Finite element errors in Stokes models of glacier evolution

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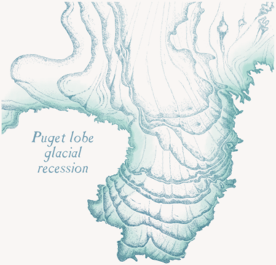
JMM 2025, Seattle

Puget Sound glaciation



Washington State Dept of Natural Resources

glacier evolution in space-time

a) what is true within the ice?

b) what is true on bare land?

c) what is true on free boundary?

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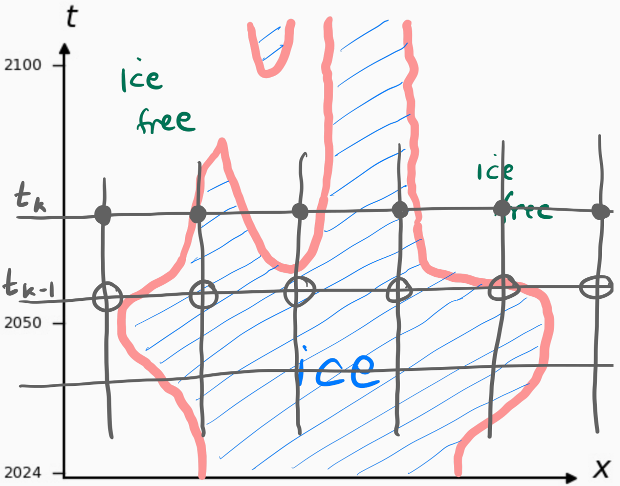
b) what is true on bare land?

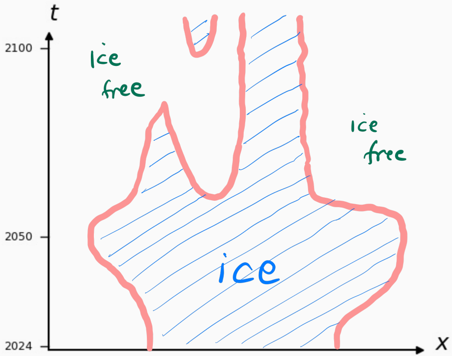
c) what is true on free boundary?

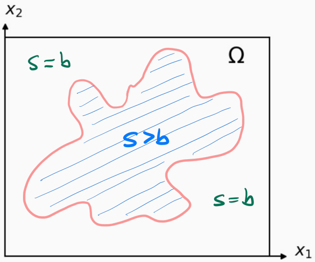


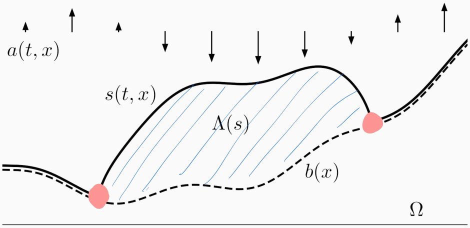
Outline

1. introduction to the standard glacier model
2. well-posed implicit steps?
3. finite element approximation

the 3 views

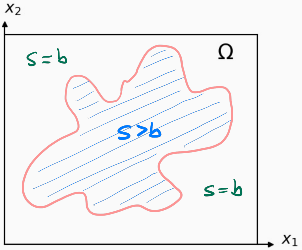

t
2100
2050
2024
x
ice
free
ice
free
ice


x_2
x_1
Ω
s=b
s>b
s=b


$a(t,x)$
$s(t,x)$
$\Lambda(s)$
$b(x)$
Ω

what is true everywhere in $[0, T] \times \Omega$?

