

## HIGH POINTS OF CALCULUS

This is a review of some calculus you will need for differential equations. However, this first Mini-Project is also a test of whether you are able to successfully submit your work in this online course!

*Logistics.* Please fill in the blanks below with complete and legible answers. You may do this either by

- electronically-editing the PDF or
- printing this blank version and writing your answers with pencil or pen.

In the first case, save your completed document as a PDF. In the second case you should scan or photograph your completed document and then figure out how to save it as a PDF. In any case *you must produce a PDF for submission*. Then submit it by uploading using the Google Form link.

### 1. Chain rule. Recall the chain rule

$$[f(g(x))]' = f'(g(x)) g'(x)$$

(a) Identify an outer function  $f(x)$  and an inner function  $g(x)$  if  $f(g(x)) = \sqrt{\tan x + 3x}$ . Then compute the following derivative.

$$\left[ \sqrt{\tan x + 3x} \right]' =$$

(b) Construct your own chain rule example:

$$f(x) =$$

$$g(x) =$$

$$[f(g(x))]' =$$

(Make it different from (a) and non-trivial but not too complicated. In particular, neither  $f(x)$  nor  $g(x)$  should be as simple as a linear function, i.e.  $ax + b$ . Remember that  $x^k, e^x, \ln x, b^x, \log_b x, \sin x, \cos x, \tan x, \sec x, \arcsin x$  are common functions from calculus which you must be able to correctly differentiate! Use this problem to practice those that are least familiar?)

2. *Integration by substitution.* Remember that the indefinite integral just means “anti-derivative,” so  $\int f(x) dx = F(x) + C$  means exactly the same thing as  $(F(x))' = f(x)$ .

In any case you can do some integrals just by recognizing a derivative, perhaps with some fiddling with constants.

(a) Compute the indefinite integral:

$$\int \frac{1}{x} dx =$$

(b) Compute the indefinite integral:

$$\int 3^{2x} dx =$$

Substitution is **the chain rule in reverse**. For example, from 1 (a) we have

$$\begin{aligned} \int \frac{\sec^2 x + 3}{\sqrt{\tan x + 3x}} dx &= \int \frac{du}{\sqrt{u}} && [\text{with } u = \tan x + 3x] \\ &= 2u^{1/2} + C = 2\sqrt{\tan x + 3x} + C \end{aligned}$$

(It is common to need to fiddle with constant factors like the “2” here.) In general:

$$\int f'(g(x))g'(x) dx = \int f'(u) du = f(u) + C = f(g(x)) + C$$

(c) Turn your chain rule example in 1 (b) into an integration by substitution.

**3. Product rule and integration-by-parts.** The product rule

$$[u(x)v(x)]' = u'(x)v(x) + u(x)v'(x)$$

can be used in reverse too. The indefinite integral of both sides of the above gives

$$u(x)v(x) = \int u'(x)v(x) dx + \int u(x)v'(x) dx.$$

The main use of this is to exchange one of the last two integrals for the other, which is integration-by-parts:

$$\int u(x)v'(x) dx = u(x)v(x) - \int u'(x)v(x) dx.$$

(You probably have this memorized as  $\int u dv = uv - \int v du$ .)

**(a)** Construct your own product rule example:

$$u(x) =$$

$$v(x) =$$

$$[u(x)v(x)]' =$$

(Again, make it non-trivial but not too complicated.)

**(b)** Turn the above example into an integration-by-parts example.

4. *Fundamental Theorem of Calculus (FTC)*. When you compute a definite integral by hand you usually use a form of the FTC:

$$\int_a^b f(x) dx = F(b) - F(a) \quad \text{where } F'(x) = f(x)$$

This says that doing an integral is the same as un-doing a derivative.

Recall that if you do a definite integral by substitution then you can change the limits:

$$\int_{x=a}^{x=b} f(g(x)) g'(x) dx = \int_{u=g(a)}^{u=g(b)} f(u) du = F(g(b)) - F(g(a))$$

- (a) Compute

$$\int_{\pi/6}^{\pi/2} \frac{\cos x}{1 + 9 \sin^2 x} dx =$$

There is another form of the FTC, often called “FTC I.” It says that a derivative un-does an integral:

$$\frac{d}{dx} \left( \int_a^x f(t) dt \right) = f(x)$$

The integral inside the parentheses computes the area under the curve  $y = f(t)$  from  $t = a$  to  $t = x$ . One should think of this area varying as  $x$  changes, thus defining a function  $g(x) = \int_a^x f(t) dt$ . One may answer some questions about  $g(x)$  even when an antiderivative of the integrand is not known.

- (b) Suppose we define

$$g(x) = \int_2^x \sin(e^t) dt$$

Compute the exact value of  $g(2)$ .

- (c) For the same function  $g(x)$ , find  $g'(x)$ .