2.6 A Numerical Method (Euler's Method)

a lesson for MATH F302 Differential Equations

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for textbook: D. Zill, *A First Course in Differential Equations with Modeling Applications*, 11th ed.

where we stand

- we now have methods for generating by-hand solutions to first-order differential equations:
  2.2 separable equations: $y' = g(x)h(y)$
  2.3 linear equations: $y' + P(x)y = f(x)$
  2.4 exact equations: $M\,dx + N\,dy = 0$ where $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$
- there are further methods … such as in section 2.5
  - but we are skipping §2.5; its methods are weak
- where do we stand?:
  - *there are some problems we can do …*
  - *but often our by-hand calculus/algebra techniques don't work*
- this situation is permanent

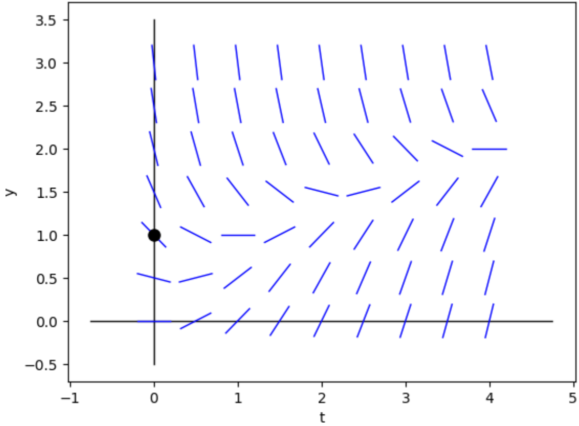
- Example 1. solve the initial value problem

$$\frac{dy}{dt} = t - y^2, \qquad y(0) = 1$$

in particular, find $y(4)$

Solution version 0: *Explain why 2.3–2.5 methods don't apply.*

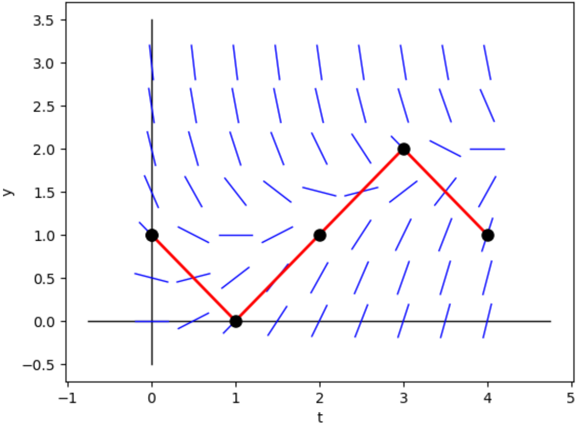
Solution version 1: *Solve it using a direction field and a pencil.*



- this is only approximate

example 1, cont. cont.

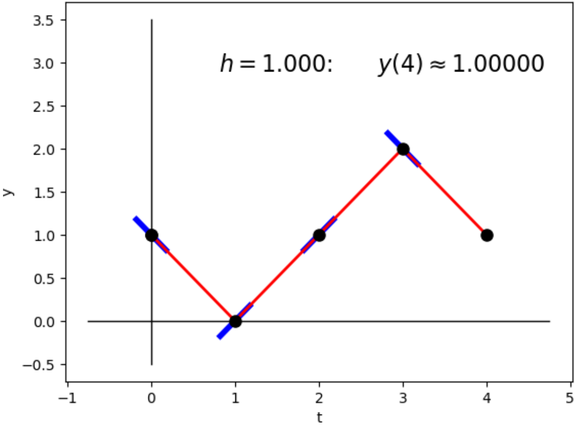
Solution version 2: *Make a computer follow the direction field.*



- this is still only approximate because we go straight

example 1, cont. cont. cont.

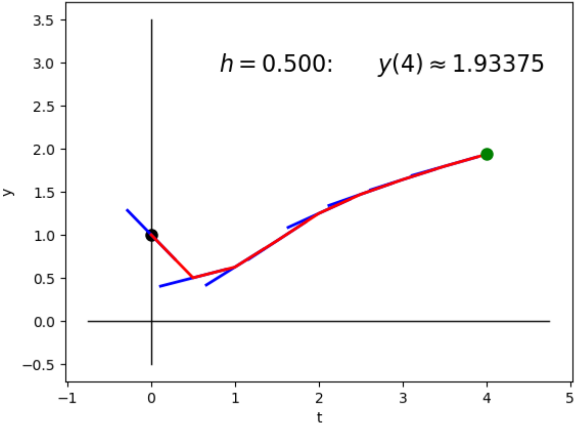
Solution version 3: *The direction field is not actually needed.*



- this is the same as previous

example 1, cont. cont. cont. cont.

