8.1 Linear systems of first-order ODEs: basics and forms

a lecture for MATH F302 Differential Equations

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for textbook: D. Zill, *A First Course in Differential Equations with Modeling Applications*, 11th ed.

first-order systems

- we have already seen the most general form of a system of ODEs (§3.3):

$$\begin{aligned}
\frac{dx_1}{dt} &= g_1(t, x_1, x_2, \dots, x_n) \\
\frac{dx_2}{dt} &= g_2(t, x_1, x_2, \dots, x_n) \\
&\vdots \\
\frac{dx_n}{dt} &= g_n(t, x_1, x_2, \dots, x_n)
\end{aligned}$$

  - my claim in §3.3: everything is modeled this way
- Chapter 8 is about a special case:

  suppose variables x_i only appear with first powers

first-order *linear* systems

- a first-order system of linear ODEs is

$$
\begin{aligned}
\frac{dx_1}{dt} &= a_{11}(t)x_1 + a_{12}(t)x_2 + \cdots + a_{1n}(t)x_n + f_1(t)\\
\frac{dx_2}{dt} &= a_{21}(t)x_1 + a_{22}(t)x_2 + \cdots + a_{2n}(t)x_n + f_2(t)\\
&\vdots\\
\frac{dx_n}{dt} &= a_{n1}(t)x_1 + a_{n2}(t)x_2 + \cdots + a_{nn}(t)x_n + f_n(t)
\end{aligned}
$$

  - the book calls this the *normal form* of the system
  - $a_{ij}(t)$ functions are the *coefficients*
    - if $a_{ij}(t)$ are independent of time then we say it is a *constant-coefficient* system
  - $f_i(t)$ are the *source functions*
    - if all $f_i = 0$ then the system is *homogeneous*

examples

- consider three example systems from the §3.3 slides and video

instructions: identify the coefficients $a_{ij}(t)$ and source functions $f_i(t)$

- *example 1.*

$$
\begin{aligned}
\frac{dx}{dt} &= -2x \\
\frac{dy}{dt} &= x - y
\end{aligned}
$$

- *example 2.*

$$
\begin{aligned}
\frac{dx_1}{dt} &= -0.04x_1 + 0.02x_2 \\
\frac{dx_2}{dt} &= 0.04x_1 - 0.07x_2 + 0.03x_3 \\
\frac{dx_3}{dt} &= 0.05x_2 - 0.05x_3
\end{aligned}
$$

examples, cont.

- *example 3.*

$$
\begin{aligned}
y' &= u \\
u' &= v \\
v' &= w \\
w' &= 4w - 7v - 10u + y + \sin(3t)
\end{aligned}
$$

matrix form

- a first-order linear system

$$\frac{dx_1}{dt} = a_{11}(t)x_1 + a_{12}(t)x_2 + \cdots + a_{1n}(t)x_n + f_1(t)$$

$$\vdots$$

$$\frac{dx_n}{dt} = a_{n1}(t)x_1 + a_{n2}(t)x_2 + \cdots + a_{nn}(t)x_n + f_n(t)$$

- is usually written

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix}$$

- or

$$\mathbf{X}' = \mathbf{AX} + \mathbf{F}$$

a matrix times a vector

- so: recall matrix-vector multiplication!
- *example 4.* compute the product

$$\begin{pmatrix} 2 & -3 & -2 \\ 1 & 0 & 5 \\ 4 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} =$$

- *example 5.* compute

$$\begin{pmatrix} 3 & -2 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \end{pmatrix} + \begin{pmatrix} 4 \\ 2 \end{pmatrix} =$$

example matrix forms

instructions: write the linear systems in matrix form $\mathbf{X}' = \mathbf{AX} + \mathbf{F}$ (what is $\mathbf{X}$? $\mathbf{A}$? $\mathbf{F}$?)

- *example 6.*

$$\frac{dx}{dt} = -2x$$
$$\frac{dy}{dt} = x - y$$

- *example 7.*

$$\frac{dx_1}{dt} = -0.04x_1 + 0.02x_2$$
$$\frac{dx_2}{dt} = 0.04x_1 - 0.07x_2 + 0.03x_3$$
$$\frac{dx_3}{dt} = 0.05x_2 - 0.05x_3$$

example matrix forms, cont.

