3 standard glacier models: are they well-posed?

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Glaciers Seminar

Well-posed problem

Article   Talk

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In mathematics, a **well-posed problem** is one for which the following properties hold:[a]

1. The problem has a solution (existence)
2. The solution is unique
3. The solution's behavior changes continuously with the data of the problem

Jacques Salomon Hadamard

ex 1: well posed!

$$\boxed{\begin{aligned} -u''(x) &= f(x) \\ u(0) = 0, &\ u(1) = 0 \end{aligned}}$$

best-known proof that it is well-posed is via minimizing

$$J(u) = \int_0^1 \frac{1}{2} u'(x)^2 - f(x)\,u(x)\,dx$$

over functions satisfying the b.c.s

solution:

$$u(x) = \int_0^1 g(x,y) f(y)\,dy \quad \text{where} \quad g(x) = \begin{cases} y(1-x), & x<y \\ x(1-y), & y>x \end{cases}$$

ex 2: usually well-posed

$$\boxed{\begin{gathered} -u''(x) + a\,u(x) = f(x) \\ u(0) = 0, \quad u(1) = 0 \end{gathered}}$$

issue: if e.g. $a = -\pi^2$ then $u(x) + c\sin(\pi x)$ is also a solution, for any c

well-posed if $a \neq -\pi^2, -4\pi^2, -9\pi^2, \dots$

otherwise multiple solutions

Uniqueness can fail

ex 3: rarely well-posed ← or "usually ill-posed"?

$$\boxed{\begin{aligned} &u'(x) = f(x) \\ &u(0)=0, \quad u(1)=0 \end{aligned}}$$

issue: $u(x) = \int_0^x f(y)\,dy$,

but then only rarely (for special f) does

$$0 = u(1) = \int_0^1 f(y)\,dy$$

hold

existence can fail

but this is not a
math class ...

Outline

<u>problem 1</u>: Glen-Stokes model to find $\vec{u}, p$ for a fixed glacier geometry

well-posed!

<u>problem 2</u>: shallow ice approximation for steady glacier geometry

uniqueness?

<u>problem 3</u>: time-dependent Glen-Stokes for evolving glacier geometry

open!

$\boxed{\text{problem 1}}$ is to use the Glen-Stokes equations, within ice of <u>known</u> geometry, and with dynamical boundary conditions, to determine $\vec{u}$, p

Symbols (for all problems)

$x = (x_1, x_2)$ horizontal coordinate(s)

z vertical coordinate

s ice surface elevation

b bedrock elevation

$\vec{u}$ ice flow velocity

p ice fluid pressure

a surface mass balance

$D\vec{u}$ strain rate tensor

$|D\vec{u}|$ Frobenius norm of $D\vec{u}$ $\left(\mathrm{II}_{\vec{u}} = \frac{1}{2}|D\vec{u}|^2\right)$

$\nu = \nu(D\vec{u})$ ice viscosity (assumed isothermal)

ρ_i ice density

$\vec{g}$ gravity

ν_0 viscosity scale

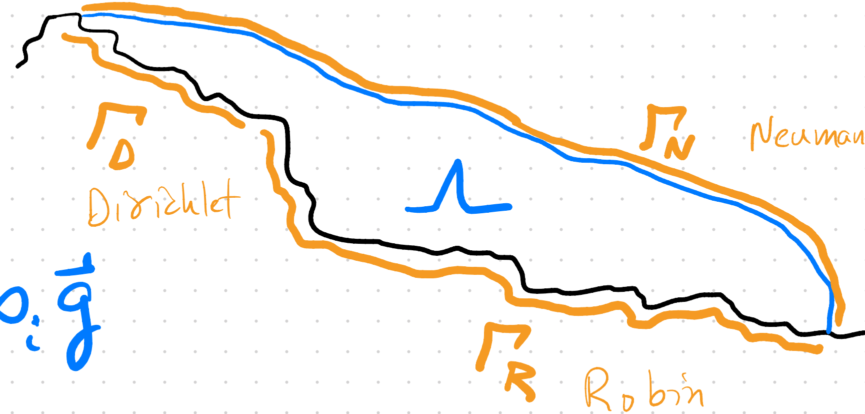
$n=3$ Glen exponent

equations within ice Λ:

$$\nu(D\vec{u}) = \nu_0 \left(|D\vec{u}|^2 + \epsilon^2\right)^{-2/3}$$

$$-\nabla\cdot\left(2\nu(D\vec{u})\,D\vec{u}\right) + \nabla p = \rho_i \vec{g}$$

$$\nabla\cdot\vec{u} = 0$$



on boundary

$\sigma\,\hat{n} = 0$ on Γ_N

$\vec{u} = 0$ on Γ_D

$$\left.\begin{aligned}(\sigma\,\hat{n})_t + \beta\,\vec{u}_t &= 0\\ \vec{u}\cdot\hat{n} &= 0\end{aligned}\right\}\ \text{on } \Gamma_R$$

