

Numerical
modelling

Ed Bueler

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Numerical modelling of ice sheets, streams, and shelves

Ed Bueler

Dept of Mathematics and Statistics
and Geophysical Institute
University of Alaska, Fairbanks

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Karthaus Summer School

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slogans:

- focus on approximating ice flow
- example numerical codes that actually work

scope:

- continuum models
 - shallow ice approximation (SIA) in 2D
 - shallow shelf approximation (SSA) in 1D
 - mass continuity & surface kinematical equations
- numerical ideas
 - finite difference schemes
 - solving algebraic systems from stress balances
 - verification

notation: see printed notes

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not covered here:

- Stokes and “higher order” flow equations
- thermomechanical coupling or polythermal ice
- subglacial hydrology/processes
- mass balance and snow/firn processes
- constitutive relations other than Glen isotropic
- grounding lines, calving fronts, ocean interaction
- paleo-climate and “spin-up”
- earth deformation under ice sheet load
- other numerics: FEM, spectral, multigrid, parallel, ...
- etc.

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- lectures are structured around 16 Matlab codes
- they also work in Octave
- each is 1/2 to one page
- .zip and .tar.gz forms available from memory stick
- online:

<http://www.dms.uaf.edu/~bueler/karthaus/>

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- my goal is to get to an equation for which I can say:

*numerically solve just this equation, and you've got a usable
model for a flowing ice sheet*

- a “usable” model is *understood* more than it is *correct*
- to get to my goal I will quickly recall some continuum mechanics

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- we describe fluids primarily by a *velocity field* $\mathbf{u}(t, x, y, z)$
 - if the ice fluid were
 - faster-moving than it actually is, and
 - linearly-viscous like liquid water
- then ice flow would be a “typical” fluid
- we would use the Navier-Stokes equations as our flow model:

$$\nabla \cdot \mathbf{u} = 0 \quad \textit{incompressibility}$$

$$\rho (\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u}) = -\nabla p + \nu \nabla^2 \mathbf{u} + \rho \mathbf{g} \quad \textit{force balance}$$

- so, to numerically model our fluid, we grab a textbook on “computational fluid dynamics”
- right?

hmmm . . . does not sound like glaciology to me!

is numerical ice flow modeling a part of computational fluid dynamics?

- yes
- large scale like atmosphere/ocean
- . . . but it is a weird one
- consider what makes atmosphere/ocean flow modeling exciting:
 - turbulence
 - convection
 - coriolis force
 - density variation
 - chemistry (methane, ozone, . . .)
- none of the above is very relevant to ice flow
- so what could be interesting about the flow of slow, cold, stiff, laminar, inert old ice?

. . . it's *ice dynamics!*

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- our fluid is

slow:

$$\rho(\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u}) \approx 0$$

non-Newtonian: viscosity ν is not constant

- “slow”:

$$\rho(\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u}) \approx 0$$

 \iff

(forces of inertia)
are negligible

- “non-Newtonian” flow: it is “shear-thinning”, so larger strain rate means smaller viscosity
- so the standard ice flow model is Glen-law ($n = 3$) Stokes:

$$\nabla \cdot \mathbf{u} = 0$$

incompressibility

$$0 = -\nabla p + \nabla \cdot \boldsymbol{\tau}_{ij} + \rho \mathbf{g}$$

force balance

$$\mathbf{D}\mathbf{u}_{ij} = A\tau^2 \boldsymbol{\tau}_{ij}$$

flow law

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- equations on previous slide are true at every instant
- *geometry, boundary stress, and ice viscosity determine velocity field instantaneously*
- a time-stepping ice sheet code recomputes the velocity field at every time step, without requiring velocity from the previous step¹
- thus no memory of previous momentum/velocity, so velocity is a “diagnostic” output of an ice flow model

¹to be a weatherman you've got to know which way the wind blows . . . but don't expect that from a glaciologist

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- in the x, z plane flow case the Stokes equations say

$$u_x + w_z = 0 \quad \text{incompressibility}$$

$$p_x = \tau_{11,x} + \tau_{13,z} \quad \text{stress balance (x)}$$

$$p_z = \tau_{13,x} - \tau_{11,z} - \rho g \quad \text{stress balance (z)}$$

$$u_x = A\tau^2 \tau_{11} \quad \text{flow law (diagonal)}$$

$$u_z + w_x = 2A\tau^2 \tau_{13} \quad \text{flow law (off-diagonal)}$$

- x, z subscripts are partial derivatives
- τ_{13} is a “vertical” shear stress
- τ_{11} and $\tau_{33} = -\tau_{11}$ are deviatoric longitudinal stresses
- we have five equations in five unknowns ($u, w, p, \tau_{11}, \tau_{13}$)
- complicated enough ...
- what about in a simplified situation?

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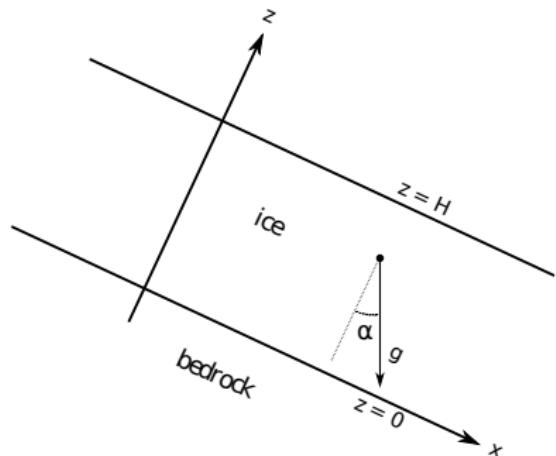
- rotated coordinates:

$$\mathbf{g} = g \sin \alpha \hat{x} - g \cos \alpha \hat{z}$$

- so p_x, p_z equations are now:

$$p_x = \tau_{11,x} + \tau_{13,z} + \rho g \sin \alpha$$

$$p_z = \tau_{13,x} - \tau_{11,z} - \rho g \cos \alpha$$



- for a slab-on-a-slope there is *no variation in x*
- so equations simplify:

$$w_z = 0$$

$$0 = \tau_{11}$$

$$\tau_{13,z} = -\rho g \sin \alpha$$

$$u_z = 2A\tau^2 \tau_{13}$$

$$p_z = -\rho g \cos \alpha$$

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- add some boundary conditions:

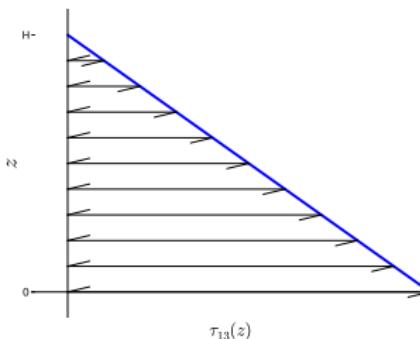
$$w(\text{base}) = 0, \quad p(\text{surface}) = 0, \quad u(\text{base}) = u_0$$

- by integrating vertically, get :

$$w = 0, \quad p = \rho g \cos \alpha (H - z), \quad \tau_{13} = \rho g \sin \alpha (H - z)$$

- and from " $u_z = 2A\tau^2\tau_{13}$ " get

$$\begin{aligned} u(z) &= u_0 + 2A(\rho g \sin \alpha)^3 \int_0^z (H - z')^3 dz' \\ &= u_0 + \frac{1}{2} A(\rho g \sin \alpha)^3 (H^4 - (H - z)^4) \end{aligned}$$



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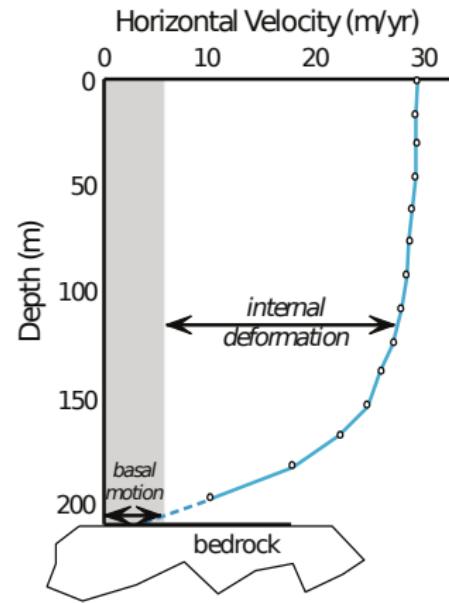
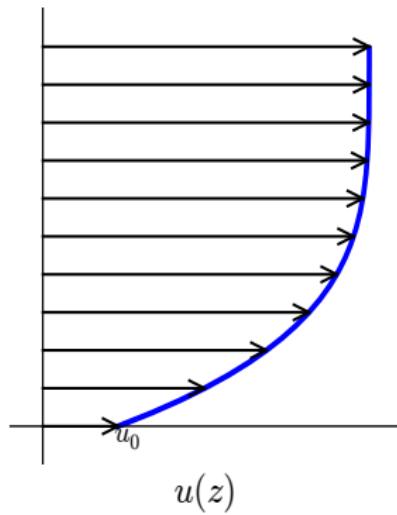
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- do we believe these equations?
- velocity on last slide (and below) was from a *formula*
- compare to observations at right



velocity profile of the Athabasca
Glacier from inclinometry
(Savage and Paterson, 1963)

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- now we know the velocity $u = u(t, x, z) \dots$ so what?
- suppose our slab has variable thickness $H(t, x)$
- compute the vertical average of velocity:

$$\bar{u}(x, t) = \frac{1}{H} \int_0^H u(t, x, z) dz$$

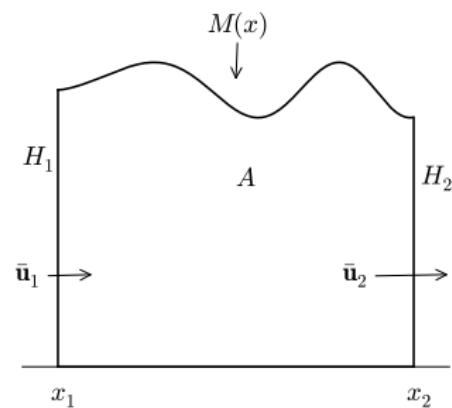
- consider change of area (ice volume in 3D) in the figure:

$$\frac{dA}{dt} = \int_{x_1}^{x_2} M(x) dx + \bar{u}_1 H_1 - \bar{u}_2 H_2$$

- but, assuming width $dx = x_2 - x_1$ is small, $A \approx dx H$; divide by dx and get

$$H_t = M - (\bar{u}H)_x$$

- this is a *mass continuity equation*



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rough explanation of “shallow ice approximation” (SIA)

- consider only $u_0 = 0$ case for now (“non-sliding SIA”)
- from slab-on-slope velocity formula

$$\begin{aligned}\bar{u}H &= \int_0^H \frac{1}{2} A(\rho g \sin \alpha)^3 (H^4 - (H-z)^4) \, dz \\ &= \frac{1}{2} A(\rho g \sin \alpha)^3 \left(\frac{4}{5} H^5 \right) \\ &= \frac{2}{5} A(\rho g \sin \alpha)^3 H^5\end{aligned}$$