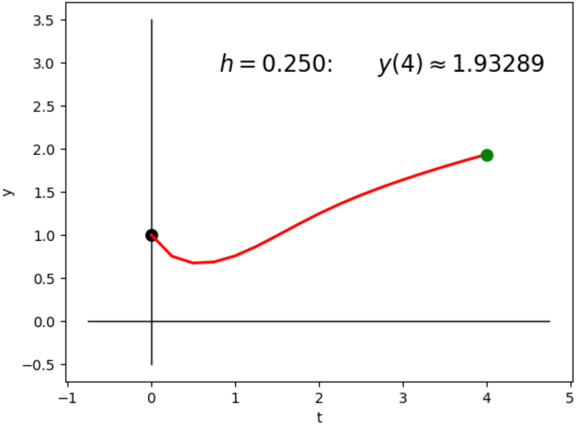
Solution version 4: *Do it more accurately by smaller steps*



- the blue slope lines are not really needed ...

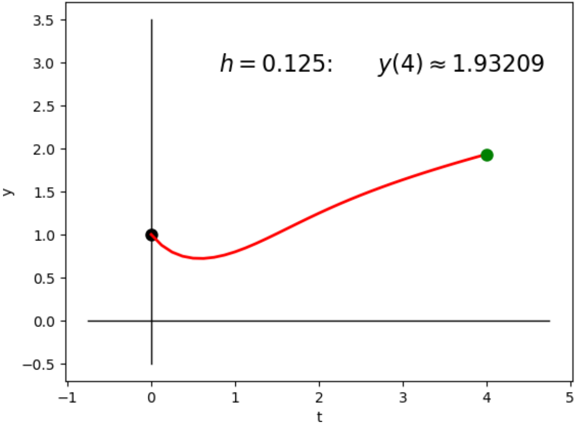
example 1, cont.[5]

Solution version 5: *Smaller steps.*



- this is *still* only approximate

Solution version 6: *Smaller. (Make the computer do more work.)*



- this looks like a *solution* not a direction field

Euler's method

- the idea of following the direction field, in a straight line for a short distance, and repeating, is *Euler's method*
- for the general DE $\frac{dy}{dx} = f(x, y)$, Euler's method is

$$y_{n+1} = y_n + h\, f(x_n, y_n) \qquad (*)$$

  - $h \neq 0$ is a step size *you must choose*
  - the next x-value is always h away from the last: $x_{n+1} = x_n + h$
  - $(*)$ is a formula to understand *and* memorize
  - ... and put in computer programs
- in the previous slides we had $f(x, y) = x - y^2$, starting values $(x_0, y_0) = (0, 1)$, and four values of h: $h = 1, 0.5, 0.25, 0.125$

a derivation of Euler's method

easy to derive it from the direction field of $\frac{dy}{dx} = f(x, y)$, as follows:

- suppose we are at a point (x_n, y_n)
  - this might be the initial point (x_0, y_0)
- the slope is $m = f(x_n, y_n)$ so the line we want is

$$y - y_n = f(x_n, y_n)(x - x_n)$$

- we want to move to a new location $x_{n+1} = x_n + h$ so $x - x_n = h$ and $y = y_{n+1}$
- thus

$$y_{n+1} - y_n = f(x_n, y_n)\, h$$

- i.e. $y_{n+1} = y_n + h\, f(x_n, y_n)$

measuring accuracy

- assume we are solving an ODE IVP: $\frac{dy}{dx} = f(x, y)$, $y(x_0) = y_0$
- if know the exact solution $y(x)$ then we can measure (evaluate) the error in the approximation, i.e.

$$y_n \approx y(x_n)$$

  - "y_n" is the number produced by Euler's method
  - "$y(x_n)$" is the exact solution at the x-value x_n
- there are two common ways to report the error:
  1. *absolute error* $= |y(x_n) - y_n|$
  2. *relative error* $= \frac{|y(x_n) - y_n|}{|y(x_n)|}$
- absolute error is the distance between actual value and approximation
- relative error divides this by the actual value

big caveat about measuring accuracy

- you can only compute absolute or relative error *if* the exact solution is known
- ...but often the reason we use a numerical method like Euler's is because the exact solution is *not* known
- so examples where the absolute or relative error is computed are automatically "toy examples"

example 2

- this is exercise #3 in §2.6 . . . a "toy example"
- Example 2: for the ODE IVP

$$y' = y, \quad y(0) = 1$$

  (a) use Euler's method to get a 4-decimal approximation of $y(1)$
    - use $h = 0.1$ first, and then $h = 0.05$

  (b) find the exact solution

  (c) show in a table: x_n, y_n, the exact value $y(x_n)$, the absolute error, and the relative error

example 2, cont.

- so one can proceed by hand, but its tedious work . . .
- and it is an original purpose for which electronic computers were designed
- I used the Matlab/Octave code below
  - see simpleeuler.m at the "other" tab on the course webpage

```
h = 0.1;                 % change to e.g. h=0.05
N = 10;                  % change to e.g. N=20
x = 0;
y = 1;
for n = 1:N+1
    exact = exp(x);
    [x, y, exact, abs(y-exact), 100*abs(y-exact)/abs(exact)]
    y = y + h * y;       % this IS Euler's method
    x = x + h;
end
```

example 2, cont. cont.