- $\Omega \subset \mathbb{R}^2$ fixed simulation domain
- $x = (x_1, x_2) \in \Omega$
- $a(t, x)$ surface mass balance (SMB)
- $b(x)$ bed elevation
- $s(t, x)$ *solution* surface elevation
- $\mathbf{n}_s = (-\nabla s, 1)$ surface-normal vector
- $\mathbf{u}|_s(t, x)$ surface value of ice velocity, extended by zero to bare land



- nonlinear complementarity problem (NCP) in $[0, T] \times \Omega$:

$$
\begin{aligned}
s - b &\geq 0 \\
\frac{\partial s}{\partial t} - \mathbf{u}|_s \cdot \mathbf{n}_s - a &\geq 0 \\
(s - b)\left(\frac{\partial s}{\partial t} - \mathbf{u}|_s \cdot \mathbf{n}_s - a\right) &= 0
\end{aligned}
$$

the free-boundary problem for glacier surface elevation

- surface kinematical equation (SKE) in glaciology:

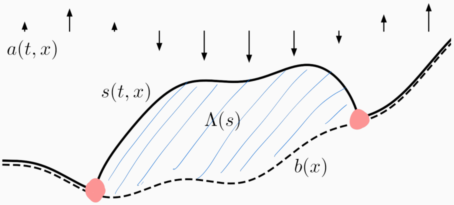
$$\frac{\partial s}{\partial t} - \mathbf{u}|_s \cdot \mathbf{n}_s - a = 0$$

- our time-dependent, nonlinear complementarity problem (NCP) is the underlying *free-boundary* meaning of the SKE:

$$\begin{aligned} s - b &\ge 0 \\ \frac{\partial s}{\partial t} - \mathbf{u}|_s \cdot \mathbf{n}_s - a &\ge 0 \\ (s-b)\left(\frac{\partial s}{\partial t} - \mathbf{u}|_s \cdot \mathbf{n}_s - a\right) &= 0 \end{aligned}$$

  - applies regardless of dynamical model within the ice
  - this NCP appears in (Calvo et al 2003), but only for shallow ice
- state-of-the-art numerically: first-order explicit time-stepping

what is true within the ice?



- fix t and define:

$$\Lambda(s) = \{(x,z)\,:\, b(x) < z < s(t,x)\} \qquad \subset \mathbb{R}^3$$

- Glen-Stokes equations in $\Lambda(s)$ with $\mathrm{p} = (\mathrm{n}+1)/\mathrm{n} \approx 4/3$:

$$-\nabla \cdot (2\nu(D\mathbf{u})\, D\mathbf{u}) + \nabla p = \rho_\mathrm{i}\mathbf{g}$$

$$\nabla \cdot \mathbf{u} = 0$$

$$\nu(D\mathbf{u}) = \nu_0 |D\mathbf{u}|^{\mathrm{p}-2} \qquad \leftarrow \text{needs regularization}$$

- boundary conditions:

$$(2\nu(D\mathbf{u})D\mathbf{u} - pI)\,\mathbf{n}_s = \mathbf{0} \qquad \text{on } \Gamma_s \subset \partial\Lambda(s)$$

$$\mathbf{u} = \mathbf{0} \;\text{ or }\; f(\mathbf{u}, D\mathbf{u}) = 0 \qquad \text{on } \Gamma_b \subset \partial\Lambda(s)$$

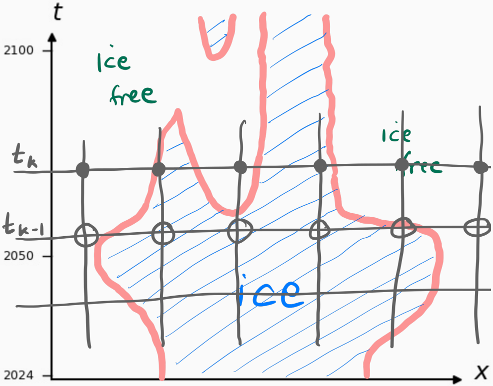
the standard model for glacier evolution