- note $\sin \alpha \approx \tan \alpha = -h_x$
- combine with mass continuity $H_t = M - (\bar{u}H)_x$:

$$H_t = M + \left(\frac{2}{5} (\rho g)^5 A H^5 |h_x|^2 h_x \right)_x .$$

- this is SIA evolution equation for ice thickness
- I'll return to this topic ...

slow, non-Newtonian, some basal slip, and shallow

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- ice sheets have four outstanding properties as *fluids*:
 - 1 slow
 - 2 non-Newtonian
 - 3 shallow
 - 4 contact slip (sometimes)
- the first model will capture the first three properties

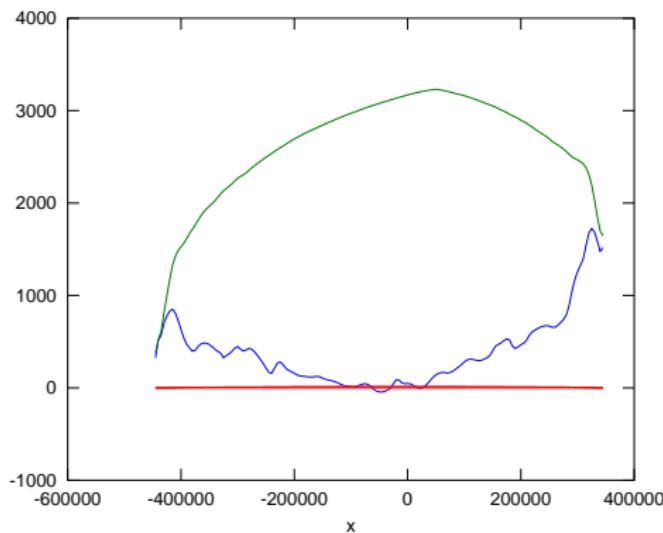
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- below in red is a no-vertical-exaggeration cross section of Greenland at 71°
 - green and blue: standard vertically-exaggerated cross section



- you can scale Stokes equation using smallness of $\epsilon = [H]/[L]$, where $[H]$ is a typical thickness of an ice sheet and $[L]$ is a typical horizontal dimension, ...

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non-sliding, isothermal shallow ice approximation = (SIA)

a model which applies to

- shallow grounded ice sheets
- on not-too-rough bed topography,
- whose flow is not dominated by sliding and/or liquid water at the base or margin



"Polaris Glacier," northwest Greenland, photo 122, Post & LaChapelle (2000)

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- here we “derive” the SIA by the simple slogan:

the SIA uses the formulas from slab-on-a-slope

- shear stress approximation:

$$(\tau_{13}, \tau_{23}) \approx -\rho g(h - z) \nabla h$$

- notation: $\mathbf{U} = (u, v)$ = the horizontal velocity
- we further approximate

$$\begin{aligned}\mathbf{U}_z &\approx 2A|(\tau_{13}, \tau_{23})|^{n-1}(\tau_{13}, \tau_{23}) \\ &= -2A(\rho g)^n(h - z)^n|\nabla h|^{n-1}\nabla h\end{aligned}$$

- by integrating vertically, in the non-sliding case,

$$\mathbf{U} = -\frac{2A(\rho g)^n}{n+1} [H^{n+1} - (h - z)^{n+1}] |\nabla h|^{n-1} \nabla h$$

- but mass continuity remains, $H_t = M - (\overline{\mathbf{U}} H)_x$

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- combine last two equations on last slide
- get the non-sliding, isothermal shallow ice approximation for thickness changes:

$$H_t = M + \nabla \cdot (\Gamma H^{n+2} |\nabla h|^{n-1} \nabla h)$$

- where H is ice thickness, h is ice surface elevation, b is bed elevation ($h = H + b$)
- M combines surface and basal mass (im)balance:
accumulation if $M > 0$, ablation if $M < 0$
- n is the exponent in the Glen flow law
- $\Gamma = 2A(\rho g)^n / (n+2)$ is a positive constant

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- numerically solve this equation

$$H_t = M + \nabla \cdot (\Gamma H^{n+2} |\nabla h|^{n-1} \nabla h) \quad (1)$$

and you've got a usable model for . . . *the Barnes ice cap*
(Mahaffy, 1976)

good questions:

- ① how to solve equation (1)
numerically?
- ② how to *think* about it?



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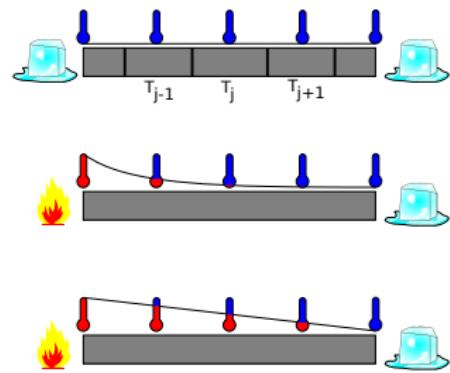
- for understanding the SIA, recall the heat equation describing conduction
- here's one quick way to derive it ...
- Newton's law of cooling for each segment of the rod:

$$\begin{aligned}\frac{dT_j}{dt} &= -C \left(T_j - \frac{1}{2}(T_{j-1} + T_{j+1}) \right) \\ &= \frac{C}{2} (T_{j-1} - 2T_j + T_{j+1})\end{aligned}$$

- the limit as segments shrink $\Delta x \rightarrow 0$:

$$T_t = D T_{xx}$$

- D = diffusivity, with units $\text{m}^2 \text{s}^{-1}$
- $T_{xx} > 0 \implies T(t)$ decreases
- $T_{xx} < 0 \implies T(t)$ increases
- so $T(x)$ becomes smoother with time



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- $T(t, x, y)$ is temperature in a 2D object at position x, y and time t
- Fourier rewrote Newton's law as a rule for heat flux: $\mathbf{q} = -k \nabla T$
- allow an additional heat source f
- coefficients: ρ is density, c is specific heat, k is conductivity
 - let's assume ρ, c are constant
- by conservation of energy:

$$\rho c T_t = f + \nabla \cdot (k \nabla T)$$

- define the diffusivity $D = k/(\rho c)$ and also $F := f/(\rho c)$
- for 2D object (e.g. a plate), the heat equation:

$$T_t = F + \nabla \cdot (D \nabla T) \tag{2}$$

- the temperature T solves a *diffusive, time-evolving partial differential equation (PDE)* . . . just like thickness H in SIA

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- side-by-side comparison:

SIA: $H(t, x, y)$ is ice thickness

$$H_t = M + \nabla \cdot (\Gamma H^{n+2} |\nabla h|^{n-1} \nabla h)$$

heat: $T(t, x, y)$ is temperature

$$T_t = F + \nabla \cdot (D \nabla T)$$

- we identify the diffusivity in the SIA:

$$D = \Gamma H^{n+2} |\nabla h|^{n-1}$$

- *non-sliding shallow ice flow diffuses the ice sheet*

- some issues with this analogy:

- D depends on solution $H(t, x, y)$
- $D \rightarrow 0$ at margin, where $H \rightarrow 0$
- $D \rightarrow 0$ at divides/domes, where $|\nabla h| \rightarrow 0$

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- numerical schemes for heat equation are good start for SIA
- for differentiable $f(x)$ and any Δ , *Taylor's theorem* says

$$f(x + \Delta) = f(x) + f'(x)\Delta + \frac{1}{2}f''(x)\Delta^2 + \frac{1}{3!}f'''(x)\Delta^3 + \dots$$

- you can replace " Δ " by its multiples, e.g.:

$$f(x - \Delta) = f(x) - f'(x)\Delta + \frac{1}{2}f''(x)\Delta^2 - \frac{1}{3!}f'''(x)\Delta^3 + \dots$$

$$f(x + 2\Delta) = f(x) + 2f'(x)\Delta + 2f''(x)\Delta^2 + \frac{4}{3}f'''(x)\Delta^3 + \dots$$

- basic finite difference idea:

combine expressions like these to give approximations of derivatives from values of $f(x)$ on a grid

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- we want partial derivative expressions, for example with some function $u = u(t, x)$:

$$u_t(t, x) = \frac{u(t + \Delta t, x) - u(t, x)}{\Delta t} + O(\Delta t),$$

$$u_t(t, x) = \frac{u(t + \Delta t, x) - u(t - \Delta t, x)}{2\Delta t} + O(\Delta t^2),$$

$$u_x(t, x) = \frac{u(t, x + \Delta x) - u(t, x)}{\Delta x} + O(\Delta x),$$

$$u_{xx}(t, x) = \frac{u(t, x + \Delta x) - 2u(t, x) + u(t, x - \Delta x)}{\Delta x^2} + O(\Delta x^2)$$

- sometimes we want a derivative in-between grid points:

$$u_x(t, x + (\Delta x/2)) = \frac{u(t, x + \Delta x) - u(t, x)}{\Delta x} + O(\Delta x^2)$$

- “ $+O(\Delta^2)$ ” is better than “ $+O(\Delta)$ ” if Δ is a small number

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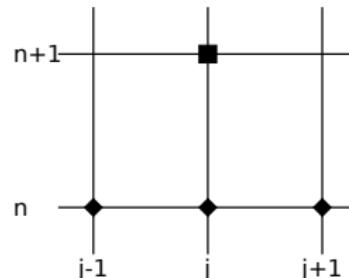
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- recall 1D heat equation $T_t = DT_{xx}$
- an *explicit* scheme:

$$\frac{T(t + \Delta t, x) - T(t, x)}{\Delta t} = D \frac{T(t, x + \Delta x) - 2T(t, x) + T(t, x - \Delta x)}{\Delta x^2}$$

- the difference between the equation $T_t = DT_{xx}$ and the scheme is $O(\Delta t, \Delta x^2)$
- notation: (t_n, x_j) and $T_j^n \approx T(t_n, x_j)$
- let $\mu = D\Delta t / (\Delta x)^2$, so

$$T_j^{n+1} = \mu T_{j+1}^n + (1 - 2\mu) T_j^n + \mu T_{j-1}^n$$



- scheme has stencil at right →

explicit scheme in two space dimensions

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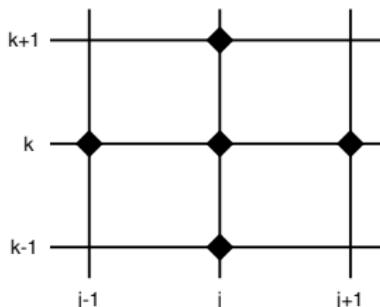
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- recall heat equation in 2D: $T_t = D(T_{xx} + T_{yy})$
- in two spatial variables we write $T_{jk}^n \approx T(t_n, x_j, y_k)$
- so the 2D explicit scheme is

$$\frac{T_{jk}^{n+1} - T_{jk}^n}{\Delta t} = D \left(\frac{T_{j+1,k}^n - 2T_{jk}^n + T_{j-1,k}^n}{\Delta x^2} + \frac{T_{j,k+1}^n - 2T_{jk}^n + T_{j,k-1}^n}{\Delta y^2} \right)$$



