Surface elevation errors in finite element Stokes models for glacier evolution

Ed Bueler, University of Alaska Fairbanks

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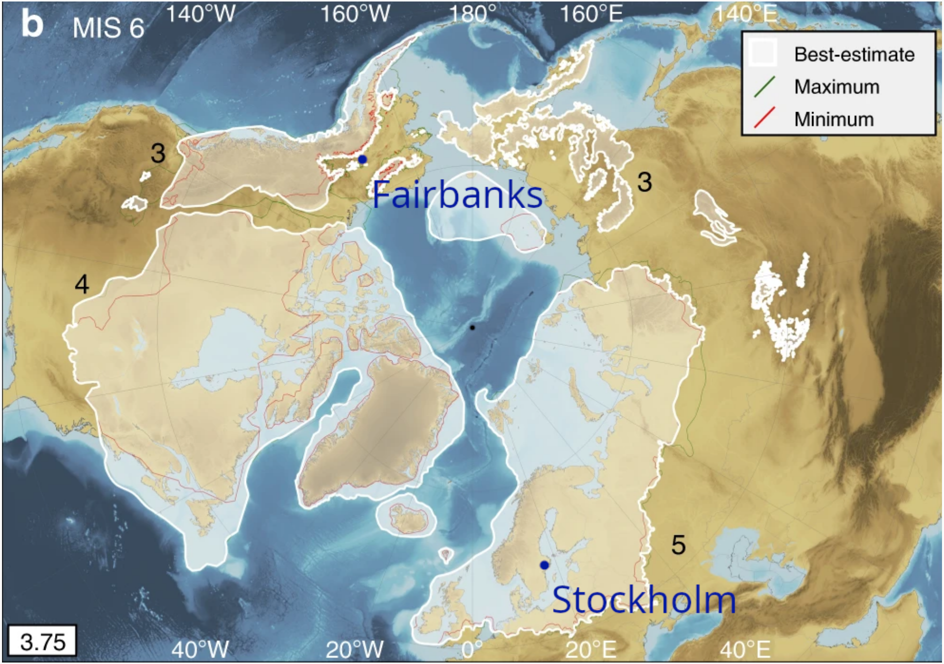


Fig. 1 from Batchelor (2019), *The configuration of Northern Hemisphere ice sheets through the Quaternary*

motivating question about glaciers and ice sheets

- in a given climate and with a given topography,

  what is the extent of glaciation?

  - what is the geometry, namely the surface elevation, of the ice?
  - this is a coupled climate-and-ice-flow problem

- my motivation for finite element solutions of variational inequalities is this free-boundary problem for the surface elevation of glaciers

Outline

1. variational inequalities (VIs)

   (and coercivity)

2. a new *a priori* error bound for finite element methods on VIs

3. the standard glacier model

4. are implicit steps of the standard model well-posed?

   (core issue: is the surface motion q-coercive?)

5. application: *a priori* bound on surface elevation errors

reminder: coercivity for gradients

- let's recall a famous inequality from convex optimization
- let $J : \mathbb{R}^n \to \mathbb{R}$ be a smooth objective function
- assume the Hessian $H(x) = \nabla^2 J(x)$ is uniformly symmetric positive definite (SPD), and thus J is convex

proposition

the gradient $F = \nabla J$ is coercive: there exists $\alpha > 0$ so that

$$(F(x) - F(y)) \cdot (x - y) \geq \alpha \|x - y\|^2$$

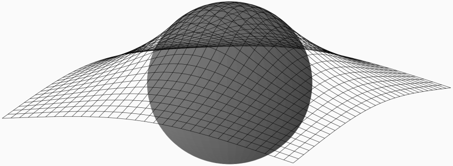
proof. By Taylor expansion, and $H \geq \alpha I$ with $\alpha = \min_x \lambda_{\min}(H(x))$,

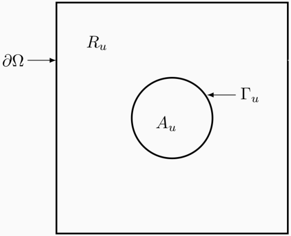
$$J(x) - J(y) \geq F(y) \cdot (x - y) + \frac{\alpha}{2}\|x - y\|^2$$

$$J(y) - J(x) \geq F(x) \cdot (y - x) + \frac{\alpha}{2}\|y - x\|^2$$

Add the above: $\quad 0 \geq (F(y) - F(x)) \cdot (x - y) + \alpha\|x - y\|^2.$ $\square$

