

## Assignment 7

### Due Friday 19 April 2024

This Assignment is based on sections 3.2, 3.3, 3.4, 4.1, and 4.2 of our textbook, Borthwick (2020) *Spectral Theory: Basic Concepts and Applications*, Springer.

PLEASE DO THE FOLLOWING EXERCISES.

**P33.** The adjoint of a linear map between complex Hilbert spaces  $\mathcal{H}_1$  and  $\mathcal{H}_2$  can be defined. Here we consider bounded operators only. For  $T \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$  the (Hilbert space) adjoint  $T^*$  is the unique linear map  $T^* \in \mathcal{L}(\mathcal{H}_2, \mathcal{H}_1)$  so that

$$\langle v, Tu \rangle_2 = \langle T^*v, u \rangle_1 \quad \text{for all } u \in \mathcal{H}_1 \text{ and } v \in \mathcal{H}_2.$$

When  $\mathcal{H}_1 = \mathcal{H}_2$  this is the same definition as in section 3.2.

- (a) Show that  $(T^*)^* = T$  and that  $(ST)^* = T^*S^*$ .
- (b) Show that if  $T$  is invertible with  $T^{-1} \in \mathcal{L}(\mathcal{H}_2, \mathcal{H}_1)$  then  $(T^*)^{-1} = (T^{-1})^*$ .
- (c) Suppose that  $Q \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$  satisfies  $Q^*Q = I_1$  and  $QQ^* = I_2$  where  $I_i$  is the identity map on  $\mathcal{H}_i$ . Show that  $Q$  is unitary.

*Hints.* For part (a) assume that  $S \in \mathcal{L}(\mathcal{H}_2, \mathcal{H}_3)$ . For part (b) use  $TT^{-1} = T^{-1}T = I$ , then (a). For part (c) use the definition on page 17 of the text:  $U$  is *unitary* if it is a bijective isometry.

**P34.** Suppose  $\{\phi_n\}$  is an orthonormal basis, of a complex Hilbert space  $\mathcal{H}$ , consisting of eigenvectors of some (unbounded) linear operator  $T$  on  $\mathcal{H}$ . Specifically,  $\phi_n \in \mathcal{D}(T)$  and  $T\phi_n = \lambda_n\phi_n$  for  $\lambda_n \in \mathbb{C}$ .

- (a) Define the map  $Q \in \mathcal{L}(\mathcal{H}, \ell^2)$ , where  $\ell^2 = \ell^2(\mathbb{N})$ , by  $(Qf)_n = \langle \phi_n, f \rangle_{\mathcal{H}}$ . Give a formula for  $Q^*$ . Show that  $Q$  is unitary.
- (b) Show that  $Q$  unitarily diagonalizes  $T$  in the sense that

$$QTQ^* = M$$

defines a multiplication operator on  $\ell^2$ .

*Hints.* For part (a) you may use **P33(c)**, though that is not the only way. For part (b), make sure to define the action of  $M$  on arbitrary elements of  $\ell^2$ .

**P35.** (a) Let  $\mathcal{H} = L^2(\mathbb{R})$ . Define  $(M_{x^2}v)(x) = x^2v(x)$ , an unbounded multiplication operator with domain  $\mathcal{D}(M_{x^2}) = \{v \in \mathcal{H} : x^2v(x) \in \mathcal{H}\}$ . Define  $(Tv)(x) = v''(x)$ , an unbounded second derivative operator with domain  $\mathcal{D}(T) = C_0^\infty(\mathbb{R})$ . Show that these operators have no eigenvalues.

**(b)** Let  $\mathcal{H} = L^2(0, \pi)$ . Define  $(M_{x^2} v)(x) = x^2 v(x)$ , a multiplication operator with domain  $\mathcal{D}(M_{x^2}) = \{v \in \mathcal{H} : x^2 v(x) \in \mathcal{H}\}$ . Show that  $M_{x^2}$  is actually bounded, but that it has no eigenvalues.

**(c)** Let  $\mathcal{H} = L^2(0, \pi)$ . Define  $(Tv)(x) = v''(x)$ , a second derivative operator with domain  $\mathcal{D}(T) = \{v \in \mathcal{H} : v \in C^2[0, \pi] \text{ and } v(0) = v(\pi) = 0\}$ . Show that  $T$  is unbounded, and that  $\phi_k(x) = \sin(kx)$  is an eigenfunction for any  $k \in \mathbb{N}$ . Find the corresponding eigenvalues.

*Hints.* For part **(a)** you may use results in Example 3.3. For part **(b)** you may use the result in Example 2.8.

*Comments.* You do not need to prove self-adjointness or spectrum. However, textbook Examples 3.2, 3.5, and 3.22 show both  $M_{x^2}$  are self-adjoint. Example 3.26 shows that  $T$  in part **(a)** is essentially self-adjoint. Example 3.20 sketches why  $T$  in part **(c)** is essentially self-adjoint. See Theorems 4.5 for the spectrum of both  $M_{x^2}$ , thus by unitary-equivalence for the closure of  $T$  in part **(a)** also. Use **P34(b)** for the spectrum of the closure of  $T$  in part **(b)**.

**P36.** Let  $\mathcal{H}$  be a complex Hilbert space. Recall that if  $A$  is a symmetric operator on  $\mathcal{H}$  then  $v \in \mathcal{D}(A)$  implies  $\langle v, Av \rangle \in \mathbb{R}$ . We will write  $A - z$  for  $A - zI$ .

**(a)** Suppose  $A$  is a symmetric operator on  $\mathcal{H}$ . Show that if  $z \in \mathbb{C}$  then

$$\operatorname{Im} \langle v, (A - z)v \rangle = -\operatorname{Im}(z) \|v\|^2.$$

**(b)** If furthermore  $z \in \mathbb{C}$  is strictly complex, i.e.  $\operatorname{Im} z \neq 0$ , then

$$\|v\| \leq \frac{\|(A - z)v\|}{|\operatorname{Im}(z)|}.$$

In this situation, show that  $A - z$  is injective.

**P37.** Let  $\mathcal{H}$  be a complex Hilbert space. Recall that  $\mathcal{L}(\mathcal{H})$  is a normed vector space with norm  $\|T\| = \sup_{\|v\|=1} \|Tv\|$ .

**(a)** Suppose  $T \in \mathcal{L}(\mathcal{H})$  and  $z \in \mathbb{C}$ . If  $|z| > \|T\|$  then

$$\sum_{k=0}^{\infty} z^{-k} T^k$$

converges in norm, i.e. absolutely, in  $\mathcal{L}(\mathcal{H})$ . By Theorems 2.4 and 2.10, the sum defines a bounded operator  $S \in \mathcal{L}(\mathcal{H})$ .

**(b)** Under the same assumptions, show that

$$S(T - zI) = (T - zI)S = -zI.$$

Explain why this shows  $z \in \rho(T)$ .