notation

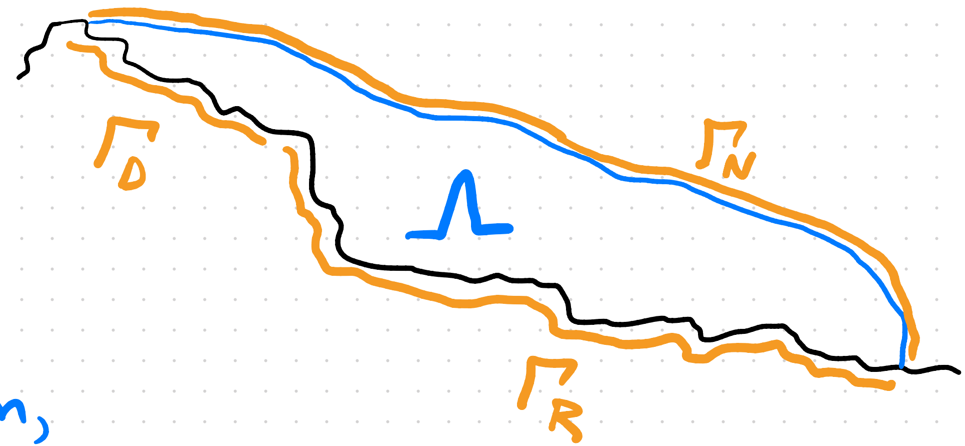
$\sigma = 2\nu(D\vec{u})D\vec{u} - pI$

$\hat{n} =$ (unit outward normal)

$\vec{w}_t = (I - \hat{n}\hat{n}^T)\,\vec{w}$

$=$ (tangential part of $\vec{w}$)

- key idea in problem 1 is that the domain Λ itself is the "input data" to the model



- the surface elevation, defining Γ_N, is <u>data in problem 1</u>

Theorem (Jouvet & Rappaz 2011, Isaac et al 2015)

if $s(x)$ and $b(x)$ are continuous and Lipschitz, and if $\varepsilon > 0$, and if either Γ_D is positive measure, or if the sliding law on Γ_R gives sufficient resistance, then there are unique fields $\vec{u}, p$ satisfying the weak form of the Glen-Stokes model

Lip: $|s(x) - s(y)| \le L \|x - y\|$

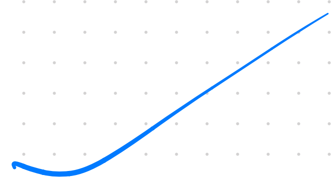
What, physically-speaking, are these caveats about?

"$\varepsilon > 0$": avoid ∞ viscosity at zero strain.
(and uniqueness)

"the sliding law on Γ_B gives sufficient resistance":

so the glacier doesn't slide off

"$s(x)$ and $b(x)$ are continuous and Lipschitz":