example variational inequality: classical obstacle problem



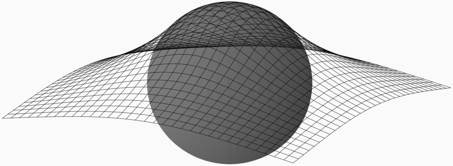


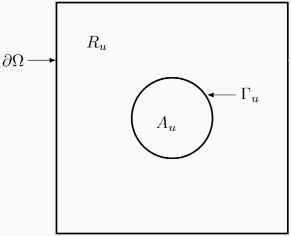
- given: domain $\Omega \subset \mathbb{R}^2$, obstacle ψ, Dirichlet condition g, source φ
- the *admissible set*: $\mathcal{K} = \left\{ v \in H^1(\Omega) \,:\, v\big|_{\partial\Omega} = g \text{ and } v \geq \psi \right\}$
- let $J(v) = \frac{1}{2}\int_\Omega |\nabla v|^2 - \varphi v$, and consider $\min_{v\in\mathcal{K}} J(v)$
- the *variational inequality* (VI) is to find $u \in \mathcal{K}$ so that

$$\int_\Omega \nabla u \cdot \nabla(v-u) - \varphi(v-u) \geq 0 \quad \text{for all } v \in \mathcal{K}$$

  - the weak form operator is $F(u)[v] = \int_\Omega \nabla u \cdot \nabla v - \varphi v$
  - F is coercive on $H^1(\Omega)$

example variational inequality: classical obstacle problem





- the solution defines *active* $A_u = \{u = \psi\}$ and *inactive* $R_u = \{u > \psi\}$ subsets of Ω, and a *free boundary* $\Gamma_u = \partial R_u \cap \Omega$
- the intuitive/naive strong form poses the problem in terms of its solution, a kind of nonsense:

$$-\nabla^2 u = \varphi \quad \text{on } R_u, \qquad \text{and} \qquad u = \psi \quad \text{on } A_u$$

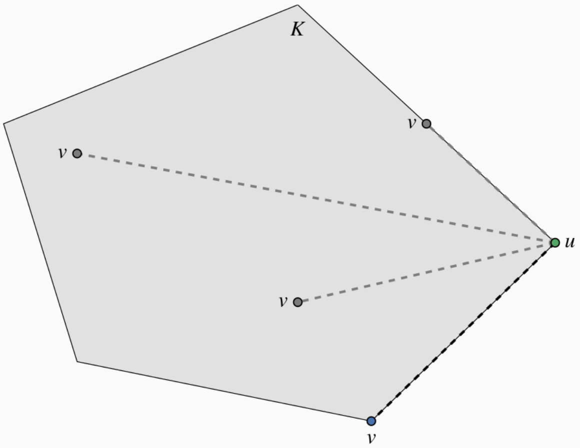
general variational inequalities

- let $\mathcal{X}$ be a real, reflexive Banach space
- let $\mathcal{K} \subset \mathcal{X}$ be a closed and convex subset
- suppose $F : \mathcal{K} \to \mathcal{X}'$ is a continuous operator
    - F may not be the gradient of any objective function J
    - F may be defined only on $\mathcal{K}$
    - F may be nonlinear
- suppose $\ell \in \mathcal{X}'$
- denote the general variational inequality problem as VI(F,ℓ,$\mathcal{K}$):

$$F(u)[v-u] \geq \ell[v-u] \quad \text{for all } v \in \mathcal{K}$$

- if $\mathcal{K}$ is nontrivial, VI(F,ℓ,$\mathcal{K}$) is nonlinear, even when F is linear

why "$v - u$" in the VI?



P. Farrell figure

- $v - u \in \mathcal{X}$ is a vector pointing feasibly into $\mathcal{K}$
- $(F(u) - \ell)[v - u] \geq 0 \quad \Longleftrightarrow \quad$ "$F(u) - \ell$ is within 90° of any $v - u$"

complementarity, a key idea about VIs

- if $u \in \mathcal{K}$ solves the VI

$$F(u)[v-u] \geq \ell[v-u] \quad \text{for all } v \in \mathcal{K},$$

  then generally the residual $F(u) - \ell$ is nonzero
  - if $\mathcal{K} = \mathcal{X}$ (unconstrained) then $F(u) - \ell = 0$
- for a unilateral obstacle problem, with $\mathcal{K} = \{v \in \mathcal{X} \,:\, v \geq \psi\}$, the residual $F(u) - \ell$ is at least nonnegative
  - the residual is a positive measure $d\mu_u = F(u) - \ell$ supported in the active set $A_u = \{x \,:\, u(x) = \psi(x)\}$
- for classical obstacle problem, the strong form statement of complementarity is:

$$u \geq \psi, \quad -\nabla^2 u - \varphi \geq 0, \quad (u-\psi)(-\nabla^2 u - \varphi) = 0$$

variational inequality = constrained equation

unconstrained optimization: $$\min_{u \in \mathcal{X}} J(u)$$	constrained optimization: $$\min_{u \in \mathcal{K}} J(u)$$
equation for $u \in \mathcal{X}$: $$F(u) = \ell$$	?

