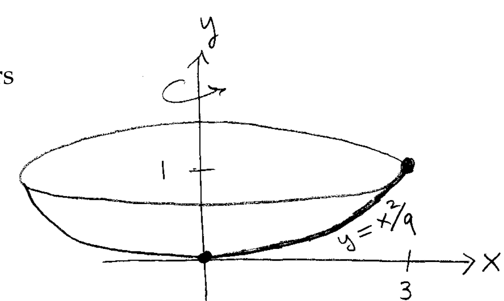
$$y = b\sqrt{1 - \frac{x^2}{a^2}} \qquad \text{and so} \qquad \frac{dy}{dx} = \frac{b}{2}\left(1 - \frac{x^2}{a^2}\right)^{-1/2} \frac{-2x}{a^2} = \frac{-bx}{a^2\sqrt{1-\frac{x^2}{a^2}}} = \frac{-bx}{a\sqrt{a^2 - x^2}}$$

Now the length is an integral with respect to x:

$$L = \int_{-a}^{a} \sqrt{1 + \frac{b^2x^2}{a^2(a^2-x^2)}}\, dx$$

(*There are various ways to simplify, but no simplification was needed. Also, there are other correct answers that use an integral with respect to y.*)

3. **(a)** See sketch at right.



(b) The area is $A = \int 2\pi x\, ds$. There are two correct answers depending on which variable you integrate with respect to:

$$A = \int_0^3 2\pi\, x \sqrt{1 + \frac{4}{81}x^2}\, dx$$
$$A = \int_0^1 2\pi\, 3\sqrt{y} \sqrt{1 + \frac{9}{4}\frac{1}{y}}\, dy$$

In the first we use $y = x^2/9$ so $dy/dx = (2/9)x$. In the second we use $x = 3\sqrt{y}$ so $dx/dy = (3/2)y^{-1/2}$.

(c) Based on the integral you wrote in **(b)**, you should do either a u-substitution $u = 1 + (4/81)x^2$, or first simplify the square root and then do the u-substitution $u = y + 9/4$, respectively. Thus:

$$A = 2\pi\frac{81}{8}\int_1^{13/9} \sqrt{u}\, du = \frac{81\pi}{4}\frac{2}{3}u^{3/2}\Big]_1^{13/9} = \frac{27\pi}{2}\left(\left(\frac{13}{9}\right)^{3/2} - 1\right)$$

or

$$A = 6\pi \int_0^1 \sqrt{y + \frac{9}{4}}\, dy = 6\pi \int_{9/4}^{13/4} \sqrt{u}\, du = 6\pi\frac{2}{3}u^{3/2}\Big]_{9/4}^{13/4} = 4\pi\left((13/4)^{3/2} - (9/4)^{3/2}\right)$$

(*With extra work you could check that these are the same, and with a calculator that* $A \approx 31.215$.)