6.1 Series solutions about ordinary points

a lecture for MATH F302 Differential Equations

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for textbook: D. Zill, *A First Course in Differential Equations with Modeling Applications*, 11th ed.

series solutions of DEs

- these slides contain three gory exercises solving *linear, homogeneous 2nd-order DEs* by power series methods
  - two are DEs we could not previously solve
- recall the main idea of using series to solve DEs:

  1. substitute a series with unknown coefficients into the DE
  2. find coefficients by matching on either side

- see/do §6.1 first . . . or these slides will not make sense!

ordinary points

- in §6.2 we only use *ordinary* base points for our series:

  definition. Assume $a_2(x), a_1(x), a_0(x)$ are continuous, smooth, and well-behaved functions.[1] If $a_2(x_0) \neq 0$ then the point $x = x_0$ is an *ordinary point* of the DE

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$$

- we often write the same DE as

$$y'' + P(x)y'' + Q(x)y = 0$$

  where $P(x) = a_1(x)/a_2(x)$ and $Q(x) = a_0(x)/a_2(x)$
    - $x = x_0$ is ordinary point if $P(x)$ and $Q(x)$ are analytic there
    - ...don't divide by zero
- a point which is not ordinary is *singular* ...see §6.3 & 6.4

[1] Precisely: *analytic* functions.

summation notation realization

- in these slides we do 2nd-order DEs only
- so consider y' and y'':

$$y(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots = \sum_{n=0}^{\infty} c_n x^n = \sum_{k=0}^{\infty} c_k x^k$$

$$y'(x) = c_1 + 2c_2 x + 3c_3 x^2 + \cdots = \sum_{n=0}^{\infty} n c_n x^{n-1} = \sum_{k=0}^{\infty} (k+1) c_{k+1} x^k$$

$$y''(x) = 2c_2 + 3(2)c_3 x + \cdots = \sum_{n=0}^{\infty} n(n-1) c_n x^{n-2}$$

$$= \sum_{k=0}^{\infty} (k+2)(k+1) c_{k+2} x^k$$

- these forms make summation notation an effective tool!

an Airy equation

exercise 1. find the general solution by series:

$$y'' + xy = 0$$

$$\boxed{\begin{gathered} 2 \cdot 1 \cdot c_2 = 0 \\ 3 \cdot 2 \cdot c_3 = -c_0 \\ 4 \cdot 3 \cdot c_4 = -c_1 \\ 5 \cdot 4 \cdot c_5 = -c_2 \\ 6 \cdot 5 \cdot c_6 = -c_3 \\ 7 \cdot 6 \cdot c_7 = -c_4 \\ \vdots \end{gathered}}$$

exercise 1, cont.

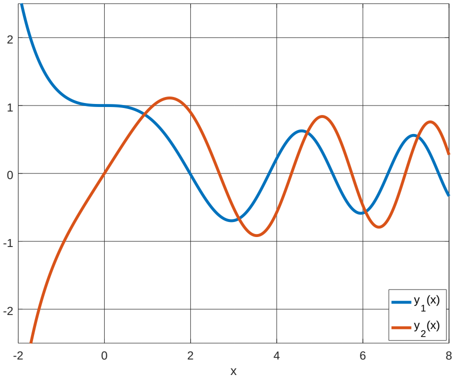
$$\boxed{\begin{aligned} y_1(x) &= 1 - \frac{1}{3\cdot 2}x^3 + \frac{1}{6\cdot 5\cdot 3\cdot 2}x^6 - \frac{1}{9\cdot 8\cdot 6\cdot 5\cdot 3\cdot 2}x^9 + \dots \\ y_2(x) &= x - \frac{1}{4\cdot 3}x^4 + \frac{1}{7\cdot 6\cdot 4\cdot 3}x^7 - \frac{1}{10\cdot 9\cdot 7\cdot 6\cdot 4\cdot 3}x^{10} + \dots \end{aligned}}$$

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

exercise 1, cont.[2]

- what do these Airy[2] functions look like?
  - I wrote a code to plot approximations to $y_1(x), y_2(x)$
  - … by summing first twenty terms of the series
- Airy functions smoothly connect a kind of exponential growth (left side of figure) to sinusoid-ish stuff (right side)

$$y'' + xy = 0$$



[2] George Airy was an astronomer: en.wikipedia.org/wiki/Airy_function.

we already know how to solve it!

exercise 2. $$y'' + 3y' - 4y = 0, \quad y(0) = 1, \quad y'(0) = 1$$

(a) solve the IVP by any means you want

exercise 2, cont.

