Assignment 4

Due Wednesday 6 March 2024

This Assignment is based primarily on sections 2.6, 2.7, and 3.1 of our textbook, Borthwick (2020) *Spectral Theory: Basic Concepts and Applications*, Springer.

PLEASE DO THE FOLLOWING EXERCISES.

P19. *This is Exercise 2.10, but clarified. One way to show* **(a)** *is to argue that* $\operatorname{range}(P) = \ker(Q)$ *for some bounded operator* $Q \in \mathcal{L}(\mathcal{H})$ *built from* P.

Suppose $P \in \mathcal{L}(\mathcal{H})$ satisfies $P^2 = P$ and $\langle Pv, w\rangle = \langle v, Pw\rangle$ for all $v, w \in \mathcal{H}$.

(a) Show that $\operatorname{range}(P)$ is a closed subspace of $\mathcal{H}$.

(b) Show that $\mathcal{H} = \operatorname{range}(P) \oplus \ker(P)$.

P20. *This is Exercise 2.6, asking for the proof of Corollary 2.29. Hint: First show that for fixed* $w \in \mathcal{H}$*, the functional* $\ell(v) = \eta(w, v)$ *is in* $\mathcal{H}'$*. Remember that* $\langle \cdot, \cdot \rangle$ *and* $\eta(\cdot, \cdot)$ *are only conjugate linear in the first spot. Please prove the linearity and boundedness of* T.

Given a bounded sesquilinear form $\eta : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$, show that there is a unique operator $T \in \mathcal{L}(\mathcal{H})$ so that

$$\eta(v, w) = \langle v, Tw\rangle$$

for all $v, w \in \mathcal{H}$.

P21. Carefully prove Corollary 2.36, which is Parseval's theorem and equality, from Theorems 2.34 and 2.35.

P22. *In Example 2.32 on page 28 our textbook casually says that a "standard argument using the Dirichlet kernel" shows that a certain set of orthogonal complex exponentials is in fact a basis. This exercise begins this quite substantial "standard argument," actually perhaps the greatest triumph of 19th century analysis, by defining the Dirichlet kernel. Note I use* $\mathcal{H} = L^2(-\pi, \pi)$ *here, instead of* $L^2(0, 2\pi)$*, but this is a detail.*

(a) Let $\phi_k(\theta) = \dfrac{1}{\sqrt{2\pi}} e^{ik\theta}$ for $k \in \mathbb{Z}$. Since these functions are continuous and bounded they are in $\mathcal{H} = L^2(-\pi, \pi)$. Show that $\{\phi_k\}_{k\in\mathbb{Z}}$ is an orthonormal set.

(b) For $n \in \mathbb{N}$, let $D_n(\theta) = \displaystyle\sum_{j=-n}^{n} e^{ij\theta}$ be the Dirichlet kernel. Show that $D_n(0) = 2n + 1$ (*easy!*). For $\theta \neq 0$ show by a geometric series argument that

$$D_n(\theta) = \frac{\sin\left((n + 1/2)\theta\right)}{\sin(\theta/2)}.$$

(c) Use a computer to plot $D_2(\theta)$, $D_8(\theta)$, and $D_{20}(\theta)$ in a single figure, on the interval $-\pi \le \theta \le \pi$.

(d) Compute $\displaystyle\int_{-\pi}^{\pi} D_n(\theta)\,d\theta$. Conjecture what the integral $\displaystyle\int_{-\pi}^{\pi} D_n(\theta) f(\theta)\,d\theta$ yields when $f(x)$ is continuous at $x = 0$, and if n is very large?

(e) Show that if $f \in \mathcal{H} = L^2(-\pi, \pi)$ then

$$\int_{-\pi}^{\pi} D_n(\theta - x) f(\theta)\,d\theta = 2\pi \sum_{j=-n}^{n} \langle \phi_j, f\rangle\, \phi_j(x).$$

Observe—there is nothing to prove here—that this says that

$$(P_n f)(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} D_n(\theta - x) f(\theta)\,d\theta$$

is the orthogonal projection of $f \in \mathcal{H}$ onto $\operatorname{span}\{\phi_{-n}, \dots, \phi_n\}$; compare (2.30).

P23. *Hints. Use Cauchy-Schwarz for part* **(a)**. *An example for part* **(b)** *can be built from a power of the polar coordinate* $r = (x^2 + y^2)^{1/2}$.

(a) Let $\Omega = (0,1)^2 \subset \mathbb{R}^2$. Suppose $K \in L^2(\Omega)$, that is,

$$\int_0^1 \int_0^1 |K(x,y)|^2\,dx\,dy < \infty.$$

Show that the integral operator with kernel K, namely,

$$(T_K f)(x) = \int_0^1 K(x,y) f(y)\,dy$$

is linear and bounded ($T_K \in \mathcal{L}(\mathcal{H})$) on $\mathcal{H} = L^2(0,1)$.

(b) Find $K \in L^2(\Omega)$ which is *not* a bounded function.

(c) For a finite sequence $a_j \in \mathbb{C}$, $j = 1, \dots, n$, let

$$K_a(x,y) = \sum_{j=1}^{n} a_j e^{i\,2\pi j(x-y)}.$$

This is a continuous and bounded function and thus $K_a \in L^2(\Omega)$. Find all of the eigenvalues and eigenfunctions of the operator $T_{K_a} \in \mathcal{L}(\mathcal{H})$.