HIGH POINTS OF CALCULUS

This is a review of some calculus you will need for differential equations. However, this first Mini-Project is also a test of whether you are able to successfully submit your work in this online course!

Logistics. Please fill in the blanks below with complete and legible answers. You may do this either by

- electronically-editing the PDF or

- printing this blank version and writing your answers with pencil or pen.

In the first case, save your completed document as a PDF. In the second case you should scan or photograph your completed document and then figure out how to save it as a PDF. In any case *you must produce a PDF for submission.* Then submit it by uploading using the Google Form link.

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1. *Chain rule.* Recall the chain rule

$$[f\,(g(x))]' = f'\,(g(x))\,g'(x)$$

(a) Identify an outer function $f(x)$ and an inner function $g(x)$ if $f(g(x)) = \sqrt{\tan x + 3x}$. Then compute the following derivative.

$$\left[\sqrt{\tan x + 3x}\right]' =$$

(b) Construct your own chain rule example:

$$f(x) =$$

$$g(x) =$$

$$[f\,(g(x))]' =$$

*(Make it different from **(a)** and non-trivial but not too complicated. In particular, neither $f(x)$ nor $g(x)$ should be as simple as a linear function, i.e. $ax + b$. Remember that $x^k, e^x, \ln x, b^x,$ $\log_b x, \sin x, \cos x, \tan x, \sec x, \arcsin x$ are common functions from calculus which you must be able to correctly differentiate! Use this problem to practice those that are least familiar?)*

2. *Integration by substitution.* Remember that the indefinite integral just means "anti-derivative," so $\int f(x)\,dx = F(x) + C$ means exactly the same thing as $(F(x))' = f(x)$.

In any case you can do some integrals just by recognizing a derivative, perhaps with some fiddling with constants.

(a) Compute the indefinite integral:

$$\int \frac{1}{x}\,dx =$$

(b) Compute the indefinite integral:

$$\int 3^{2x}\,dx =$$

Substitution is **the chain rule in reverse**. For example, from **1 (a)** we have

$$\int \frac{\sec^2 x + 3}{\sqrt{\tan x + 3x}}\,dx = \int \frac{du}{\sqrt{u}} \qquad\qquad [\text{with } u = \tan x + 3x]$$
$$= 2u^{1/2} + C = 2\sqrt{\tan x + 3x} + C$$

(It is common to need to fiddle with constant factors like the "2" here.) In general:

$$\int f'(g(x))g'(x)\,dx = \int f'(u)\,du = f(u) + C = f(g(x)) + C$$

(c) Turn your chain rule example in **1 (b)** into an integration by substitution.

3. *Product rule and integration-by-parts.* The product rule

$$[u(x)v(x)]' = u'(x)v(x) + u(x)v'(x)$$

can be used in reverse too. The indefinite integral of both sides of the above gives

$$u(x)v(x) = \int u'(x)v(x)\, dx + \int u(x)v'(x)\, dx.$$

The main use of this is to exchange one of the last two integrals for the other, which is integration-by-parts:

$$\int u(x)v'(x)\, dx = u(x)v(x) - \int u'(x)v(x)\, dx.$$

(You probably have this memorized as $\int u\, dv = uv - \int v\, du$.)

(a) Construct your own product rule example:

$$u(x) =$$

$$v(x) =$$

$$[u(x)v(x)]' =$$

(*Again, make it non-trivial but not too complicated.*)

(b) Turn the above example into an integration-by-parts example.

4. *Fundamental Theorem of Calculus (FTC).* When you compute a definite integral by hand you usually use a form of the FTC:

$$\int_a^b f(x)\, dx = F(b) - F(a) \qquad \text{where } F'(x) = f(x)$$

This says that doing an integral is the same as un-doing a derivative.

Recall that if you do a definite integral by substitution then you can change the limits:

$$\int_{x=a}^{x=b} f(g(x))\, g'(x)\, dx = \int_{u=g(a)}^{u=g(b)} f(u)\, du = F(g(b)) - F(g(a))$$

(a) Compute

$$\int_{\pi/6}^{\pi/2} \frac{\cos x}{1 + 9\sin^2 x}\, dx =$$

There is another form of the FTC, often called "FTC I." It says that a derivative un-does an integral:

$$\frac{d}{dx}\left(\int_a^x f(t)\, dt \right) = f(x)$$

The integral inside the parentheses computes the area under the curve $y = f(t)$ from $t = a$ to $t = x$. One should think of this area varying as x changes, thus defining a function $g(x) = \int_a^x f(t)\, dt$. One may answer some questions about $g(x)$ even when an antiderivative of the integrand is not known.

(b) Suppose we define

$$g(x) = \int_2^x \sin(e^t)\, dt$$

Compute the exact value of $g(2)$.

(c) For the same function $g(x)$, find $g'(x)$.