2.3 Linear Equations

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D. Zill, *A First Course in Differential Equations with Modeling Applications*, 11th ed.

a *linear* ordinary differential equation has a first power on both dy/dx and y *and* it can be put in the form

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

or

$$\frac{dy}{dx} + P(x)y = g(x)$$

main idea: we can write solutions to such equations in terms of integrals

linear equation standard form:

$$\frac{dy}{dx} + P(x)y = g(x)$$

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$$\frac{dy}{dx} + P(x)y = g(x)$$

- **Example**.

$$\frac{dy}{dx} + y = x + 3$$

Here $P(x) = 1$ and $g(x) = x + 3$.

linear equation standard form:

$$\frac{dy}{dx} + P(x)y = g(x)$$

- **Example.**

  $$tz' = z + \cos t$$

  which is the same as

  $$\frac{dz}{dt} + \frac{-1}{t}z = \frac{\cos t}{t}$$

  with $P(t) = -1/t$ and $g(t) = \cos t/t$.

linear equation standard form:

$$\frac{dy}{dx} + P(x)y = g(x)$$

- **Not an example**.

$$y\frac{dy}{dx} = x + e^x$$

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  ... but it is *separable* (section 2.2)

before giving general formulas, here's how the method works on an example

- **Example**.

$$\frac{dy}{dx} + y = x + 3$$

to solve a first-order, linear ordinary differential equation $y' + Py = g$ we multiply by a factor which allows us to *un*do the product rule for $y' + Py = g$:

1. find $\mu(x)$ so that $\mu'(x) = P(x)\mu(x)$
2. multiply both sides by μ:

$$\mu y' + \mu P y = \mu g$$

3. recognize product rule:

$$(\mu y)' = \mu g$$

4. integrate:

$$\mu(x)y(x) = \int \mu(x)g(x)\,dx$$

5. solve for y:

$$y(x) = \mu(x)^{-1} \int \mu(x)g(x)\,dx$$

there is a formula for the *integrating factor* $\mu(x)$:

$$\mu(x) = e^{\int P(x)\,dx}$$

- **Example with initial condition**

$$\frac{dy}{dx} + y = x + 3, \qquad y(0) = 3$$

visualization of

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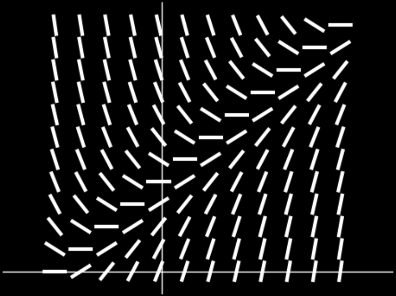

visualization of

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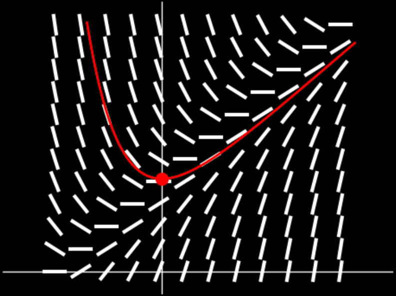
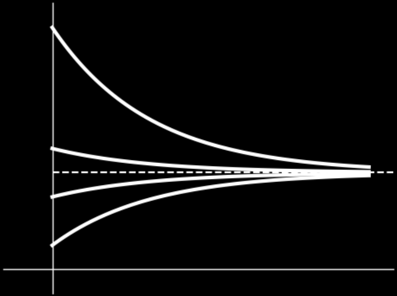


- **Example**. Newton's law of cooling

$$\frac{dT}{dt} = k(T_m - T), \qquad T(0) = T_0$$

where k, T_m, T_0 are constants



- **Example.**

$$x^2y' + x(x+2)y = e^x$$

Expectations

- *read* section 2.3, including
  - an example with piece-wise functions
  - the error function as an example
- *doing exercises* is essential