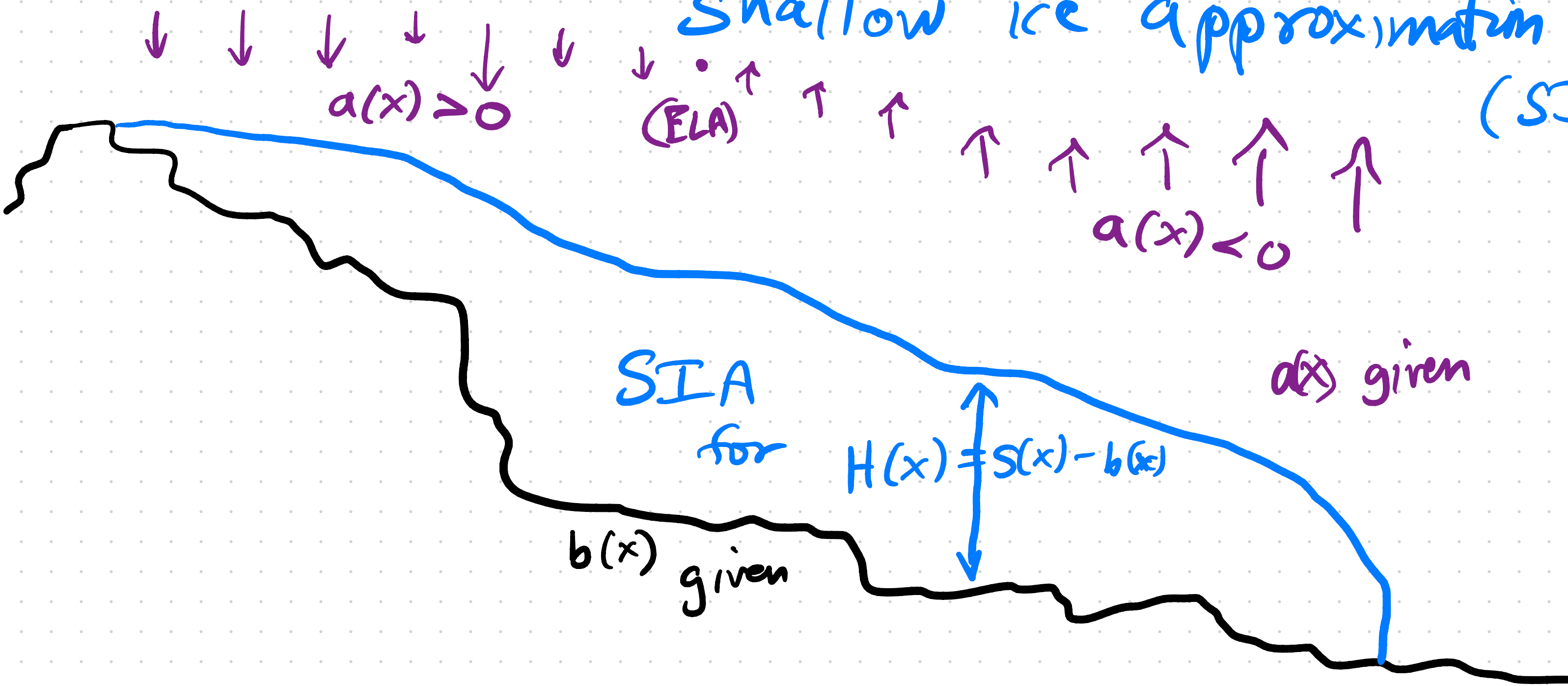
problem 1 summary

if you know the geometry of a glacier, and choose a reasonable sliding law, then you can find its velocity and pressure, thus its stress field

Comments:

- problem 1 is instantaneous, but it is not steady state
- if you solve problem 1 numerically, your implementation may make errors, but you won't (shouldn't?) be confused about their sources

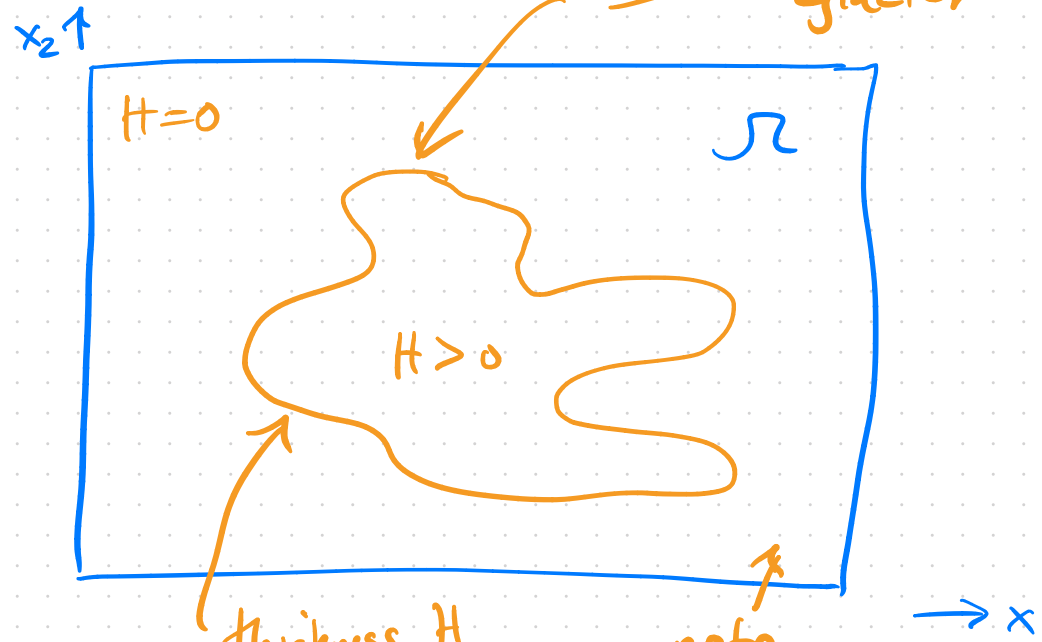
$\boxed{\text{problem 2}}$ is to find the steady state thickness, in a steady climate (SMB $a(x)$), of a glacier modeled using the isothermal shallow ice approximation (SIA)



domain and data for problem 2

solution is the glacier

$a(x)$ and $b(x)$ are given on all of the "map" $\Omega \subset \mathbb{R}^2$



thickness H and flux $\vec{q}$ go to zero along free boundary

note $\vec{q}=0$ where ice free

equations and inequalities on "map" Ω

$$H \geq 0$$

$$\vec{q} = -\Gamma\, H^{n+2} \left|\nabla(H+b)\right|^{n-1} \nabla(\underbrace{H+b}_{=s,}) \Bigg\}$$

vertically-integrated ice flux

the surface elevation

$$\nabla \cdot \vec{q} - a \geq 0$$

$$H(\nabla \cdot \vec{q} - a) = 0$$

where

$\Gamma > 0$ constant, a rescaled ice softness

note: this is the non-sliding version, but certain sliding laws are possible...

Theorem (Jouvet & Bueler, 2012)

if $b(x)$ has an integrable gradient over Ω, and if $a(x)$ is (merely) integrable, then there exists a steady thickness $H(x)$, for which $H(x)^{8/3}$ ← (n=3 case) is a well-behaved function with an integrable gradient, satisfying the weak form of problem Z

What, physically-speaking, are these caveats about?

