

Conservation
in
free-boundary fluid layer
models

Ed Bueler

Dept of Mathematics and Statistics, and Geophysical Institute
University of Alaska Fairbanks

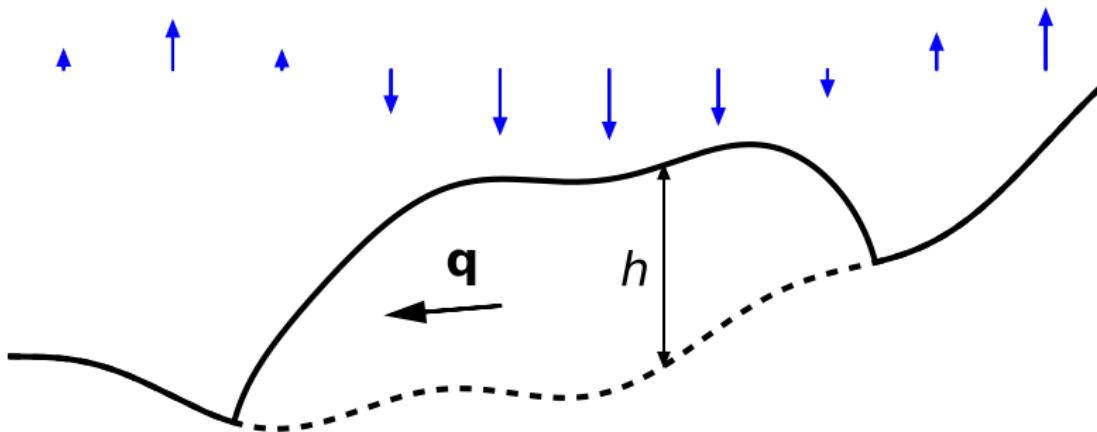
AGU 2014

Outline

- ① The problem I'm worried about:
Time-stepping free-boundary fluid layer models.

- ② Practical consequences:
Limitations in reporting discrete conservation.
Numerical weak free boundary solution needed.

A fluid layer in a climate



- mass conservation for a layer:

$$h_t + \nabla \cdot \mathbf{q} = f$$

- h is a thickness: $h \geq 0$
- mass conservation PDE applies *only where* $h > 0$
- \mathbf{q} is flow (vertically-integrated)
- source f is “climate”; $f > 0$ shown downward

Examples



glaciers



ice shelves & sea ice



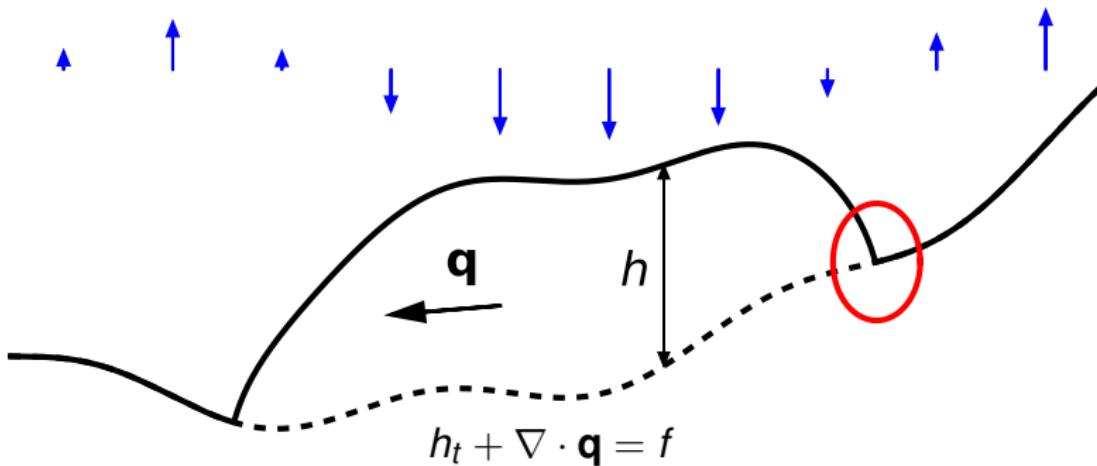
tidewater marsh

and subglacial hydrology, supraglacial runoff, surface hydrology, ...



