31 May, 2010 [AUTHOR: Maxwell/Bueler]

Real Analysis Comprehensive Exam

Complete **EIGHT** of the following ten problems. It is better to fully complete fewer problems than to earn partial credit on many problems.

1. Compute, with justification,
$$\lim_{n\to\infty} \int_0^1 \frac{nx}{1+n^2x^2}\, dx.$$
Hint: $(1-nx)^2 \geq 0$.

2. Give examples of the following
   (a) A sequence (f_n) in $C[0,1]$ that converges pointwise to 0 but such that $\int_0^1 f_n \not\to 0$.
   (b) A bounded sequence in ℓ_1 that has no convergent subsequence.
   (c) A sequence in $C[0,1]$ that converges pointwise to a discontinuous function.

3. Let (f_n) be a bounded sequence of functions in $C[0,1]$. Define
$$F_n(x) = \int_0^x f_n(s)\, ds.$$
Show that (F_n) has a uniformly convergent subsequence.

4. (a) State the Axiom of Completeness for $\mathbb{R}$.
   (b) Prove that a monotone increasing sequence of real numbers converges if and only if it is bounded above.

5. Let m^* denote Lebesgue outer measure. Suppose that E is a subset of $\mathbb{R}$ such that $m^*(E\cap(a,b)) \leq 3/4(b-a)$ for every interval (a,b). Prove that $m^*(E) = 0$.

6. Let (a_n) be a bounded sequence. Show that $\sum_{n=1}^{\infty} a_n e^{-nx}$ defines a continuous function on $[1,2]$.

7. Suppose that X is compact and $f : X \to \mathbb{R}$ is continuous. Prove that f is uniformly continuous.

8. Suppose $f \in L^1(\mathbb{R})$ is uniformly continuous. Show that $\lim_{x\to\infty} f(x) = 0$.

9. Let $f \in L^1[-\pi,\pi]$. Show that
$$\lim_{n\to\infty} \int_{-\pi}^{\pi} f(x)\cos(kx)\, dx = 0.$$
Hint: First consider the case where f is the characteristic function of an interval.

10. Let $f : [0,1] \to \mathbb{R}$ be an increasing, but not necessarily continuous function. Show that f is Borel measurable.