variational inequality = constrained equation

unconstrained optimization:	constrained optimization:
$$\min_{u\in\mathcal{X}} J(u)$$	$$\min_{u\in\mathcal{K}} J(u)$$
equation for $u \in \mathcal{X}$:	variational inequality for $u \in \mathcal{K}$:
$$F(u)[v] = \ell[v] \quad \forall v \in \mathcal{X}$$	$$F(u)[v-u] \ge \ell[v-u] \quad \forall v \in \mathcal{K}$$

q-coercivity and well-posedness

continuum VI problem

VI for $u \in \mathcal{K}$:

$$F(u)[v-u] \geq \ell[v-u] \quad \forall v \in \mathcal{K}$$

definition

$F : \mathcal{K} \to \mathcal{X}'$ is q-coercive for $\mathrm{q} > 1$ if

$$\left(F(v) - F(w)\right)[v-w] \geq \alpha \|v-w\|^{\mathrm{q}} \qquad \forall v, w \in \mathcal{K}$$

theorem (well-posedness; Kinderlehrer & Stampaccia (1980))

if F is q-coercive and continuous then the continuum VI problem is well-posed

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   (and coercivity)

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   (core issue: is the surface motion q-coercive?)

5. application: *a priori* bound on surface elevation errors

finite element (FE) approximation of the VI

continuum problem

VI for $u \in \mathcal{K}$:

$$F(u)[v-u] \ge \ell[v-u] \quad \forall v \in \mathcal{K}$$

finite element problem

VI for $u_h \in \mathcal{K}_h$:

$$F_h(u_h)[v_h-u_h] \ge \ell[v_h-u_h] \quad \forall v_h \in \mathcal{K}_h$$

conforming assumptions:

- $\mathcal{K}_h \subset \mathcal{X}_h \subset \mathcal{X}$

finite element (FE) approximation of the VI

continuum problem

VI for $u \in \mathcal{K}$:

$$F(u)[v-u] \ge \ell[v-u] \quad \forall v \in \mathcal{K}$$

finite element problem

VI for $u_h \in \mathcal{K}_h$:

$$F_h(u_h)[v_h-u_h] \ge \ell[v_h-u_h] \quad \forall v_h \in \mathcal{K}_h$$

conforming assumptions?

- $\mathcal{K}_h \subset \mathcal{X}_h \subset \mathcal{X}$
- generally $F_h \neq F$

finite element (FE) approximation of the VI

continuum problem

VI for $u \in \mathcal{K}$:

$$F(u)[v-u] \geq \ell[v-u] \quad \forall v \in \mathcal{K}$$

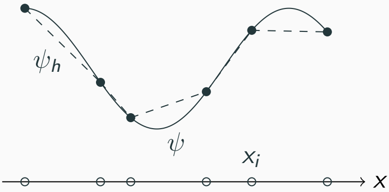
finite element problem

VI for $u_h \in \mathcal{K}_h$:

$$F_h(u_h)[v_h-u_h] \geq \ell[v_h-u_h] \quad \forall v_h \in \mathcal{K}_h$$

conforming assumptions?

- $\mathcal{K}_h \subset \mathcal{X}_h \subset \mathcal{X}$
- generally $F_h \neq F$
- generally $\mathcal{K}_h \not\subseteq \mathcal{K}$



a priori error bound

theorem (B '24)

assume F is q-coercive with $\mathrm{q} > 1$ and $\alpha > 0$, and Lipschitz continuous. then there is $c = c(\|u\|, \|u_h\|, \alpha) > 0$ so that

$$\|u - u_h\|^{\mathrm{q}} \leq \frac{2}{\alpha}\left[\inf_{v \in \mathcal{K}} (F(u) - \ell)[v - u_h] + \inf_{v_h \in \mathcal{K}_h} (F(u) - \ell)[v_h - u]\right] + \frac{2}{\alpha}(F(u_h) - F_h(u_h))[u_h] + c \inf_{v_h \in \mathcal{K}_h} \|v_h - u\|^{\mathrm{q}'}$$

