

Real Analysis Comprehensive Exam

Complete **EIGHT** of the following ten problems. It is better to fully complete fewer problems than to earn partial credit on many problems.

1. Compute, with justification,

$$\lim_{n \rightarrow \infty} \int_0^1 \sqrt{\frac{nx}{1+n^2x^2}} dx.$$

2. Give examples of the following

- (a) A sequence (f_n) in $C[0, 1]$ that converges pointwise to 0 but such that $\int_0^1 f_n \not\rightarrow 0$.
- (b) A bounded sequence in ℓ_1 that has no convergent subsequence.
- (c) A sequence in $C[0, 1]$ that converges pointwise to a discontinuous function.

3. Let (f_n) be a bounded sequence of functions in $C[0, 1]$. Define

$$F_n(x) = \int_0^x f_n(s) ds.$$

Show that (F_n) has a uniformly convergent subsequence.

4. (a) State the Axiom of Completeness for \mathbb{R} .
(b) Prove that a monotone increasing sequence of real numbers converges if and only if it is bounded above.
5. Let m^* denote Lebesgue outer measure. Suppose that E is a subset of \mathbb{R} such that $m^*(E \cap (a, b)) \leq 3/4(b - a)$ for every interval (a, b) . Prove that $m^*(E) = 0$.
6. Let (a_n) be a bounded sequence. Show that $\sum_{n=1}^{\infty} a_n e^{-nx}$ defines a continuous function on $[1, 2]$.
7. Suppose that X is compact and $f : X \rightarrow \mathbb{R}$ is continuous. Prove that f is uniformly continuous.
8. Suppose $f \in L^1(\mathbb{R})$ is uniformly continuous. Show that $\lim_{x \rightarrow \infty} f(x) = 0$.
9. Let $f \in L^1[-\pi, \pi]$. Show that

$$\lim_{n \rightarrow \infty} \int_{-\pi}^{\pi} f(x) \cos(nx) dx = 0.$$

Hint: First consider the case where f is the characteristic function of an interval.

10. Let $f : [0, 1] \rightarrow \mathbb{R}$ be an increasing, but not necessarily continuous function. Show that f is Borel measurable.