Name: SOLUTIONS

SAMPLE Midterm 1

Proctored. 90 minutes. 100 points total. No textbook or notes or calculator. When it makes sense to do so, please circle your final answer(s).

1. (*10 pts*) Show that $6y^{1/3} - x^2 = 7$ defines an implicit solution to the ODE

$$\frac{dy}{dx} = xy^{2/3}$$

implicit differentiation: $6 \cdot \frac{1}{3} y^{-2/3} \frac{dy}{dx} - 2x = 0$

$$\frac{dy}{dx} = \frac{2x}{2y^{-2/3}} = xy^{2/3} \checkmark$$

Extra Credit. (*3 pts*) Note $6y^{1/3} - x^2 = 0$ also defines a solution to the DE in the above problem. Show this solution passes through $(0,0)$. Then find another solution to the same DE which also passes through $(0,0)$. Explain why this situation is *not* a violation of the theorem on existence and uniqueness for initial value problems.

$y_1(x) = \left(\frac{x^2}{6}\right)^3 = \frac{x^{2/3}}{6^3}$ $\therefore$ $y(0) = \frac{0}{6} = 0$ $\therefore$ hits $(0,0)$

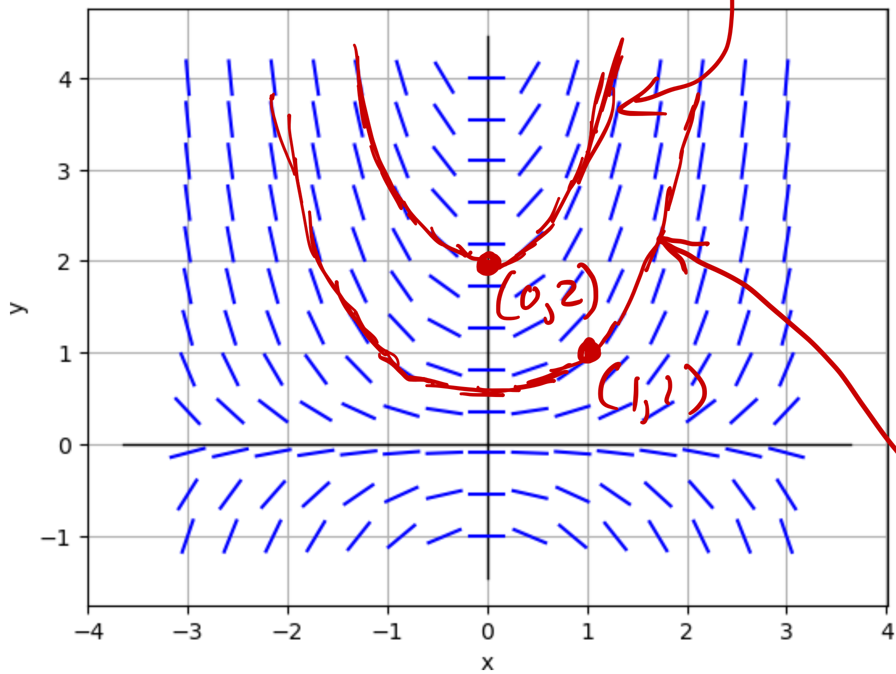
but $y_2(x) = 0$ also solves $\frac{dy}{dx} = xy^{2/3}$ $(0 = x \cdot 0 \checkmark)$

however, $f(x,y) = xy^{2/3}$ does not have continuous derivative $\partial f/\partial y = \frac{2}{3} x y^{-1/3}$ at $y=0$, so theorem is not violated

2. **(a)** (*5 pts*) The ODE

$$\frac{dy}{dx} = xy$$

has the direction field shown below. Sketch the solution to this ODE which passes through $(x_0, y_0) = (0, 2)$.



(b) (*10 pts*) Solve the following initial value problem—*give an exact formula* $y(x)$ *for the solution!*—and then sketch the solution you find on the direction field above:

$$\frac{dy}{dx} = xy, \qquad y(1) = 1.$$

separable: $\int \frac{dy}{y} = \int x\,dx$

$\ln|y| = \frac{1}{2}x^2 + C$

$y = Ae^{x^2/2}$

$1 = y(1) = Ae^{1/2} \quad \therefore \quad A = e^{-1/2} \approx 1.8$

$\therefore \quad y(x) = e^{-1/2}e^{x^2/2} = e^{(x^2-1)/2}$

3. **(a)** (*5 pts*) Show that the following equation is exact:

$$\frac{t}{y}\,dy + (1 + \ln y)\,dt = 0.$$

if solving for $y(t)$: $M\,dt + N\,dy = (1+\ln y)\,dt + \frac{t}{y}\,dy = 0$

$\therefore \quad \frac{1}{y} = M_y = N_t = \frac{1}{y}$ ✓ $\quad \therefore$ exact

(b) (*10 pts*) Solve the ordinary differential equation in part **(a)**. Write the solution as an explicit formula for $y(t)$.