- in the unconstrained case ($\mathcal{K} = \mathcal{X}$), with no variational crimes ($F_h = F$), this is Cea's lemma in a Banach space:

$$\|u - u_h\|^{\mathrm{q}} \leq c \inf_{v_h \in \mathcal{X}_h} \|v_h - u\|^{\mathrm{q}'}$$

a priori error bound

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- if $\mathcal{X}$ is a Hilbert space, $F(u)[v] = (Au, v)$ is linear, $\mathcal{X} \subset \mathcal{H}$ for some Hilbert space $\mathcal{H}$, $\mathrm{q} = 2$, and with no variational crimes ($F_h = F$), then this is Falk's (1974) theorem for VIs:

$$\|u - u_h\|^2 \le \frac{2}{\alpha}\|Au - \ell\|_{\mathcal{H}'}\left(\inf_{v \in \mathcal{K}} \|v - u_h\|_{\mathcal{H}} + \inf_{v_h \in \mathcal{K}_h} \|v_h - u\|_{\mathcal{H}}\right) + c \inf_{v_h \in \mathcal{K}_h} \|v_h - u\|^2$$

a priori error bound

theorem (B '24)

assume F is q-coercive with $\mathrm{q} > 1$ and $\alpha > 0$, and Lipschitz continuous. then there is $c = c(\|u\|, \|u_h\|, \alpha) > 0$ so that

$$\|u - u_h\|^{\mathrm{q}} \le \frac{2}{\alpha}\left[\inf_{v \in \mathcal{K}} (F(u) - \ell)[v - u_h] + \inf_{v_h \in \mathcal{K}_h} (F(u) - \ell)[v_h - u]\right] + \frac{2}{\alpha}(F(u_h) - F_h(u_h))[u_h] + c \inf_{v_h \in \mathcal{K}_h} \|v_h - u\|^{\mathrm{q}'}$$

- for unilateral obstacle problems $\mathcal{K} = \{v \ge \psi\}$, with $d\mu_u = F(u) - \ell$ a positive measure supported in the exact active set A_u, this says

$$\|u - u_h\|^{\mathrm{q}} \le \frac{2}{\alpha}\left(\inf_{v \in \mathcal{K}} \int_{A_u} v - u_h \, d\mu_u + \inf_{v_h \in \mathcal{K}_h} \int_{A_u} v_h - u \, d\mu_u\right) + \frac{2}{\alpha}(F(u_h) - F_h(u_h))[u_h] + c \inf_{v_h \in \mathcal{K}_h} \|v_h - u\|^{\mathrm{q}'}$$

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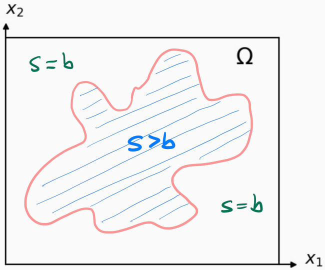
the free-boundary problem for glacier surface elevation

- $\Omega \subset \mathbb{R}^2$ fixed domain
- $a(t,x)$ surface mass balance data
- $b(x)$ bed elevation data
- $s(t,x)$ surface elevation (solution)
- $\mathbf{n}_s = (-\nabla s, 1)$ surface-normal vector
- $\mathbf{u}|_s(t,x)$ surface value of ice velocity, extended by zero to bare land



- an obstacle problem, in strong form, holds in $[0,T] \times \Omega$:

$$
\begin{aligned}
s - b &\geq 0 \\
\frac{\partial s}{\partial t} - \mathbf{u}|_s \cdot \mathbf{n}_s - a &\geq 0 \\
(s-b)\left(\frac{\partial s}{\partial t} - \mathbf{u}|_s \cdot \mathbf{n}_s - a\right) &= 0
\end{aligned}
$$

the free-boundary problem for glacier surface elevation

- free surface equation:

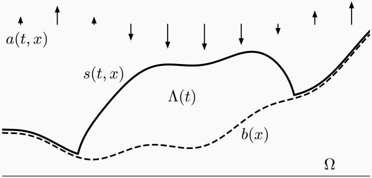
$$\frac{\partial s}{\partial t} - \mathbf{u}|_s \cdot \mathbf{n}_s - a = 0$$

- the complementarity problem is the true free-boundary meaning of the free surface equation:

$$\begin{aligned} s - b &\geq 0 \\ \frac{\partial s}{\partial t} - \mathbf{u}|_s \cdot \mathbf{n}_s - a &\geq 0 \\ (s - b)\left(\frac{\partial s}{\partial t} - \mathbf{u}|_s \cdot \mathbf{n}_s - a\right) &= 0 \end{aligned}$$

  - this applies regardless of dynamical model within the ice
  - the free surface equation holds where there is ice
  - $a \leq 0$ where there is no ice
- this free-boundary problem appears first in (Calvo et al 2003), but only for shallow ice

