

Assignment 1

Due Wednesday 24 January 2024, at the start of class

This Assignment is based primarily on the “Definitions and facts” handout, but also on sections 2.1 and 2.2 of our textbook

D. Borthwick (2020). *Spectral Theory: Basic Concepts and Applications*, Springer

DO THE FOLLOWING EXERCISES.

P1. Suppose $(V, \|\cdot\|)$ is a normed vector space. Let Y be any finite set of points. Show that Y is closed.

P2. Let $(V, \|\cdot\|)$ be a normed vector space. Let $S = \{x \in V : \|x\| = 1\}$. Recalling that B_1 is the open ball of radius one, show S is exactly the boundary set of B_1 : $\partial B_1 = S$.

P3. Let X_k be the set of RNA sequences of length k . All you need to know about such things is that an RNA sequence has one character from $\{A, C, G, U\}$ in each location. That is, an element of X_k is a string of k letters, each of which is A, C, G, U . Let $\delta(a, b) = 0$ if $a = b$ and $\delta(a, b) = 1$ if $a \neq b$. For $x, y \in X_k$ let

$$d(x, y) = \sum_{j=1}^k \delta(x_j, y_j)$$

where x_j is the j th letter in the RNA sequence x .

(a) Show rigorously that (X_k, d) is a metric space. That is, show $d(x, y)$ computes a valid distance between any two RNA sequences of the same length.

(b) Can (X_k, d) be regarded as a subset of a real normed vector space $(V, \|\cdot\|)$? Argue, at least informally, that this is true. You will need to construct V and sketch the embedding. Formally, one says that there is an *isometric* embedding $X_k \hookrightarrow V$. (Hint. I used $\|\cdot\|_\infty$ on each letter.)

P4. Show that a convergent sequence in a metric space cannot converge to two different limits.

P5. (a) Is $f(x) = \frac{1}{\sqrt{x}}$ in $L^1(0, 1)$? In $L^2(0, 1)$?

(b) Let $s \geq 0$ and define $g_s(x) = \frac{1}{x^s}$. For $1 \leq p \leq \infty$, determine exactly which spaces $L^p(0, 1)$ contain g_s .

P6. The definition of the spaces $\ell^p = \ell^p(\mathbb{N})$ in section 2.2 of the textbook is too terse. This problem at least includes a clear definition for $p = \infty$. Make sure to observe that infinite sequences are the same as functions from \mathbb{N} to \mathbb{C} .

The normed vector space $(\ell^\infty, \|\cdot\|_\infty)$ is the set of infinite sequences $a = (a_k) = (a_1, a_2, a_3, \dots)$ with complex entries ($a_k \in \mathbb{C}$) which are bounded, that is, so that there is $M > 0$ so that $|a_k| < M$ for all $k \in \mathbb{N}$. The norm is the supremum of the absolute values of the entries:

$$\|a\|_\infty = \sup_{k=1,2,\dots} |a_k|.$$

- (a) Show that $(\ell^\infty, \|\cdot\|_\infty)$ is a Banach space.
- (b) Consider the set of complex-valued sequences which have a limit of zero:

$$Y = \{(a_k) : a_k \rightarrow 0\}.$$

Show that $Y \subset \ell^\infty$.

- (c) Show that $(Y, \|\cdot\|_\infty)$ is a Banach space. (*Hint.* You may do this directly or use part (a). For less than obvious reasons, this subset Y is often instead called c_0 .)

P7. For any set X and measure μ , the normed vector space $L^1(X, d\mu)$ is defined in section 2.2 of our textbook. For such a space the Minkowski (triangle) inequality (2.5) can be proved directly, without reference to Appendix A.2. Do so.