$$\frac{\partial f}{\partial t} = M = 1 + \ln y$$

$$f(t,y) = t + t\ln y + g(y)$$

$$\cancel{\frac{t}{y}} = N = \frac{\partial f}{\partial y} = 0 + \cancel{t\frac{1}{y}} + g'(y)$$

$$\therefore\ g'(y) = 0 \quad \therefore\ g(y) = 0$$

[any antiderivative suffices]

$$\therefore \quad t + t\ln y = c$$

$$\ln y = \frac{c-t}{t} = \frac{c}{t} - 1$$

$$y = e^{\frac{c}{t}-1}$$

4. (*15 pts*) Find the solution to the initial value problem:

$$\frac{dy}{dx} + 2y = e^{-x}, \qquad y(-1) = e.$$

1st-order linear: $\mu = e^{2x}$

$$\frac{d}{dx}(e^{2x} y) = e^{2x} \cdot e^{-x} = e^{x}$$

$$e^{2x} y(x) = \int e^{x} dx = e^{x} + c$$

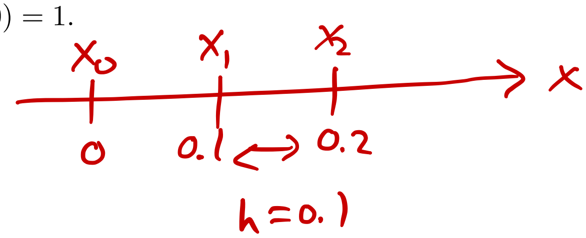
$$y(x) = e^{-x} + c e^{-2x}$$

$$e = y(-1) = e^{+1} + c e^{+2} \quad \therefore \quad c e^{2} = 0 \quad \therefore \quad c = 0$$

$$y(x) = e^{-x}$$

5. (*15 pts*) Use Euler's method to approximate the solution to the initial value problem at the points $x = 0.1$ and $x = 0.2$, with steps of size $h = 0.1$:

$$\frac{dy}{dx} = x + y, \qquad y(0) = 1.$$



$y_0 = 1$

$$y_1 = y_0 + h\, f(x_0, y_0)$$
$$= 1 + 0.1\,(0 + 1) = 1.1 \approx y(0.1)$$

$$y_2 = y_1 + h\, f(x_1, y_1)$$
$$= 1.1 + 0.1\,(0.1 + 1.1) = 1.1 + 0.1\,(1.2)$$
$$= 1.1 + 0.12 = 1.22 \approx y(0.2)$$

6. Recall that Newton's law of cooling is the DE

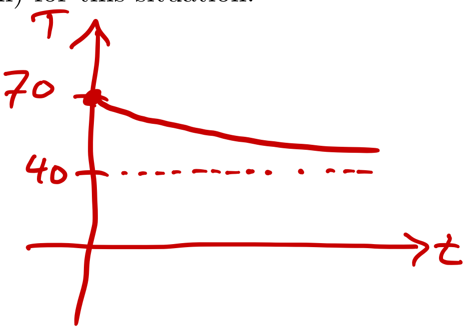
$$\frac{dT}{dt} = k(T_m - T) \tag{1}$$

where $T(t)$ is the temperature of the object, T_m is the ambient temperature, and k is a positive constant.

(a) (*5 pts*) Suppose that at $t = 0$ a glass of water at room temperature, say 70 °F, is taken outside where it is 40 °F. Suppose that for this glass of water, $k = 0.01$, and that we measure time in minutes. Including (1), write a concrete ODE IVP (i.e. initial value problem) for this situation.

$$\frac{dT}{dt} = 0.01(40 - T),$$

$$T(0) = 70$$



(b) (*10 pts*) A more realistic situation is that though we know the ambient temperature—again assume it is 40 °F—we don't know k for the glass of water. However, we measure that the water is at temperature 58 °F after 3 minutes. Determine k. (*Hint.* You will need to solve (1). You can leave your answer as a formula for a specific number; a calculator would give a decimal value.)

$$\frac{dT}{dt} = k(40 - T)$$

$$\int \frac{dT}{40-T} = \int k\,dt \longrightarrow -\ln|40-T| = kt + C$$

$$40 - T = Ae^{-kt}$$

$$T(t) = 40 - Ae^{-kt}$$

$$T(0) = 70 \quad \therefore \quad A = -30$$

$$[\ T(t) = 40 + 30e^{-kt}\ ]$$

$$T(3) = 58 = 40 + 30e^{-k\cdot 3}$$

$$\frac{18}{30} = e^{-3k} \longrightarrow e^{3k} = \frac{5}{3}$$

$$3k = \ln(5/3)$$

$$k = \frac{1}{3}\ln(5/3)$$

(units: $\frac{1}{\text{minute}}$)

7. (*15 pts*) Find the general solution

$$\frac{dz}{dt} = \frac{t-5}{z^2 - z}$$

(*Hint.* You may write the solution implicitly.)

Separable:

$$\int (z^2 - z)\,dz = \int (t-5)\,dt$$

$$\frac{z^3}{3} - \frac{z^2}{2} = \frac{t^2}{2} - 5t + C$$