$$
\begin{aligned}
s - b &\geq 0 && \text{in } \Omega \\
\frac{\partial s}{\partial t} - \mathbf{u}|_s \cdot \mathbf{n}_s - a &\geq 0 \\
(s-b)\left(\frac{\partial s}{\partial t} - \mathbf{u}|_s \cdot \mathbf{n}_s - a\right) &= 0 \\
-\nabla \cdot \left(2\nu_0 |D\mathbf{u}|^{\mathrm{p}-2}\, D\mathbf{u}\right) + \nabla p &= \rho_{\mathrm{i}} \mathbf{g} && \text{in } \Lambda(s) \\
\nabla \cdot \mathbf{u} &= 0 \\
\left(2\nu(D\mathbf{u})D\mathbf{u} - pI\right)\mathbf{n}_s &= \mathbf{0} && \text{on } \Gamma_s \subset \partial\Lambda(s) \\
\mathbf{u} = \mathbf{0} \quad \text{or} \quad f(\mathbf{u}, D\mathbf{u}) &= 0 && \text{on } \Gamma_b \subset \partial\Lambda(s)
\end{aligned}
$$

standard model

an NCP in $[0,T] \times \Omega$, for fixed $\Omega \subset \mathbb{R}^2$, coupled to a Glen-Stokes problem within the ice, in $\Lambda(s) \subset \mathbb{R}^3$

the standard model wants "fully-implicit" time-stepping



standard model as a dynamical system

an inequality-constrained **differential algebraic system** in ∞ dimensions

Outline

introduction to the standard glacier model

well-posed implicit steps?

finite element approximation

implicit time-step problem is a variational inequality

- weak form of time-discretized NCP; details found in preprint*
- $\mathcal{X}$ denotes Banach space of elevation functions on Ω; $b \in \mathcal{X}$
- for $\Delta t > 0$ define:

$$\mathcal{K} = \{r \in \mathcal{X} \,:\, r \geq b \text{ a.e. } \Omega\}$$

$$\ell^k(x) = s(t_{k-1}, x) + \int_{t_{k-1}}^{t_k} a(t, x)\, dt$$

$$F(s)[q] = \int_\Omega (s - \Delta t\, \mathbf{u}|_s \cdot \mathbf{n}_s) q$$

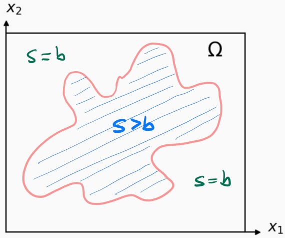
Definition

the *backward Euler time-step problem*, for $t_k \in [0, T]$, is to find the surface elevation $s \approx s(t_k, x) \in \mathcal{K}$ solving the variational inequality (VI)

$$F(s)[r - s] \geq \ell^k[r - s] \quad \text{for all } r \in \mathcal{K}$$

*Bueler (2024). *Surface elevation errors in finite element Stokes models for glacier evolution*, arxiv 2408.06470

well-posedness is only conjectural



Conjecture (B '24)

for some $\mathrm{q} > 2$, *with* $\mathcal{X} = W^{1,\mathrm{q}}(\Omega)$, *and defining the admissible surface elevation subset*

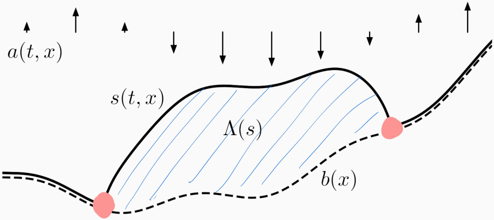
$$\mathcal{K} = \{r \geq b\} \subset \mathcal{X},$$