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```
function T = heat(D,J,K,dt,N)

dx = 2 / J;      dy = 2 / K;
[x,y] = meshgrid(-1:dx:1, -1:dy:1);
T = exp(-30*(x.*x + y.*y));

mu_x = dt * D / (dx*dx);
mu_y = dt * D / (dy*dy);
for n=1:N
    T(2:J,2:K) = T(2:J,2:K) + ...
        mu_x * ( T(3:J+1,2:K) - 2 * T(2:J,2:K) + T(1:J-1,2:K) ) + ...
        mu_y * ( T(2:J,3:K+1) - 2 * T(2:J,2:K) + T(2:J,1:K-1) );
end

surf(x,y,T), shading('interp'), xlabel x, ylabel y
```

heat.m

- solves $T_t = D(T_{xx} + T_{yy})$ on square $-1 < x < 1, -1 < y < 1$
- uses gaussian initial condition: $T(0, x, y) = e^{-30r^2}$
- uses “colon notation” to remove loops over spatial variables
- » `heat(1.0,30,30,0.001,20)`
approximates T on 30×30 spatial grid, with $D = 1$ and $N = 20$ steps of $\Delta t = 0.001$

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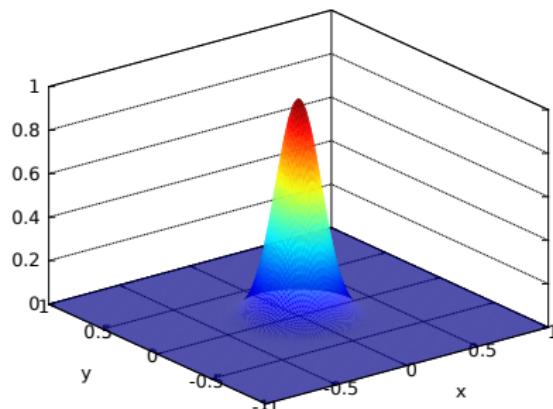
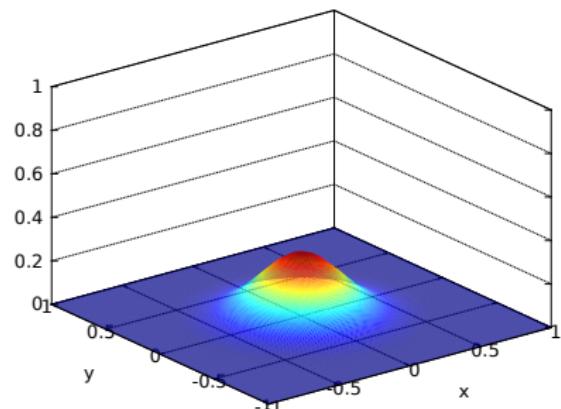
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- solving $T_t = T_{xx} + T_{yy}$ on 30×30 grid

initial condition $T(0, x, y)$ approximate solution $T(t, x, y)$ at
 $t = 0.02$ with $\Delta t = 0.001$ 

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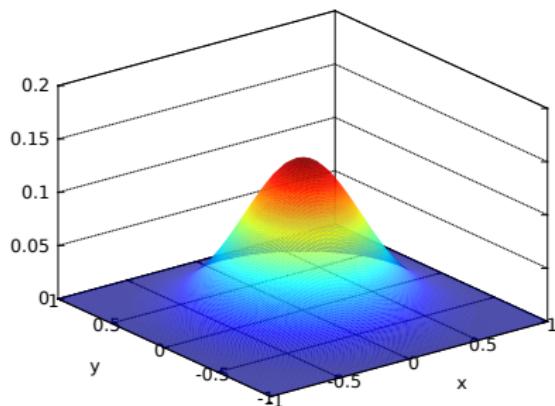
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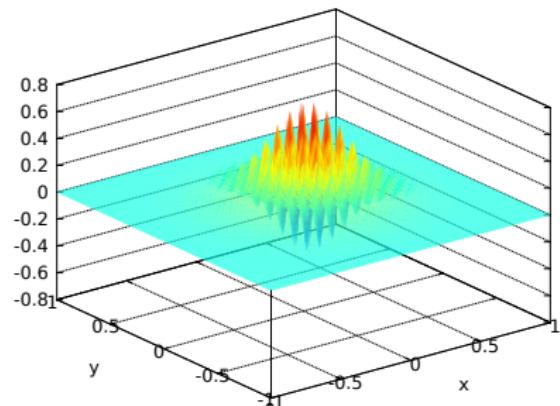
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- figures below are from solving $T_t = T_{xx} + T_{yy}$ on the same space grid, but with slightly different time steps



$$\frac{D\Delta t}{\Delta x^2} = 0.2$$



$$\frac{D\Delta t}{\Delta x^2} = 0.4$$

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- recall 1D explicit scheme has form

$$T_j^{n+1} = \mu T_{j+1}^n + (1 - 2\mu) T_j^n + \mu T_{j-1}^n$$

- thus the new value T_j^{n+1} is an *average* of the old values, *if the middle coefficient is positive*:

$$1 - 2\mu \geq 0 \iff \frac{D\Delta t}{\Delta x^2} \leq \frac{1}{2} \iff \Delta t \leq \frac{\Delta x^2}{2D}$$

- averaging is always stable because averaged wiggles are always smaller than the original wiggles
- this condition is a sufficient *stability criterion*
- the result was unstable because the time step was too big*
- in 2D case with $\Delta x = \Delta y$ the condition is

$$\frac{D\Delta t}{\Delta x^2} \leq \frac{1}{4}$$

adaptive implementation: guaranteed stability

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```
function T = heatadapt(D,J,K,tf)

dx = 2 / J;      dy = 2 / K;
[x,y] = ndgrid(-1:dx:1, -1:dy:1);
T = exp(-30*(x.*x + y.*y));

t = 0.0;      count = 0;
while t < tf
    dt0 = 0.25 * min(dx,dy)^2 / D;
    dt = min(dt0, tf - t);
    mu_x = dt * D / (dx*dx);      mu_y = dt * D / (dy*dy);
    T(2:J,2:K) = T(2:J,2:K) + ...
        mu_x * ( T(3:J+1,2:K) - 2 * T(2:J,2:K) + T(1:J-1,2:K) ) + ...
        mu_y * ( T(2:J,3:K+1) - 2 * T(2:J,2:K) + T(2:J,1:K-1) );
    t = t + dt;
    count = count + 1;
end

surf(x,y,T), shading('interp'), xlabel x, ylabel y
```

heatadapt.m

- same as heat.m except

choose time step from stability criterion

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- explicit scheme is only “conditionally stable”
 - the adaptive implementation uses the condition
- there are **implicit** methods which are stable for *any* Δt
- an implicit scheme for the heat equation is *Crank-Nicolson*
 - it has smaller error too: $O(\Delta t^2, \Delta x^2)$
- *but* you have to solve systems of equations at each time step
- in nonlinear case you may end up spending much more programmer time to implement implicit methods
 - this can impose a big opportunity cost
- Donald Knuth has advice for ice sheet modelers:
We should forget about small efficiencies . . . : premature optimization is the root of all evil.

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- recall the analogy: (SIA) \leftrightarrow (heat eqn)
- the SIA has a diffusivity $D(x, y)$ which varies in space
- and it has both H and $h = H + b$
- so consider a more general heat equation:

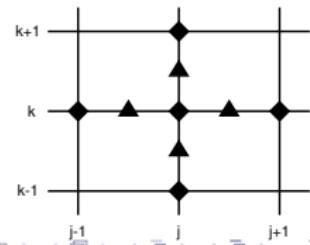
$$T_t = F + \nabla \cdot (D \nabla(T + b)) \quad (*)$$

- the best explicit method for (*) evaluates diffusivity D at **staggered** grid points:

$$\begin{aligned} \nabla \cdot (D \nabla X) \approx & \frac{D_{j+1/2,k}(X_{j+1,k} - X_{j,k}) - D_{j-1/2,k}(X_{j,k} - X_{j-1,k})}{\Delta x^2} \\ & + \frac{D_{j,k+1/2}(X_{j,k+1} - X_{j,k}) - D_{j,k-1/2}(X_{j,k} - X_{j,k-1})}{\Delta y^2} \end{aligned}$$

- best = just as stable as previous
- in stencil at right:

diamonds: $X = T + b$
triangles: D



general diffusion equation code

```
function [T,dtav] = diffusion(Lx,Ly,J,K,Dup,Ddown,Dright,Dleft,T0,tf,F,b)

dx = 2 * Lx / J;      dy = 2 * Ly / K;
[x,y] = ndgrid(-Lx:dx:Lx, -Ly:dy:Ly);
T = T0;
if nargin < 11, F = zeros(size(T0)); end
if nargin < 12, b = zeros(size(T0)); end

t = 0.0;    count = 0;
while t < tf
    maxD = [max(max(Dup)) max(max(Ddown)) max(max(Dleft)) max(max(Dright))];
    maxD = max(maxD);
    if maxD <= 0.0
        dt = tf - t;
    else
        dt0 = 0.25 * min(dx,dy)^2 / maxD;
        dt = min(dt0, tf - t);
    end
    mu_x = dt / (dx*dx);    mu_y = dt / (dy*dy);
    Tb = T + b;
    T(2:J,2:K) = T(2:J,2:K) + ...
        mu_y * Dup .* ( Tb(2:J,3:K+1) - Tb(2:J,2:K) ) - ...
        mu_y * Ddown .* ( Tb(2:J,2:K) - Tb(2:J,1:K-1) ) + ...
        mu_x * Dright .* ( Tb(3:J+1,2:K) - Tb(2:J,2:K) ) - ...
        mu_x * Dleft .* ( Tb(2:J,2:K) - Tb(1:J-1,2:K) );
    T = T + F * dt;
    t = t + dt;    count = count + 1;
end
dtav = tf / count;
```

diffusion.m

- solves abstract diffusion equation $T_t = \nabla \cdot (D \nabla(T + b))$
- user supplies diffusivity D on staggered grid

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- before getting to ice flow, one more heat equation topic . . .
- many *exact* solutions to the heat equation are known
- I'll show the “Green's function”
- . . . also known as “heat kernel”
- it starts at time $t = 0$ with a “delta function” of heat at the origin $x = 0$ and then it spreads out over time
- we find it by a method which generalizes to the SIA

