8.4 The matrix exponential solves systems

a lesson for MATH F302 Differential Equations

Ed Bueler, Dept. of Mathematics and Statistics, UAF

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for textbook: D. Zill, *A First Course in Differential Equations with Modeling Applications*, 11th ed.

solving the simplest ODEs

- *simplest scalar ODE:*

$$y' = ay \qquad \text{has solution} \qquad y(t) = ce^{at}$$

- *simplest system of ODEs:*

$$\mathbf{X}' = \mathbf{A}\mathbf{X} \qquad \text{has solution} \qquad \mathbf{X}(t) = e^{\mathbf{A}t}\mathbf{C}$$

  - the last formula is new in §8.4

what does $e^{\mathbf{A}t}$ mean?

- what does $e^{\mathbf{A}t}$ mean?
  - what does e^{at} mean?
    - what does e^x mean? $\longleftarrow$ we know this one!

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$
$$= \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

- *definition.* if $\mathbf{A}$ is a square matrix and t is any number then

$$e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t + \mathbf{A}^2\frac{t^2}{2} + \mathbf{A}^3\frac{t^3}{3!} + \mathbf{A}^4\frac{t^4}{4!} + \dots$$
$$= \sum_{k=0}^{\infty} \mathbf{A}^k \frac{t^k}{k!}$$

  - note $\mathbf{A}^0 = \mathbf{I}$ makes sense if we believe $x^0 = 1$
  - also recall $0! = 1$

like exercise #1 in §8.4

- $e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t + \mathbf{A}^2 \frac{t^2}{2} + \mathbf{A}^3 \frac{t^3}{3!} + \mathbf{A}^4 \frac{t^4}{4!} + \dots$
- *example 1.* use the above series to compute $e^{\mathbf{A}t}$ and $e^{-\mathbf{A}t}$, in simplified form, if

$$\mathbf{A} = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}$$

like exercise #4 in §8.4

- *example 2.* use the series definition to compute $e^{\mathbf{A}t}$ and $e^{-\mathbf{A}t}$, in simplified form, if

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ -1 & 3 & 0 \end{pmatrix}$$

like exercise #3 in §8.4

- *example 3.* use the series definition to compute $e^{\mathbf{A}t}$, in simplified form, if

$$\mathbf{A} = \begin{pmatrix} -3 & -3 & -3 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{pmatrix}$$

last two were special

- *example 2* and *example 3* had unusual matrices with the property that some power was the zero matrix:

$$\mathbf{A}^k = \mathbf{0}$$

- this is not typical; for most $\mathbf{A}$:
  - $e^{\mathbf{A}t}$ has infinitely-many nonzero terms
  - the pattern is hard to see
- but we can show the matrix exponential $e^{\mathbf{A}t}$ gets it done!
  - next slide

- *fact 1.*

$$\frac{d}{dt}\left(e^{\mathbf{A}t}\right) = \mathbf{A}e^{\mathbf{A}t}$$

Proof.

$$\begin{aligned}\frac{d}{dt}\left(e^{\mathbf{A}t}\right) &= \frac{d}{dt}\left(\mathbf{I} + \mathbf{A}t + \mathbf{A}^2\frac{t^2}{2} + \mathbf{A}^3\frac{t^3}{3!} + \mathbf{A}^4\frac{t^4}{4!} + \dots\right)\\ &= \mathbf{0} + \mathbf{A} + \mathbf{A}^2\frac{2t}{2} + \mathbf{A}^3\frac{3t^2}{3!} + \mathbf{A}^4\frac{4t^3}{4!} + \dots\\ &= \mathbf{A}\left(\mathbf{I} + \mathbf{A}t + \mathbf{A}^2\frac{t^2}{2} + \mathbf{A}^3\frac{t^3}{3!} + \dots\right) = \mathbf{A}e^{\mathbf{A}t}\end{aligned}$$

$\square$

- *fact 2.* if $\mathbf{X}(t) = e^{\mathbf{A}t}\mathbf{C}$ then $\mathbf{X}' = \mathbf{A}\mathbf{X}$

Proof.