the backward Euler time-step VI problem,

$$F(s)[r - s] \geq \ell^k[r - s] \quad \textit{for all } r \in \mathcal{K},$$

is well-posed for $s \in \mathcal{K}$

what's behind the conjecture? 1



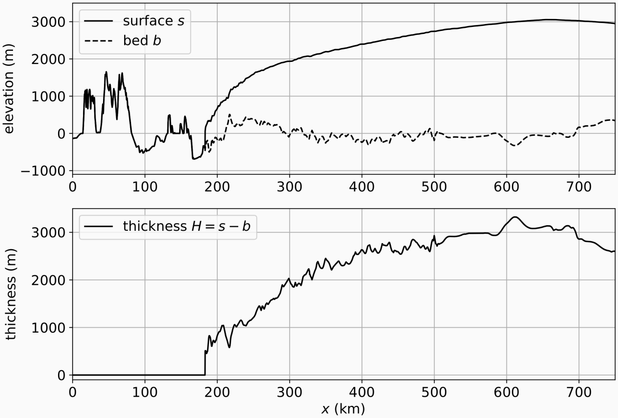
Theorem (Jouvet & Rappaz, 2011)

over a fixed domain Λ, the Glen-Stokes problem

$$
\begin{aligned}
-\nabla\cdot\left(2\nu_0|D\mathbf{u}|^{\mathrm{p}-2}\,D\mathbf{u}\right)+\nabla p &= \rho_i\mathbf{g} && \textit{in } \Lambda \\
\nabla\cdot\mathbf{u} &= 0 \\
\left(2\nu(D\mathbf{u})D\mathbf{u}-pI\right)\mathbf{n}_s &= \mathbf{0} && \textit{on } \Gamma_s\subset\partial\Lambda \\
\mathbf{u}=\mathbf{0}\quad \textit{or}\quad f(\mathbf{u},D\mathbf{u}) &= 0 && \textit{on } \Gamma_b\subset\partial\Lambda
\end{aligned}
$$

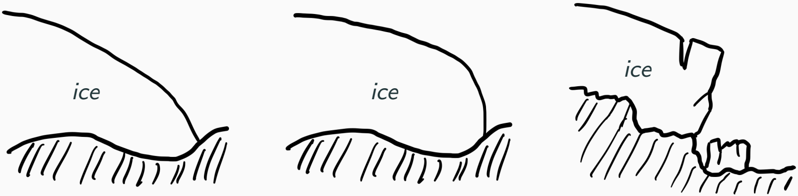
is well-posed for $(\mathbf{u},p)\in W_0^{1,\mathrm{p}}(\Omega;\mathbb{R}^3)\times L^{\mathrm{p}'}(\Omega)$

what's behind the conjecture? 2



- question: use ice surface elevation s or thickness $H = s - b$ for geometry in the standard model?
- answer: prefer s for smoothness reasons

what's behind the conjecture? 3



- the conjecture is that s exists in $\mathcal{X} = W^{1,\mathrm{q}}(\Omega)$ for some $\mathrm{q} > 2$
- however, margin shape of glaciers is a hard modeling problem

what's behind the conjecture? 4

Conjecture (actual; B '24)

For $s \in \mathcal{K} = \{r \geq b\} \subset \mathcal{X} = W^{1,\mathrm{q}}(\Omega)$, for some $\mathrm{q} > 2$, let $\Lambda(s) = \{b < z < s\} \subset \mathbb{R}^3$, and suppose $\mathbf{u}$ solves Glen-Stokes over $\Lambda(s)$. Define

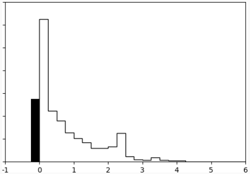
$$\Phi(s) = -\mathbf{u}|_s \cdot \mathbf{n}_s,$$

a map $\Phi : \mathcal{K} \to \mathcal{X}'$. Then Φ is q-coercive: there is $\alpha > 0$ so that

$$(\Phi(s) - \Phi(r))\,[r - s] \geq \alpha \|r - s\|_{\mathcal{X}}^{\mathrm{q}} \qquad \text{for all } r, s \in \mathcal{K}$$

numerical evidence?

