4.2 Reduction of order
a lesson for MATH F302 Differential Equations

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for textbook: D. Zill, *A First Course in Differential Equations with Modeling Applications*, 11th ed.

a 2nd-order example

- start with an example for which the methods of section 4.3 *do not* work, but where we *can* see the solution anyway
- *example 1*: find the general solution

$$xy'' + y' = 0$$

the §4.3 rule needing explanation

- the next example is one we *do* know how to solve, but I will use a rule which requires justification
- *example 2*: find the general solution using §4.3 rules

$$y'' - 6y' + 9y = 0$$

reducing the order 1: first illustration

- *reduction of order* is a technique:
  - substitute $y(x) = u(x)y_1(x)$
  - derive a DE for u which has no zeroth-order term
  - solve a first-order equation for $w = u'$
- key understanding: the purpose is to find another linearly-independent solution given you have $y_1(x)$
- *example 3*: we know $y_1(x) = e^{3x}$ is a solution; find another

$$y'' - 6y' + 9y = 0$$

reducing the order 2: general case

suppose $y_1(x)$ is a solution to 2nd-order homogeneous DE

$$y'' + P(x)y' + Q(x)y \stackrel{*}{=} 0,$$

and we seek another solution of the form $y(x) = u(x)y_1(x)$:

- compute $y' = u'y_1 + uy_1'$ and $y'' = u''y_1 + 2u'y_1' + uy_1''$

  check it: $y'' =$

- substitute into $*$:

$$(u''y_1 + 2u'y_1' + uy_1'') + P(u'y_1 + uy_1') + Quy_1 = 0$$

- group by derivatives on u:

$$y_1 u'' + (2y_1' + Py_1)u' + (y_1'' + Py_1' + Qy_1)u = 0$$

- term in green is zero (*why?*) so u solves

$$y_1 u'' + (2y_1' + Py_1)u' = 0$$

reducing the order 3: a first-order equation

- we are seeking a solution of the form $y = uy_1$, and u solves

$$y_1 u'' + (2y_1' + Py_1)u' = 0$$

- there is no zeroth-order term so we can solve it

- the equation is first-order and separable for $w = u'$:

$$y_1 w' + (2y_1' + Py_1)w = 0$$
$$\frac{dw}{dx} = -\frac{(2y_1' + Py_1)w}{y_1}$$
$$\frac{dw}{w} = -\left(2\frac{y_1'}{y_1} + P\right) dx$$
$$\int \frac{dw}{w} = -2 \int \frac{y_1'(x)}{y_1(x)} dx - \int P(x) dx$$

reducing the order 4: the second solution

- continuing:

$$\int \frac{dw}{w} = -2 \int \frac{y_1'(x)}{y_1(x)} \, dx - \int P(x) \, dx$$

$$\ln |w(x)| = -2 \ln |y_1(x)| - \int P(x) \, dx + C$$

$$w(x) = c_1 \frac{e^{-\int P(x) \, dx}}{y_1(x)^2}$$

- recall $u' = w$; thus integrating again gives

$$u(x) = c_1 \int \frac{e^{-\int P(x) \, dx}}{y_1(x)^2} \, dx + c_2$$

- the second solution is the new part of $y = uy_1$:

$$y_2(x) = y_1(x) \int \frac{e^{-\int P(x) \, dx}}{y_1(x)^2} \, dx$$

example 4

- in this example we tell a complete story: we *guess* a first solution and then derive a second one by reduction of order
- *example 4*: find the general solution (for $x > 0$)

$$x^2 y'' + 5xy' + 4y = 0$$

example 4, finished

$$y(x) = c_1 x^{-2} + c_2 x^{-2} \ln x$$

differential equations *must be* hard

- differential equations *must be* hard, and sometimes impossible

Proof.
The simple (separable) DE

$$\frac{dy}{dx} = f(x)$$

has solution

$$y(x) = \int f(x)\, dx.$$

Integration is hard, for example, try $f(x) = e^{-x^2}$ in the above.
But the set of possible problems in DEs is *bigger* than the set of
hard integrals, so DEs must be hard. $\square$

exercise 17 in §4.2

- you can use reduction order on nonhomogeneous problems, if you are careful in understanding where you are going
- *exercise 17*: given that $y_1 = e^{-2x}$ solves the homogeneous equation $y'' - 4y = 0$, find the general solution of the nonhomogeneous equation

$$y'' - 4y = 2$$

exercise 17, finished

$$y(x) = c_1 e^{-2x} + c_2 e^{2x} - \frac{1}{2}$$

expectations

- just watching this video is *not* enough!
  - see "found online" videos at
    bueler.github.io/math302/week7.html
  - *read* §4.2 in the textbook
  - *do* the WebAssign exercises for §4.2
- note example 4 is a *Cauchy-Euler* type of differential equation
  - covered in §4.7 ... which we will skip
- to do reduction of order on a quiz or exam you have a choice
- do you
  1. memorize $\boxed{y_2(x) = y_1(x) \int \frac{e^{-\int P(x)\,dx}}{y_1(x)^2}\,dx}$?

  2. or substitute $y(x) = u(x)y_1(x)$ and see how it comes out?