A space-time view of good glacier models

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only a view point

- this is an informal view-point talk
- not a results talk

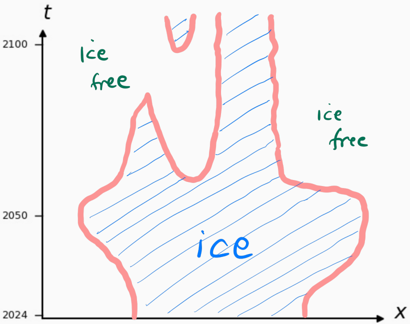
common goals

I am assuming certain goals/attitudes in common with this audience:

1. we care about how the glaciated area evolves in a numerical model
2. good glacier models need to balance membrane/longitudinal stresses

the life of a glacier in space-time

the 3 views


t
2100
2050
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x
ice
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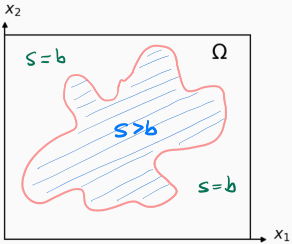
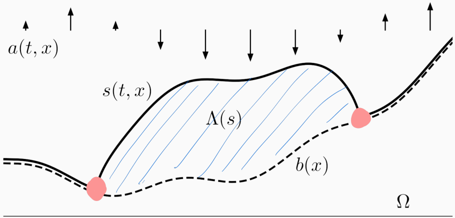

x_2
x_1
Ω
s=b
s>b
s=b


$a(t,x)$
$s(t,x)$
$\Lambda(s)$
$b(x)$
Ω

the (viscous) mathematical model

$$
\begin{aligned}
s - b &\geq 0 && \text{in } \Omega \\
\frac{\partial s}{\partial t} - \mathbf{u}|_s \cdot \mathbf{n}_s - a &\geq 0 && \text{in } \Omega \\
(s-b)\left(\frac{\partial s}{\partial t} - \mathbf{u}|_s \cdot \mathbf{n}_s - a\right) &= 0 && \text{in } \Omega \\
-\nabla \cdot (2\nu(D\mathbf{u})\, D\mathbf{u}) + \nabla p &= \rho_{\text{i}}\mathbf{g} && \text{in } \Lambda(s) \\
\nabla \cdot \mathbf{u} &= 0 && \text{in } \Lambda(s) \\
\nu(D\mathbf{u}) &= \nu_n \left(|D\mathbf{u}|^2 + \epsilon\right)^{q_n} && \text{in } \Lambda(s) \\
(2\nu(D\mathbf{u})D\mathbf{u} - pI)\,\mathbf{n}_s &= \mathbf{0} && \text{on } \Gamma_s \subset \partial\Lambda(s) \\
\mathbf{u} &= \mathbf{0} \text{ or } f(\mathbf{u}, D\mathbf{u}) = 0 && \text{on } \Gamma_b \subset \partial\Lambda(s)
\end{aligned}
$$

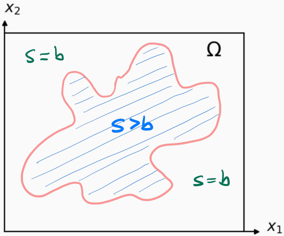
what is true at different points (t, x)?

a) what is *true* in the ice?

b) what is *true* on bare land?

c) what is *true* at the free boundary?

what is true everywhere in Ω?



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(s - b)\left(\frac{\partial s}{\partial t} - \mathbf{u}|_s \cdot \mathbf{n}_s - a\right) &= 0 && \text{in } \Omega
\end{aligned}
$$

this nonlinear complementarity problem first appears in (Calvo et al 2003), for SIA

result (and last slide): an FE error theorem

- you will be disappointed: "result" here means a theorem about size of finite element (FE) errors
- this theorem needs "reasonable" assumptions for a (continuum) geometry-evolving, Stokes model for glaciers … including **conjectured** well-posedness
- *Theorem (Bueler, 2024).* the FE error in computing an updated surface elevation, using an implicit time step, comes from 3 terms:

$$\begin{aligned}\|s_h - s\|^r \le \quad & \frac{c_1}{\Delta t}\int_{\Omega_A(s)} (b-\ell)(b_h - b) \\ & + \Gamma\big\|\mathbf{u}_h - \mathbf{u}\big\| \\ & + c_0\|\Pi_h(s) - s\|^q\end{aligned}$$

- this separates the causes of surface elevation errors:
  1. discretizing the bed elevation (b_h versus exact b)
  2. numerically solving the Stokes equations ($\mathbf{u}_h$ versus exact $\mathbf{u}$)
  3. Cea's lemma for the surface elevation (s_h versus exact s)
     - s necessarily projected to be admissible with respect to b_h

