

Free boundaries and conservation equations in ice sheet models

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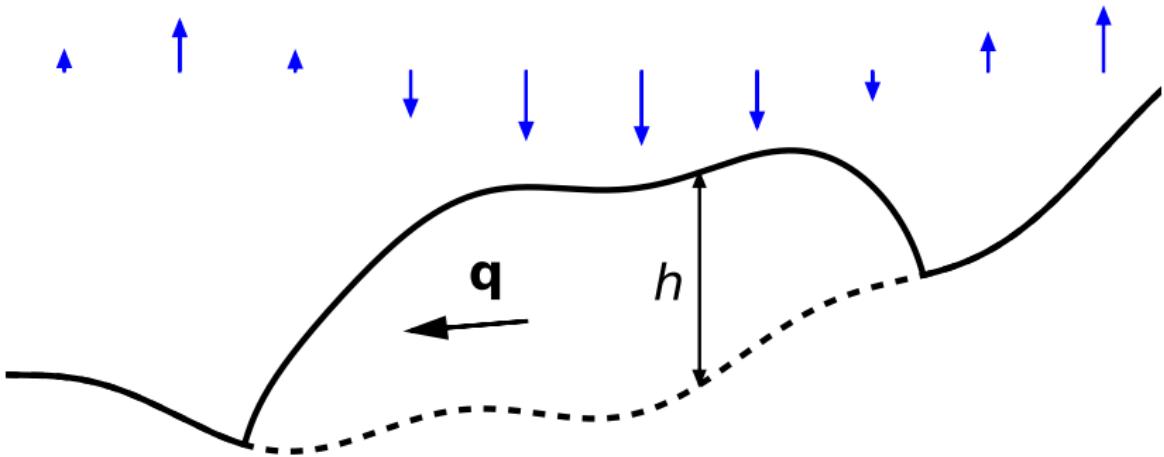
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Outline

- 1 The problem I'm worried about:
 - Time-stepping free-boundary fluid layer models.
- 2 Practical consequences:
 - Limitations to discrete conservation.
 - Need for weak numerical free boundary solutions.

A fluid layer in a climate

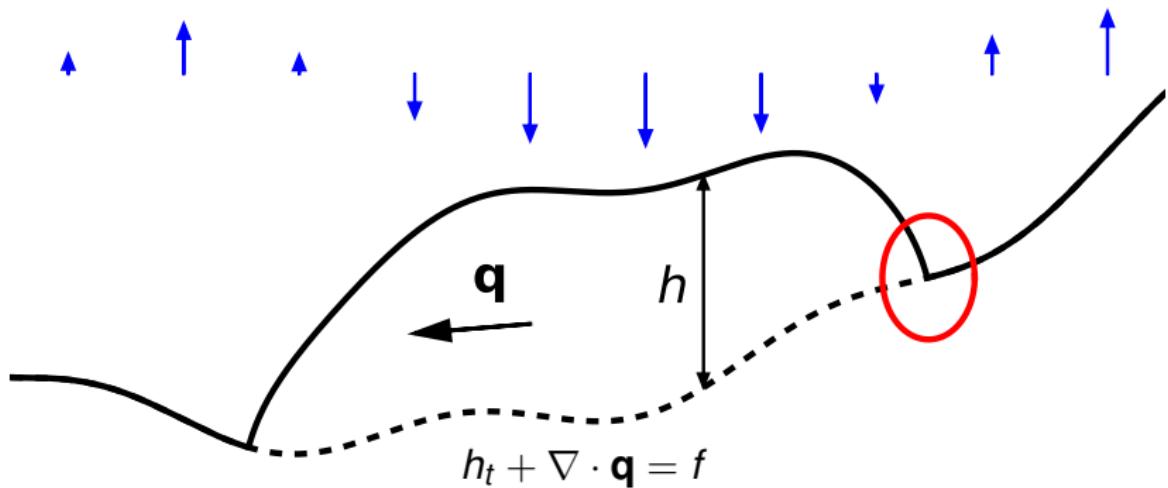


- mass conservation PDE for a layer:

$$h_t + \nabla \cdot \mathbf{q} = \mathbf{f}$$

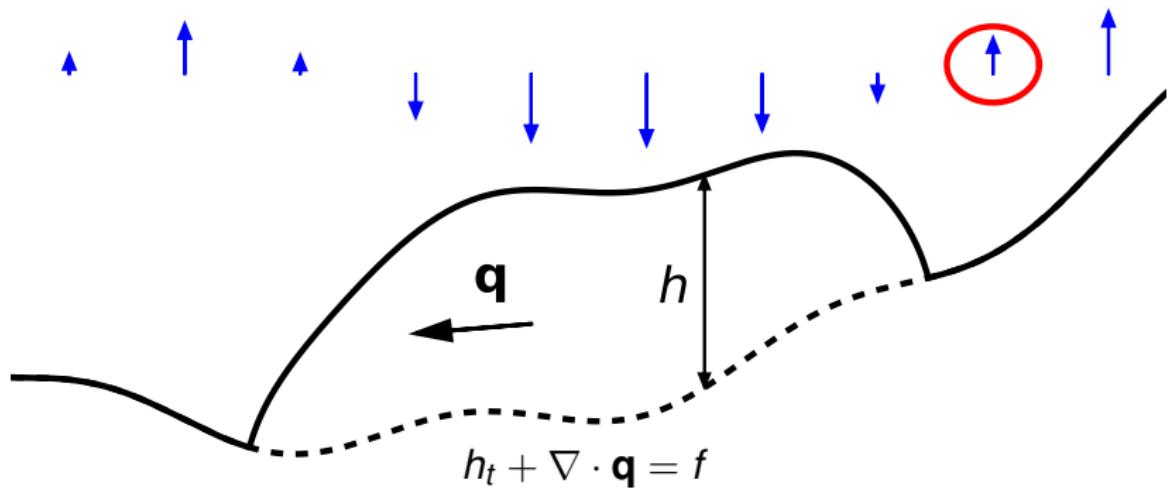
- h is a thickness: $h \geq 0$
- mass conservation PDE applies *only where* $h > 0$
- \mathbf{q} is flow (vertically-integrated)
- source \mathbf{f} is “climate”; $\mathbf{f} > 0$ shown downward

A fluid layer in a climate: *the troubles*



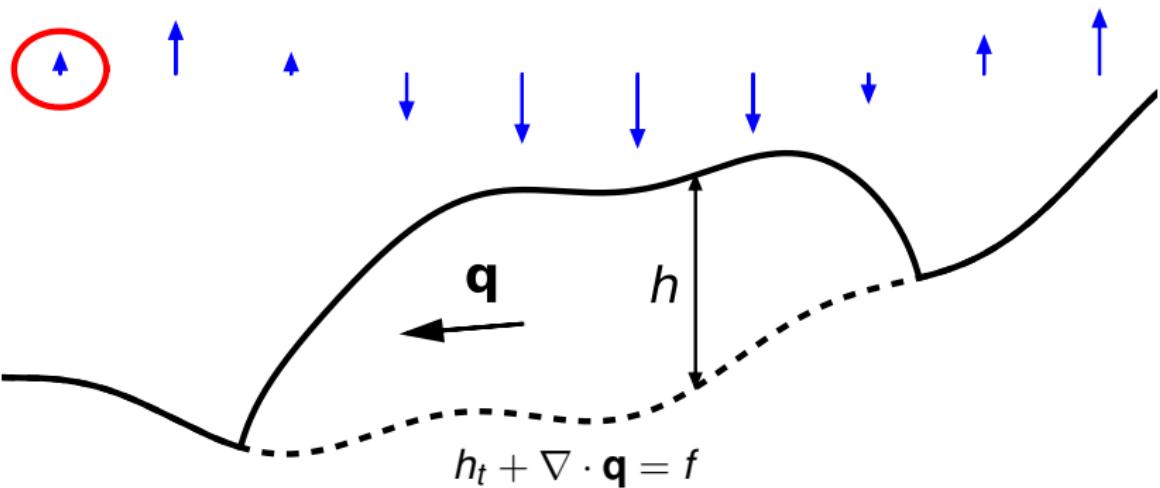
- $h = 0$ and what else at free boundary?
 - shape at free boundary depends on both \mathbf{q} and f
- $f < 0$ not “detected” by model where $h = 0$
 - how to do mass conservation accounting?
- $f \approx 0$ threshold behavior
 - $h > 0$ as soon as $f < 0$ switches to $f > 0$

