

Assignment 5

Due Wednesday 27 March 2024

This Assignment is based primarily on sections 3.1, 3.2, and 3.3 of our textbook, Borthwick (2020) *Spectral Theory: Basic Concepts and Applications*, Springer.

PLEASE DO THE FOLLOWING EXERCISES.

P24. Here is why one always assumes that $\mathcal{D}(T) = \mathcal{H}$ for a bounded operator.

After the text defines “operator” (Definition 3.1; page 36) it says that “a bounded operator admits a unique continuous extension to the full space \mathcal{H} , since $\mathcal{D}(T)$ is dense.” Prove this.

P25. This is a basic exercise for Definition 3.4, of the operator adjoint.

(a) Consider the unbounded multiplication operator M_a on $\mathcal{H} = \ell^2 = \ell^2(\mathbb{N})$, defined for $x = (x_1, x_2, x_3, \dots)$ as

$$M_a x = (x_1, 2x_2, 3x_3, \dots),$$

with a domain consisting of sequences which are eventually zero:

$$\mathcal{D}(M_a) = \{x \in \ell^2 : \text{there is } N \text{ so that if } k \geq N \text{ then } x_k = 0\}.$$

Show that $\mathcal{D}(M_a)$ is a dense subspace of \mathcal{H} .

(b) Directly from Definition 3.4, find a formula for the adjoint M_a^* , and its domain.

(c) Let M_b be the unbounded multiplication operator with the same formula $M_b x = (x_1, 2x_2, 3x_3, \dots)$ but on the larger domain

$$\mathcal{D}(M_b) = \left\{x \in \ell^2 : \sum_{k=1}^{\infty} k^2 |x_k|^2 < \infty\right\}.$$

Show that $\mathcal{D}(M_b)$ is dense and that M_b is self-adjoint (Definition 3.19).

P26. Please read Example 3.6 on page 39. In this exercise you prove a crucial aspect.

(a) Let $\mathcal{V} = C^1[0, 1]$ be a normed vector space with norm $\|v\| = \left(\int_0^1 |v(x)|^2 dx\right)^{1/2}$. Define $\omega : \mathcal{V} \rightarrow \mathbb{C}$ by $\omega(v) = v(0)$. Show that ω is linear, but also show that it is *not* bounded.

In the next part you do not need to prove the assertion that $C^1[0, 1]$ is dense in $L^2[0, 1]$, nor do you need to prove that ℓ is linear.

(b) Let $\mathcal{H} = L^2[0, 1]$ and $\mathcal{D}(T) = C^1[0, 1]$ for $T = d/dx$ the first derivative. Since $\mathcal{D}(T)$ is dense in \mathcal{H} , thus T is an operator by Definition 3.1. Fix $u \in \mathcal{D}(T)$ and define the linear functional $\ell : \mathcal{D}(T) \rightarrow \mathbb{C}$ by

$$\ell(v) = -\langle Tu, v \rangle + \overline{u(1)}v(1) - \overline{u(0)}v(0).$$

Show that if ℓ is bounded in the \mathcal{H} norm then $u(0) = u(1) = 0$.

P27. This is a simplification of Exercise 3.12. For both parts below note that f , being merely measurable, could be unbounded, and indeed it could go to infinity anywhere in $(0, 1)$, but at least $f(x) \in \mathbb{C}$ is well-defined for every $x \in (0, 1)$. Also, note that we may conclude from part **(b)** that M_f is self-adjoint if and only if f is real a.e.

Suppose $f : (0, 1) \rightarrow \mathbb{C}$ is measurable. Let M_f be the multiplication operator on $\mathcal{H} = L^2(0, 1)$ with domain

$$\mathcal{D}(M_f) = \{v \in L^2(0, 1) : fv \in L^2(0, 1)\}$$

and formula

$$(M_f g)(x) = f(x)g(x)$$

for $g \in \mathcal{H}$.

(a) Show that $\mathcal{D}(M_f)$ is dense in \mathcal{H} .

(b) Show that $\mathcal{D}(M_f) = \mathcal{D}(M_f^*)$ and that $M_f^* = M_{\bar{f}}$.