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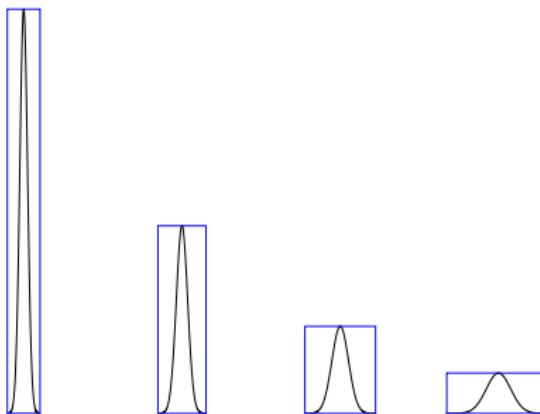
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- the solution is “self-similar” over time
- with time it changes shape by
 - shrinking the output (vertical) axis and
 - lengthening the input (horizontal) axis
- ... but otherwise it is the same shape
- the integral over x is independent of time

*increasing time →*

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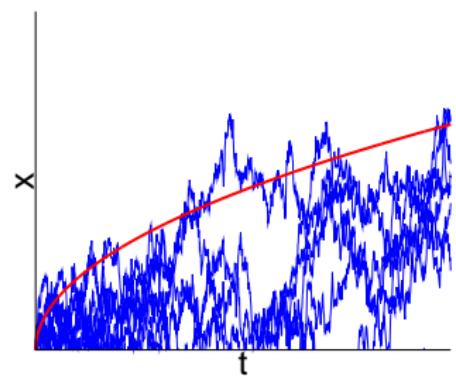
- Green's function of 1D heat equation ($T_t = DT_{xx}$) is

$$T(t, x) = C t^{-1/2} e^{-x^2/(4Dt)}$$

- “similarity” variables for 1D heat equation are

$$\begin{array}{ccc} \text{input scaling} & & \text{output scaling} \\ s & = & t^{-1/2}x, \quad T(t, x) & = & t^{-1/2}\phi(s) \end{array}$$

- *historical note:* in 1905 Einstein saw that the average distance traveled by particles in thermal motion scales like \sqrt{t} , so $s = t^{-1/2}x$ is an invariant



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- jump forward to 1981
- P. Helffer found the similarity solution of the SIA in the case of flat bed and no surface mass balance
- Helffer's 2D solution for Glen flow law with $n = 3$ has scalings

$$\begin{array}{ccc} \textit{input scaling} & & \textit{output scaling} \\ s & = & t^{-1/18}r, \quad H(t, r) & = & t^{-1/9}\phi(s) \end{array}$$

- so: the nonlinear diffusion of the SIA quickly slows down the rate of change of the profile as the shape flattens out

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frames from $t = 4$ months to $t = 10^6$ years, equal spaced in *exponential* time

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- for $n = 3$ the solution formula is:

$$H(t, r) = H_0 \left(\frac{t_0}{t} \right)^{1/9} \left[1 - \left(\left(\frac{t_0}{t} \right)^{1/18} \frac{r}{R_0} \right)^{4/3} \right]^{3/7}$$

- the “characteristic time” is

$$t_0 = \frac{1}{18\Gamma} \left(\frac{7}{4} \right)^3 \frac{R_0^4}{H_0^7}$$

if H_0 , R_0 are central height and ice cap radius at $t = t_0$

- you choose H_0 and R_0 and then determine t_0
- it is a simple formula to use for verification!

is the Halfar solution *good for any modelling?*

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- John Nye and others (2000) compared long-time consequences of different flow laws for the Mars polar caps
- they evaluated CO₂ ice versus H₂O ice parameters
- ... by comparing long-time behavior of the Halfar solutions
- conclusions:

... none of the three possible [CO₂] flow laws will allow a 3000-m cap, the thickness suggested by stereogrammetry, to survive for 10⁷ years, indicating that the south polar ice cap is probably not composed of pure CO₂ ice ... the south polar cap probably consists of water ice, with an unknown admixture of dust.

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- recall that the SIA is

$$H_t = M + \nabla \cdot (D \nabla h) \quad \text{where} \quad D = \Gamma H^{n+2} |\nabla h|^{n-1}$$

- thus the diffusivity “degenerates”, $D \rightarrow 0$, when either $H \rightarrow 0$ or $\nabla h \rightarrow 0$
- summary:

	why $D \rightarrow 0$	so what?
domes	$\nabla h \rightarrow 0$	H and ∇h are continuous but $\nabla^2 h$ is singular
margins	$H \rightarrow 0$	H is continuous but ∇h is singular

- in terms of numerical error, margin is worse than dome
- degenerate diffusion equations are automatically free boundary problems

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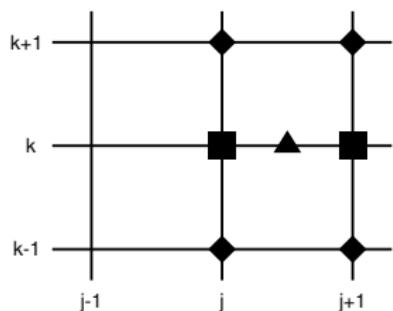
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- for numerical stability we compute $D = \Gamma H^{n+2} |\nabla h|^{n-1}$ on the staggered grid
 - various schemes proposed
 - all schemes involve
 - averaging H
 - differencing h
 - in a “balanced” way, for better accuracy,
- to get the diffusivity on staggered grid

- Mahaffy's scheme has stencil →



SIA implementation: flat bed case

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```

function [H,dtlist] = siaflat(Lx,Ly,J,K,H0,deltat,tf)

g = 9.81;      rho = 910.0;      secpera = 31556926;
A = 1.0e-16/secpera;      Gamma = 2 * A * (rho * g)^3 / 5;
H = H0;

dx = 2 * Lx / J;      dy = 2 * Ly / K;
N = ceil(tf / deltat);      deltat = tf / N;
j = 2:J;      k = 2:K;
nk = 3:K+1;      sk = 1:K-1;      ej = 3:J+1;      wj = 1:J-1;

t = 0;      dtlist = [];
for n=1:N
    Hup = 0.5 * ( H(j,nk) + H(j,k) );      Hdn = 0.5 * ( H(j,k) + H(j,sk) );
    Hrt = 0.5 * ( H(ej,k) + H(j,k) );      Hlt = 0.5 * ( H(j,k) + H(wj,k) );
    a2up = (H(ej,nk) + H(ej,k) - H(wj,nk) - H(wj,k)).^2 / (4*dx)^2 + ...
            (H(j,nk) - H(j,k)).^2 / dy.^2;
    a2dn = (H(ej,k) + H(ej,sk) - H(wj,k) - H(wj,sk)).^2 / (4*dx)^2 + ...
            (H(j,k) - H(j,sk)).^2 / dy.^2;
    a2rt = (H(ej,k) - H(j,k)).^2 / dx.^2 + ...
            (H(ej,nk) + H(j,nk) - H(ej,sk) - H(j,sk)).^2 / (4*dy)^2;
    a2lt = (H(j,k) - H(wj,k)).^2 / dx.^2 + ...
            (H(wj,nk) + H(j,nk) - H(wj,sk) - H(j,sk)).^2 / (4*dy)^2;
    Dup = Gamma * Hup.^5 .* a2up;      Ddn = Gamma * Hdn.^5 .* a2dn;
    Drt = Gamma * Hrt.^5 .* a2rt;      Dlt = Gamma * Hlt.^5 .* a2lt;
    [H,dtadapt] = diffusion(Lx,Ly,J,K,Dup,Ddn,Drt,Dlt,H,deltat);
    t = t + deltat;      dtlist = [dtlist dtadapt];
end

```

siaflat.m

- solves the $M = 0$, flat bed SIA: $H_t = \nabla \cdot (\Gamma H^{n+2} |\nabla H|^{n-1} \nabla H)$
- calls diffusion.m at each major time step

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- how do we make sure a numerical scheme is correct?
- how do we make sure an *implemented* numerical scheme is correct?
 - *technique 1*: don't make any mistakes
 - *technique 2*: compare your model with others, and hope that the outliers are the ones with errors
 - *technique 3*: build-in a comparison to an exact solution, and actually measure the numerical error
- technique 3 is called **verification**

where to get exact solutions for ice flow models?

- textbook: Greve and Blatter (2009)
- similarity solutions to SIA (Halfar 1983; Bueler et al 2005)
- manufactured solutions to thermo-coupled SIA (Bueler et al 2007)
- flowline and cross-flow SSA solutions (van der Veen, 1985; Schoof, 2006)
- flowline Blatter solutions (Glowinski and Rappaz 2003)
- flowline Stokes solutions for constant viscosity (Ladyzhenskaya 1963; Balise and Raymond 1985)
- manufactured solutions to the Stokes equations (Sargent and Fastook 2010; Jouvet and Rappaz 2011)

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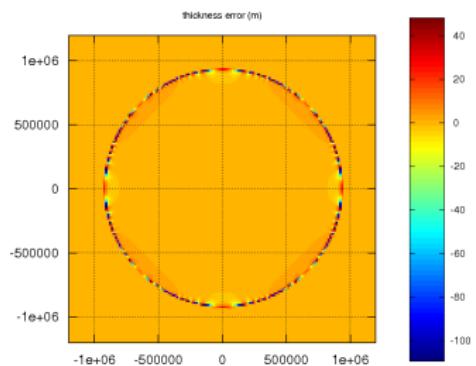
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test program `verifysia.m` calls `siaflat.m`