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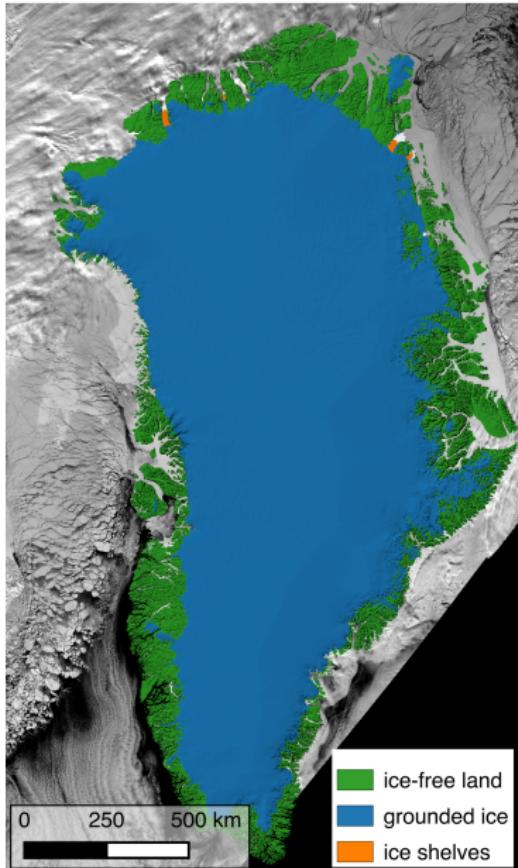
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A concern driven by practical modeling

- the icy region is nearly-fractal and disconnected
- currently in PISM*:
 - explicit time-stepping
 - free boundary by truncation
- want for PISM:
 - implicit time steps
 - better conservation accounting to user



*= Parallel Ice Sheet Model, pism-docs.org

Examples



glaciers



ice shelves & sea ice



tidewater marsh

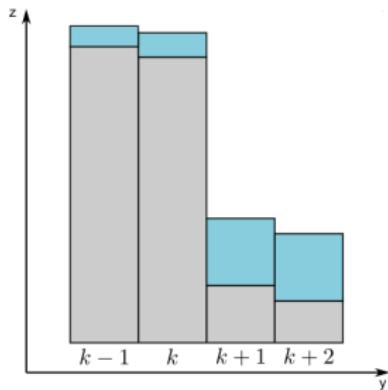
and subglacial hydrology, supraglacial runoff, surface hydrology, ...



tsunami inundation

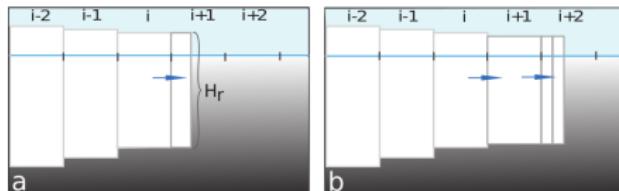
Anyone numerically-solved these problems before?

- yes, of course!
 - generic result: *ad hoc* schemes near the free boundary

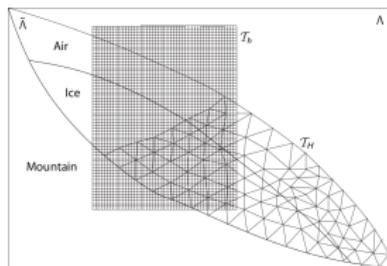


glacier ice
on steep terrain

(Jarosch, Schoof, Anslow, 2013)



volume-of-fluid method at ice shelf fronts
(Albrecht et al, 2011)



volume-of-fluid method at glacier surface
(Jouvet et al 2008)

New goals

- I don't mind "if ...then ..." in my code, *but* I want to know what mathematical problem is behind it
 - maintaining code with those *ad hoc* schemes scares the #!*& out of me
- my goals:
 - redefine the problem so free boundary is part of solution
 - tell the model user what is going on at the free boundary
 - find numerical schemes which automate the details

Numerical models *must* discretize time

$$h_t + \nabla \cdot \mathbf{q} = f \quad \rightarrow \quad \frac{H_n - H_{n-1}}{\Delta t} + \nabla \cdot \mathbf{Q}_n = F_n$$

- semi-discretize in time: $H_n(x) \approx h(t_n, x)$
- the new equation is a “single time-step problem”
 - a PDE in space **where $H_n > 0$**
 - called the “strong form”
- details of flux \mathbf{Q}_n and source F_n come from time-stepping scheme

1D time-stepping examples

same:

- equation

$$\frac{H_n - H_{n-1}}{\Delta t} + \nabla \cdot \mathbf{Q}_n = f$$

- climate f

- bed shape

- constrained-
Newton scheme

how different
are the fluxes
 \mathbf{Q}_n ?