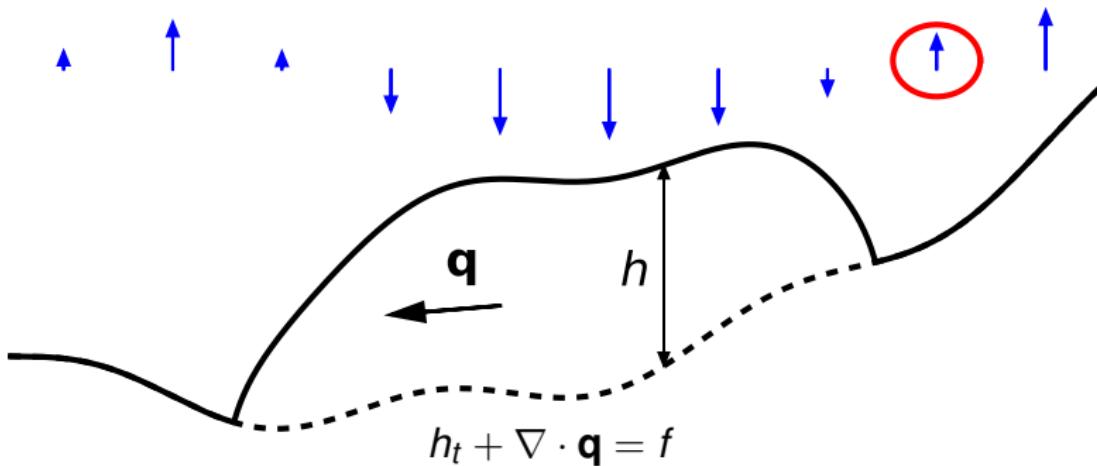
tsunami inundation

A fluid layer in a climate: *the troubles*



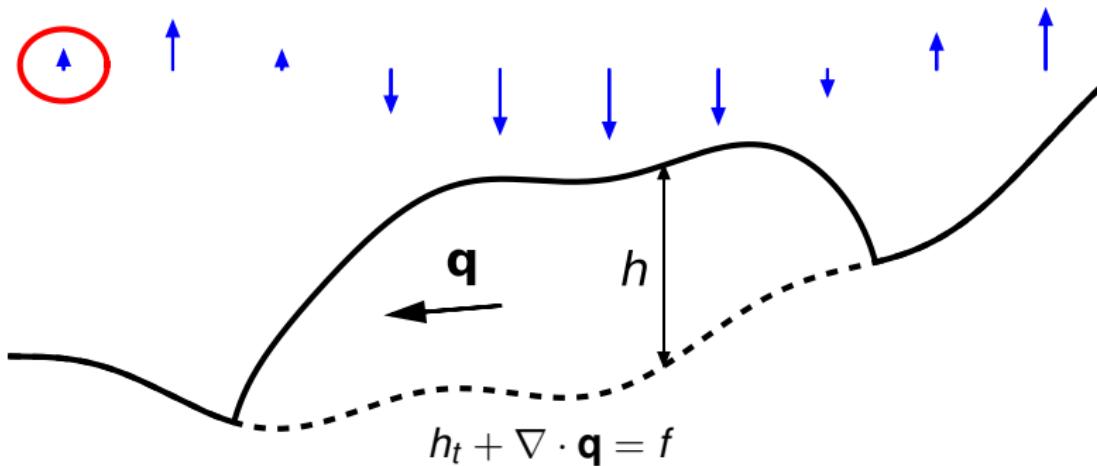
- $h = 0$ and what else at free boundary?
 - shape at free boundary depends on both \mathbf{q} and f
- $f < 0$ not “detected” by model where $h = 0$
 - how to do mass conservation accounting?
- $f \approx 0$ threshold behavior
 - $h > 0$ as soon as $f < 0$ switches to $f > 0$

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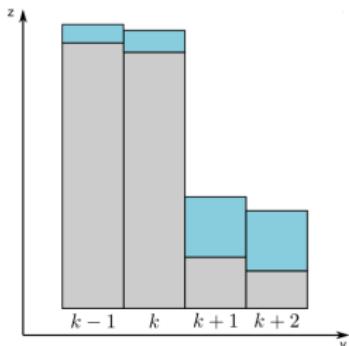
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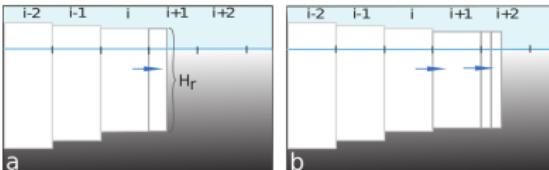
Anyone faced these problems before?