```

octave:40> verifysia(20)
average abs error = 22.310
maximum abs error = 227.849
octave:41> verifysia(40)
average abs error = 9.490
maximum abs error = 241.470
octave:42> verifysia(80)
average abs error = 2.800
maximum abs error = 155.796
octave:43> verifysia(160)
average abs error = 1.059
maximum abs error = 109.466

```



Trust but verify.
(Ronald Reagan)

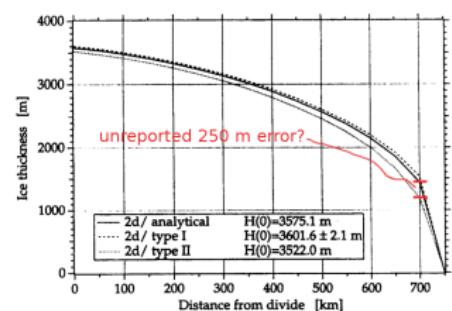


figure 2 in Huybrechts et al. (1996)

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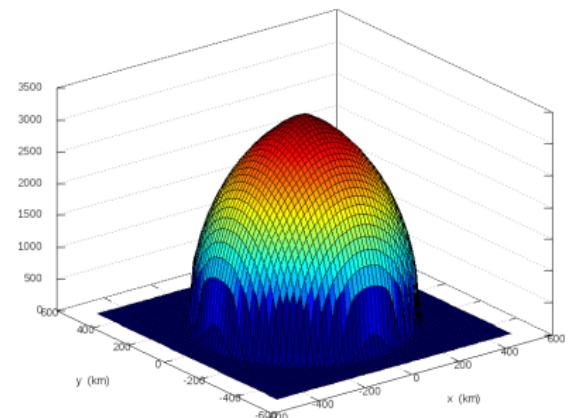
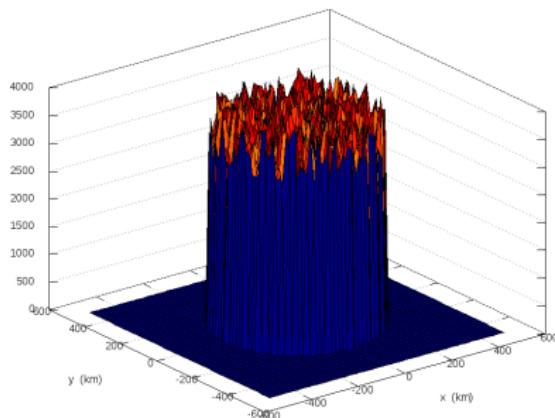
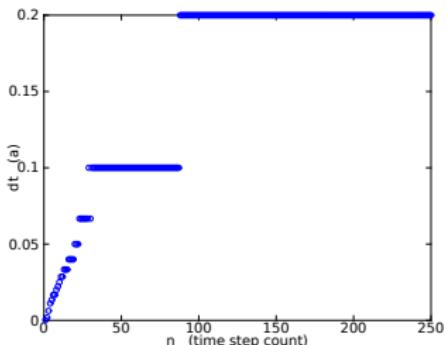
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- test program `roughice.m` calls `siaflat.m`
- sets up the nasty initial state (below left)
- runs for 50 years
- get final state (below right)
- time steps adapt (upper right)



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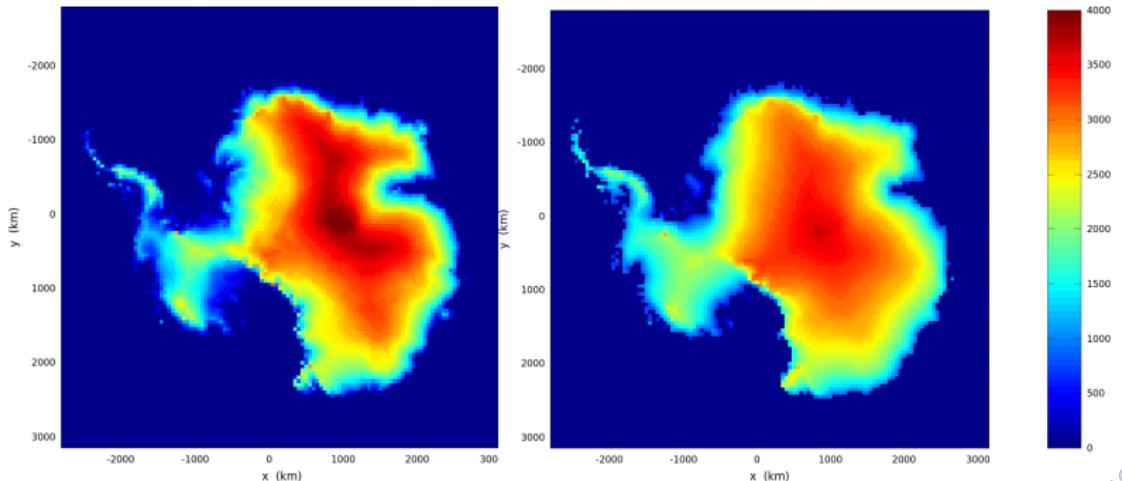
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model of the Antarctic ice sheet

- careful-but-small modifications of `siaflat.m` give `siageneral.m`
 - allow non-flat bed
 - calve floating ice (*SIA can't handle it*)
- build a *toy* Antarctic flow model
 - observed accumulation as surface mass balance, sans melt
 - data from ALBMAPv1, a NetCDF file
- results from a 40 ka model run on a $\Delta x = 50 \text{ km}$ grid below
- runtime four minutes on laptop



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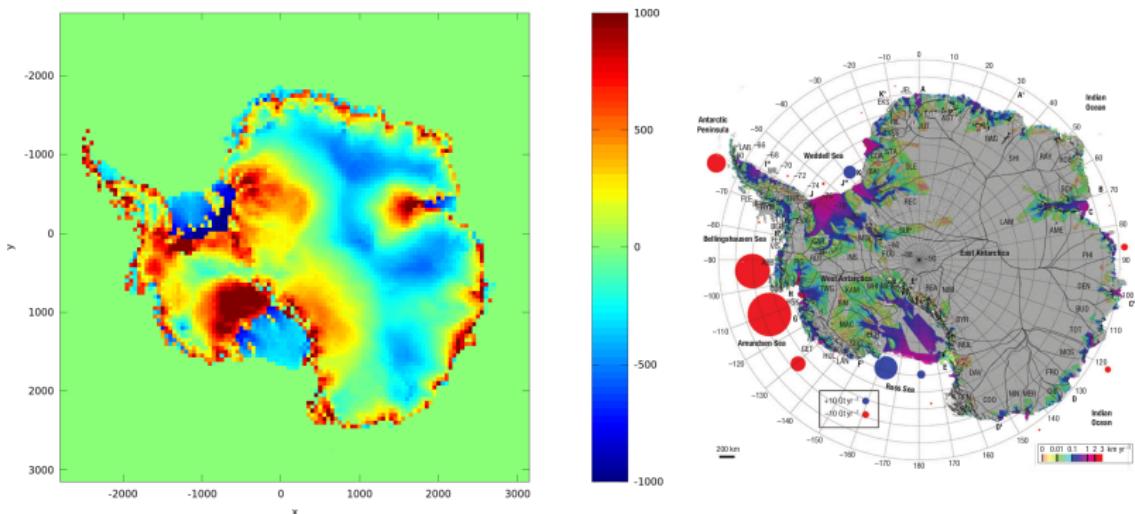
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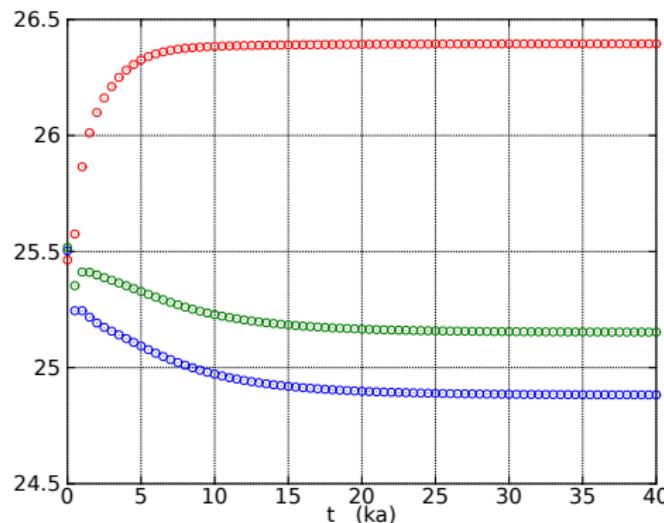
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- volume time series from **50 km**, **25 km**, **20 km** runs
- units of 10^6 km^3
- conclusion: look at your results on *multiple grid resolutions* before interpreting your results (e.g. their parameter-dependence)



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the most basic shallow assumption

- there are many shallow theories:
SIA, SSA, hybrids, Blatter, ...
- *all* make one assumption not required in the (non-shallow)
Stokes theory:

the surface and base of the ice are given by differentiable functions
 $z = h(t, x, y)$ and $z = b(t, x, y)$

- surface overhang is not allowed



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- what does this “most basic shallow assumption” get you?
- *answer:* a map-plane mass continuity equation
- consider these three equations:
 - the surface kinematical equation
 - the base kinematical equation
 - the map-plane mass continuity equation
- under the “most basic shallow assumption”,
any two imply the third
- these three equations are on the next slide

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- a is surface mass balance ($a > 0$ is accumulation)
- s is basal melt rate ($s > 0$ is basal melting)
- $M = a - s$
- define the map-plane flux of ice,

$$\mathbf{q} = \int_b^h (u, v) dz = \bar{\mathbf{U}} H$$

- the three equations are:

surface kinematical

$$h_t = a - u|_h h_x - v|_h h_y + w|_h \quad (3)$$

base kinematical

$$b_t = s - u|_b b_x - v|_b b_y + w|_b \quad (4)$$

mass continuity

$$H_t = M - \nabla \cdot \mathbf{q} \quad (5)$$

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- how are the three equations related to each other?
- start with the incompressibility of ice

$$u_x + v_y + w_z = 0 \quad \iff \quad w_z = -\nabla \cdot \mathbf{U}$$

- integrate vertically:

$$w|_h - w|_b = \int_b^h w_z dz = - \int_b^h \nabla \cdot \mathbf{U} dz$$

- now be careful and use the Leibniz rule for differentiating integrals:

$$\frac{d}{dx} \left(\int_{A(x)}^{B(x)} F(x, y) dy \right) = B'(x)F(x, B(x)) - A'(x)F(x, A(x)) + \int_{A(x)}^{B(x)} F_x(x, y) dy$$

- which implies that

$$-\nabla \cdot \int_b^h \mathbf{U} dz = -\nabla h \cdot \mathbf{U}|_h + \nabla b \cdot \mathbf{U}|_b - \int_b^h \nabla \cdot \mathbf{U} dz$$

- so

$$w|_h - w|_b = -\nabla \cdot \mathbf{q} + \nabla h \cdot \mathbf{U}|_h - \nabla b \cdot \mathbf{U}|_b$$

- and so on

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- why am I telling you this?
- because glaciology articles are filled with recapitulation of these equivalences
- my main points regarding mass continuity:
 - most ice sheet models use the mass continuity equation to describe change in ice sheet geometry
 - but they could instead use the surface kinematical equation
 - and they almost always use, and greatly simplify, the basal kinematical equation

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- the ingredients of a typical ice sheet model:
 - ① numerical implementation of a stress balance: gives velocity $\mathbf{u} = (u, v, w)$
 - ② from the horizontal velocity $\mathbf{U} = (u, v)$, or the horizontal flux \mathbf{q} , and the climatic mass (surface) balance M , do a time-step of the mass continuity equation to get $\Delta H = H_t \Delta t$
 - ③ update upper (and perhaps lower) surface elevation:
$$h \mapsto h + H_t \Delta t$$
 - ④ decide on next time-step Δt , and repeat at 1.
- the non-sliding SIA is slightly atypical because we can write $\mathbf{q} = -D \nabla h$ in addition to $\mathbf{q} = \bar{\mathbf{U}} H$

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shallow shelf approximation (SSA) stress balance

SSA model applies very well to **ice shelves**

- ... for parts away from grounding lines
- ... and away from calving fronts

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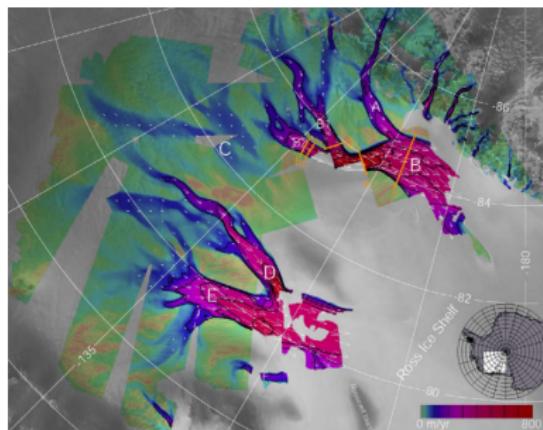
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shallow shelf approximation (SSA) stress balance 2

SSA also applies reasonably well to **ice streams**

- ... with modest bed topography
- ... and weak bed strength
- ... but not so good near shear margins & grounding lines
- energy conservation (= ice temperature and basal melt) is a major aspect of ice stream flow, but not addressed here



RADARSAT-derived surface velocity for Siple Coast ice streams, Antarctica

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what is, *and is not*, an ice stream?

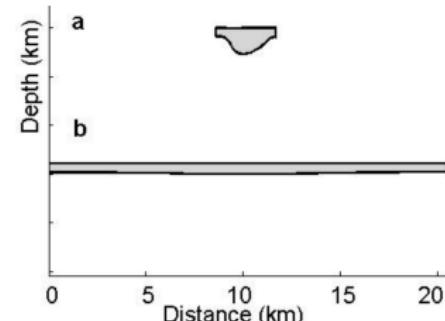
- ice streams

- slide (100 to 1000 m a^{-1})
- have concentration of vertical shear in a thin layer near base
- liquid water at bed has critical role

- “outlet glaciers”

- have fast surface speed (up to 10 km a^{-1})
- uncertain how much is true sliding
- substantial vertical shear “up” in the ice column
- not-at-all flat bed topography
- soft, temperate ice plays a role

- therefore **few simplifying assumptions are appropriate for outlet glaciers**



