Assignment 6

Due Wednesday 10 April 2024

This Assignment is based primarily on sections 3.1, 3.2, 3.3, 3.4, and 4.1 of our textbook, Borthwick (2020) *Spectral Theory: Basic Concepts and Applications*, Springer. Note that 3.5 is skipped for now.

PLEASE DO THE FOLLOWING EXERCISES.

P28. *Hint. You did Exercise 2.5, and you may use its result.*

Show that if $T \in \mathcal{L}(\mathcal{H})$ then $\|T\| = \|T^*\|$ and $\|T^*T\| = \|T\|^2$.

P29. *Recall that $U \in \mathcal{L}(\mathcal{H})$ is* unitary *if it is bijective (one-to-one and onto) and an isometry. Note that we will be able to show that the spectrum of $U = M_f$ in parts* **(c)** *and* **(d)** *is the entire unit circle $\mathbb{S}$. Regarding part* **(d)**, *Section 4.3 defines the phrase "approximate eigenvalue" for this situation, where $Uv_n \approx zv_n$.*

(a) Note $\mathcal{H}$ is a complex Hilbert space. Use the correct[1] polarization identity to show that if U is unitary then $\langle Ux, Uy\rangle = \langle x, y\rangle$ for all x, y in $\mathcal{H}$.

(b) Suppose $\lambda \in \mathbb{C}$ is an eigenvalue of a unitary operator U. Show that λ is on the unit circle $\mathbb{S} = \{z \in \mathbb{C} : |z| = 1\}$.

(c) Let $\mathcal{H} = L^2(0, 1)$ and fix the function $f(x) = e^{2\pi i x}$. Show that the multiplication operator $U = M_f$ is bounded and unitary. (*Hint. Bounded is easy; use Example 2.8 from the textbook. Unitary is also easy.*) Show that this U has no eigenvalues.

(d) For U from part **(c)**, and each $z \in \mathbb{S}$, find a sequence of functions $v_n \in \mathcal{H}$ so that $\|v_n\| = 1$ and $\lim_{n\to\infty} \|(U - z)v_n\| = 0$.

P30. *I am deliberately suppressing most details of* closed *operators (section 3.3). However, this exercise uses Definition 3.8, and suggests what "closed" is saying.*

(a) Let T be an operator on $\mathcal{H}$, so $\mathcal{D}(T)$ is a dense subspace. Show that

$$\|u\|_T = \left(\|u\|^2 + \|Tu\|^2\right)^{1/2}$$

defines a norm on $\mathcal{D}(T)$, called the *graph norm* for T.

(b) Show that T is closed as an operator on $\mathcal{H}$ if and only if the normed vector space $(\mathcal{D}(T), \|\cdot\|_T)$ is complete.

[1] See Assignment # 3 for the correct form which you proved. The textbook's version, (2.17) on page 17, has sign errors.

P31. *Page 47 of the text is important as it defines* self-adjoint, symmetric, *and* positive *for unbounded operators. Hint. Consider vectors of the form $u + iv$ for $u, v \in \mathcal{D}(A)$. Also, if a number is nonnegative then it is real!*

Prove that a positive operator A on a complex Hilbert space $\mathcal{H}$ is symmetric.

P32. *Hint. You will need the fact that $v' \in L^2(0,1)$ implies $v' \in L^1(0,1)$ implies $v \in C[0,1]$ implies $v(0)$ is well-defined. And, of course, integration by parts.*

Let $\mathcal{H} = L^2(0,1)$. Consider $T = d/dx$, an unbounded operator on $\mathcal{H}$, with domain $\mathcal{D}(T) = \{v \in C^1[0,1] \,:\, v(1) = 0\}$. Show that

$$\mathcal{D}(T^*) = \{v \in \mathcal{H} \,:\, v' \in \mathcal{H} \text{ and } v(0) = 0\}$$

and that $T^* = -d/dx$.