

Assignment 6

Due Wednesday 10 April 2024

This Assignment is based primarily on sections 3.1, 3.2, 3.3, 3.4, and 4.1 of our textbook, Borthwick (2020) *Spectral Theory: Basic Concepts and Applications*, Springer. Note that 3.5 is skipped for now.

PLEASE DO THE FOLLOWING EXERCISES.

P28. *Hint. You did Exercise 2.5, and you may use its result.*

Show that if $T \in \mathcal{L}(\mathcal{H})$ then $\|T\| = \|T^*\|$ and $\|T^*T\| = \|T\|^2$.

P29. *Recall that $U \in \mathcal{L}(\mathcal{H})$ is unitary if it is bijective (one-to-one and onto) and an isometry. Note that we will be able to show that the spectrum of $U = M_f$ in parts (c) and (d) is the entire unit circle \mathbb{S} . Regarding part (d), Section 4.3 defines the phrase “approximate eigenvalue” for this situation, where $Uv_n \approx zv_n$.*

(a) Note \mathcal{H} is a complex Hilbert space. Use the correct¹ polarization identity to show that if U is unitary then $\langle Ux, Uy \rangle = \langle x, y \rangle$ for all x, y in \mathcal{H} .

(b) Suppose $\lambda \in \mathbb{C}$ is an eigenvalue of a unitary operator U . Show that λ is on the unit circle $\mathbb{S} = \{z \in \mathbb{C} : |z| = 1\}$.

(c) Let $\mathcal{H} = L^2(0, 1)$ and fix the function $f(x) = e^{2\pi i x}$. Show that the multiplication operator $U = M_f$ is bounded and unitary. (*Hint. Bounded is easy; use Example 2.8 from the textbook. Unitary is also easy.*) Show that this U has no eigenvalues.

(d) For U from part (c), and each $z \in \mathbb{S}$, find a sequence of functions $v_n \in \mathcal{H}$ so that $\|v_n\| = 1$ and $\lim_{n \rightarrow \infty} \|(U - z)v_n\| = 0$.

P30. *I am deliberately suppressing most details of closed operators (section 3.3). However, this exercise uses Definition 3.8, and suggests what “closed” is saying.*

(a) Let T be an operator on \mathcal{H} , so $\mathcal{D}(T)$ is a dense subspace. Show that

$$\|u\|_T = (\|u\|^2 + \|Tu\|^2)^{1/2}$$

defines a norm on $\mathcal{D}(T)$, called the *graph norm* for T .

(b) Show that T is closed as an operator on \mathcal{H} if and only if the normed vector space $(\mathcal{D}(T), \|\cdot\|_T)$ is complete.

¹See Assignment # 3 for the correct form which you proved. The textbook’s version, (2.17) on page 17, has sign errors.

P31. Page 47 of the text is important as it defines self-adjoint, symmetric, and positive for unbounded operators. Hint. Consider vectors of the form $u + v$ and $u - iv$ for $u, v \in \mathcal{D}(A)$. Also, if a number is nonnegative then it is real!

Prove that a positive operator A on a complex Hilbert space \mathcal{H} is symmetric.

P32. Hint. You will need the fact that $v' \in L^2(0, 1)$ implies $v' \in L^1(0, 1)$ implies $v \in C[0, 1]$ implies $v(0)$ is well-defined. And, of course, integration by parts.

Let $\mathcal{H} = L^2(0, 1)$. Consider $T = d/dx$, an unbounded operator on \mathcal{H} , with domain $\mathcal{D}(T) = \{u \in C^1[0, 1] : u(1) = 0\}$. Show that

$$\mathcal{D}(T^*) = \{v \in \mathcal{H} : v' \in \mathcal{H} \text{ and } v(0) = 0\}$$

and that $T^* = -d/dx$.