(BUELER; JUNE 12, 2012)

PROJECT 10: MINIMAL MODELS OF SUBGLACIAL HYDROLOGY

Summary

The Kennicott glacier by McCarthy shows how important subglacial liquid water is to how ice flow varies in time and space. Combining a subglacial water layer model with an ice flow model, though too hard for a project this week, at least requires a model of subglacial water pressure. Here we "extreme" models of basal water pressure, which are nevertheless representative of ideas in the literature. We consider their numerical implementation and consequences, using realistic ice geometry but in isolation from the ice flow.

Any reasonable model of the aquifer has at least these two elements: liquid water is conserved and water flows from high to low hydraulic potential ("head"). Additionally, physical processes control the geometry of the aquifer/layer (e.g. cavities open by sliding, cavities/channels close by creep, channels open by melting, sediment moves, ...), but we do not model these. Instead I suggest we model water pressure by two highly-simplified models which I think of as being at the extremes. On the one extreme, we can assume the water pressure is equal to the overburden pressure, and we can look at the consequences. On the other extreme we can assume the water pressure is given by a power law relating to the water amount. How do we solve these extreme models numerically? How do we test these numerical models? Do these models have steady states, and if so, what are the water distributions?

The following is a outline by which the project might be approached. Different paths are encouraged!

A) This project depends minimally on the content of my lectures, which occur late in the school (on Monday June 18). But you can get from me a copy of the lecture slides `lecture.pdf`, this document `project10.pdf`, and the Matlab codes. All of these are inside the archive

`mccarthy-bueler.zip`

Please browse the slides and refer back to them as needed. Talk with me about the ideas in these materials, especially the discussions of numerically solving the heat equation and the SIA equation for thickness (which is based on mass continuity). These ideas are most relevant to "Extreme Model 2" below.

B) Listen to Bob Anderson's talks about glacial hydrology, and read the chapter of his book which was distributed to all students.

C) Read the supplied PDF, a wonderful review paper:

G. K. C. Clarke, 2005. *Subglacial processes*, Annu. Rev. Earth Planet. Sci. **33**, pp 247–276

Especially pay attention to sections 3.1 and 4.2.1. Note that he does not say much about what the liquid water pressure depends on; that is a hard question! There is also a modeling paper which I will connect to below:

G. E. Flowers and G. K. C. Clarke, 2002. *A multicomponent coupled model of glacier hydrology 1. Theory and synthetic examples*, J. Geophys. Res. 107 (B11), 2287, doi:10.1029/2001JB001122

D) Describe in physical terms how these extremes might be achieved:
  1. the water pressure is equal to the overburden pressure (equation (4)), and
  2. the water pressure is a power of the water amount (equation (5)).

Where in the cryosphere might they be achieved? (Do you regard these as extremes?)

E) *Subglacial hydrology element 1*: Water is conserved.

If we regard the water as being in a layer, whether thick or thin, we can keep track of its thickness $W(t,x)$ in one spatial dimension by the equation

$$W_t + Q_x = \Phi \tag{1}$$

where Q is the water flux, with SI units $\mathrm{m}^2\,\mathrm{s}^{-1}$, and Φ is a source term with units $\mathrm{m\,s}^{-1}$. The layer thickness W here is only likely to be meaningful, however, if it is regarded as an average over a horizontal scale of tens to thousands of meters, because of the fine spatial variation which one is unlikely to be able to model. We will attempt only to model spatially-averaged versions of water amount and water pressure.

Justify why equation (1) is a conservation statement. Solve equation (1) in the case $Q = 0$ and $\Phi(x)$ is known. (That is: no lateral flux of water.) On the other hand, describe the steady state case where $W_t = 0$; how does a "flux boundary condition" come in?

F) (*Just a comment.*) We might separate the water sources between the melt on the lower surface of the glacier and the en- or supra-glacial drainage origin,

$$\Phi = \rho_w^{-1}\,(m + S)$$

where ρ_w is the density of fresh liquid water, m is the rate at which basal melting (refreeze) of ice adds (removes) water, and S is the rate at which surface runoff or englacial drainage adds water. But this split is not really critical; for the first part of this project let's take Φ to be constant, $\Phi = \Phi_0$. Note m and S have units $\mathrm{kg\,m}^{-2}\,\mathrm{s}^{-1}$ while Φ has units $\mathrm{m\,s}^{-1}$. (Here the density is mass-per-volume and the fluxes m, S are mass rates per area. This gives Φ in the right units whether or not we have flow-line geometry.)

G) *Subglacial hydrology element 2*: There is a hydraulic potential.

We will relate the water flux Q in equation (1) to the gradient of the hydraulic potential $\psi(t,x)$. By definition, ψ combines the actual water pressure

$P(t,x)$ and the gravitational potential corresponding to that mass of water at the location on the bed of the glacier,

(2) $$\psi = P + \rho_w g\, b.$$

Here $z = b(x)$ is the bedrock elevation. We will assume that both the ice thickness $H(x)$ and the bed elevation $b(x)$ are given, time-independent data.

