2.2 Separable Equations

a lecture for MATH F302 Differential Equations

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for textbook: D. Zill, *A First Course in Differential Equations with Modeling Applications*, 11th ed.

regarding chapters 1 and 2

- the purpose of textbook chapter 1 is to sketch the language and meaning of differential equations
- the purpose of chapter 2 is to actually solve some differential equations by hand
    - section 2.2 is the first example: separable equations
    - warning: by-hand methods can be difficult or impossible on a given differential equation
    - however, the examples where we know how to solve are often important in practice

separable differential equations

Definition

a *separable* differential equation can be put in the form

$$\boxed{\frac{dy}{dx} = g(x)h(y)}$$

- Example.
$$\frac{dy}{dx} = \frac{y\cos(x)}{1+y^2}$$
- **Not** an example.
$$\frac{dy}{dx} = \cos(x) + y$$
- Also **not** an example.
$$\frac{dy}{dx} = \sin(x+y^2)$$

emphasis on *can*

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- Example.

$$p(y)\frac{dy}{dx} = g(x)$$

emphasis on *can*

Definition

a *separable* differential equation can be put in the form

$$\boxed{\frac{dy}{dx} = g(x)h(y)}$$

- Example.

$$p(y)\frac{dy}{dx} = g(x)$$

define $h(y) = 1/p(y)$ to make into standard separable form

how to solve separable equations?

- move y stuff to left and x to right:

$$\frac{dy}{dx} = g(x)h(y)$$

$$\frac{1}{h(y)}\frac{dy}{dx} = g(x)$$

$$\frac{1}{h(y)}\,dy = g(x)\,dx$$

how to solve separable equations?

- move y stuff to left and x to right:

$$\frac{dy}{dx} = g(x)h(y)$$

$$\frac{1}{h(y)}\frac{dy}{dx} = g(x)$$

$$\frac{1}{h(y)}\,dy = g(x)\,dx$$

- integrate both sides:

$$\int \frac{1}{h(y)}\,dy = \int g(x)\,dx$$

how to solve separable equations?

- alternative appearance with $p(y) = 1/h(y)$:

$$p(y)\frac{dy}{dx} = g(x)$$

how to solve separable equations?

- alternative appearance with $p(y) = 1/h(y)$:

$$p(y)\frac{dy}{dx} = g(x)$$

- move y stuff to left and x to right:

$$p(y)\, dy = g(x)\, dx$$

- integrate both sides:

$$\int p(y)\, dy = \int g(x)\, dx$$

why does it work?

- the method works because of the chain rule
- the integrals you are really doing are both with respect to x:

$$\int p(y(x))\,\frac{dy}{dx}\,dx = \int g(x)\,dx$$

how do you finish up?

- once you do the integrals

$$\int p(y)\, dy = \int g(x)\, dx$$

  then solve for y, if possible, to get an explicit solution
- if you cannot solve for y then the solution remains implicit

example 1

- example: find $y(x)$ if

$$\frac{dy}{dx} = xy^2$$

example 2

- some familiar equations are also separable
- example: find $y(x)$ if

$$\frac{dy}{dx} = -5y$$

example 3

- what if there are initial conditions?
- example: find $z(t)$ if $z(4) = 1$ and

$$z' = \frac{e^{-z}}{t}$$

example 4

- you may end up only knowing the solution implicitly
- example: find $y(x)$ if

$$\frac{dy}{dx} = \frac{x(1-x)}{y(2+y)}$$

standard expectations

to learn this material, just listening to a lecture is *not* enough

- please *read* section 2.2 in the textbook
- please *do* the Homework for section 2.2
- search "separable ODEs" at YouTube to see more examples