# Revision of Surface elevation errors in finite element Stokes models for glacier evolution

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This paper explores the well posedness of the implicit time step model for glacier surface elevation, namely

$$s = \Delta t\, \boldsymbol{u}_n(s) + \Delta t\, a + s^{n-1}, \tag{1a}$$

$$s \geq b. \tag{1b}$$

Here, $\boldsymbol{u}_n(s) = \boldsymbol{u} \cdot \boldsymbol{n}|_s$ is the normal velocity along the upper surface $s = s(x)$ at time $t^n$. The velocity field $\boldsymbol{u}(s)$ results from solving the Stokes equations over a domain with upper surface $s$w. In order to establish the well-posedness of this problem, the author makes two conjectures: Conjectures 2.2 and 3.1.

This problem is of great importance in glaciology. Given the lack of any results in this direction, I think this contribution could be welcomed. However, I am highly skeptical of the validity of conjecture 3.1. On the one hand, the author himself makes it clear that the numerical results in section 4 are not compelling evidence for conjecture 3.1. On the other hand, the coercivity condition (54) in the paper is a very restrictive inequality. In fact, I suspect that Conjecture 2.2 (or a variant of it) should be sufficient to establish the well-posedness of our problem. Moreover, since the numerical evidence for Conjecture 2.2 is much more solid, the paper would be of much higher quality if it could drop Conjecture 3.1.

Equation (1a) can be written as the fixed point of the operator

$$T(s) = \Delta t\, \boldsymbol{u}_n(s) + \Delta t\, a + s^{n-1}.$$

We see that

$$\|T(s) - T(r)\| = \Delta t \|\boldsymbol{u}_n(s) - \boldsymbol{u}_n(r)\| \tag{2}$$

A consequence of conjecture 2.2 might be that there exists a $C > 0$ such that

$$\|\boldsymbol{u}_n(s) - \boldsymbol{u}_n(r)\| \leq C\|s - r\|. \tag{3}$$

Perhaps one might need some assumptions on $r, s$ as in Lemma 2.3 of the paper. Or maybe a new conjecture is needed all together. In any case, from (2) and (3) it is clear that for sufficiently small $\Delta t$, our mapping is a contraction mapping. Therefore, by Banach's fixed point theorem, (1a) has a unique solution. The next step is to see if this unique solution satisfies (1b). For this final step, we probably need additional assumptions on $a$.

Approaches that use a fixed point theorem are commonplace when proving the well-posedness of these kind of problems. There are proofs of the Picard-Lindelof theorem on Banach spaces along these lines. Given the weakness of the numerical evidence for Conjecture 3.1, I strongly believe that this is the adequate approach to the analysis presented by the author.

I will leave the revision at this stage for now. If the author is able to convince me that this approach does not work and one must assume Conjecture 3.1, I will present a more complete revision of the paper.