3.3 Systems of first-order ODEs are models of everything

a lecture for MATH F302 Differential Equations

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Fall 2023

for textbook: D. Zill, *A First Course in Differential Equations with Modeling Applications*, 11th ed.

first-order systems

- a *system of two first-order equations*:

$$\frac{dx}{dt} = f(t, x, y)$$
$$\frac{dy}{dt} = g(t, x, y)$$

  - the solution is the pair of functions $x(t), y(t)$
  - we say system is *coupled* if f depends on y or g depends on x

- f and g can be any formulas; here's a silly example:

$$\frac{dx}{dt} = t^5 + x^6 + y^7$$
$$\frac{dy}{dt} = \arctan(y + \sin(x + \cos(t)))$$

easily-solvable example

- *example 1.* find the general solution to

$$\frac{dx}{dt} = -2x$$
$$\frac{dy}{dt} = x - y$$

solution.

special properties: (1) one-way coupled and (2) linear and (3) homogeneous

the system can be any size

- notation for two equations:

$$\frac{dx_1}{dt} = g_1(t, x_1, x_2)$$
$$\frac{dx_2}{dt} = g_2(t, x_1, x_2)$$

- system of n equations:

$$\frac{dx_1}{dt} = g_1(t, x_1, x_2, \dots, x_n)$$
$$\frac{dx_2}{dt} = g_2(t, x_1, x_2, \dots, x_n)$$
$$\vdots$$
$$\frac{dx_n}{dt} = g_n(t, x_1, x_2, \dots, x_n)$$

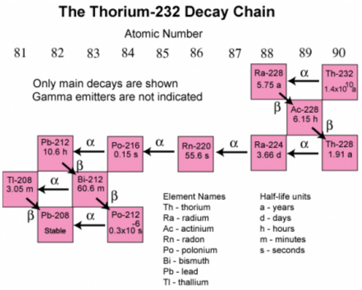
  - solution is set of n functions $x_1(t), x_2(t), \dots, x_n(t)$
  - in practical, modern fluids simulations: $n \geq 10^6$
  - such systems are also the physics in video games

most math models are systems of DEs

- systems of ODEs are common
- ...because most real things involve
  - many parts $x_1, \dots, x_n$
  - changing in time $\frac{dx_i}{dt} = g_i(\dots)$
  - interacting with each other g_i depends on x_j
- everything is modeled this way:
  1. populations of hares and lynx
  2. the galaxy
  3. your body

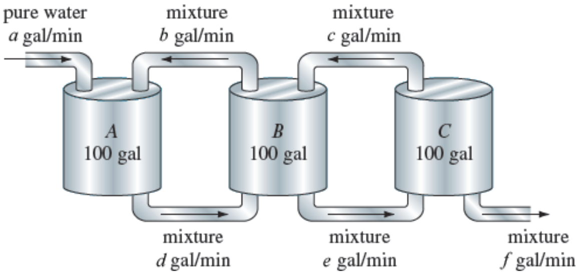
radioactive decay series

- read about it in §3.3
    - often one-way coupled
    - simple cases can be easy/solvable (e.g. example 1)



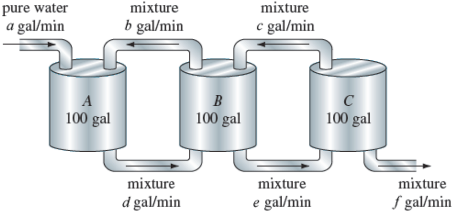
- *example 2.* Three 100 gallon tanks have brine solutions and are connected as shown. The tanks are always full. $x_1(t), x_2(t), x_3(t)$ pounds of salt are in each tank, respectively.

  (a) What equations must hold for the flow rates a, b, c, d, e, f?
  (b) Suppose $a = 2, d = 4, e = 5$ in gal/min. Compute b, c, f.
  (c) Write a first-order ODE system for $x_1(t), x_2(t), x_3(t)$.



connected tanks, cont.

solution.



given $a = 2, d = 4, e = 5$

higher order equations become systems

- any individual (a.k.a. *scalar*) ODE can be turned into a first-order system
- for example, a damped nonlinear pendulum for $\theta(t)$:

$$m\ell\theta'' + \beta\theta' + mg\sin\theta = 0$$

becomes this system:

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= -\left(\frac{\beta}{m\ell}\right)x_2 - \left(\frac{g}{\ell}\right)\sin(x_1) \end{aligned}$$

  - just name θ as x_1 and name θ' as x_2
  - solve for the derivative because that is the standard form

a 4th order ODE as a system

- *example 3.* write the following fourth-order ODE as a first-order system:

$$y^{(4)} - 4y''' + 7y'' + 10y' - y = \sin(3t)$$

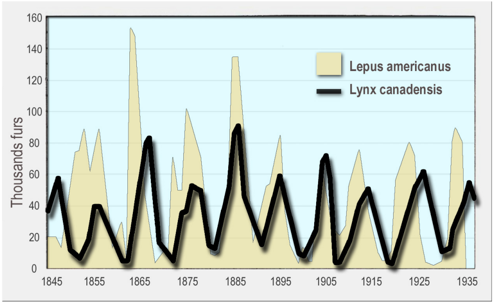
solution.