- yes, of course!
 - generic result: *ad hoc* schemes for finite volume/difference mass conservation near the free boundary

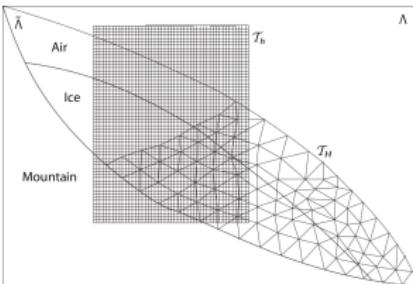


glacier ice
on steep terrain

(Jarosch, Schoof, Anslow, 2013)



volume-of-fluid method at ice shelf fronts
(Albrecht et al, 2011)



volume-of-fluid method (on fine grid) at
glacier surface
(Jouvet et al 2008)

Anyone faced these problems before?

- yes, of course!
 - generic result: *ad hoc* schemes for finite volume/difference mass conservation near the free boundary
- I don't mind “`if ... then ...`” in my code, *but* I want to know what mathematical problem it reflects!
- my goals:
 - redefine the problem so free boundary is part of solution
 - use numerical schemes which automate the details

Numerical models *must* discretize time

$$h_t + \nabla \cdot \mathbf{q} = f \quad \rightarrow \quad \frac{H_n - H_{n-1}}{\Delta t} + \nabla \cdot \mathbf{Q}_n = F_n$$

- semi-discretize in time: $H_n(x) \approx h(t_n, x)$
- the new equation is a “single time-step problem”
 - a PDE in space where $H_n > 0$
 - this PDE is the “strong form”
- details of flux \mathbf{Q}_n and source F_n come from time-stepping scheme
 - forward/backward Euler, trapezoid, RK all o.k.
 - note low regularity of $h(t, x)$ for x near margin

Weak form incorporates constraint

- define:

$$\mathcal{K} = \left\{ v \in W^{1,p}(\Omega) \mid v \geq 0 \right\} = \text{admissible thicknesses}$$

- define: $H_n \in \mathcal{K}$ solves the **weak single time-step problem** if

$$\int_{\Omega} H_n(v - H_n) - \Delta t \mathbf{Q}_n \cdot \nabla(v - H_n) \geq \int_{\Omega} (H_{n-1} + \Delta t F_n)(v - H_n)$$

for all $v \in \mathcal{K}$

- derive this *variational inequality* from:
 - ◊ the strong form *and*
 - ◊ integration-by-parts *and*
 - ◊ arguments about $H_n = 0$ areas

Theorem: weak solves strong

Theorem. Assume $\mathbf{Q}_n = 0$ when $H_n = 0$. Assume $H_n \in \mathcal{K}$ solves weak single time-step problem and is smooth. Then

- ① “interior condition” on set where $H_n > 0$:

$$\frac{H_n - H_{n-1}}{\Delta t} + \nabla \cdot \mathbf{Q}_n = F_n$$

- ② on set where $H_n = 0$:

$$H_{n-1} + \Delta t F_n \leq 0$$

re part 2:

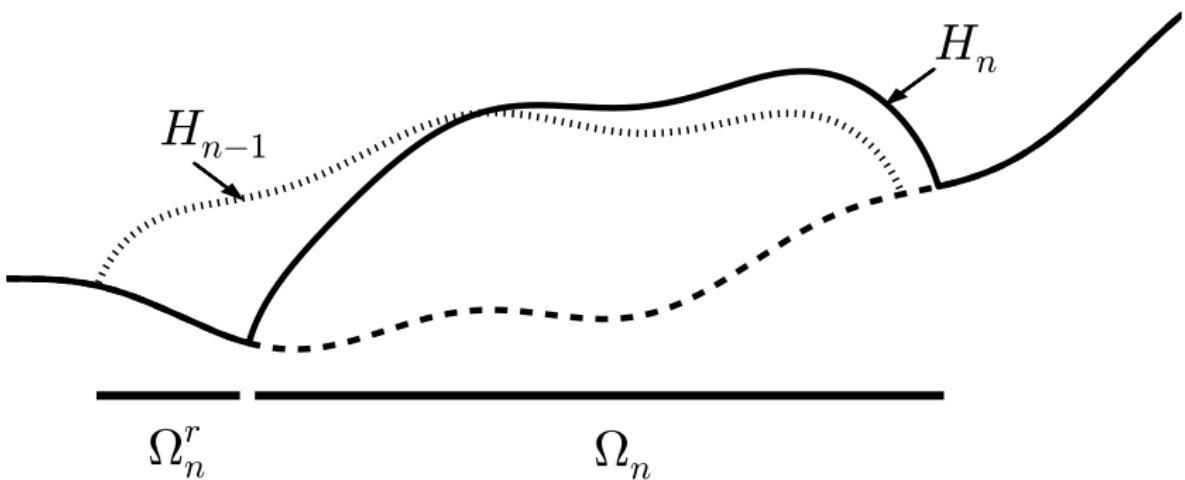
- “climate is negative enough to remove old thickness”
- assumption “ $\mathbf{Q}_n = 0$ when $H_n = 0$ ” is needed
 - ... yes, we are talking about a *layer*

Subsets for reporting conservation

- suppose H_n solves the weak single time-step problem
- define

$$\Omega_n = \text{supp } H_n = \{H_n(x) > 0\}$$

$$\Omega_n^r = \left\{ H_n(x) = 0 \text{ and } H_{n-1}(x) > 0 \right\} \quad \leftarrow \text{retreat set}$$



Reporting discrete conservation

- define:

$$M_n = \int_{\Omega} H_n(x) dx \quad \text{mass at time } t_n$$

- then

$$\boxed{\Delta t (-\nabla \cdot \mathbf{Q}_n + F_n)}$$

$$\begin{aligned} M_n - M_{n-1} &= \int_{\Omega_n} H_n - H_{n-1} dx + \int_{\Omega_n^r} 0 - H_{n-1} dx \\ &= \Delta t \left(0 + \int_{\Omega_n} F_n dx \right) - \int_{\Omega_n^r} H_{n-1} dx \end{aligned}$$

- new term:

$$R_n = \int_{\Omega_n^r} H_{n-1} dx \quad \text{retreat loss during step } n$$

Reporting discrete conservation: *claim*

- we want to “balance the books” for the model user
- the retreat loss R_n is not balanced by the climate
 - yes, R_n is caused by the climate, but we don’t know what *computable integral* it balances
- a numerical model must track three time series:
 - mass at time t_n : $M_n = \int_{\Omega} H_n(x) dx$
 - climate over fluid-covered region:

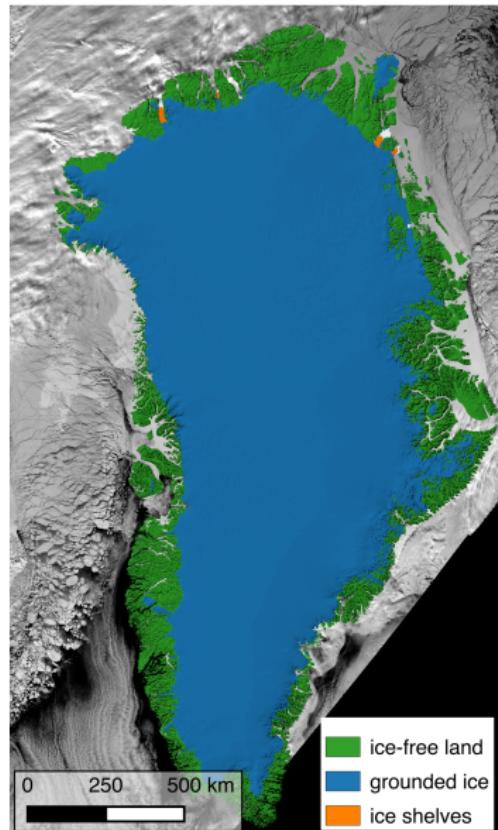
$$C_n = \Delta t \int_{\Omega_n} F_n dx \approx \int_{t_{n-1}}^{t_n} \int_{\Omega_n} f(t, x) dx dt$$