Cross sections of Jakobshavn Isbrae (**a**) and Whillans Ice Stream (**b**). Plotted without vertical exaggeration. (Figure 1 in Truffer and Echelmeyer (2003), *Of isbrae and ice streams*)

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- only plane flow case (“flow line”) in these lectures
- the stress balance equation which determines velocity:

$$\left(2BH|u_x|^{1/n-1} u_x \right)_x - C|u|^{m-1} u = \rho g H h_x \quad (6)$$

- the **red term** inside parentheses is the vertically-integrated “longitudinal” or “membrane” stress
- the **blue term** is basal resistance; $C = 0$ in an ice shelf
- the right-hand side is driving stress; see *next slide*
- *how to think about this equation?*
- *how do you solve it numerically?*

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- the inequality " $\rho H < -\rho_w b$ " is the **flotation criterion**
- at the grounding line $x = x_g$ the inequality switches
- driving stress:
 - on the grounded side $\rho H > -\rho_w b$, so
$$\rho g H h_x = \rho g H (H_x + b_x)$$
 - on the floating side $\rho H < -\rho_w b$, so $h = (1 - \rho/\rho_w)H$, and so
$$\rho g H h_x = \rho(1 - \rho/\rho_w)g H H_x$$
- best numerical models for moving grounding lines still an open question (e.g. MISMIP and all that)

from stream to shelf across grounding line

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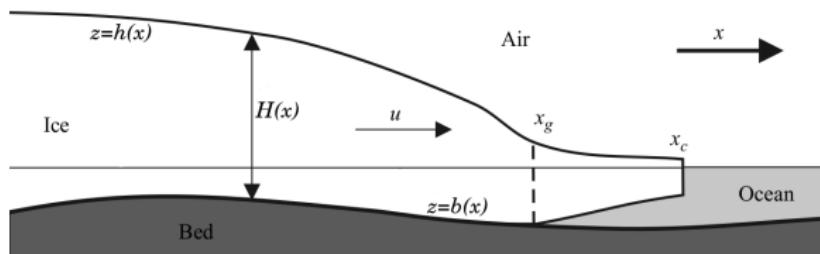
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$$u = u_0 \quad \text{at } x = 0$$

$$\left. \begin{aligned} (2BH|u_x|^{1/n-1}u_x)_x - C|u|^{m-1}u &= \rho g H h_x \\ h &= H + b \end{aligned} \right\} \quad \text{on } 0 < x < x_g$$

$$\left. \begin{aligned} (2BH|u_x|^{1/n-1}u_x)_x + 0 &= \rho g H h_x \\ h &= (1 - \rho/\rho_w)H \end{aligned} \right\} \quad \text{on } x_g < x < x_c$$

$$2BH|u_x|^{1/n-1}u_x = \frac{1}{2}\rho(1 - \rho/\rho_w)gH^2 \quad \text{at } x = x_c$$



exact velocity and thickness for steady ice shelf

Numerical
modelling

Ed Bueler

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- limited goal here: describe a steady state, 1D ice shelf
- there is this nice **by-hand** result: the thickness and velocity in the ice shelf can be completely determined in terms of the
 - ice thickness H_g at the grounding line and
 - ice velocity u_g at the grounding line
- derived by van der Veen (1983)
- shown on next slide
- we will use this to
 - understand the SSA better
 - verify a numerical SSA code

exact velocity and thickness for steady ice shelf 2

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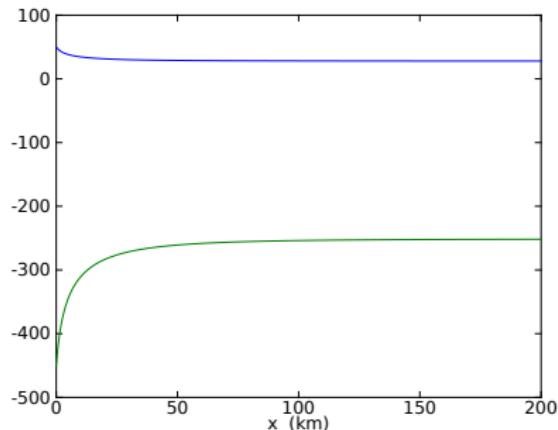
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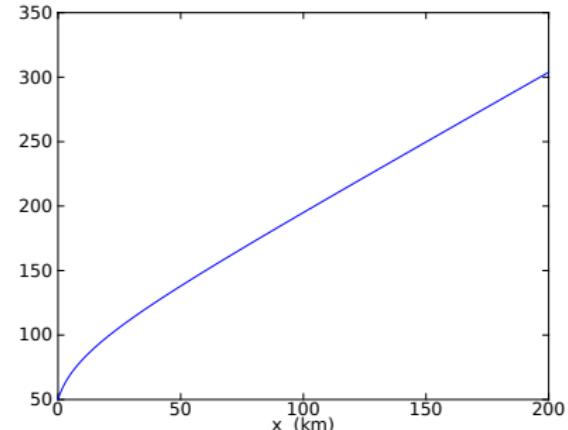
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see `testshelf.m`:

surface and base elevation (m)



velocity (m/a)



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- now we fix ice thickness $H(x)$ and find the velocity numerically
- the stress balance is a nonlinear equation in the velocity:

$$\left(2BH|u_x|^{1/n-1}u_x\right)_x - C|u|^{m-1}u = \rho g H h_x$$

- nonlinear so **iteration is needed**

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- coefficient $\bar{\nu} = B|u_x|^{1/n-1}$ is the “effective viscosity”:

$$(2\bar{\nu} Hu_x)_x - C|u|^{m-1}u = \rho g H h_x$$

- simplest iteration idea:* use initial velocity estimate to get effective viscosity estimate, then solve for new velocity, and repeat until things stop changing
 - this is “Picard” iteration
 - Newton iteration is a superior alternative
- the iteration in formulas:

- previous iterate $u^{(k-1)}$
- define $W^{(k-1)} = 2\bar{\nu}H = 2B|u_x^{(k-1)}|^{1/n-1}H$
- solve for current iterate (unknown) $u^{(k)}$:

$$(W^{(k-1)} u_x^{(k)})_x - C|u^{(k-1)}|^{m-1}u^{(k)} = \rho g H h_x$$

- repeat at 1

where do you get an initial velocity $u^{(0)}$?

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- *for floating ice*, an initial guess assumes uniform strain rate:

$$u^{(0)}(x) = \gamma(x - x_g) + u_g$$

where γ is the value of u_x from calving front stress imbalance

- *for grounded ice*, an initial guess assumes ice is held by basal resistance only:

$$u^{(0)}(x) = (-C^{-1} \rho g H h_x)^{1/m}$$

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- abstract the problem:

$$(W(x) u_x)_x - \alpha(x) u = \beta(x)$$

on $0 < x < L$, with boundary conditions

$$u(0) = V, \quad u_x(L) = \gamma$$

- an *linear*, elliptic PDE boundary value problem
- $W(x), \alpha(x), \beta(x)$ are known functions in the SSA context:
 - both $W(x)$ and $\alpha(x)$ come from previous iteration
 - $\beta(x)$ is driving stress

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- equally-spaced grid x_1, x_2, \dots, x_{J+1}
 - $x_1 = x_g$ and $x_{J+1} = x_c$ are endpoints
- the finite difference approximation to the linear PDE is:

$$\frac{W_{j+1/2}(u_{j+1} - u_j) - W_{j-1/2}(u_j - u_{j-1})}{\Delta x^2} - \alpha_j u_j \stackrel{*}{=} \beta_j$$

- so $W(x)$ is needed on the staggered grid
- left-hand boundary condition: $u_1 = V$ given
- right-hand boundary condition is $u_x(L) = \gamma$
- to put right-hand b.c. into system:
 - introduce notional point x_{J+2}
 -
$$\frac{u_{J+2} - u_J}{2\Delta x} = \gamma$$
- using equation $*$ in $j = J + 1$ case, eliminate u_{J+2} variable by-hand before solving system

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- thus discretized stress balance has form $\mathbf{Ax} = \mathbf{b}$:

$$\begin{bmatrix} 1 & & & \\ W_{3/2} & A_{22} & W_{5/2} & \\ & W_{5/2} & A_{33} & \\ & & \ddots & \ddots \\ & W_{J-1/2} & A_{JJ} & W_{J+1/2} \\ & & A_{J+1,J} & A_{J+1,J+1} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_J \\ u_{J+1} \end{bmatrix} = \begin{bmatrix} 0 \\ \beta_2 \Delta x^2 \\ \beta_3 \Delta x^2 \\ \vdots \\ \beta_J \Delta x^2 \\ b_{J+1} \end{bmatrix}$$

- with specific formulas on diagonal (A_{jj}) and in the last equation
 - see printed notes
- this is a *tridiagonal* system
- hand the system to a linear algebra black box

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```
function u = flowline(L,J,gamma,W,alpha,beta,V0)

dx = L / J;
rhs = dx^2 * beta(:);
rhs(1) = V0;
rhs(J+1) = rhs(J+1) - 2 * gamma * dx * W(J+1);

A = sparse(J+1,J+1);
A(1,1) = 1.0;
for j=2:J
    A(j,j-1:j+1) = [ W(j-1), -(W(j-1) + W(j) + alpha(j) * dx^2), W(j) ];
end
A(J+1,J) = W(J) + W(J+1);
A(J+1,J+1) = - (W(J) + W(J+1) + alpha(J+1) * dx^2);

scale = full(max(abs(A),[],2));
for j=1:J+1, A(j,:) = A(j,:). ./ scale(j); end
rhs = rhs ./ scale;

u = A \ rhs;
```