$\square$

- *fact 3.* if $\mathbf{X}(t) = e^{\mathbf{A}t}\mathbf{C}$ then $\mathbf{X}(0) = \mathbf{C}$

the matrix exponential solves systems

in summary:

1. for the ODE
$$\mathbf{X}' = \mathbf{A}\mathbf{X}$$
the general solution is
$$\mathbf{X}(t) = e^{\mathbf{A}t}\mathbf{C}$$
where $\mathbf{C} = \langle c_1, c_2, \dots, \rangle$ is a vector of unknown constants
2. for the ODE IVP
$$\mathbf{X}' = \mathbf{A}\mathbf{X}, \qquad \mathbf{X}(0) = \mathbf{C}$$
the solution is
$$\mathbf{X}(t) = e^{\mathbf{A}t}\mathbf{C}$$

use a computer ...

- for the ODE IVP

$$\mathbf{X}' = \mathbf{A}\mathbf{X}, \qquad \mathbf{X}(0) = \mathbf{C}$$

  suppose you want the solution at time T:

$$\mathbf{X}(T) = e^{\mathbf{A}T}\mathbf{C}$$

- with MATLAB/OCTAVE:

```
>> A = [...];              % enter square matrix A
>> C = [...];              % enter column vector C
>> expm(A*T) * C           % bam! done!
```

- `expm()` computes the matrix exponential
  - be careful ... `exp()` is *not* what you want

- *example 4*. solve the initial value problem for $x(2), y(2), z(2)$:

$$\begin{aligned} x' &= 2x - 5y + z & x(0) &= -2 \\ y' &= -x + y + 3z & y(0) &= 0 \\ z' &= x - 2y - z & z(0) &= 3 \end{aligned}$$

solution.

```
>> A = [2 -5 1; -1 1 3; 1 -2 -1];
>> C = [-2; 0; 3];
>> expm(A*2) * C
ans =
    -1227.9
     68.564
    -381.69
```

so: $x(2) = -1227.9$, $y(2) = 68.564$, $z(2) = -381.69$

can you check it?

- what tool would help us quickly check previous slide result?
    - consider all the tools in the whole course . . .
- *answer.* the most general tool is numerical approximation

- *example 4, cont.* check the solution on the previous slide

solution.

```
>> f = @(t,U) [2*U(1)-5*U(2)+U(3); ...
               -U(1)+U(2)+3*U(3); ...
               U(1)-2*U(2)-U(3)];
>> [tt,UU] = ode45(f,[0,2],[-2;0;3]);
>> UU(end,:)
ans =
    -1227.9      68.565      -381.7
```

like exercise #8 in §8.4

- as long as we can compute $e^{\mathbf{A}t}$ by hand, we can solve an ODE by hand using the matrix exponential
- *example 5.* use $\mathbf{X}(t) = e^{\mathbf{A}t}\mathbf{C}$ to find the general solution of the given system

$$\mathbf{X}' = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ -1 & 3 & 0 \end{pmatrix} \mathbf{X}$$

the two views in §8.2 and §8.4

§8.2: the ODE system $\mathbf{X}' = \mathbf{A}\mathbf{X}$ has solutions

$$\mathbf{X}_j(t) = \mathbf{K}_j e^{\lambda_j t}$$

where λ_j is an eigenvalue of $\mathbf{A}$ and $\mathbf{K}_j$ is a corresponding eigenvector,

$$\mathbf{A}\mathbf{K}_j = \lambda_j \mathbf{K}_j$$

and the general solution is

$$\begin{aligned} \mathbf{X}(t) &= c_1 \mathbf{X}_1(t) + \cdots + c_n \mathbf{X}_j(t) \\ &= c_1 \mathbf{K}_1 e^{\lambda_1 t} + \cdots + c_n \mathbf{K}_n e^{\lambda_n t} \end{aligned}$$

§8.4: the ODE system $\mathbf{X}' = \mathbf{A}\mathbf{X}$ has general solution

$$\mathbf{X}(t) = e^{\mathbf{A}t}\mathbf{C}$$

expectations

- just watching this video is *not* enough!
  - see "found online" videos and stuff at

    `bueler.github.io/math302/week14.html`
  - *read* §8.4
  - *do* the WebAssign exercises for section 8.4
    - these exercises are about cases where the matrix exponential reduces to a simple pattern or just a few nonzero terms
    - like examples 1–3 and 5 above

- I hope you have learned something from this course!
  - perhaps even found it interesting?
  - note that
    - MATH 302 discrete mathematics
    - MATH 314 linear algebra
    - MATH 310 numerical analysis

    are courses of more-or-less comparable level to MATH 302 (differential equations)