Glen-Stokes equations within the ice



- define $\Lambda(t) = \{(x, z) \,:\, b(x) < z < s(t, x)\}$
- Glen-Stokes equations in $\Lambda(t)$ with $\mathrm{p} = (\mathrm{n}+1)/\mathrm{n} \approx 4/3$:

$$-\nabla \cdot (2\nu(D\mathbf{u})\, D\mathbf{u}) + \nabla p = \rho_{\mathsf{i}}\mathbf{g}$$
$$\nabla \cdot \mathbf{u} = 0$$

  where $D\mathbf{u} = \frac{1}{2}\left(\nabla\mathbf{u} + \nabla\mathbf{u}^\top\right)$ is the strain-rate tensor
  and $\nu(D\mathbf{u}) = \nu_0 |D\mathbf{u}|^{\mathrm{p}-2}$ is the viscosity
- stress boundary conditions:

$$(2\nu(D\mathbf{u})D\mathbf{u} - pI)\,\mathbf{n}_s = \mathbf{0} \qquad \text{on } \Gamma_s \subset \partial\Lambda(t)$$
$$\beta(\mathbf{u}, D\mathbf{u}) = 0 \qquad \text{on } \Gamma_b \subset \partial\Lambda(t)$$

the standard model for glacier evolution?

$$
\begin{aligned}
s - b &\geq 0 && \text{in } \Omega \\
\frac{\partial s}{\partial t} - \mathbf{u}|_s \cdot \mathbf{n}_s - a &\geq 0 \\
(s-b)\left(\frac{\partial s}{\partial t} - \mathbf{u}|_s \cdot \mathbf{n}_s - a\right) &= 0 \\
-\nabla \cdot \left(2\nu_0 |D\mathbf{u}|^{\mathrm{p}-2}\, D\mathbf{u}\right) + \nabla p &= \rho_{\mathrm{i}} \mathbf{g} && \text{in } \Lambda(t) \\
\nabla \cdot \mathbf{u} &= 0 \\
\left(2\nu_0 |D\mathbf{u}|^{\mathrm{p}-2} D\mathbf{u} - pI\right) \mathbf{n}_s &= \mathbf{0} && \text{on } \Gamma_s \subset \partial\Lambda(t) \\
\beta(\mathbf{u}, D\mathbf{u}) &= 0 && \text{on } \Gamma_b \subset \partial\Lambda(t)
\end{aligned}
$$

standard model

a complementarity problem (obstacle problem) in $[0, T] \times \Omega$, over a fixed $\Omega \subset \mathbb{R}^2$, coupled to a Glen-Stokes problem for $\mathbf{u}$ and p, within the s-dependent ice domain $\Lambda(t) \subset \mathbb{R}^3$; the solution is a triple $(s, \mathbf{u}, p)$

mathematical knowledge about the standard model

- ... is limited, for now, to the fixed-domain Stokes problem

theorem (Jouvet & Rappaz, 2011)

over a fixed C^1 domain $\Lambda \subset \mathbb{R}^3$, the $\mathrm{p} > 1$ Stokes problem

$$
\begin{aligned}
-\nabla \cdot \left(2\nu_0 |D\mathbf{u}|^{\mathrm{p}-2}\, D\mathbf{u}\right) + \nabla p &= \rho_i \mathbf{g} && \textit{in } \Lambda \\
\nabla \cdot \mathbf{u} &= 0 && \\
\left(2\nu_0 |D\mathbf{u}|^{\mathrm{p}-2} D\mathbf{u} - pI\right) \mathbf{n}_h &= \mathbf{0} && \textit{on } \Gamma_s \subset \partial\Lambda \\
\beta(\mathbf{u}, D\mathbf{u}) &= 0 && \textit{on } \Gamma_b \subset \partial\Lambda
\end{aligned}
$$

is well-posed for the solution $(\mathbf{u}, p) \in W^{1,\mathrm{p}}(\Omega; \mathbb{R}^3) \times L^{\mathrm{p}'}(\Omega)$

- thus, at each instant t, if the surface elevation s is known and smooth, then the velocity $\mathbf{u}$ and pressure p are uniquely-determined

the standard model wants implicit time-stepping

$$
\begin{aligned}
s - b &\geq 0 && \text{in } \Omega \\
\frac{\partial s}{\partial t} - \mathbf{u}|_s \cdot \mathbf{n}_s - a &\geq 0 && \\
(s - b)\left(\frac{\partial s}{\partial t} - \mathbf{u}|_s \cdot \mathbf{n}_s - a\right) &= 0 && \\
-\nabla \cdot \left(2\nu_0 |D\mathbf{u}|^{\mathrm{p}-2}\, D\mathbf{u}\right) + \nabla p &= \rho_{\mathrm{i}} \mathbf{g} && \text{in } \Lambda(t) \\
\nabla \cdot \mathbf{u} &= 0 && \\
\left(2\nu_0 |D\mathbf{u}|^{\mathrm{p}-2} D\mathbf{u} - pI\right) \mathbf{n}_s &= \mathbf{0} && \text{on } \Gamma_s \subset \partial\Lambda(t) \\
\beta(\mathbf{u}, D\mathbf{u}) &= 0 && \text{on } \Gamma_b \subset \partial\Lambda(t)
\end{aligned}
$$

standard model (as a dynamical system)

the model is an inequality-constrained differential algebraic equation (DAE) system in ∞ dimensions

