

4.1 Higher-order linear equations: first examples and preliminaries

a lesson for MATH F302 Differential Equations

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February 9, 2019

for textbook: D. Zill, *A First Course in Differential Equations with Modeling Applications*, 11th ed.

outline

plan for these slides

- a bit of review of first-order linear equations (§2.3)
- a first look at how to solve constant-coefficient, second-order linear equations (from §4.3)
- a whole bunch of new language for higher-order linear equations
 - basically, §4.1 is a lot of new words

first-order linear DEs: a review

- recall first-order linear DEs:

$$a_1(x)y' + a_0(x)y = g(x)$$

- one may divide by the leading coefficient:

$$y' + P(x)y = f(x)$$

- this requires leading coefficient $a_1(x)$ to *not* be zero on the interval where we are solving
- special case 1 (*easiest to solve*): constant-coefficient and homogeneous

$$y' + by = 0$$

- *homogeneous* means the right-hand side is zero
 - *constant-coefficient* means b is constant
 - the solution is (“by inspection”?)

$$y(x) = Ae^{-bx}$$

first-order linear review cont.

- special case 2: homogeneous (but otherwise general)

$$y' + P(x)y = 0$$

- now we need an integrating factor $\mu(x) = e^{Q(x)}$ where $Q(x) = \int P(x) dx$ is any antiderivative of $P(x)$
- multiplying by μ the equation becomes $(\mu(x)y(x))' = 0$
- thus

$$e^{Q(x)}y(x) = A$$

- thus the solution is

$$y(x) = Ae^{-Q(x)}$$

- homogeneous: a multiple of a solution is still a solution

first-order linear review cont.²

- general *nonhomogeneous* case: first-order linear

$$y' + P(x)y = f(x)$$

- need same integrating factor; multiplying by $\mu = e^{Q(x)}$ yields $(\mu(x)y(x))' = \mu(x)f(x)$
- integrate:

$$e^{Q(x)}y(x) = A + \int_a^x e^{Q(t)}f(t) dt$$

- where $Q(x) = \int P(x) dx$ is *any* antiderivative of $P(x)$
- written to emphasize right side has a free constant A
- thus the solution is

$$y(x) = Ae^{-Q(x)} + e^{-Q(x)} \int_a^x e^{Q(t)}f(t) dt$$

- solution is the homogeneous solution plus a particular solution

higher-order linear DEs: overview

for n th-order linear equations

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = g(x)$$

new versions of all four comments in red on the previous slides still apply

overview cont.

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y \stackrel{*}{=} g(x)$$

- ① if $a_n(x) \neq 0$ then we can divide by it:

$$y^{(n)} + b_{n-1}(x)y^{(n-1)} + \cdots + b_1(x)y' + b_0(x)y = f(x)$$

- ② easiest case (§4.3) is homogeneous and constant coefficient

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0$$

- ③ for the associated homogeneous equation to *,

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = 0$$

any multiple of, or sum of, solutions is again a solution

- ④ solutions of * are always solutions of the homogeneous equation plus a particular solution

solutions exist

Theorem

- Consider the linear DE

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = g(x)$$

If the functions $a_j(x)$ and $g(x)$ are continuous on some interval, and if $a_n(x) \neq 0$ on that interval, then solutions exist.

- Furthermore, if x_0 is in that interval then there is exactly one solution which satisfies the initial values*

$$y(x_0) = y_0$$

$$y'(x_0) = y_1$$

$$\vdots$$

$$y^{(n-1)}(x_0) = y_{n-1}$$

linear, homogeneous, constant-coefficient

- furthermore, linear DEs which are **homogeneous and constant-coefficient** always have exponential solutions
 - you can always find *at least one solution* $y = e^{mx}$
 - *and* multiples and sums of solutions are solutions
- *example 1:* solve, by trying $y(x) = e^{mx}$, the equation

$$y'' + 4y' - 5y = 0$$

fundamental set of solutions:

general solution:

example 2

- *example 2*: solve, by trying $y(x) = e^{mx}$, the equation

$$y''' + 3y'' - y' - 3y = 0$$

fundamental set of solutions:

general solution:

linear combination

- examples 1 and 2 are from §4.3 but they let me illustrate the language introduced in §4.1 [← read this section!](#)
- for example,

Theorem

If $y_1(x), y_2(x), \dots, y_n(x)$ solve a linear and homogeneous DE

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \dots + a_1(x)y' + a_0(x)y = 0$$

then any linear combination

$$y(x) = c_1y_1(x) + c_2y_2(x) + \dots + c_ny_n(x)$$

is also a solution.