1D time-stepping examples

$$\mathbf{Q}_n = v_0 H_n$$

hyperbolic

(constant velocity)

$$\mathbf{Q}_n = -\Gamma |H_n|^{n+2}$$

$$\cdot |\nabla h_n|^{n-1} \nabla h_n$$

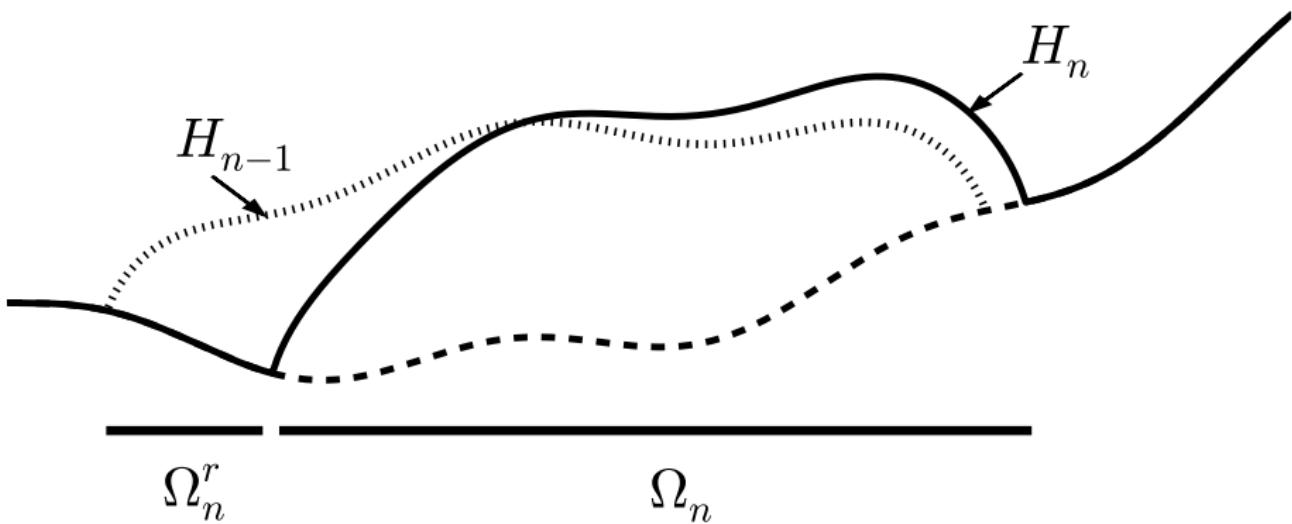
highly-nonlinear diffusion

Subsets for time-stepping and conservation

- suppose H_n solves the single time-step problem
- define

$$\Omega_n = \{H_n(x) > 0\}$$

$$\Omega_n^r = \{H_n(x) = 0 \text{ and } H_{n-1}(x) > 0\} \quad \leftarrow \text{retreat set}$$



Reporting discrete conservation

- define:

$$M_n = \int_{\Omega} H_n(x) dx \quad \text{mass at time } t_n$$

- then

$$\boxed{\Delta t (-\nabla \cdot \mathbf{Q}_n + F_n)}$$

$$\begin{aligned} M_n - M_{n-1} &= \int_{\Omega_n} H_n - H_{n-1} dx + \int_{\Omega_n^r} 0 - H_{n-1} dx \\ &= \Delta t \left(0 + \int_{\Omega_n} F_n dx \right) - \int_{\Omega_n^r} H_{n-1} dx \end{aligned}$$