- implicit methods are the usual recommendation for the infinitely-stiff limit of ODE systems, namely DAEs

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a single implicit time step of the standard model

- let $\Delta t > 0$ and denote $s \approx s(t_k)$ and $s^{k-1} \approx s(t_{k-1})$
- change notation: $\Lambda(s) = \{(x,z)\,:\, b(x) < z < s(x)\}$
- consider a backward Euler time step of the non-sliding model:

$$
\begin{aligned}
s - b &\geq 0 && \text{in } \Omega \\
s - \Delta t\, \mathbf{u}|_s \cdot \mathbf{n}_s - \ell &\geq 0 && \\
(s-b)\,(s - \Delta t\, \mathbf{u}|_s \cdot \mathbf{n}_s - \ell) &= 0 && \\
-\nabla \cdot \left(2\nu_0 |D\mathbf{u}|^{\mathrm{p}-2}\, D\mathbf{u}\right) + \nabla p &= \rho_{\mathrm{i}} \mathbf{g} && \text{in } \Lambda(s) \\
\nabla \cdot \mathbf{u} &= 0 && \\
\left(2\nu_0 |D\mathbf{u}|^{\mathrm{p}-2} D\mathbf{u} - pI\right) \mathbf{n}_s &= \mathbf{0} && \text{on } \Gamma_s \subset \partial\Lambda(s) \\
\mathbf{u} &= 0 && \text{on } \Gamma_b \subset \partial\Lambda(s)
\end{aligned}
$$

- with source term

$$\ell(x) = s^{k-1}(x) + \Delta t \int_{t_{k-1}}^{t_k} a(t,x)\, dt$$

- the solution, if it exists, is a triple $(s, \mathbf{u}, p)$ for the new time t_k

the surface motion term

- regarding the questions of well-posedness and surface elevation errors, we focus on the key term

definition

the surface motion in the standard model: $\Phi(s) = -\mathbf{u}_s \cdot \mathbf{n}_s$

- ...and on the key question

question

is the surface motion $\Phi(s)$ q*-coercive?*

are there $\mathrm{q} > 1$ *and* $\alpha > 0$ *so that*

$$(\Phi(s) - \Phi(\sigma))[s - \sigma] \overset{?}{\geq} \alpha \|s - \sigma\|_{\mathcal{X}}^{\mathrm{q}}$$

for all $s, \sigma \in \mathcal{K} = \{\omega \in \mathcal{X} \,:\, \omega\big|_{\partial\Omega} = b\big|_{\partial\Omega}$ *and* $\omega \geq b\}$, *where* $\mathcal{X}$ *is a Banach space to be determined?*

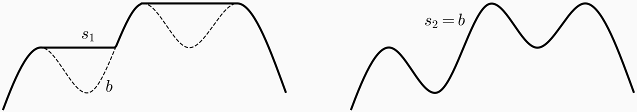
is the surface motion coercive?

question

$$(\Phi(s) - \Phi(\sigma))[s - \sigma] \overset{?}{\geq} \alpha \|s - \sigma\|_{\mathcal{X}}^{q}$$

- the answer is no for general bumpy beds
- for surfaces $s_1, s_2 \in \mathcal{K}$ below we compute

$$\Phi(s_1) = 0$$
$$\Phi(s_2) = 0$$
$$\|s_1 - s_2\|_{\mathcal{X}} > 0$$



- neither surface generates flow when we solve Stokes over $\Lambda(s_i)$
  - note $\Lambda(s_2) = \emptyset$

is the regularized surface motion coercive?

question

$$(\Phi^{\epsilon}(s) - \Phi^{\epsilon}(\sigma))[s - \sigma] \overset{?}{\geq} \alpha^{\epsilon} \|s - \sigma\|_{\mathcal{X}}^{q}$$

- for $\epsilon > 0$ small and H_0 comparable to ice thickness, define

$$\Phi^{\epsilon}(s) = (u|_s, v|_s) \cdot \nabla s - (1 - \epsilon) w|_s - \epsilon \nabla \cdot \left(\Gamma H_0^5 |\nabla s|^2 \nabla s\right)$$

- this regularization of the surface value of the vertical velocity, using the shallow ice approximation formula, breaks the symmetry on the last slide
    - it prefers flat ice surfaces
- $\epsilon = 0$ case returns $\Phi(s)$:

$$\Phi^{0}(s) = (u|_s, v|_s) \cdot \nabla s - w|_s = -\mathbf{u}_s \cdot \mathbf{n}_s$$

is the regularized surface motion coercive?

