# Assignment 7

## Due Friday 19 April 2024

This Assignment is based on sections 3.2, 3.3, 3.4, 4.1, and 4.2 of our textbook, Borthwick (2020) *Spectral Theory: Basic Concepts and Applications*, Springer.

PLEASE DO THE FOLLOWING EXERCISES.

**P33.** The adjoint of a linear map between complex Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ can be defined. Here we consider bounded operators only. For $T \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ the *(Hilbert space) adjoint* $T^*$ is the unique linear map $T^* \in \mathcal{L}(\mathcal{H}_2, \mathcal{H}_1)$ so that

$$\langle v, Tu \rangle_2 = \langle T^* v, u \rangle_1 \qquad \text{for all } u \in \mathcal{H}_1 \text{ and } v \in \mathcal{H}_2.$$

When $\mathcal{H}_1 = \mathcal{H}_2$ this is the same definition as in section 3.2.

**(a)** Show that $(T^*)^* = T$ and that $(ST)^* = T^* S^*$.

**(b)** Show that if $T$ is invertible with $T^{-1} \in \mathcal{L}(\mathcal{H}_2, \mathcal{H}_1)$ then $(T^*)^{-1} = (T^{-1})^*$.

**(c)** Suppose that $Q \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ satisfies $Q^* Q = I_1$ and $QQ^* = I_2$ where $I_i$ is the identity map on $\mathcal{H}_i$. Show that $Q$ is unitary.

*Hints.* For part **(a)** assume that $S \in \mathcal{L}(\mathcal{H}_2, \mathcal{H}_3)$. For part **(b)** use $TT^{-1} = I_2$ and $T^{-1}T = I_1$, and then apply **(a)**. For part **(c)** use the definition on page 17 of the text: $U$ is *unitary* if it is a bijective isometry.

**P34.** **(a)** Suppose $\{\phi_n\}$ is an orthonormal basis of a complex Hilbert space $\mathcal{H}$. Define the map $Q \in \mathcal{L}(\mathcal{H}, \ell^2)$, where $\ell^2 = \ell^2(\mathbb{N})$, by $(Qf)_n = \langle \phi_n, f \rangle_{\mathcal{H}}$ for $f \in \mathcal{H}$. Give a formula for $Q^*$. Show that $Q$ is unitary.

**(b)** Let $T$ be a closed (unbounded) linear operator on $\mathcal{H}$. Suppose $\phi_n \in \mathcal{D}(T)$ and $T\phi_n = \lambda_n \phi_n$, for $n \in \mathbb{N}$ and $\lambda_n \in \mathbb{C}$. If $\{\phi_n\}$ is an orthonormal basis of $\mathcal{H}$ then $Q$ in part **(a)** unitarily diagonalizes $T$ in the sense that

$$QTQ^* = M$$

defines an unbounded multiplication operator on $\ell^2$.

*Hints.* For part **(a)** you may use **P33(c)**, though that is not the only way. For part **(b)**, make sure to define the domain of $M$ and the action of $M$ on elements of $\mathcal{D}(M)$.

**P35.** **(a)** Let $\mathcal{H} = L^2(\mathbb{R})$. Define $(M_{x^2}\, v)(x) = x^2 v(x)$, an unbounded multiplication operator with domain $\mathcal{D}(M_{x^2}) = \{v \in \mathcal{H} \,:\, x^2 v(x) \in \mathcal{H}\}$. Define $(Tv)(x) = v''(x)$, an unbounded second derivative operator with domain $\mathcal{D}(T) = C_0^\infty(\mathbb{R})$. Show that these operators have no eigenvalues.

**(b)** Let $\mathcal{H} = L^2(0,\pi)$. Define $(M_{x^2}\, v)\,(x) = x^2 v(x)$, a multiplication operator with domain $\mathcal{D}(M_{x^2}) = \{v \in \mathcal{H} \,:\, x^2 v(x) \in \mathcal{H}\}$. Show that $M_{x^2}$ is actually bounded, but that it has no eigenvalues.

**(c)** Let $\mathcal{H} = L^2(0,\pi)$. Define $(Tv)(x) = v''(x)$, a second derivative operator with domain $\mathcal{D}(T) = \{v \in \mathcal{H} \,:\, v \in C^2[0,\pi] \text{ and } v(0) = v(\pi) = 0\}$. Show that $T$ is unbounded, and that $\phi_k(x) = \sin(kx)$ is an eigenfunction for any $k \in \mathbb{N}$. Find the corresponding eigenvalues.

*Hints.* For part **(a)** you may use results in Example 3.3. For part **(b)** you may use the result in Example 2.8.

*Comments.* You do not need to prove self-adjointness or spectrum. However, textbook Examples 3.2, 3.5, and 3.22 show both $M_{x^2}$ are self-adjoint. Example 3.26 shows that $T$ in part **(a)** is essentially self-adjoint. Example 3.20 sketches why $T$ in part **(c)** is essentially self-adjoint. See Theorems 4.5 for the spectrum of both $M_{x^2}$, thus by unitary-equivalence for the closure of $T$ in part **(a)** also. Use **P34(b)** for the spectrum of the closure of $T$ in part **(b)**.

**P36.** Let $\mathcal{H}$ be a complex Hilbert space. Recall that if $A$ is a symmetric operator on $\mathcal{H}$ then $v \in \mathcal{D}(A)$ implies $\langle v, Av\rangle \in \mathbb{R}$. We will write $A - z$ for $A - zI$.

**(a)** Suppose $A$ is a symmetric operator on $\mathcal{H}$. Show that if $z \in \mathbb{C}$ then

$$\operatorname{Im}\langle v, (A-z)v\rangle = -\operatorname{Im}(z)\|v\|^2.$$

**(b)** If furthermore $z \in \mathbb{C}$ is strictly complex, i.e. $\operatorname{Im} z \neq 0$, then

$$\|v\| \leq \frac{\|(A-z)v\|}{|\operatorname{Im}(z)|}.$$

In this situation, show that $A - z$ is injective.

**P37.** Let $\mathcal{H}$ be a complex Hilbert space. Recall that $\mathcal{L}(\mathcal{H})$ is a normed vector space with norm $\|T\| = \sup_{\|v\|=1}\|Tv\|$, and also recall Theorem 2.10.

**(a)** Suppose that $T \in \mathcal{L}(\mathcal{H})$, $z \in \mathbb{C}$, and $|z| > \|T\|$. Show that

$$\sum_{k=0}^{\infty} z^{-k} T^k$$

converges to $S \in \mathcal{L}(\mathcal{H})$.

**(b)** Under the same assumptions, show that

$$S(T - zI) = (T - zI)S = -zI.$$

Explain why this shows $z \in \rho(T)$.