H) *Subglacial hydrology element 3*: Water flows from high to low hydraulic potential. The simplest such is to assume a water sheet, which gives equation

(3) $$Q = -\frac{K\,W}{\rho_w g}\psi_x$$

(Clarke 2005). Here, ρ_w is the water density, g the gravitational acceleration and K the effective hydraulic conductivity. The transmissivity of the system is really the product KW. Notice that the system transmits more water for a given head gradient if either the holes are bigger (K is larger) or the water sheet is thicker (W is larger). The main point here: By (3), water flows from high to low fluid potential.

I) (*Another comment*.) From equations (2) and (3), flow depends on both horizontal gradients in the water pressure and on the bedrock slope. However, because the bedrock elevation comes from rough data in practice, the hydraulic potential is not actually very smooth. The gradient $\nabla\psi$ will be seen to have very large spatial variability in practice.

J) Combine equations (1), (2), and (3) in a single equation, eliminating symbols Q and ψ, to get a new equation $W_t + \dots = \dots$. If you had initial and boundary values for W and P, and all other (remaining) symbols are chosen constants, could you make predictions from this new equation?

K) *Extreme model 1*: Water pressure P equals the overburden pressure.

The ice is a fluid which has a pressure field of its own. At the base of the ice we denote this as P_i, the *overburden pressure*. We make, without worrying further[1], that this ice pressure is hydrostatic ("cryostatic"):

$$P_i = \rho_i g H.$$

Here ρ_i is the density of ice and H is the ice thickness. The extreme model is

(4) $$P = P_i$$

L) Combine equations (1), (2), (3), and (4) in a single equation, eliminating symbols Q, P and ψ. If we identify the water amount W as the primary unknown, what order—i.e. how many partial derivatives?—is this differential equation? What are the correct boundary conditions in a one-dimensional version of the new equation?

[1] You should rethink this issue if you are solving the Stokes equations in highly-variable ice flows.

M) In fact, let's now choose some data for the the ice sheet surface and bed elevation. For East Antarctica I have written a mfile `getEAIStransect.m`, which calls modest data processing scripts `buildant.m`, `netcdf.m` to extract and plot surface elevation and bed elevation. You will also need a 100Mb NetCDF file called `Antarctica_5km_dev1.0.nc`, which I can supply. As an admittedly-modest alteration, modify these to plot overburden pressure for these geometry data. Compute the hydraulic potential assuming extreme model (4), and then show Q in some manner (arrows?). If the flow is down the overburden pressure gradient, where will lakes occur?

N) *Extreme model 2*: More water implies higher pressure, and W and P are related by a power law.

The extreme model here is from (Flowers and Clarke, 2002), for example,

$$P = P_0 \left(\frac{W}{W_0} \right)^{\gamma} \tag{5}$$

Let's choose the parameters P_0, W_0, and $\gamma > 1$ all to be constant for this project.[2] The meaning of W_0 is a "critical water thickness", so that if $W > W_0$ then the pressure grows very fast, at least if $\gamma > 1$. Typical values (Flowers and Clarke, 2002) are $\gamma = 7/2$ and $W_0 = 10$ cm.

What are the units of P_0 and γ? Explain physically: in what kinds of subglacial systems would pressure P increase with increasing water amount W?

O) Combine equations (1), (2), (3), and (5) in a single equation, eliminating symbols Q, P and ψ. If we identify the water amount W as the primary unknown, what order—i.e. how many partial derivatives?—is this differential equation? What are the correct boundary conditions in a one-dimensional version of the new equation?

P) Build a numerical model of the equation from Extreme Model 1. It might be very helpful to consider the abstract form

$$u_t + (vu)_x = f.$$

for $v(x)$ and $f(x)$ given. How would you test your numerical scheme? (*Here we actually get our hands dirty in numerical methods. Talk to me about finite difference schemes and boundary conditions.*)

Q) Using representative values, use Extreme Model 1 to compute a solution using the EAIS transect geometry. Are there steady states?

R) Build a numerical model of the equation from Extreme Model 2. It might be very helpful to consider the abstract form

$$u_t + C_0(u(u^{\gamma})_x)_x = f$$

[2] See below for the more realistic version with $P_0 = P_i$.

(which is a good analogy only with flat bed). You can even change variables $Y = W^{\gamma+1}$ and consider the abstract form

$$(u^{1/\gamma})_t + C_1(u)_{xx} = f.$$

How would you test your numerical scheme? (*I.e. get our hands dirty in numerical methods. Talk to me.*)

S) Get from me the Haut Glacier d'Arolla data for surface and bed elevation, on a 100 m grid; this data was used in the ISMIP-HOM intercomparison but here we just use it as representative mountain glacier geometry. Compute a steady state from Extreme Model 2 using reasonable boundary conditions.

T) I have been a bit unfair to "Extreme Model 2". Really we should make the pressure scale with the overburden pressure,

$$P = P_i \left(\frac{W}{W_0}\right)^{\gamma} \tag{6}$$

What are the consequences to Extreme Model 2 if we use equation (6) instead of (5)?

U) Present your results on Tuesday June 19.