flowline.m

- solves

$$(W(x) u_x)_x - \alpha(x) u = \beta(x)$$

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testing the linear PDE solver

- before proceeding to solve nonlinear SSA problem, we can test the “abstracted” code `flowline.m`
- test by “manufacturing” solutions
 - see `testflowline.m`; not shown
- results: converges at optimal rate $O(\Delta x^2)$

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```
function [u,u0] = ssaflowline(p,J,H,b,ug,initchoice)

if nargin ~= 6, error('exactly 6 input arguments required'), end

dx = p.L / J;
x = (0:dx:p.L)';
xstag = (dx/2:dx:p.L+dx/2)';

alpha = p.C * ones(size(x));
h = H + b;
hx = regslope(dx,h);
beta = p.rho * p.g * H .* hx;
gamma = ( 0.25 * p.A^(1/p.n) * (1 - p.rho/p.rhow) *...
          p.rho * p.g * H(end) )^p.n;

u0 = ssainit(p,x,beta,gamma,initchoice);
u = u0;

Hstag = stagav(H);
tol = 1.0e-14;
eps_reg = (1.0 / p.secpera) / p.L;
maxdiff = Inf;
W = zeros(J+1,1);
iter = 0;
while maxdiff > tol
    uxstag = stagslope(dx,u);
    sqr_ux_reg = uxstag.^2 + eps_reg.^2;
    W(1:J) = 2 * p.A^(-1/p.n) * Hstag .* sqr_ux_reg.^(((1/p.n)-1)/2.0);
    W(J+1) = W(J);