hares and lynx

- consider this "Lotka-Volterra" model

$$\frac{dx}{dt} = 0.7x - 1.3xy$$
$$\frac{dy}{dt} = xy - y$$

  - $x(t)$ is the number of prey
  - $y(t)$ is the number of predators
  - constants merely representative

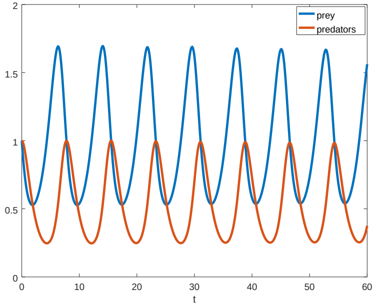


- *example 4.* solve numerically for $0 \leq t \leq 60$:

$$\frac{dx}{dt} = 0.7x - 1.3xy \qquad x(0) = 1$$
$$\frac{dy}{dt} = xy - y \qquad y(0) = 1$$

solution.

```
>> f = @(t,z) [0.7*z(1)-1.3*z(1)*z(2); z(1)*z(2)-z(2)];
>> [tt,zz] = ode45(f,0:.1:60,[1;1]);
>> plot(tt,zz),  xlabel t
>> legend('prey','predators')
```



phase plane: a different view

- a different view is to plot $x = z_1$ versus $y = z_2$

```
>> figure(2)
>> plot(zz(:,1),zz(:,2),'k')    % curve in black
>> xlabel('x(t)  prey'), ylabel('y(t)  predators')
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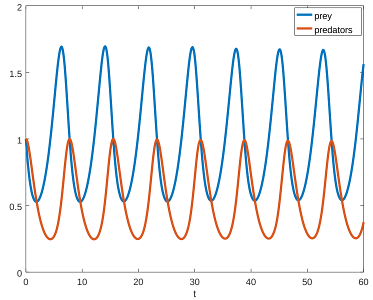
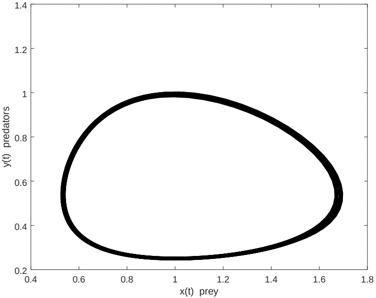




- we will get back to this view in Chapter 8

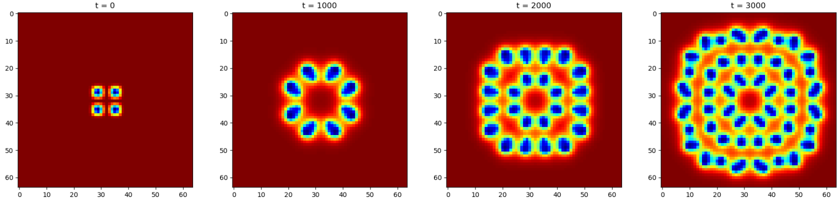
beyond: PDEs and pattern generation

- consider this system of *partial* differential equations:

$$\frac{\partial u}{\partial t} = D_u \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) - uv^2 + \phi(1-u)$$
$$\frac{\partial v}{\partial t} = D_v \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) + uv^2 - (\phi+\kappa)v$$

  - D_u, D_v, ϕ, κ are all constants

- it is a model[1] of a reaction between two chemicals u and v, similar to Lotka-Volterra (predator-prey) system
- and a model of how the leopard got her spots



[1] J. E. Pearson (1993). *Complex patterns in a simple system*, Science, 261, 189–192

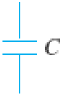
ODE systems from circuits

- the voltage $v(t)$ and current $i(t)$ in an electrical circuits changes in time
- each element in a circuit (network) has a little model:

resistor $v = iR$

inductor $v = L\frac{di}{dt}$

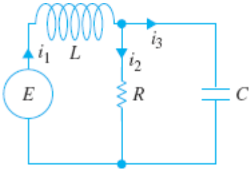
resistor $v = \frac{q}{C}$



- *Kirchoff's laws* allow you to assemble systems of ODEs from these elements
- building such models is the heart of electical engineering

a linear ODE system for an RLC circuit

- I'll do an example, but you are *not* responsible for doing this!
- *example 5.* construct a system of first-order ODEs for the currents i_1, i_2, i_3 in this electical circuit



expectations

to learn this material, just listening to a lecture is *not* enough

- *read* section 3.3
  - what are you actually responsible for? be able to do computations like in examples 1–4
  - ... *and* be able to do radioactive decay series examples
    - read the section!
  - you are *not* responsible for electrical circuits as in example 5
- do Homework 3.3