"$H(x)^{8/3}$ is a well-behaved function...": } next slide

Raviart (1970)

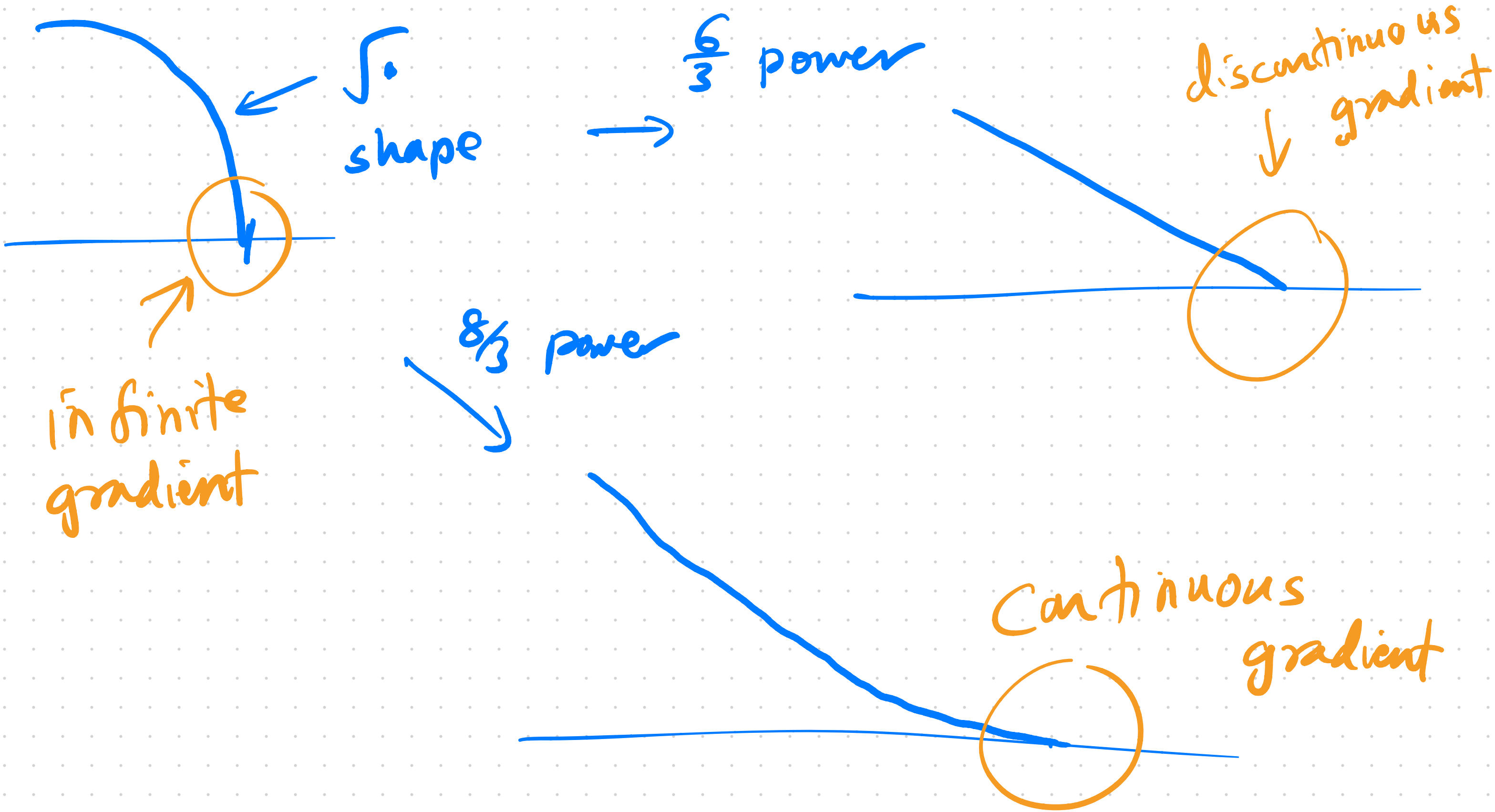
"$b(x)$ has an integrable gradient over Ω":

2 slides ahead

"there exists ...":

uniqueness not known!

re garding 8/3 power:



regarding irregular $b(x)$:

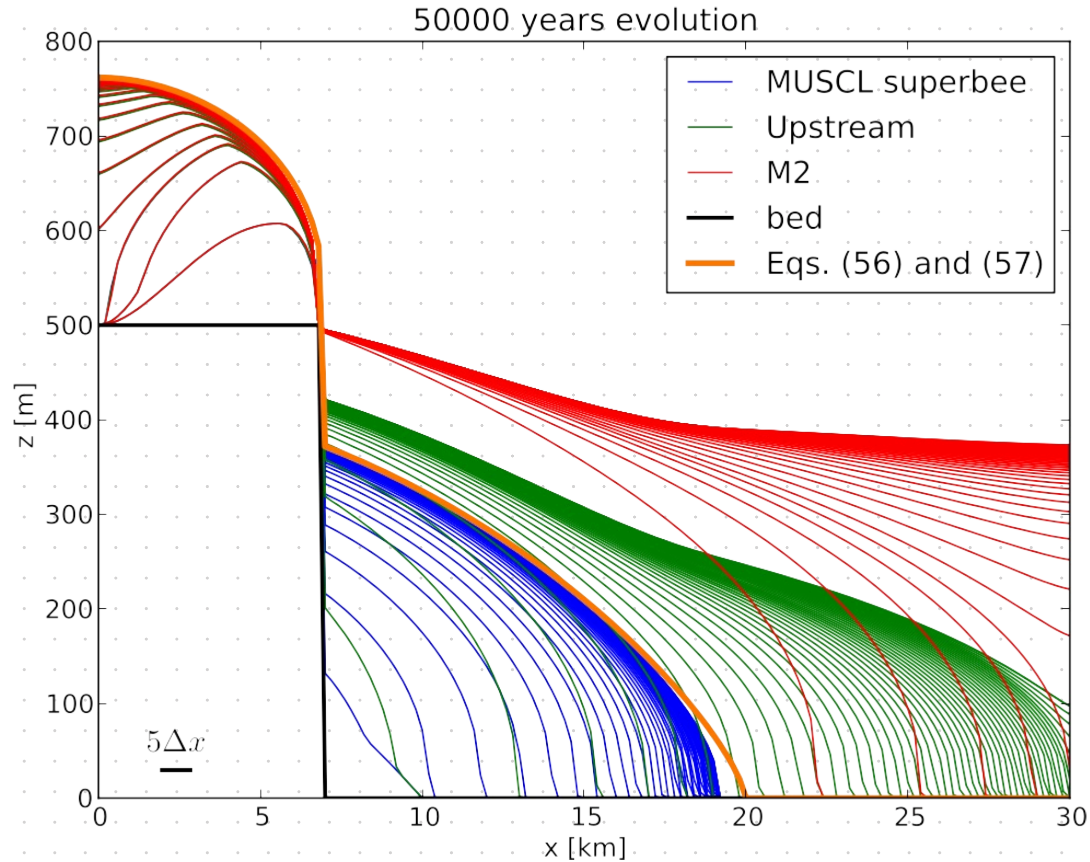


figure from Jarosch et al (2013)

numerical confusion?

- problem 2 remains hard to solve numerically on sufficiently-rough bed topography
- the methods which work best must agressively enforce edge continuity of the flux $\vec{q}$ (Brinkerhoff...)
- I think the underlying numerical issue is the "same" as the uniqueness issue

- how about the time dependent case?

Theorem (Calvo et al 2003 in 1D, Piersanti & Temam 2023 in 2D)

if the bed is flat, and if $a(t,x)$ is integrable, then there is a unique $H(t,x)$, with $H(t,x)^{8/3}$ well-behaved, satisfying the time dependent version of problem 2

- this is flat-bed and only SIA

- note

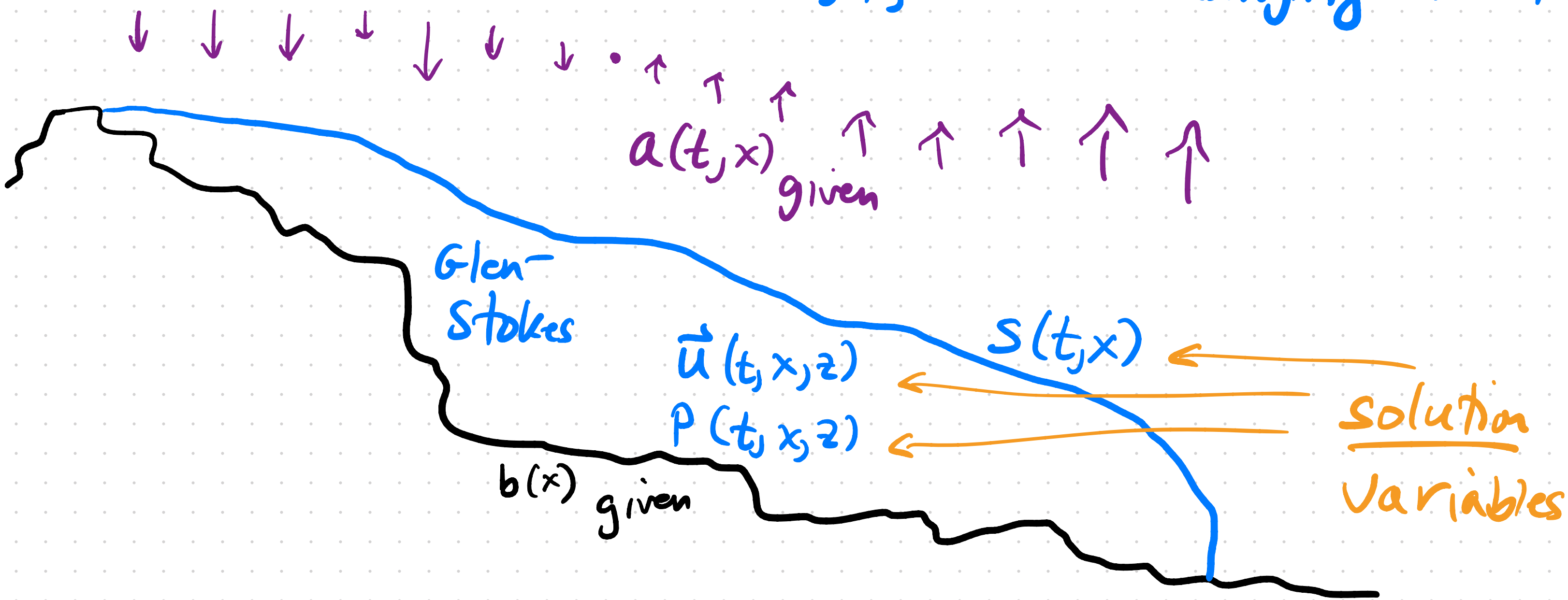
$$a = a(x)$$

in problem 2, is not

$$a = a(x, z) = a(x, s)$$

- the fact that uniqueness is open in problem 2 is not about elevation-accumulation feed back (for which actual non-uniqueness can be shown)