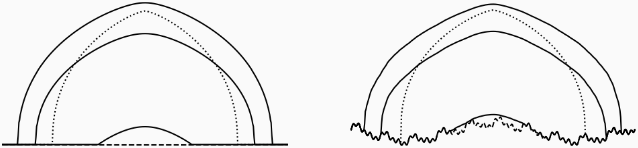
question

$$(\Phi^\epsilon(s) - \Phi^\epsilon(\sigma))[s-\sigma] \overset{?}{\geq} \alpha^\epsilon \|s-\sigma\|_{\mathcal{X}}^{\mathrm{q}}$$

- a numerical experiment, computing these ratios for $\epsilon = 0.1$ and $H_0 = 1000$ m,

$$\frac{(\Phi^\epsilon(s_1) - \Phi^\epsilon(s_1))[s_1 - s_2]}{\|s_1 - s_2\|_{\mathcal{X}}^4}$$

  gives evidence of $\mathrm{q} = 4$-coercivity
- randomly chosen pairs $s_1, s_2 \in W^{1,4}(\Omega)$ from 10^3 states which were generated using FSSA acceleration (Löfgren et al. 2022)

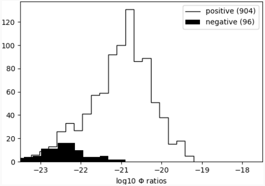


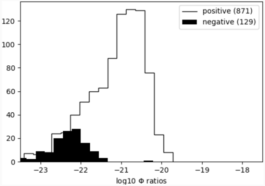
is the regularized surface motion coercive?

question

$$(\Phi^{\epsilon}(s) - \Phi^{\epsilon}(\sigma))[s - \sigma] \overset{?}{\geq} \alpha^{\epsilon}\|s - \sigma\|_{\mathcal{X}}^{q}$$

- ratios without regularization:





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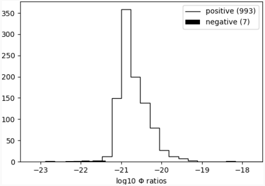

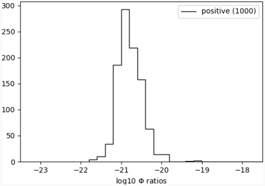

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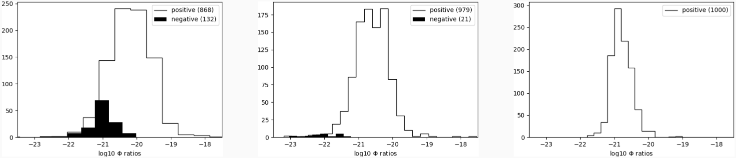
$$(\Phi^{\epsilon}(s) - \Phi^{\epsilon}(\sigma))[s - \sigma] \overset{?}{\geq} \alpha^{\epsilon}\|s - \sigma\|_{\mathcal{X}}^{q}$$

- mesh refinement ($\Delta x = 2$ km, 1 km, 500 m) eliminates negative ratios:



- perhaps $\alpha_{\epsilon} \sim 10^{-21}\,\mathrm{m}^{9/4}\,\mathrm{s}^{-1}$?

- define

$$F^\epsilon(\sigma)[\omega] = \int_\Omega \left(\sigma + \Delta t\, \Phi^\epsilon(\sigma)\right)\omega$$

definition

the weak form backward Euler time-step problem is to find the surface elevation $s \approx s(t_k, x)$ in $\mathcal{K} = \{\sigma \,:\, \sigma \geq b \text{ and } \sigma|_{\partial\Omega} = b_{\partial\Omega}\} \subset \mathcal{X}$, where $\mathcal{X} = W^{1,4}(\Omega)$, solving the VI

$$F^\epsilon(s)[\sigma - s] \geq \ell[\sigma - s] \quad \text{for all } \sigma \in \mathcal{K}$$

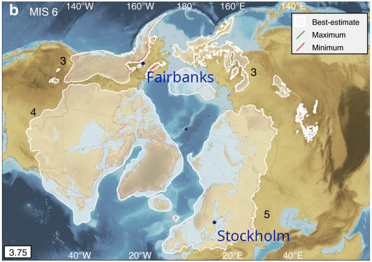
well-posedness is only conjectural

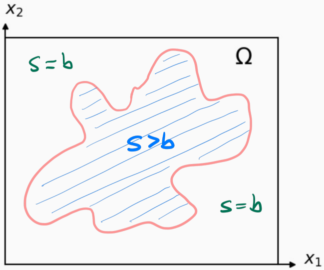
conjecture (B '24)

Φ^ϵ is 4-coercive,

so the backward Euler time-step problem is well-posed for s

- this is a license to go hunting for a numerical solution to the glaciation problem





Outline

1. variational inequalities (VIs)

   (and coercivity)

2. a new *a priori* error bound for finite element methods on VIs

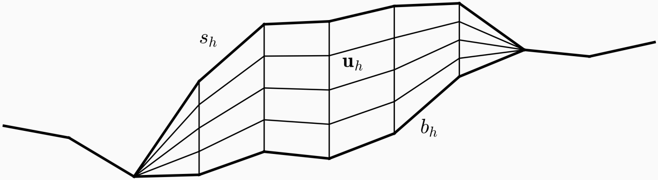
3. the standard glacier model

4. are implicit steps of the standard model well-posed?

