2.1 Solution Curves (Without a Solution)

a lesson for MATH F302 Differential Equations

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for textbook: D. Zill, *A First Course in Differential Equations with Modeling Applications*, 11th ed.

meaning of a differential equation

- start over on the meaning of a differential equation (DE):

$$\frac{dy}{dx} = f(x, y)$$

1. the left side is the *slope* of the solution $y(x)$
2. given a point (x, y), the right side computes a number $f(x, y)$

- thus a first-order DE says:

$$\begin{array}{c} \text{the slope of the} \\ \text{solution } y(x) \end{array} \quad \overset{\text{equals}}{=} \quad \begin{array}{c} \text{a known function of} \\ \text{the location } (x, y) \end{array}$$

- this literal reading of the DE means that

  we can *draw* a picture of the DE itself

  - whether or not we can do the calculus/algebra to find a formula for $y(x)$

direction field

- *main idea*: $\frac{dy}{dx} = f(x, y)$ should be read as computing a slope $m = \frac{dy}{dx}$ at each point (x, y)
- we can create a *direction field* or *slope field*:
  1. generate a grid of point in the x,y plane
  2. for each point, draw a short line segment with the slope given by $f(x, y)$ at that point
- *Example.* By hand, draw a direction field for

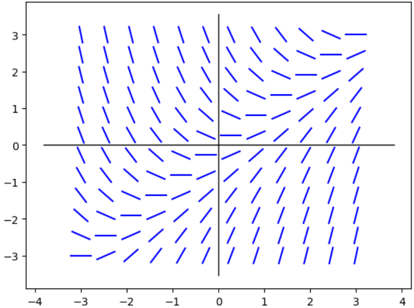
$$\frac{dy}{dx} = x - y$$

on the square $-3 \le x \le 3$, $-3 \le y \le 3$

computers are useful

- I acknowledge *happily* that this is a job for a computer
  - for computer tools, see "found online" at the Week 2 tab
  - see also: `en.wikipedia.org/wiki/Slope_field`
- *Example.* Use a computer to draw a direction field for $\frac{dy}{dx} = x - y$ on the square $-3 \le x \le 3, -3 \le y \le 3$

  Solution:




```
def f(x,y):  return x - y
dirfield(f,[-3,3,-3,3],mx=12,my=12)
```

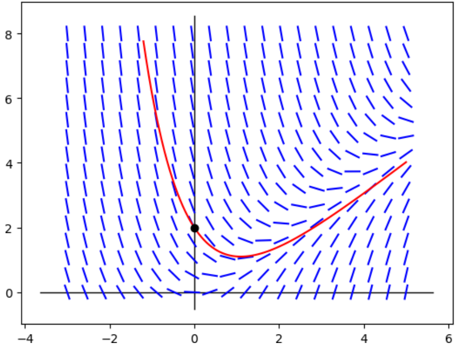
$\longleftarrow$ *from my Python code*

- recall that we are often solving initial value problems
- *next main idea*: one can *see* the solution to an ODE IVP by plotting the initial point in the plane and then following the direction field both ways from that point

- *Example.* Use the direction field for $\frac{dy}{dx} = x - y$ to sketch the solution $y(x)$ of

$$\frac{dy}{dx} = x - y, \; y(0) = 2$$

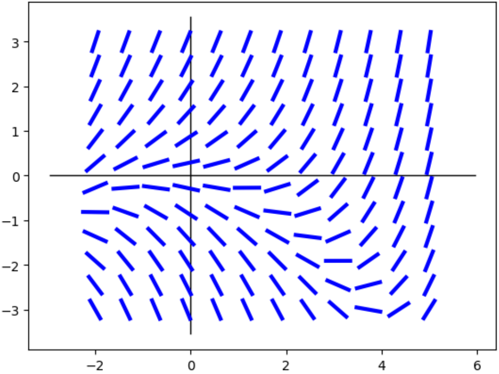
  - soon: methods in §2.3 will give a formula for $y(x)$



9. *Use computer software to obtain a direction field for the given differential equation. By hand, sketch an approximate solution curve passing through each of the given points.*

$$\frac{dy}{dx} = 0.2x^2 + y$$

(a) $y(0) = \frac{1}{2}$
(b) $y(2) = -1$




```
def f(x,y):  return 0.2*x**2 + y
dirfield(f,[-2,5,-3,3],mx=12,my=12)
```

two topics in §2.1

- there are two topics in §2.1:
  - direction fields for 1st-order DEs
  - autonomous 1st-order DEs
- equally-important topics!
- both topics are about *picturing DEs*, but "autonomous" is a special case where we can draw a simpler picture

autonomous first-order DEs

- *definition.* a first-order differential equation is *autonomous* if the function does not depend on the independent variable:

$$\frac{dy}{dx} = f(y)$$

  - "autonomous" means "independent of control"
  - ...above DE is not directly controlled by input variable x
  - ...but the *solution* $y(x)$ is still a function of x
  - a big idea: fundamental laws of nature are autonomous DEs

- Example.

$$\frac{dy}{dx} = \sqrt{\sin(y)} \quad \text{is autonomous}$$

- Example.