- idea: for linear and homogeneous DEs you can form a more general solution from any set of solutions
 - see examples 1 and 2

linear dependence and independence

- a set of functions $\{f_1(x), \dots, f_n(x)\}$ is *linearly dependent* if you can combine with constants c_1, \dots, c_n , some of which are not zero, and get the zero function:

$$c_1 f_1(x) + c_2 f_2(x) + \dots + c_n f_n(x) = 0$$

- a set is *linearly independent* if it is not linearly dependent
- *example:*

$$f_1(x) = x^2 + x, \quad f_2(x) = x^2 - x, \quad f_3(x) = 5x$$

are linearly dependent because

$$1 \cdot f_1(x) - 1 \cdot f_2(x) - \frac{2}{5} \cdot f_3(x) = 0$$

example 3

- recall from example 1 that $f_1(x) = e^x$ and $f_2(x) = e^{-5x}$ are solutions to $y'' + 4y' - 5y = 0$
- *example 3*: Find a solution of the initial value problem

$$y'' + 4y' - 5y = 0, \quad y(0) = 2, \quad y'(0) = -3$$

- this calculation works because $\{f_1(x), f_2(x)\} = \{e^x, e^{-5x}\}$ is a linearly-independent set

checking linear independence

- generally it would require linear algebra thinking to check whether a set of functions is linearly independent
- *but* there is a determinant to save you from thinking!
- definition. given functions $f_1(x), \dots, f_n(x)$ the *Wronskian* is the determinant where the rows are derivatives:

$$W(f_1, \dots, f_n) = \det \begin{pmatrix} \begin{bmatrix} f_1 & f_2 & \dots & f_n \\ f_1' & f_2' & \dots & f_n' \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \dots & f_n^{(n-1)} \end{bmatrix} \end{pmatrix}$$

- *example 4:* find the Wronskian of $\{e^{-3x}, e^{-x}, e^x\}$

Theorem

Suppose $\{y_1(x), y_2(x), \dots, y_n(x)\}$ are solutions of a homogeneous linear n th-order differential equation on some interval. Then

- *The set of solutions is linearly-independent if and only if the Wronskian $W(y_1, \dots, y_n)$ is nonzero on the interval.*
- *If the Wronskian $W(y_1, \dots, y_n)$ is nonzero at some point on the interval then it is nonzero on the whole interval.*

fundamental set

definition. a set of n linearly-independent solutions $\{y_1(x), y_2(x), \dots, y_n(x)\}$ of the homogeneous linear n th-order differential equation

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \dots + a_1(x)y' + a_0(x)y = 0$$

is a *fundamental set of solutions*

- once you have a fundamental set then the *general solution* of the above DE is

$$y(x) = c_1y_1(x) + c_2y_2(x) + \dots + c_ny_n(x)$$

- if you have fewer than n solutions, or they are not linearly independent, then the linear combination *is* a solution, but not fully general

exercise 25 in §4.1

- *exercise #25*: Verify that the functions form a fundamental set of solutions on the interval. Form the general solution.

$$y'' - 2y' + 5y = 0, \quad \{e^x \cos 2x, e^x \sin 2x\}, \quad (-\infty, \infty)$$

exercise 27 in §4.1

- *exercise #27*: Verify that the functions form a fundamental set of solutions on the interval. Form the general solution.

$$x^2 y'' - 6xy' + 12y = 0, \quad \{x^3, x^4\}, \quad (0, \infty)$$

expectations

- just watching this video is *not* enough!

- see “found online” videos at
bueler.github.io/math302/week6.html
- read section 4.1 in the textbook
 - know the meaning/definitions of:

homogeneous

nonhomogeneous

associated homogeneous equation

linear combination

superposition

linearly dependent

linearly independent

Wronskian

fundamental set of solutions

general solution

particular solution

complementary function

- we will soon focus more on nonhomogeneous equations (§4.4), but the homogeneous case is central for a while (§4.3 and then §4.2)
 - *but* during this course I will not ask questions about “boundary conditions” and “boundary value problems”
 - there is quite a bit of new language in §4.1!
- do the WebAssign exercises for section 4.1