HIGH POINTS OF CALCULUS

Below is a review of some calculus you will need for this course in differential equations. However, this first Mini-Project is also a test of whether you are able to successfully submit your work in this online course!

Logistics. Please fill in the blanks below with complete and legible answers. You may do this either by

- electronically-editing the PDF or
- printing this blank version and writing your answers with pencil or pen.

In the first case, save your completed document as a PDF. In the second case you should scan or photograph your completed document and then figure out how to save it as a PDF. In any case *you must produce a PDF for submission*. Submit it by uploading the PDF using the Google Form you were sent for uploading this Mini-Project.

1. *Chain rule.* Recall the chain rule

$$[f\,(g(x))]' = f'\,(g(x))\;g'(x)$$

(a) Compute the derivative. Identify the outer function $f(x)$ and the inner function $g(x)$ you used.

$$\left[\sqrt{\arctan x + 3x}\right]' =$$

(b) Construct your own chain rule example. (*Make it non-trivial but not too complicated. In particular, neither $f(x)$ nor $g(x)$ should be as simple as a linear function, i.e. $ax+b$.*)

$$f(x) =$$

$$g(x) =$$

$$[f\,(g(x))]' =$$

(*Remember that $x^k, e^x, \ln x, b^x, \log_b x, \sin x, \cos x, \tan x, \sec x, \arcsin x$ are some common functions from calculus which you must be able to correctly differentiate! You might use this problem to practice the ones that are least familiar.*)

2. *Integration by substitution.* Starting with basics, remember that the indefinite integral just means "anti-derivative". In fact

$$\int f(x)\,dx = F(x) + C$$

means exactly the same thing as

$$(F(x))' = f(x).$$

You can do some integrals just by recognizing a derivative, perhaps with some fiddling with constants.

(a) Compute the indefinite integral:

$$\int \frac{1}{x}\,dx =$$

(b) Compute the indefinite integral:

$$\int 3^{2x}\,dx =$$

Integration by substitution is **the chain rule in reverse**. For example, from **1 (a)** we have

$$\int \frac{\sec^2 x + 3}{\sqrt{\arctan x + 3x}}\,dx = \int \frac{du}{\sqrt{u}} \qquad \text{[with } u = \arctan x + 3x\text{]}$$
$$= 2u^{1/2} + C = 2\sqrt{\arctan x + 3x} + C$$

(It is common to need to fiddle with constant factors like the "2" here.) In general:

$$\int f'(g(x))g'(x)\,dx = \int f'(u)\,du = f(u) + C = f(g(u)) + C$$

(c) Turn **1 (b)** into an integration by substitution.

3. *Product rule and integration-by-parts.* The product rule

$$[u(x)v(x)]' = u'(x)v(x) + u(x)v'(x)$$

can be used in reverse too. The indefinite integral of both sides of the above gives

$$u(x)v(x) = \int u'(x)v(x)\,dx + \int u(x)v'(x)\,dx.$$

The main use of this is to exchange one of the last two integrals for the other; *that* is integration-by-parts:

$$\int u(x)v'(x)\,dx = u(x)v(x) - \int u'(x)v(x)\,dx.$$

You probably have it memorized as

$$\int u\,dv = uv - \int v\,du.$$

(a) Construct your own product rule example. (*Again, make it non-trivial but not too complicated;* $u(x), v(x)$ *should not be as simple linear functions.*)

$$u(x) =$$

$$v(x) =$$

$$[u(x)v(x)]' =$$

(b) Take the example in **(a)** and turn it into an integration-by-parts example.

4. *Fundamental Theorem of Calculus (FTC).* When you compute a definite integral by hand you usually use a form of the FTC:

$$\int_a^b f(x)\,dx = F(b) - F(a) \qquad \text{where } F'(x) = f(x)$$

This form of the FTC says that doing an integral is the same as un-doing a derivative. Recall that if you do such an integral by substitution then you can change the limits:

$$\int_{x=a}^{x=b} f(g(x))\,g'(x)\,dx = \int_{u=g(a)}^{u=g(b)} f(u)\,du = F(g(b)) - F(g(a))$$

(a) Compute

$$\int_{\pi/6}^{\pi/2} \frac{\cos x}{1 + 9\sin^2 x}\,dx =$$

But there is another form of the FTC, often called FTC I. It says that a derivative un-does an integral:

$$\frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x)$$

The integral inside the parentheses computes the area under the curve $y = f(t)$ from $t = a$ to $t = x$, and one should think of this area varying as x changes. Thus it defines a function, namely $g(x) = \int_a^x f(t)\,dt$. You can answer some questions about this function even if you cannot find an antiderivative of the integrand.

(b) Suppose we define

$$g(x) = \int_0^x \sin(e^t)\,dt$$

Compute the exact value of $g(0)$.

(c) For the same function $g(x)$, find $g'(x)$.