$$\frac{dy}{dx} = x - y \quad \text{is } \textit{not} \text{ autonomous}$$

classification of first-order DEs

- we will see that "autonomous" also means "easier to visualize," but not always easy to solve
- using definitions from sections 1.1 and 2.1 we already have a *classification* of first-order DEs:

	autonomous	nonautonomous
linear	$y' = c\,y + d$	$y' + P(x)y = g(x)$
nonlinear	$y' = f(y)$	$y' = f(x, y)$

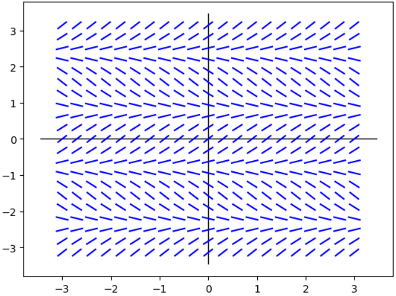
  - which can we already solve by guess-and-check?

picturing autonomous DEs

- the direction field of an autonomous DE has redundancies
- simplified picture: a one-dimensional *phase portrait*
  - a.k.a. *phase line*
  - easiest to explain by an example . . .

- Example. Use a computer to draw the direction field for $x \in [-3, 3]$ and $y \in [-\pi, \pi]$. Then draw the phase portrait.

$$\frac{dy}{dx} = \cos(2y)$$



critical points of autonomous DEs

- consider an autonomous first-order DE: $\frac{dy}{dx} = f(y)$
- a value $y = c$ is called a *critical point* if $f(c) = 0$
  - a.k.a. *equilibrium point* or *stationary point*

- if $y = c$ is a critical point then $y(x) = c$ is a solution!

- Example. $y = \frac{\pi}{4}$ is a critical point *and a solution* of

$$\frac{dy}{dx} = \cos(2y)$$

phase portrait example

- Example. By hand, sketch the phase portrait of

$$\frac{dz}{dt} = z^2 + z^3$$

and show all critical points. Then sketch the graph of solutions to the ODE IVP with the following initial values.

  (a) $z(0) = 1$

  (b) $z(0) = -1/2$

  (c) $z(0) = -1$

  (c) $z(0) = -2$

classifying critical points

- in summary, to draw a phase portrait you
  - solve $f(y) = 0$ for the critical points
  - between critical points you evaluate the sign of $f(y)$ and draw an up or down arrow accordingly
- ...and you see the idea behind the following classification
- a critical point $y = c$ is
  - *attracting* or *asymptotically stable* if

$$\lim_{x \to \infty} y(x) = c \tag{$*$}$$

    for all initial points (x_0, y_0) where y_0 is close to c,
  - *semi-stable* if $(*)$ only happens for y_0 one side of c, and
  - *repelling* or *asymptotically unstable* otherwise

- Example. Find and classify the critical points of

$$\frac{dy}{dx} = \cos(2y)$$

examples, cont.

- Example. Find and classify the critical points of

$$\frac{dz}{dt} = z^2 + z^3$$

exercise 27 in §2.1

27. *Find the critical points and phase portrait. Classify each critical point as asymptotically stable, unstable, or semi-stable. By hand, sketch typical solution curves in the regions in the xy−plane determined by the graphs of the equilibrium solutions.*

$$\frac{dy}{dx} = y \ln(y + 2)$$

exercise 40 in §2.1

40. *The autonomous differential equation*

$$m\frac{dv}{dt} = mg - kv,$$

where k is a positive constant and g is the acceleration due to gravity, is a model for the velocity v of a body of mass m that is falling under gravity. The term $-kv$, which is air resistance, implies that the velocity will not increase without bound as t increases. Use a phase portrait to find the limiting, or terminal velocity of the body.

looking ahead: next two sections 2.2, 2.3

- the first four sections of the textbook (1.1, 1.2, 1.3, 2.1) are about the *meaning* of differential equations
    - in my experience, such meaning *is* the important take-home from a course in differential equations!
- but for the next few sections we will address how to find formulas for solutions $y(x)$
- looking ahead to the next two sections:

	autonomous	nonautonomous	
linear	$y' = c\,y + d$	$y' + P(x)y = g(x)$	
nonlinear	$y' = f(y)$	separable $y' = g(x)h(y)$	nonseparable $y' = f(x, y)$

this is not a CS class

- you *don't* have to know programming to do this class
  - . . . but interacting with a computer is obligatory!
  - so you must seek-out tools such as desmos or Wolfram alpha which allow you to do particular computer jobs like generating direction fields
  - I will generally show a few lines of Matlab or Python when there is a computer-suitable job *and* I'll link to programming-free tools

expectations

to learn this material, just watching this video is *not* enough; also

- *watch* "found online" videos at

  `bueler.github.io/math302/week2.html`
- *try-out* direction-field plotters linked at the same place
- *read* section 2.1 in the textbook
  - a large new vocabulary in this section, namely the language of *qualitative* differential equations
  - I did not cover "translation property" on page 43; read that!
- *do* the WebAssign exercises for section 2.1
  - get more out of these by *not* using the internet to cheat!