$\boxed{\text{problem 3}}$ is to use the Glen-Stokes equations, and the surface kinematical equation, to find the evolution of the surface elevation s, with co-evolution of $\vec{u}$, p, in a changing climate



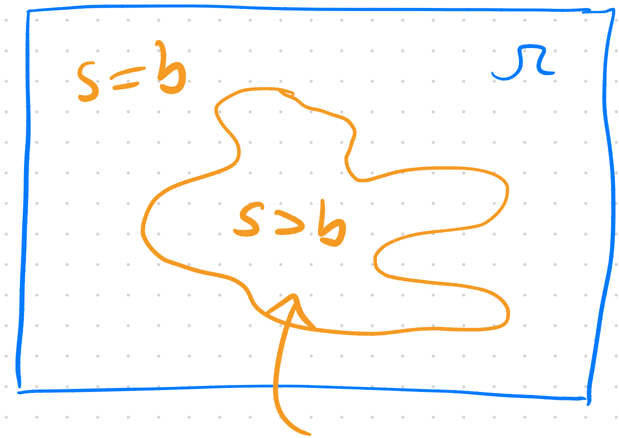
data of problem 3:

Same as time-dependent form of problem 2, at least in the non-sliding case, namely

1. $a(t,x)$ SMB
2. $b(x)$ topography,

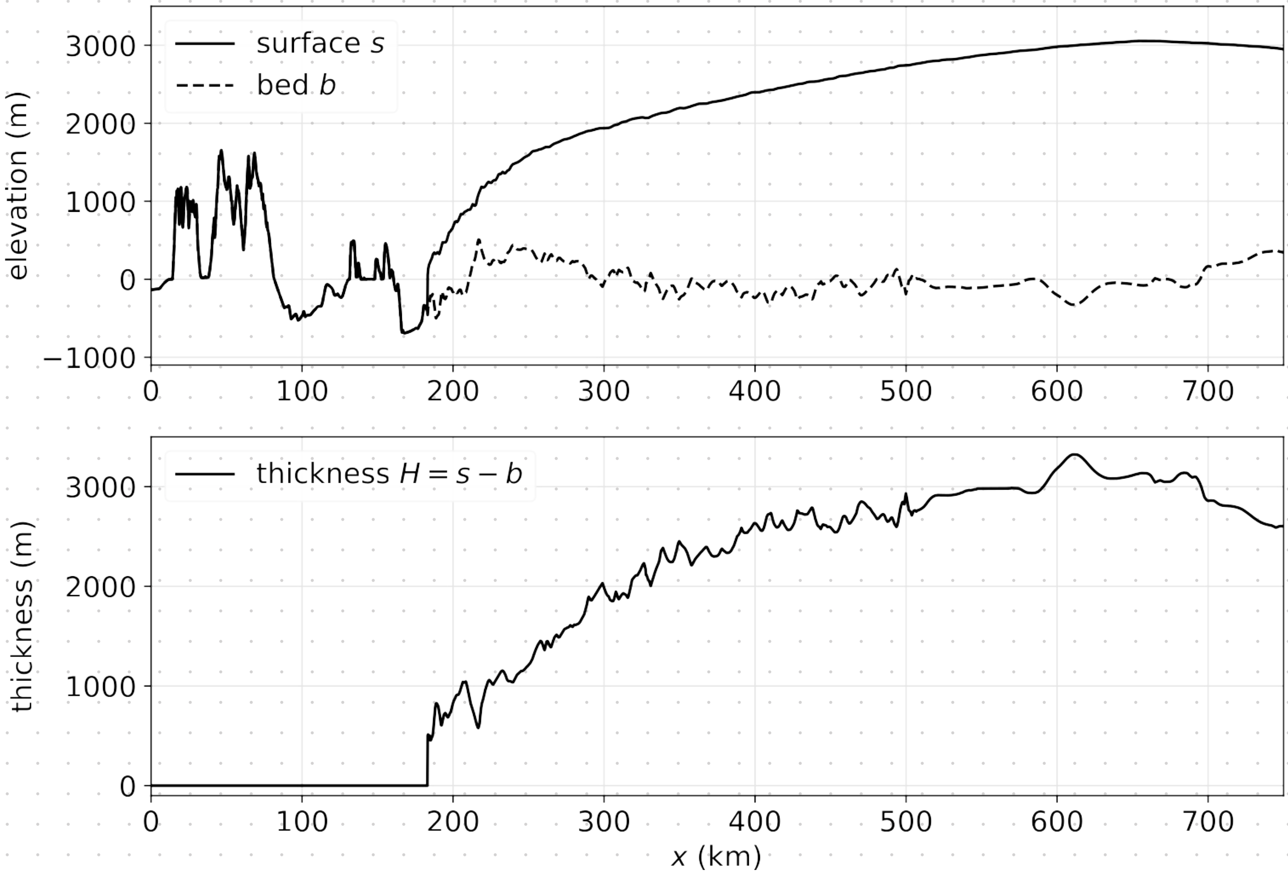
with initial surface elevation

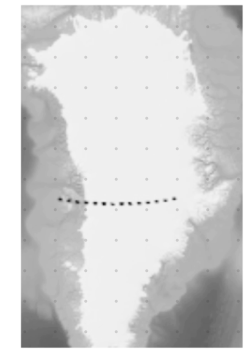
3. $s_0(x)$ satisfying $s_0 \geq b$



solution of problem 3 is the evolving glaciated area, and the evolving $s(t,x)$, and the co-evolving $\vec{u}, p$

why not phrase in terms of thickness $H = s - b$?





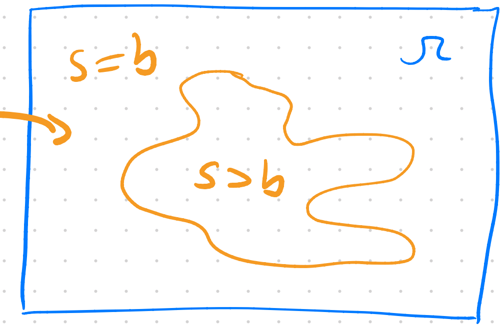