- *example 8.*

$$
\begin{aligned}
y' &= u \\
u' &= v \\
v' &= w \\
w' &= 4w - 7v - 10u + y + \sin(3t)
\end{aligned}
$$

note: *(i) examples 1,2,3 are constant coefficient, (ii) examples 1,2 are homogeneous*

- *example 9.* for the linear system

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & -7 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 4 \\ 8 \end{pmatrix}\sin t + \begin{pmatrix} t-4 \\ 2t+1 \end{pmatrix} e^{4t}$$

(a) identify $\mathbf{A}$ and $\mathbf{F}$ so it is in the form $\mathbf{X}' = \mathbf{A}\mathbf{X} + \mathbf{F}$

(b) write it *without* the use of matrices

solution.

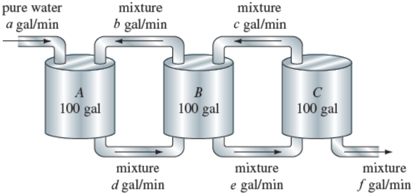
yes, but what does it look like?

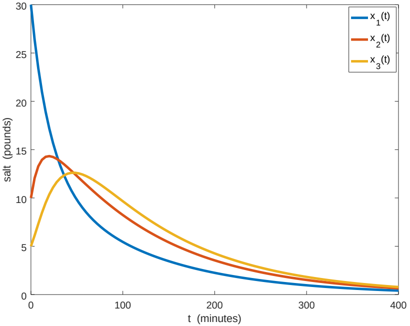
- examples 2 & 7 came from "connected tanks" in §3.3:

$$\frac{dx_1}{dt} = -0.04x_1 + 0.02x_2$$
$$\frac{dx_2}{dt} = 0.04x_1 - 0.07x_2 + 0.03x_3 \quad \Longleftrightarrow \quad \mathbf{X}' = \mathbf{A}\mathbf{X}$$
$$\frac{dx_3}{dt} = 0.05x_2 - 0.05x_3$$

$$\mathbf{A} = \begin{pmatrix} -0.04 & 0.02 & 0 \\ 0.04 & -0.07 & 0.03 \\ 0 & 0.05 & -0.05 \end{pmatrix}$$

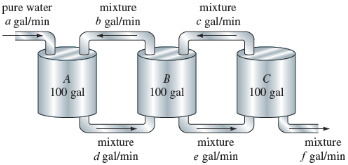
- suppose initial conditions
  $x_1(0) = 30$
  $x_2(0) = 10$
  $x_3(0) = 5$

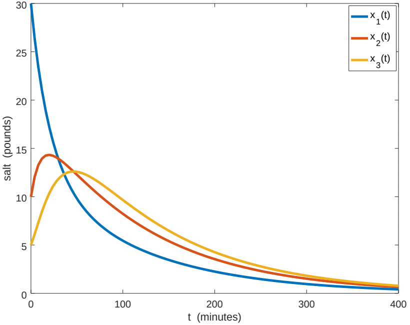


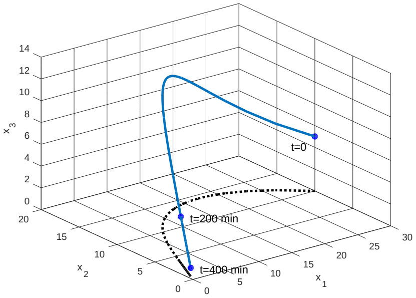


what does it look like?, cont.

- variables t, x_1, x_2, x_3 ... 4D? ... unvisualizable!
- alternate view is to suppress t and plot in 3D $= x_1, x_2, x_3$
- see code `brines.m`

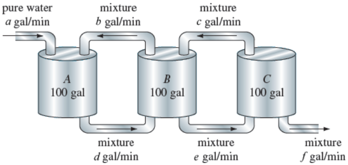


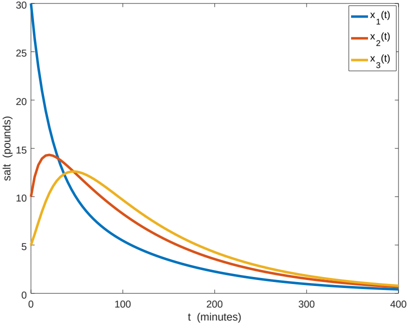


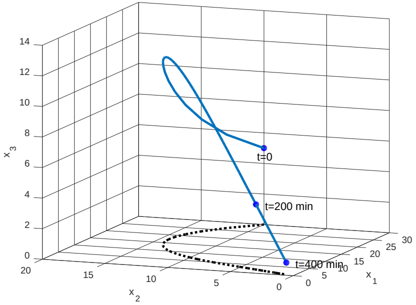


what does it look like?, cont.