- new term:

$$R_n = \int_{\Omega_n^r} H_{n-1} dx \quad \text{retreat loss during step } n$$

Reporting discrete conservation: *limitation*

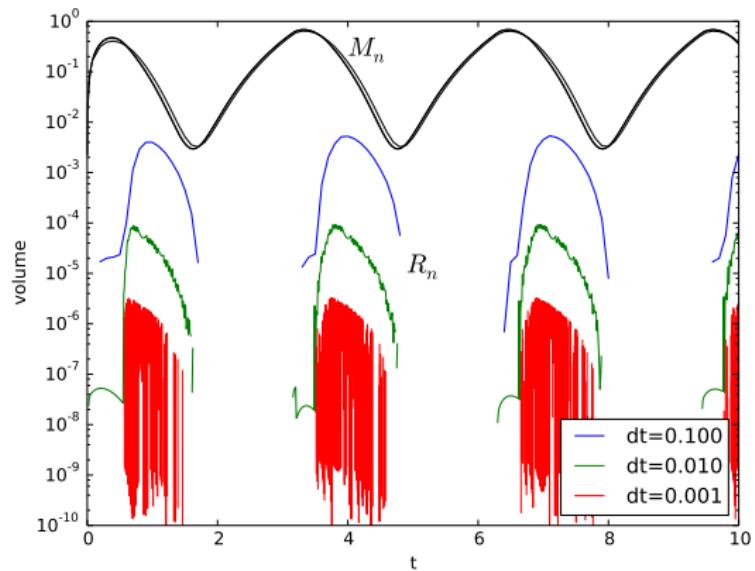
- the retreat loss R_n is not balanced by the climate
 - R_n is *caused* by the climate, but we don't know a *computable integral* to balance it
- we must track **three** time series:
 - mass at time t_n : $M_n = \int_{\Omega} H_n(x) dx$
 - climate (e.g. surface mass bal.) over current fluid-covered region:

$$C_n = \Delta t \int_{\Omega_n} F_n dx \approx \int_{t_{n-1}}^{t_n} \int_{\Omega_n} f(t, x) dx dt$$

- retreat loss during time step: $R_n = \int_{\Omega_n^r} H_{n-1} dx$
- now it balances:

$$M_n = M_{n-1} + C_n - R_n$$

Reporting discrete conservation: $R_n \rightarrow 0$ as $\Delta t \rightarrow 0$



Weak form incorporates constraint

- define:

$$\mathcal{K} = \left\{ v \in W^{1,p}(\Omega) \mid v \geq 0 \right\} = \text{admissible thicknesses}$$

- we say $H_n \in \mathcal{K}$ solves the **weak single time-step problem** if

$$\int_{\Omega} H_n(v - H_n) - \Delta t \mathbf{Q}_n \cdot \nabla(v - H_n) \geq \int_{\Omega} (H_{n-1} + \Delta t F_n)(v - H_n)$$

for all $v \in \mathcal{K}$

- derive this *variational inequality* from:
 - the strong form *and*
 - integration-by-parts *and*
 - arguments about $H_n = 0$ areas

Weak solves strong, *and* it gives more info

- assume $\mathbf{Q}_n = 0$ when $H_n = 0$
 - this means \mathbf{Q}_n describes a *layer*
- assume $H_n \in \mathcal{K}$ solves weak single time-step problem
- then
 - ➊ PDE applies on the set where $H_n > 0$:

$$\frac{H_n - H_{n-1}}{\Delta t} + \nabla \cdot \mathbf{Q}_n = F_n$$

- ➋ information on the set where $H_n = 0$:

$$H_{n-1} + \Delta t F_n \leq 0$$

- this means “mass balance was negative enough during time step to remove old thickness”

Numerical solution of the weak problem

the weak single time-step problem:

- is nonlinear because of constraint (even for \mathbf{Q}_n linear in H_n)
- can be solved by a Newton method modified for constraint
 - reduced set method
 - semismooth method
- scalable implementations are in PETSc 3.5
 - see “SNESVI” object

Summary

- layer flow model has conservation eqn. $h_t + \nabla \cdot \mathbf{q} = f$
 - long time steps wanted, but this is a free-boundary problem ...
- claim: exact discrete conservation requires tracking *retreat loss*
 - in addition to computable integrals of climate
 - it only disappears in $\Delta t \rightarrow 0$ limit
- suggestions:
 - *include* constraint on thickness: $h \geq 0$
 - pose single time-step problem *weakly* as variational inequality
 - solve it numerically by constrained-Newton method
- these are agnostic claims/suggestions, with respect to:
 - form of the flux \mathbf{q}
 - spatial discretization paradigm (i.e. finite diff./volume/element)