(b) solve it by series $\left[\quad y'' + 3y' - 4y = 0,\ y(0) = 1,\ y'(0) = 1 \quad \right]$

$$\boxed{\begin{array}{c} 2\cdot 1c_2 + 3\cdot 1c_1 - 4c_0 = 0 \\ 3\cdot 2c_3 + 3\cdot 2c_2 - 4c_1 = 0 \\ 4\cdot 3c_4 + 3\cdot 3c_3 - 4c_2 = 0 \\ 5\cdot 4c_5 + 3\cdot 4c_4 - 4c_3 = 0 \\ \vdots \end{array}}$$

exercise 2, cont.[2]

$$y(x) = 1 + x + \tfrac{1}{2}x^2 + \tfrac{1}{3\cdot 2}x^3 + \tfrac{1}{4\cdot 3\cdot 2}x^4 + \cdots = e^x$$

get radius of convergence *in advance*

- when you find a series solution you can then use the ratio test (etc.) to determine radius of convergence R
- ... but this is unwise!
- Theorem 6.2.1 on page 245 tells us that

  a minimum for R is the distance, *in the complex plane*, from the basepoint $x = x_0$ to the nearest singular point

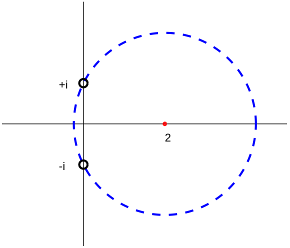
  - $a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$: anywhere $a_2(x) = 0$ is a singular point
  - $y'' + P(x)y' + Q(x)y = 0$: anywhere $P(x)$ or $Q(x)$ is not analytic is a singular point

like #2 in §6.2

exercise 3. (a) without actually solving the DE, find the minimum radius of convergence of the power series solutions about $x = 0$:

$$(x^2 + 1)y'' - 6y = 0$$

(b) same, but about $x = 2$



exercise 3, cont.

(c) find two series solutions about $x = 0$: $\quad (x^2 + 1)y'' - 6y = 0$

$$\boxed{\begin{gathered} 2 \cdot 1c_2 - 6c_0 = 0 \\ 3 \cdot 2c_3 - 6c_1 = 0 \\ 2 \cdot 1c_2 + 4 \cdot 3c_4 - 6c_2 = 0 \\ 3 \cdot 2c_3 + 5 \cdot 4c_5 - 6c_3 = 0 \\ 4 \cdot 3c_4 + 6 \cdot 5c_6 - 6c_4 = 0 \\ \vdots \end{gathered}}$$

exercise 3, cont.[2]

$$\boxed{\begin{aligned} y_1(x) &= 1 + \frac{6}{2\cdot 1}x^2 + \frac{(6-2\cdot 1)(6)}{4!}x^4 + \frac{(6-4\cdot 3)(6-2\cdot 1)(6)}{6!}x^6 + \dots \\ y_2(x) &= x + \frac{6}{3\cdot 2}x^3 + \frac{(6-3\cdot 2)(6)}{5!}x^5 + \frac{(6-5\cdot 4)(6-3\cdot 2)(6)}{7!}x^7 + \dots \end{aligned}}$$

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

was this progress?

- yes, we can solve more DEs than we could before
  - we have escaped from §4.3 constant-coefficient DEs
- *but*, to understand what you get, you must spend quality time with series-defined functions $y_1(x) = \dots$ and $y_2(x) = \dots$
- this is worthwhile in some famous cases:

$$\begin{aligned} y'' - xy = 0 &\quad\Longrightarrow\quad \text{Airy functions} \\ x^2y'' + xy' + (x^2 - \nu^2)y = 0 &\quad\Longrightarrow\quad \text{Bessel functions} \\ (1 - x^2)y'' - xy' + \alpha^2 y = 0 &\quad\Longrightarrow\quad \text{Chebyshev functions} \\ &\quad\quad\vdots \end{aligned}$$

- i.e. *special functions*

historical comment

- from about 1800 to 1950, finding new series solutions to DEs was the kind of thing that mathematicians and physicists did for a living
  - you could get your name on some new special functions!
  - e.g. Bessel, Legendre, Airy, Hermite, ... §6.4
- with powerful computers and software (since 1980?) one may/should automate the creation of series solutions
  - thus the invention of Mathematica and then Wolfram Alpha
  - naming new special functions is no longer a thing
  - the quality of approximations remains a thing

expectations

to learn this material, just listening to a lecture is *not* enough

- *read* section 6.2
- find good youtube videos on power series, for example this one from 3blue1brown:

  www.youtube.com/watch?v=3d6DsjIBzJ4

- do Homework 6.2
- we will skip §6.3 & 6.4