   (core issue: is the surface motion q-coercive?)

5. application: *a priori* bound on surface elevation errors

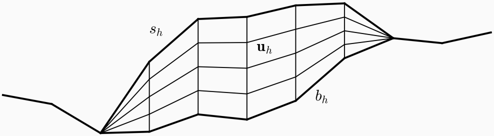
FE method

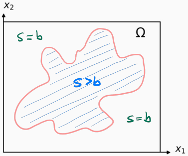


- proposed, simplest FE spaces:
    - an extruded mesh
    - $b_h, s_h \in P_1$ in 2D, over Ω
    - $\mathbf{u}_h, p_h \in P_2 \times P_1$ in 3D, over $\Lambda(s_h)$
- discrete admissible set: $\mathcal{K}_h = \{\sigma_h \,:\, \sigma_h \geq b_h \text{ and } \sigma_h|_{\partial\Omega} = b_h|_{\partial\Omega}\}$
- FE method for $s_h \in \mathcal{K}_h$:

$$F_h(s_h)[\sigma_h - s_h] \geq \ell[\sigma_h - s_h] \quad \text{for all } \sigma_h \in \mathcal{K}_h$$

bound on surface elevation errors for implicit step





theorem (B'24)

Suppose Φ^ϵ is 4-coercive in $\mathcal{X} = W^{1,4}(\Omega)$. In discretizing the bed, ensure that $b_h \geq b$. Let $\Omega_A(s)$ be the exact active set, the ice-free area. Let Π_h be interpolation and truncation $\mathcal{K} \to \mathcal{K}_h$. Then the error in the FE surface elevation $s_h \in \mathcal{K}_h$ is bounded by 3 terms:

$$\begin{aligned}\|s_h - s\|_{\mathcal{X}}^4 \leq \frac{c_0}{\Delta t} \int_{\Omega_A(s)} &(b - \ell)(b_h - b) \\ &+ c_1(s_h) \|\mathbf{u}_h - \mathbf{u}\|_{W^{1,4/3}(\Lambda(s_h))} \\ &+ c_2 \|\Pi_h(s) - s\|_{\mathcal{X}}^{4/3}\end{aligned}$$

bound on surface elevation errors for implicit step

theorem (B'24)

$$\begin{aligned}\|s_h - s\|_{\mathcal{X}}^4 \le\ &\frac{c_0}{\Delta t}\int_{\Omega_A(s)} (b-\ell)(b_h - b) \\ &+ c_1(s_h)\left\|\mathbf{u}_h - \mathbf{u}\right\|_{W^{1,4/3}(\Lambda(s_h))} \\ &+ c_2\|\Pi_h(s) - s\|_{\mathcal{X}}^{4/3}\end{aligned}$$

Proof. Apply the general *a priori* theorem. Of the four terms, the "$\inf_{v\in\mathcal{K}}$" term can be replaced by zero because $\mathcal{K}_h \subset \mathcal{K}$ from the bed construction. Estimate the "$\inf_{v_h\in\mathcal{K}_h}$" term for the residual by considering the residual measure; it simplifies to $d\mu_u = b - \ell$ in $\Omega_A(s)$. Estimate the "$(F(u_h) - F_h(u_h))[u_h]$" term by bounding the surface trace of the Stokes velocity solution. Estimate the Cea's lemma term in the usual interpolation way, but remember to truncate into $\mathcal{K}_h$. $\square$

summary

- implicit time-stepping for variational inequalities is needed for the geometry-evolving Stokes model for glaciers
    - both a differential-algebraic system and a free-boundary problem

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  - implies that each continuous-space, backward Euler time step problem is well-posed
- **theorem.** supposing the conjecture, the FE surface elevation error, in a backward Euler step of the standard glacier model, is bounded by a sum of terms:
  1. error in discretizing the bed elevation (b_h versus b)
  2. error in numerically solving the Stokes equations ($\mathbf{u}_h$ versus $\mathbf{u}$)
  3. a Cea's lemma term for the surface elevation (s_h versus $\Pi_h(s)$)

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