equations in Ω:

$$s \geq b$$

$$\frac{\partial s}{\partial t} - \vec{u}|_s \cdot \vec{n}_s - a \geq 0$$

$$(s-b)\left(\frac{\partial s}{\partial t} - \vec{u}|_s \cdot \vec{n}_s - a\right) = 0$$

set $\vec{u}|_s = 0$ where ice free

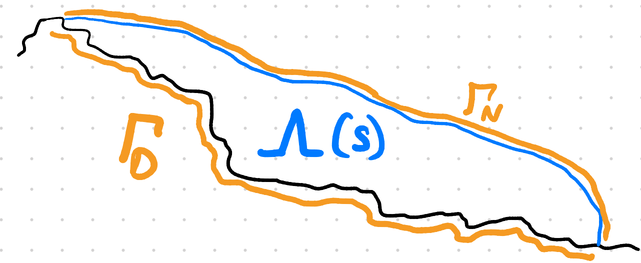


equations in $\Lambda(s) = \{x,z : b<z<s\}$

$$\nu(D\vec{u}) = \nu_0\left(|D\vec{u}|^2 + \varepsilon^2\right)^{-2/3}$$

$$-\nabla \cdot (2\nu(D\vec{u})\, D\vec{u}) + \nabla p = \rho_i \vec{g}$$

$$\nabla \cdot \vec{u} = 0$$



with nonsliding for simplicity:

$\sigma \hat{n} = 0$ on Γ_N

$\vec{u} = 0$ on Γ_D

surface kinematical equation (SKE)

$$\frac{\partial s}{\partial t} = \underbrace{\vec{u}|_s \cdot \vec{n}_s}_{①} + \underbrace{a}_{②}$$

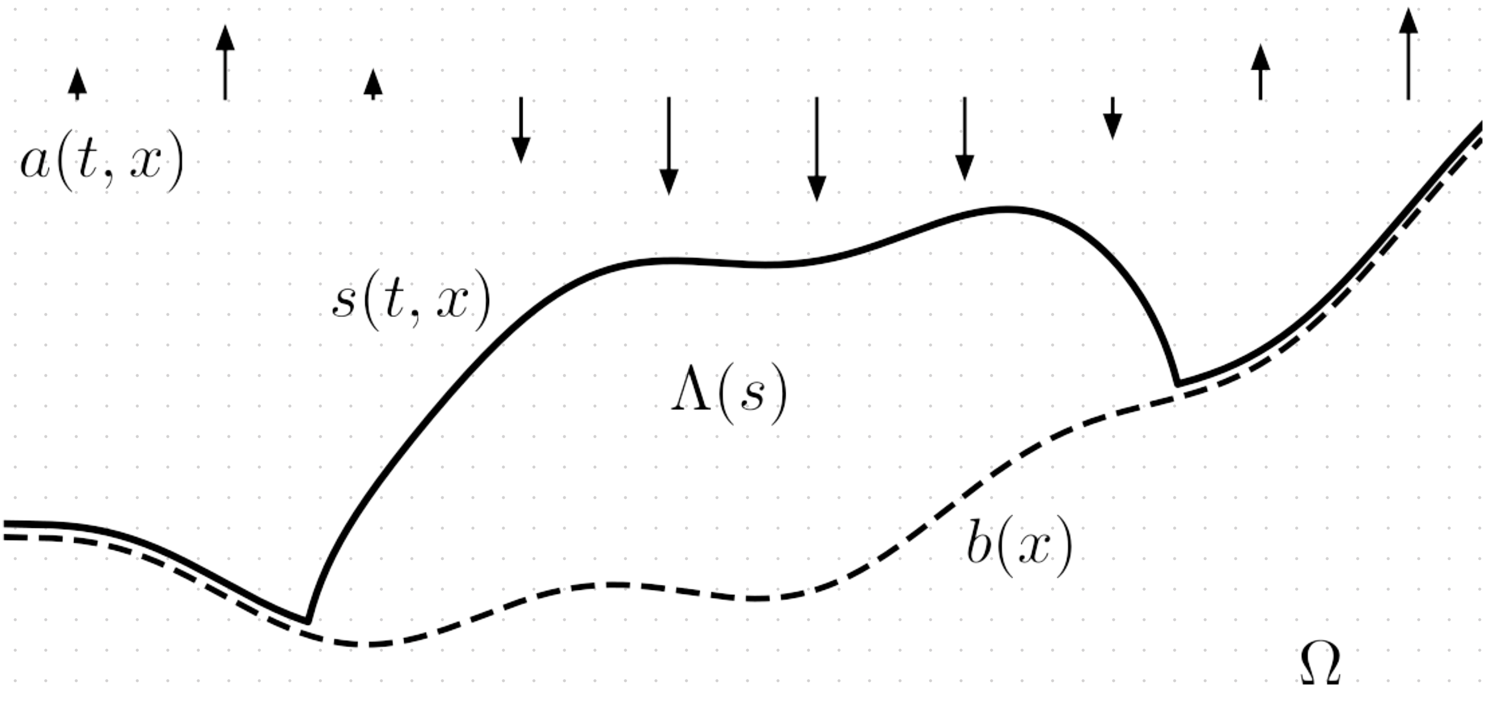
def:

$\vec{n}_s = \left(-\frac{\partial s}{\partial x}, -\frac{\partial s}{\partial y}, 1\right)$

is a surface normal

- the SKE says the surface moves up or down, in an Eulerian view, according to whether the (sloped) surface ice is moving ① and/or accumulating/ablating ②
- it is not anyone's "boundary condition"

domain of ice $\Lambda(s)$ is part of the solution



what "well-posed" would mean for problem 3

given data $s_0(x)$, $a(t,x)$, $b(x)$, then

at least for a short time (?), there

are unique solution fields

$s(t,x)$
$\vec{u}(t,x,z)$
$p(t,x,z)$,

with $s(t,x) \geq b(x)$, and $\vec{u}, p$ only defined within the ice $\left(x, z \in \Lambda(s(t,x))\right)$, which simultaneously satisfy the weak form of all inequalities, equations, and b.c.s of problem 3

- well-posedness, and existence or uniqueness as separate questions, are totally open questions for problem 3

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do you believe it is well-posed?

- if not, you should either add/subtract equations to make it so, or you should give up entirely on numerical (and ML) glacier simulations

my current strategy for problem 3

① only consider a single backward Euler time step (Bueler, 2024 arxiv):

$$s^k \geq b$$

$$\frac{s^k - s^{k-1}}{\Delta t} - \vec{u}|_{s^k} \cdot \vec{n}_{s^k} - a \geq 0$$

$$(s^k - b)\left(\frac{s^k - s^{k-1}}{\Delta t} - \vec{u}|_{s^k} \cdot \vec{n}_{s^k} - a\right) = 0$$

$s^k(x) \approx s(t_k, x)$

$(*)$

② accept that existence and/or uniqueness may be restricted to small values of $\Delta t > 0$

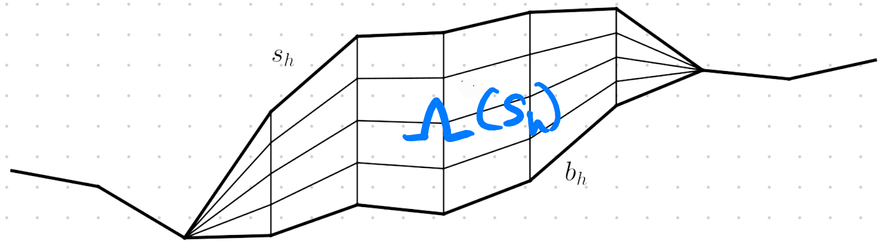
③ work numerically for now, trying to prove that the finite element (FE) approximation is well-behaved (Bueler, 2024)

④ use a mixed FE formulation of the SKE complementarity problem ⊗, in which the flux $\vec{q}$, or the surface gradient ∇s, comes from an edge-continuous "H(div)" FE space

(see Brinkerhoff, 2024 re a choice like this)

⑤ use Firedrake

⑥ use an extruded mesh for $\Lambda(s)$:



⑦ accept that a regularized SKE

$$\frac{\partial s}{\partial t} = \vec{u}|_s \cdot \vec{n}_s + a \underbrace{+ \delta \nabla^2 s}_{\text{regularization}}$$

may be necessary to put s in a reasonable function space (avoids overhangs)

⑧ try to sell such a general "fast fluid layer evolution" FE model, with optimal numerical scaling, as a useful scientific computation

⑨ ... with first demo being high-resolution, 3D Stokes simulation of all glaciers in an Alaskan mountain range thanks for your attention!

I am happy to try to answer questions

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