- from surface elevation samples r, s, compute numerical ratios $\dfrac{(\Phi(r) - \phi(s))[r - s]}{\|r - s\|_{\mathcal{X}}^{\mathrm{q}}}$



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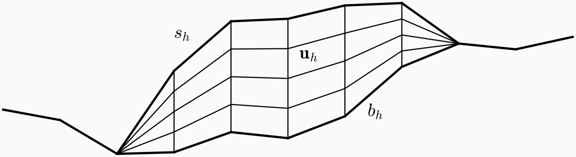
FE method



continuum problem

find $s \in \mathcal{K} = \{r \geq b\}$ *which solves the backward-Euler step VI:*

$$F(s)[r - s] \geq \ell^k[r - s] \quad \textit{for all } r \in \mathcal{K}$$

- proposed FE spaces:
  - $b_h, s_h \in P_1$
  - $\mathbf{u}_h, p_h \in P_2 \times P_1$, on an extruded mesh ← one possibility
- FE method for $s_h \in \mathcal{K}_h = \{r_h \geq b_h\}$:

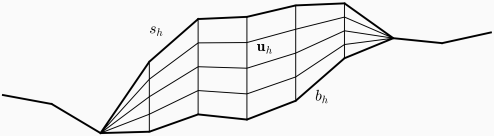
$$F_h(s_h)[r_h - s_h] \geq \ell^k[r_h - s_h] \quad \text{for all } r_h \in \mathcal{K}_h$$

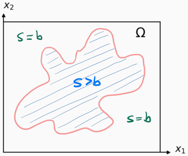
  - mostly conforming, especially if one pretends $b_h = b$

implementation . . . in progress

- extruded-mesh Stokes solver infrastructure: github.com/bueler/stokes-extrude
- solver in Firedrake/PETSc:
  - VI-adapted, reduced-space Newton method
  - for admissible surface elevation iterates $s_h^{(j)} \geq b_h$, each residual evaluation $F(s_h^{(j)})$ requires solving the Glen-Stokes model on $\Lambda(s_h^{(j)})$ to get $\mathbf{u}_h^{(j)}$
- accelerate using new FASCD nonlinear multigrid scheme (Bueler & Farrell, 2024), using above solver as the smoother

error theorem for a backward Euler step





Theorem (B '24)

Make assumptions sufficient for well-posedness. Let $\Omega_A(s)$ be the exact active set, and Π_h the interpolation into $\mathcal{K}_h = \{r_h \geq b_h\}$. Then the FE error in the new (t_k) surface elevation is bounded by 3 terms:

$$\begin{aligned}\|s_h - s\|_{W^{1,q}}^{q} \leq \frac{c_1}{\Delta t} \int_{\Omega_A(s)} &(b - \ell^k)(b_h - b) \\ &+ C(s_h)\|\mathbf{u}_h - \mathbf{u}\|_{W^{1,p}} \\ &+ c_0\|\Pi_h(s) - s\|_{W^{1,q}}^{q'}\end{aligned}$$

Proof. Make the Falk (1974) technique for VIs fully nonlinear. $\square$

summary

- implicit time-stepping makes sense for geometry-evolving, Stokes models for glaciers
  - both a differential-algebraic system and a free-boundary problem
- **conjecture.** for some $\mathrm{q} > 2$, the surface motion $-\mathbf{u}|_s \cdot \mathbf{n}_s$ from the Glen-Stokes problem is q-coercive over admissible $s \in W^{1,\mathrm{q}}(\Omega)$
- **theorem.** supposing q-coercivity, each continuous-space, backward Euler time step problem is well-posed
- **theorem.** supposing well-posedness, the surface elevation FE error is bounded by a sum of 3 terms:
  1. discretizing the bed elevation (b_h versus b)
  2. numerically solving the Stokes equations ($\mathbf{u}_h$ versus $\mathbf{u}$)
  3. Cea's lemma for the surface elevation (s_h versus $\Pi_h(s)$)

E. Bueler (2024). *Surface elevation errors in finite element Stokes models for glacier evolution*, arxiv:2408.06470

references

E. Bueler (2024). *Surface elevation errors in finite element Stokes models for glacier evolution*, arxiv:2408.06470

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