8.2 Homogeneous linear systems of first-order ODEs

a lesson for MATH F302 Differential Equations

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for textbook: D. Zill, *A First Course in Differential Equations with Modeling Applications*, 11th ed.

homogeneous linear systems of ODEs

- system of ODEs:

$$\mathbf{X}' = \mathbf{A}\mathbf{X}$$

  - in sections 8.2 and 8.4 we assume $\mathbf{A}$ is a matrix of constants
- which means

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

  - the solutions $x_1(t), \dots, x_n(t)$ are combined into a vector $\mathbf{X}(t)$
  - the coefficients a_{ij} are combined into a matrix $\mathbf{A}$

how do you solve the simplest ODEs?

- *ODE 1.* how do you solve for $y(t)$?:

$$y' = 3y$$

answer: *the solution is an exponential* $y(x) = c\, e^{3t}$

- *ODE 2.* how do you solve for $y(t)$ if p and q are constant?:

$$y'' + py' + qy = 0$$

answer: *try an exponential*

$$y(x) = e^{mt}$$

and get an auxiliary equation to determine m:

$$\begin{aligned} m^2 e^{mt} + pme^{mt} + qe^{mt} &= 0 \\ m^2 + pm + q &= 0 \end{aligned}$$

how do you solve the simplest ODEs?

- *ODE 3*. how do you solve for $\mathbf{X}(t)$ if $\mathbf{A}$ is a constant matrix?:

$$\mathbf{X}' = \mathbf{A}\mathbf{X}$$

answer: *try an exponential times a vector*

$$\mathbf{X}(t) = \mathbf{K}e^{\lambda t}$$

and get an auxiliary equation to determine λ:

[what equation goes here?]

- λ is an unknown *scalar*, like *m* before
- $\mathbf{K}$ is an unknown *vector*

the eigenvalue equation for a system

$$\mathbf{X}' = \mathbf{A}\mathbf{X}$$

- try $\quad \mathbf{X}(t) = \mathbf{K}e^{\lambda t} \quad$ so

$$\begin{aligned} &\text{left side:} && \mathbf{X}' = \mathbf{K}\lambda e^{\lambda t} \\ &\text{right side:} && \mathbf{A}\mathbf{X} = \mathbf{A}\mathbf{K}e^{\lambda t} \end{aligned}$$

- so system $\mathbf{X}' = \mathbf{A}\mathbf{X}$ becomes

$$\mathbf{K}\lambda e^{\lambda t} = \mathbf{A}\mathbf{K}e^{\lambda t}$$
$$\mathbf{K}\lambda = \mathbf{A}\mathbf{K}$$

- the *eigenvalue equation*:

$$\boxed{\mathbf{A}\mathbf{K} = \lambda\mathbf{K}}$$

the last slide

- the last slide is the main idea
- write it out yourself and understand it!

meaning of the eigenvalue equation

- eigenvalue equation is analogous to auxiliary equation:

	scalar	system
ODE	$y'' + py' + qy = 0$	$\mathbf{X}' = \mathbf{A}\mathbf{X}$
trial solution	$y(t) = e^{mt}$	$\mathbf{X}(t) = \mathbf{K}e^{\lambda t}$
equation	$m^2 + pm + q = 0$	$\boxed{\mathbf{A}\mathbf{K} = \lambda\mathbf{K}}$

- in the eigenvalue equation we are seeking *both*
  - *eigenvalues*: the exponential rates λ
  - *eigenvectors*: the "constants" $\mathbf{K}$

- "eigen" means "characteristic of" or "property of"
  - the eigenvalues of a matrix $\mathbf{A}$ are the characteristic numbers to associate to $\mathbf{A}$

forms of the eigenvalue equation

- don't forget the ODE problem we started with: $\mathbf{X}' = \mathbf{A}\mathbf{X}$

- different forms of the eigenvalue equation:

$$
\begin{aligned}
\lambda \mathbf{K} e^{\lambda t} &= \mathbf{A}\mathbf{K} e^{\lambda t} && \text{substitute } \mathbf{X} = \mathbf{K} e^{\lambda t} \text{ ... but } e^z \neq 0 \\
\mathbf{A}\mathbf{K} &= \lambda \mathbf{K} && \text{eigenvalue equation} \\
(\mathbf{A} - \lambda \mathbf{I})\mathbf{K} &= 0 && \text{because } \mathbf{I}\mathbf{K} = \mathbf{K} \\
\det(\mathbf{A} - \lambda \mathbf{I}) &= 0 && \text{because we want } \textit{nonzero} \text{ solutions } \mathbf{K}
\end{aligned}
$$