    unew = flowline(p.L,J,gamma,W,alpha,beta,ug);
    maxdiff = max(abs(unew-u));
    u = unew;
    iter = iter + 1;
end
```

ssafowline.m

- implements the Picard iteration
- calls the linear solver `flowline.m` at each iteration

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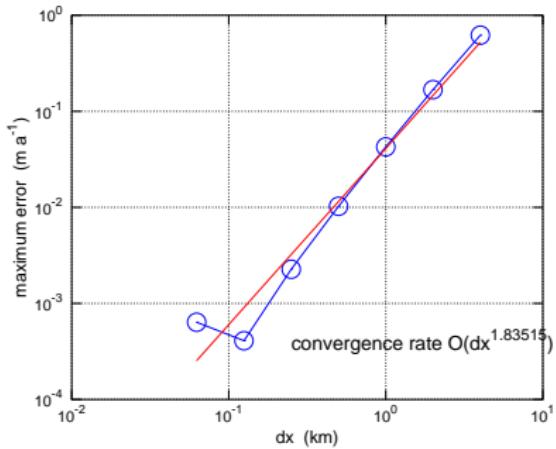
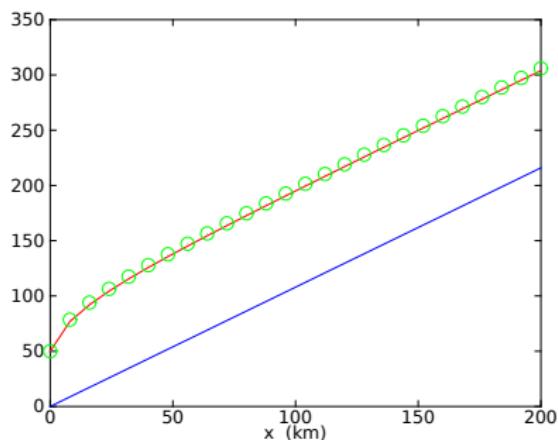
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- left: exact (red curve) and numerical (green circles) velocity
 - for very coarse 8 km grid
 - blue is initial velocity iterate
- right: convergence analysis of `ssaflowline.m`
 - error = $\max(\text{numerical} - \text{exact})$, over various grids



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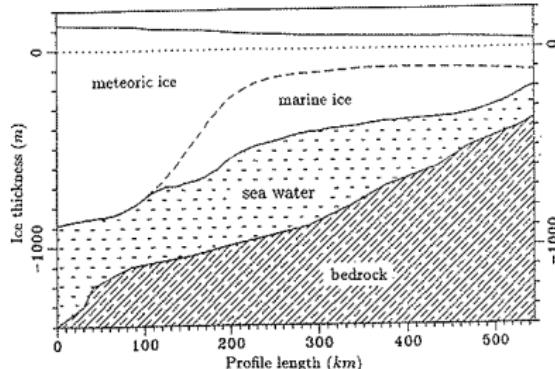
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- flow lines are never very realistic
- you can add “side drag”
 - I don’t know how to parameterize it
- also, ice shelves have surprises:
 - high basal melt near grounding lines
 - “reverse bed slope” instability
 - marine ice can freeze-on at bottom (below)



Filchner-Ronne ice shelf; from Grosfeld & Thyssen 1994

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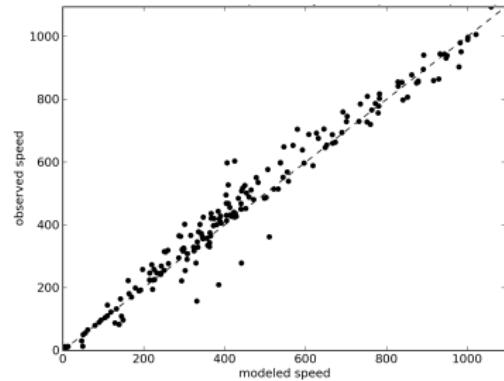
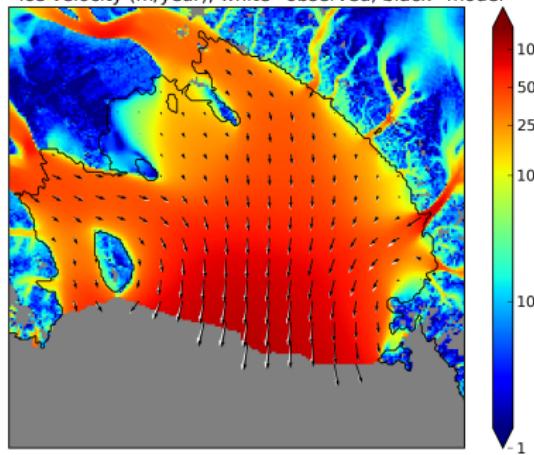
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- “diagnostic” (static geometry) ice shelf modeling in two horizontal variables has been quite successful
- observed surface velocities validate SSA stress balance
 - Ross ice shelf example below using PISM
 - many numerical models can do this

ice velocity (m/year); white=observed, black=model



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numerical solution of stress balances: a summary

- stress balance equations (e.g. SSA) determine horizontal velocity from geometry and boundary conditions
 - nonlinear so iteration is necessary
 - at each iteration a sparse matrix linear problem is solved
 - give the linear problem to a matrix solver software package

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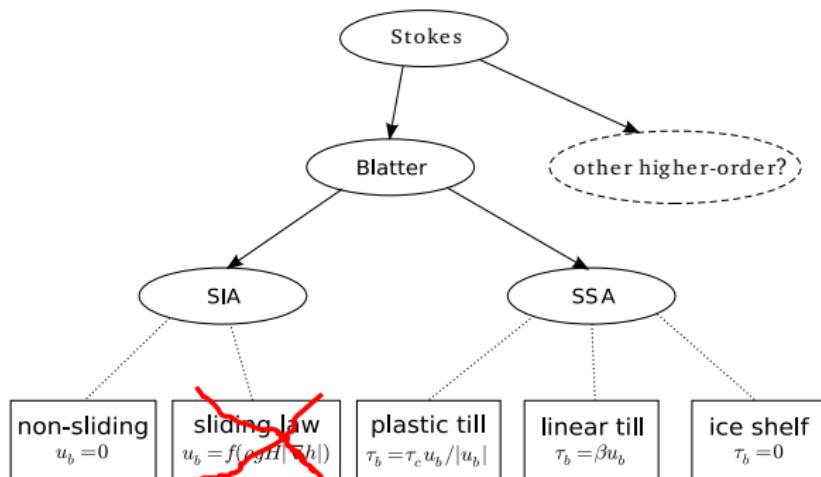
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- both the SIA and the SSA are derived by small-parameter arguments from the Stokes equations
- so: is there a *common* shallow antecedent model?
- Schoof and Hindmarsh (2009) answer: yes, the model derived by Blatter (1995) is one



- my advice: don't use the crossed-out one
 - sliding is not *locally* controlled by driving stress alone

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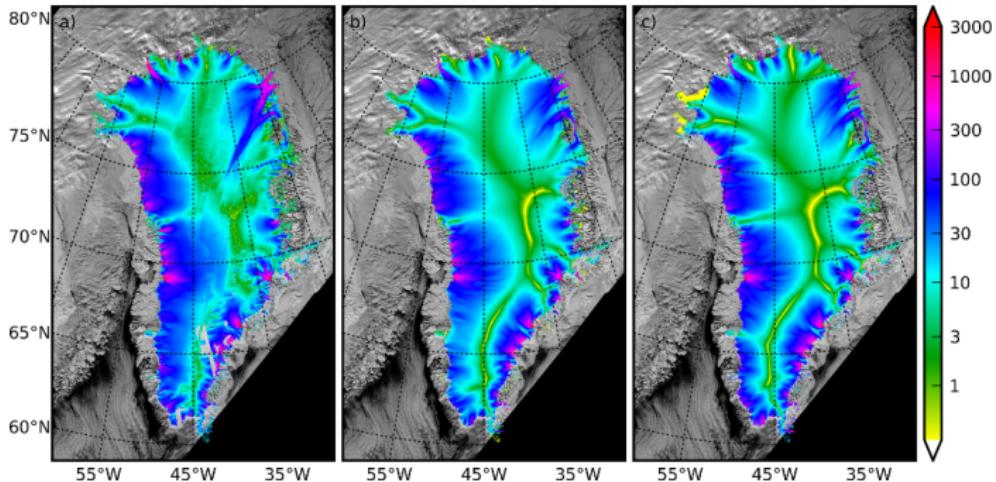
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shallow hybrid schemes

- so how about “gluing together” SIA and SSA in “hybrid”?
- there are multiple schemes (Pollard and deConto, 2007; Bueler and Brown, 2009; Goldberg 2011)
- shallow hybrids can be used *now* at high spatial resolution and paleoglacial time scales
- for example, we can compare observed (left) and 1km grid modeled (middle and right; PISM) surface velocity maps for the Greenland ice sheet



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the mass continuity equation: a summary

- the *mass continuity equation* is

$$H_t = M - \nabla \cdot (\mathbf{u}H)$$

- the numerical nature of this equation depends on the stress balance:

- the equation is a diffusion for frozen bed, large scale flows (i.e. SIA)
- it is *not* very diffusive for membrane stresses and no basal resistance (e.g. SSA for ice shelves)
- transitional for ice streams and outlet glaciers
 - that is: how much role for longitudinal stresses?
- not much helpful theory on this transport problem
- maybe *you* will help find this theory!

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- that's the end
- questions?

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POSTSCRIPT on technical skills

- next two slides are free advice on technical skills needed for numerical ice sheet modelling
 - you get what you pay for

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technical skills for numerical ice sheet modelling

- you need comfort in a technical computing environment, including
 - an editor,
 - a scripting/prototyping language (Matlab, Python, etc.),
 - a compiled language (C or Fortran),
 - a version control system (Subversion, git, etc.), and
 - some tools for NetCDF files
- you need willingness to read physics, numerical analysis, computer science, etc. books

technical skills for numerical ice sheet modelling 2

- you should *never* re-invent the wheel for basic numerics like these, except to write throw-away codes to help you understand them:
 - numerical linear algebra
 - mesh generation
 - finite element assembly and solve
- try existing open source ice flow models:
 - shallow comprehensive models:
 - GLIMMER
 - SICOPOLIS
 - PISM
 - open source higher-order/Stokes models:
 - Elmer/Ice
 - ISSM

POSTSCRIPT have I oversold diffusivity?

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- I have asserted that the default model for ice sheets, the SIA, is a *diffusion* like the heat equation
- recall the analogy:

<i>heat eqn</i>	↔	<i>SIA</i>
$T_t = \nabla \cdot (D \nabla T)$	↔	$H_t = M + \nabla \cdot (D \nabla h)$
$D = D(x, y)$	↔	$D = \Gamma H^{n+2} \nabla h ^{n-1}$

- have I oversold this diffusivity analogy?
 - possibly
 - I've acknowledged there are "issues"
- but it turns out there is some robustness to the analogy

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DIFFUSIVE IDEA 1: rough beds have the effect of *reducing* diffusivity

- define the local bed topography by removing the local mean over some range $\lambda \approx 10$ km:

$$\tilde{b}(x, \xi) = b(x + \xi) - \text{f}_{-\lambda}^{\lambda} b(x + \xi) d\xi$$

- define this average of the local bed:

$$\theta(x) = \left(\text{f}_{-\lambda}^{\lambda} \left(1 - \frac{\tilde{b}(x, \xi)}{H(x)} \right)^{-(n+2)/n} d\xi \right)^{-1/n}$$

- using a multiple-scales analysis, Schoof [2003] says you will get closer to solving the Stokes equations by making these two modifications of the SIA:
 - smooth the bed
 - but don't lose track of the smoothed-away local bed roughness; use it to reduce the diffusivity:

$$D_{\text{new}} = \theta D \quad \text{where } 0 \leq \theta \leq 1$$

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DIFFUSIVE IDEA 2: the large-scale effect of *sliding* in ice streams (*addressed in next section*), is also diffusive

- suppose that, for an ice stream modeled by the SSA equation

$$\left(2A^{-1/n}H|u_x|^{1/n-1}u_x\right)_x - C|u|^{m-1}u = \rho g H h_x$$

we assume that the basal resistance term balances the driving stress:

$$-C|u|^{m-1}u = \rho g H h_x$$

- then the ice stream geometry evolves by a non-SIA diffusion,

$$H_t = M + \nabla \cdot (D \nabla h) \quad \text{where } D = C' H^{(1/m)+1} |\nabla h|^{(1/m)-1}$$

- this “outer problem” is part of a matched asymptotic expansion that *does a good job of tracking the grounding line in a marine ice sheet* [Schoof, 2007]

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POSTSCRIPT on free boundary value problems

- ice sheet/shelf modelling means free boundary problems
- Hutter [1999] identifies some:

The physical system

a

atmosphere
climate input

cold ice

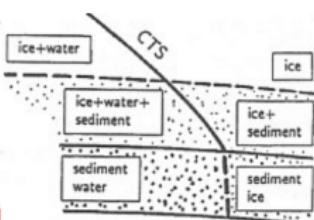
moving free
surface

moving Stefan surface, CTS

temperate ice

rock bed

b



moving front

moving ice ocean interface

ocean water

grounding line
(moving)

moving base

geothermal heat
input

fixed lower boundary

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what is a “free boundary”?

- a *free boundary* for a PDE is an unknown location at which there is a boundary condition
 - the location of the free boundary must be found at the same time as one solves the PDE problem
 - there must be enough additional information at a free boundary to determine its location
 - all free boundary problems are nonlinear, even if the PDE is linear
- classic example: consider an elastic membrane attached to a wire frame and stretched over an obstacle:

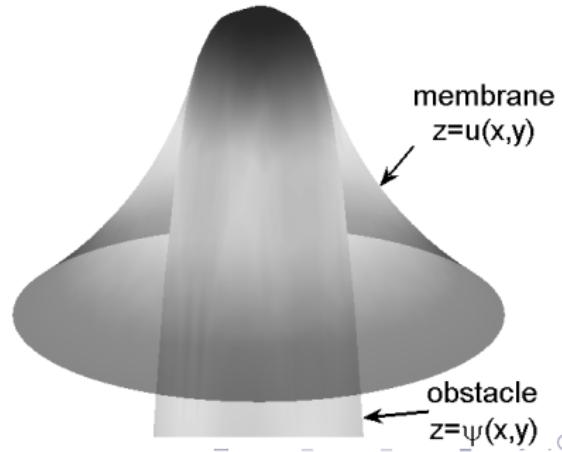
constraint:

$$u \geq \psi$$

in locations where $u > \psi$, solve:

$$\nabla^2 u = 0$$

where is the free boundary, and
what facts about u are true there?



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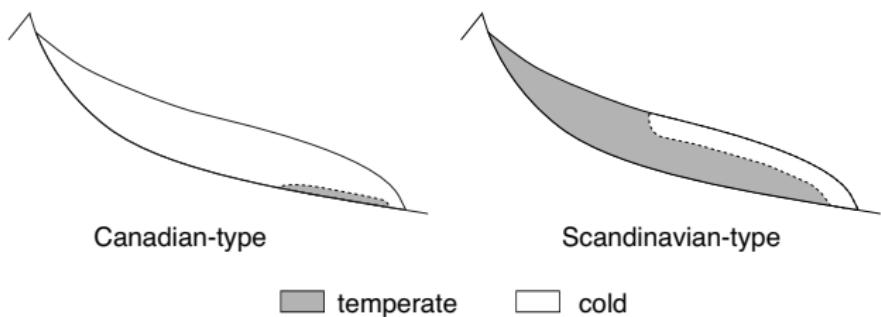
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free boundary value problem 1: polythermal ice

- by volume, majority of ice sheet is *cold* ($T < 0^\circ\text{C}$)
- ... but there is some ice which is *temperate*, where $T = 0^\circ\text{C}$ and there is a positive liquid fraction within the ice matrix
- ice sheets are *polythermal*
- boundary between cold and temperate ice is “CTS” (= cold-temperate transition surface):
 - must be found, as free boundary, when solving conservation of energy equation (“Stefan problem”)
 - can be tracked explicitly [Greve, 1997]
 - or treated as a level surface of the enthalpy variable [Aschwanden et al, 2012]



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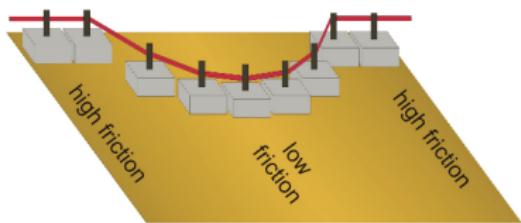
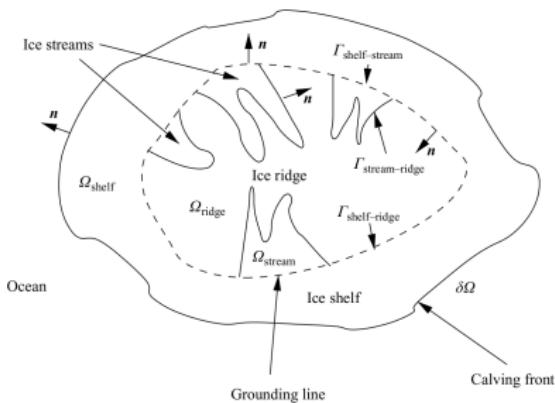
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free boundary value problem 2: for ice streaming

- Schoof's [2006] insight, for diagnostic case

$$\text{SSA} + (\text{plastic till}) = \left(\begin{array}{l} \text{well-posed free boundary problem} \\ \text{for location and velocity of sliding} \end{array} \right)$$

- "plastic till" means the basal strength (resistance) is given by a yield stress τ_c : $\vec{\tau}_b = \tau_c \mathbf{v}_b / |\mathbf{v}_b|$
- Schoof's scheme is a *whole ice sheet form* of MacAyeal's [1989] individual ice stream model



free boundary problem 3: for grounded ice sheet margin

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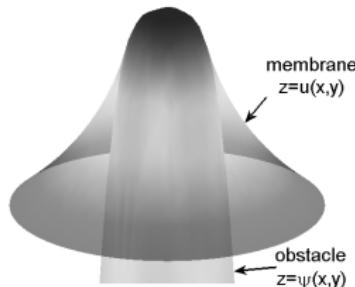
- side-by-side comparison, *classical elastic membrane problem versus steady ice sheet problem*
- Jouvet & Bueler [2012] show this is well-posed

constraint:

$$u \geq \psi$$

where $u > \psi$, solve:

$$\nabla^2 u = 0$$

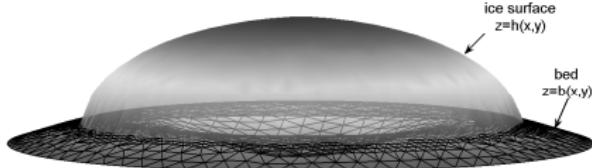


constraint:

$$h \geq b \iff H \geq 0$$

where $h > b$, solve steady SIA:

$$0 = M + \nabla \cdot \left(\Gamma H^{n+2} |\nabla h|^{n-1} \nabla h \right)$$



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free boundary problems: why do they matter?

- location of boundary may *be* the glaciological question
- free boundaries are always locations of *loss of smoothness* relative to fixed boundary solutions
 - numerical errors dominated by errors near free boundaries
 - are model results at free boundaries poor because of numerical problems or because of missing physical processes?