- the code produces the table below when $h = 0.1$ and we take $N = 10$ steps ...giving 4.58% relative error at $x = 1$

x_n	y_n	actual value	abs. error	rel. error
0.00	1.0000	1.0000	0.0000	0.00
0.10	1.1000	1.1052	0.0052	0.47
0.20	1.2100	1.2214	0.0114	0.93
0.30	1.3310	1.3499	0.0189	1.40
0.40	1.4641	1.4918	0.0277	1.86
0.50	1.6105	1.6487	0.0382	2.32
0.60	1.7716	1.8221	0.0506	2.77
0.70	1.9487	2.0138	0.0650	3.23
0.80	2.1436	2.2255	0.0820	3.68
0.90	2.3579	2.4596	0.1017	4.13
1.00	2.5937	2.7183	0.1245	4.58

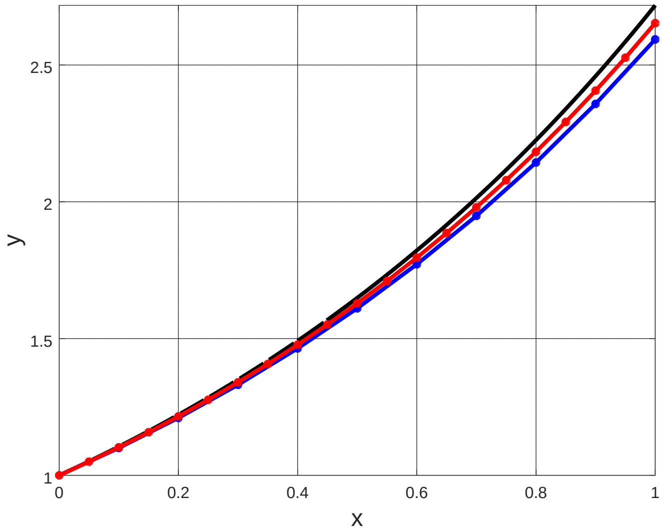
- for $h = 0.001$ and $N = 1000$ I get 0.05% rel. error:

$$y_{1000} = 2.71692 \approx 2.71828 = y(1)$$

example 2, cont.[3]; $h = 0.05$, $N = 20$ case

x_n	y_n	actual value	abs. error	rel. error
0.00	1.0000	1.0000	0.0000	0.00
0.05	1.0500	1.0513	0.0013	0.12
0.10	1.1025	1.1052	0.0027	0.24
0.15	1.1576	1.1618	0.0042	0.36
0.20	1.2155	1.2214	0.0059	0.48
0.25	1.2763	1.2840	0.0077	0.60
0.30	1.3401	1.3499	0.0098	0.72
0.35	1.4071	1.4191	0.0120	0.84
0.40	1.4775	1.4918	0.0144	0.96
0.45	1.5513	1.5683	0.0170	1.08
0.50	1.6289	1.6487	0.0198	1.20
0.55	1.7103	1.7333	0.0229	1.32
0.60	1.7959	1.8221	0.0263	1.44
0.65	1.8856	1.9155	0.0299	1.56
0.70	1.9799	2.0138	0.0338	1.68
0.75	2.0789	2.1170	0.0381	1.80
0.80	2.1829	2.2255	0.0427	1.92
0.85	2.2920	2.3396	0.0476	2.04
0.90	2.4066	2.4596	0.0530	2.15
0.95	2.5270	2.5857	0.0588	2.27
1.00	2.6533	2.7183	0.0650	2.39

example 2, cont.[4]



another derivation of Euler's method

- start with the DE
$$\frac{dy}{dx} = f(x, y)$$
- remember what a derivative is!:
$$\lim_{h \to 0} \frac{y(x + h) - y(x)}{h} = f(x, y(x))$$
- think: $y(x)$ is current value and $y(x + h)$ is next value
- drop the limit and adopt this as a method:
$$\frac{y_{n+1} - y_n}{h} = f(x_n, y_n)$$
    - at this point y_n and $y(x_n)$ mean different things!
- rewrite as Euler's method before: $y_{n+1} = y_n + hf(x_n, y_n)$

are there better methods?

- yes!
- here is a derivation, by picture, of the "explicit midpoint rule"
  - a.k.a. "modified Euler method"
  - see Wikipedia page for "midpoint method"
  - it is "second order" so it gets same accuracy in 10 steps that Euler does in 100 steps

expectations

- to learn this material, just watching this video is *not* enough!
- also:
  - *watch* "found online" videos at

    `bueler.github.io/math302/week4.html`
  - *try-out* Euler's method codes at the same link
  - *read* section 2.6 in the textbook
    - mentions the well-known "fourth order Runge-Kutta method", even better than the explicit midpoint rule above
  - *do* the WebAssign exercises for section 2.6

- by the way, Euler's and other numerical methods return in Chapter 9 . . .