  - $\mathbf{I}$ is the identity matrix
  - *fact:* a linear equation $\mathbf{M}\mathbf{Z} = 0$ has a nonzero solution only if the determinant of $\mathbf{M}$ is zero: $\det \mathbf{M} = 0$

like #1 in §8.2

- assuming eigenvalues/vectors of $\mathbf{A}$ appear when needed ...
- *example 1.* find the general solution of the system:

$$
\begin{aligned}
\frac{dx}{dt} &= 4x + 3y \\
\frac{dy}{dt} &= x + 2y
\end{aligned}
$$

like #9 in §8.2

- assuming eigenvalues/vectors of $\mathbf{A}$ appear when needed ...
- *example 2.* find the general solution of the system:

$$\mathbf{X}' = \begin{pmatrix} 1 & 7 & 0 \\ 0 & -2 & 0 \\ 1 & 6 & 4 \end{pmatrix} \mathbf{X}$$

scalar 2nd-order $\Longleftrightarrow$ system of 2 eqns

example 3.

- find the general solution of the 2nd-order scalar ODE:

  $y'' + 3y' + 2y = 0$

- convert above to a system:

- find the general solution of the system:

  $$\mathbf{X}' = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} \mathbf{X}$$

important ideas about eigenvectors

- remember the eigenvalue equation:

$$\mathbf{A}\mathbf{K} = \lambda \mathbf{K}$$

- important ideas:
  - by definition, an eigenvector $\mathbf{K}$ must be nonzero
  - any nonzero multiple of an eigenvector is also an eigenvector
  - in other words, only the *direction* of an eigenvector is important, not its magnitude

help from a machine

- once you know what you want you can get it fast by machine!
- find eigenvalues and eigenvectors in MATLAB/OCTAVE:

```
>> [V,D] = eig(A)     % diagonal of D has eigenvalues
                      % columns of V are eigenvectors
```

- *example 1, cont.* get eigenvalues and eigenvectors:

```
>> A = [4, 3; 1, 2];
>> [V,D] = eig(A)
V =
  0.94868  -0.70711
  0.31623  0.70711
D =
        5         0
        0         1
```

help from a machine, cont.

- *example 1, cont.* here's how I got cleaner vectors $\mathbf{K}_1, \mathbf{K}_2$:

```
>> K1 = V(:,1)
K1 =
  0.94868
  0.31623
>> K1 = K1/K1(1)
K1 =
  1.00000
  0.33333
>> K2 = V(:,2);  K2 = K2/K2(1)
K2 =
        1
       -1
```

- use this technique on the WebAssign problems

like #13 in §8.2

- *example 4.* solve the initial value problem:

$$\mathbf{X}' = \begin{pmatrix} 4 & 3 \\ 1 & 2 \end{pmatrix} \mathbf{X}, \qquad \mathbf{X}(0) = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

like #35 in §8.2

- eigenvalues could be complex . . . see example 6, page 351
- *example 5.* find the general solution of the system:

$$\mathbf{X}' = \begin{pmatrix} 5 & -1 \\ 5 & 1 \end{pmatrix} \mathbf{X}$$

on quizzes and exams

- on quizzes and exams:
  - I will supply the eigenvalues and eigenvectors
  - I will only ask about the distinct real eigenvalues case
  - so: examples 1–4 above could appear on quizzes/exams
  - problems like example 5 appear on WebAssign
  - actually, only the Final Exam remains ...

the main idea

- the ODE system
$$\mathbf{X}' = \mathbf{A}\mathbf{X}$$
has solutions
$$\mathbf{X}(t) = \mathbf{K}e^{\lambda t}$$
where λ is an eigenvalue of **A** and **K** is an eigenvector of **A**,
$$\mathbf{A}\mathbf{K} = \lambda\mathbf{K}$$

- in the modern world a machine provides:
```
>> [V,D] = eig(A)     % diagonal of D has eigenvalues
                      % columns of V are eigenvectors
```

- by hand:
  - solve $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$ for all the λ
  - for each λ, solve $\mathbf{A}\mathbf{K} = \lambda\mathbf{K}$ for a nonzero **K**
  - the book shows this

expectations

- just watching this video is *not* enough!
  - see "found online" videos and stuff at
    `bueler.github.io/math302/week14.html`
  - *read* §8.2
    - you are responsible for the "distinct real eigenvalues" and the "complex eigenvalues" cases for WebAssign
    - you are only responsible for "distinct real eigenvalues" on the Final Exam
    - I will give you the eigenvalues and eigenvectors on the Final
    - you are not responsible for the "repeated eigenvalues" case
  - *do* the WebAssign exercises for section 8.2