- retreat loss: $R_n = \int_{\Omega_n^r} H_{n-1} dx$
- now it is balanced:

$$M_n = M_{n-1} + C_n - R_n$$

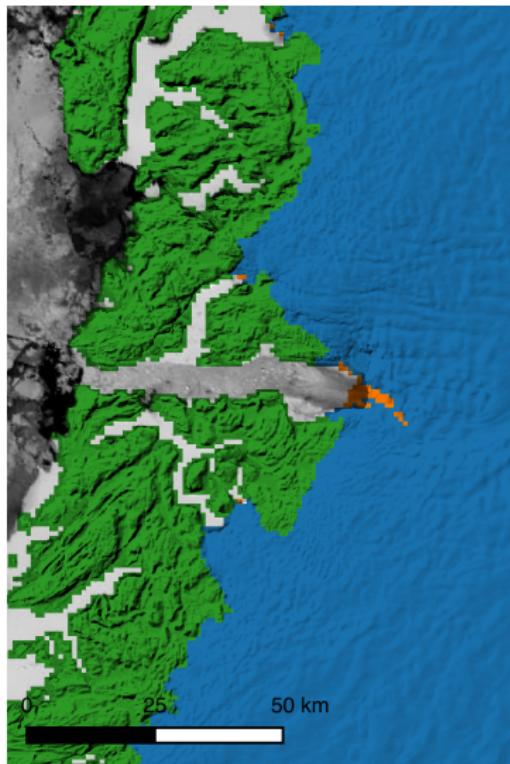
I am driven by practical modeling

- practical ice sheet modeling
(e.g. Greenland at right)
- icy region nearly-fractal and disconnected
- currently in PISM*:
 - explicit time-stepping
 - free boundary by truncation
- want for PISM:
 - implicit time steps *with free boundary*
 - better conservation accounting to user



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PISM whole Greenland at 900m
see Aschwanden C51A-0245 on 12/19

* = Parallel Ice Sheet Model, pism-docs.org

Numerical solution of the weak problem

the weak single time-step problem:

- is nonlinear because of constraint (even for \mathbf{Q}_n linear in H_n)
- can be solved by Newton method modified for constraint
 - reduced set method
 - semismooth method
- scalable implementations of both in PETSc 3.5
 - SNESVI class

Well-posedness of the weak problem

- I've been agnostic on form of \mathbf{Q}_n
 - except " $\mathbf{Q}_n = 0$ where $H_n = 0$ " (i.e. it's a layer)
- but form of \mathbf{Q}_n matters
 - for well-posedness and for numerical solutions
- cases to study:

$$\mathbf{Q}_n = \mathbf{X}(x)H_n \quad \textit{transported layer}$$

$$\mathbf{Q}_n = -k\nabla H_n \quad \textit{linear diffusion}$$

$$\mathbf{Q}_n = -\nabla(H_n^\gamma) = -\gamma H_n^{\gamma-1} \nabla H_n \quad \textit{porous medium}$$

$$\mathbf{Q}_n = -H_n^\alpha |\nabla(H_n + b)|^\beta \nabla(H_n + b) \quad \textit{shallow ice approx. & diffusive shallow water}$$

$$\mathbf{Q}_n = \text{worse (non-local)} \quad \textit{ice shelf flow & sea ice & ...}$$

- variational inequality is generally monotone
 - generally coercive if $\mathbf{Q}_n \sim -\nabla H_n$

Two numerical examples

pop quiz:

- same equation
$$\frac{H_n - H_{n-1}}{\Delta t} + \nabla \cdot \mathbf{Q}_n = f$$
- same climate f
- same bed shape
- same constrained-Newton scheme

how different
are the \mathbf{Q}_n ?

Two numerical examples

$\mathbf{Q}_n = v_0 H_n$
hyperbolic (constant vel.)

$\mathbf{Q}_n = -\Gamma |H_n|^{n+2}$
 $\cdot |\nabla h_n|^{n-1} \nabla h_n$
highly-nonlinear diffusion

Terminal slide

- I'm considering layer flow problems:
 - model has conservation eqn: $h_t + \nabla \cdot \mathbf{q} = f$
- suggestions:
 - *include* constraint on thickness: $h \geq 0$
 - pose single time-step problem *weakly* as variational inequality
 - solve single time-step problem numerically by constrained-Newton method
- claim: for *any* numerical approach,
 - exact discrete conservation requires tracking *retreat loss*
 - ◊ in addition to computable integrals of climate
 - but it *isn't really possible* except in $\Delta t \rightarrow 0$ limit