conclusion

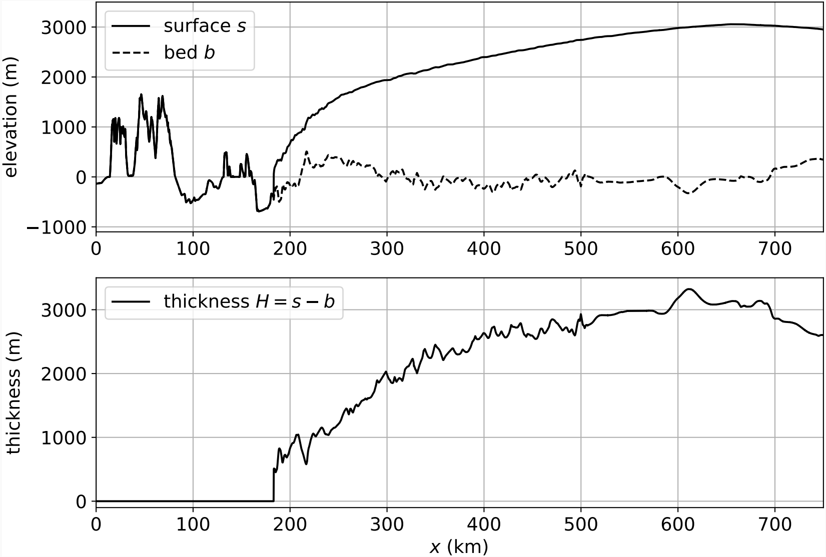
- thank you for listening!
- I'll appreciate any help I can get ...

references

- E. Bueler (2022). *Performance analysis of high-resolution ice-sheet simulations*, J. Glaciol. 69 (276), 930–935 doi:10.1017/jog.2022.113
- E. Bueler (2024). *Surface elevation errors in finite element Stokes models for glacier evolution*, submitted arxiv:2408.06470
- N. Calvo and others (2003). *On a doubly nonlinear parabolic obstacle problem modelling ice sheet dynamics*, SIAM J. Appl. Math. 63 (2), 683–707 doi:10.1137/S0036139901385345
- G. Jouvet & E. Bueler (2012). *Steady, shallow ice sheets as obstacle problems: well-posedness and finite element approximation*, SIAM J. Appl. Math. 72 (4), 1292–1314 doi:10.1137/110856654
- G. Jouvet & J. Rappaz (2011). *Analysis and finite element approximation of a nonlinear stationary Stokes problem . . .*, Adv. Numer. Analysis 2011 (164581) doi:10.1155/2011/164581
- A. Löfgren, J. Ahlkrona & C. Helanow (2022). *Increasing stable time-step sizes of the free-surface problem arising in ice-sheet simulations*, J. Comput. Phys.: X 16 (100114) doi:=10.1016/j.jcpx.2022.100114

extra slides

why is s better than $H = s - b$?

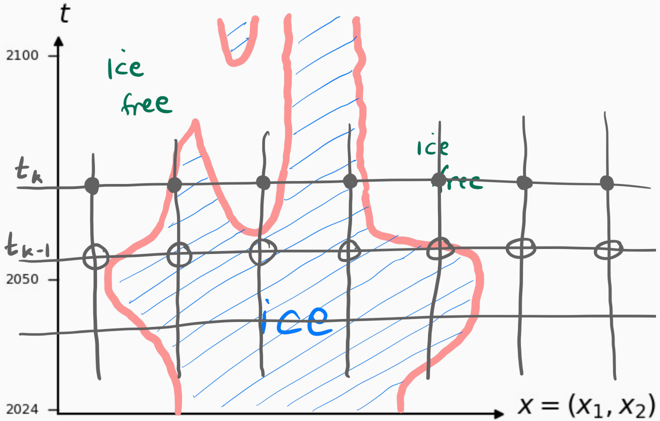


3 benefits of the space-time view

1. (numerical time-stepping) you can see *what "fully-implicit" must mean*
2. (practical modeling) you can see *why modeled surface mass balance must be available on space & time adjacent ice-free land*
3. (mathematical ignorance) you can consider *what determines the evolution of the glaciated area*, and how unclear that is at present

benefit 1: numerical time-stepping

what does "fully-implicit" time-stepping mean?

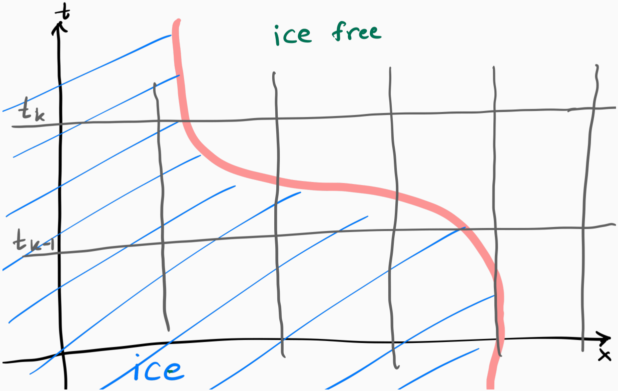


this view appears in (Bueler, 2022)

note appealing semi-implicit idea in (Löfgren et al 2022)

benefit 2: practical modeling

why must modeled surface mass balance be available on *space & time* adjacent ice-free land?



see finite element considerations in (Bueler 2024)

benefit 3: mathematical ignorance

what equations and inequalities, *precisely and mathematically*, determine the evolution of the glaciated area, when using Stokes dynamics?

FIXME

(insert answer from
NWG 2042 participants)

this view appears in (Calvo et al 2003) *and* (Jouvet & Bueler 2012), *but for SIA*

needs fixed-domain Stokes well-posedness from e.g. (Jouvet & Rappaz 2011)

see conjectures in (Bueler 2024)