- variables t, x_1, x_2, x_3 ... 4D? ... unvisualizable!
- alternate view is to suppress t and plot in 3D $= x_1, x_2, x_3$
- see code `brines.m`
  - uses ode45
  - *rotatable* figure







these problems have solutions

Theorem

Consider a linear system with initial values:

$$\mathbf{X}' = \mathbf{A}\mathbf{X} + \mathbf{F}, \quad \mathbf{X}(t_0) = \mathbf{X}_0$$

Assume the entries in $\mathbf{A}(t)$ *and* $\mathbf{F}(t)$ *are continuous. Assume* $\mathbf{X}_0$ *is a given vector. Then there is one solution* $\mathbf{X}(t)$*.*

- so what?
- it is a big deal!
- you can make predictions
  from knowledge of current state $\quad \mathbf{X}(t_0) = \mathbf{X}_0$
  and laws about how things change $\quad \mathbf{X}' = \mathbf{A}\mathbf{X} + \mathbf{F}$
  to create one prediction $\quad \mathbf{X}(t)$

these problems have *general* solutions

Theorem

Consider a homogeneous linear system:

$$\mathbf{X}' = \mathbf{A}\mathbf{X}$$

There is a fundamental set of solutions $\mathbf{X}_1(t), \mathbf{X}_2(t), \ldots, \mathbf{X}_n(t)$ *so that any solution of the linear system is a linear combination:*

$$\mathbf{X}(t) = c_1\mathbf{X}_1(t) + c_2\mathbf{X}_2(t) + \cdots + c_n\mathbf{X}_n(t)$$

these problems have *general* solutions 2

Theorem

Consider a nonhomogeneous linear system:

$$\mathbf{X}' = \mathbf{A}\mathbf{X} + \mathbf{F}$$

Suppose $\mathbf{X}_p(t)$ *is one solution of this system. Let* $\mathbf{X}_c(t)$ *be the general solution to the associated homogeneous system* $\mathbf{X}' = \mathbf{A}\mathbf{X}$*,*

$$\mathbf{X}_c(t) = c_1\mathbf{X}_1(t) + c_2\mathbf{X}_2(t) + \cdots + c_n\mathbf{X}_n(t)$$

Then the general solution is

$$\mathbf{X}(t) = \mathbf{X}_c(t) + \mathbf{X}_p(t)$$

like #12 in §8.1

- §8.2 shows how to solve homogeneous systems $\mathbf{X}' = \mathbf{AX}$
- for now, you will be asked to *check* solutions, as follows
- *example 10.* verify that $\mathbf{X}(t) = \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-2t}$ is a solution of the linear system

$$\mathbf{X}' = \begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix} \mathbf{X}$$

solution.

like #15 in §8.1

- *example 11.* verify that $\mathbf{X}(t) = \begin{pmatrix} -1 \\ -6 \\ 13 \end{pmatrix}$ is a solution of the linear system

$$\mathbf{X}' = \begin{pmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{pmatrix} \mathbf{X}$$

solution.

linear independent solutions

- *definition.* if $\mathbf{X}_1(t), \mathbf{X}_2(t), \ldots, \mathbf{X}_n(t)$ are linearly-independent then we say they form a *fundamental set*
- you can check linear independence by checking that the *Wronskian* is nonzero:

$$W\left(\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n\right) = \det \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & & x_{2n} \\ \vdots & & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{pmatrix} \neq 0$$

- above uses notation for entries:

$$\mathbf{X}_1 = \begin{pmatrix} x_{11} \\ x_{21} \\ \vdots \\ x_{n1} \end{pmatrix}, \mathbf{X}_2 = \begin{pmatrix} x_{12} \\ x_{22} \\ \vdots \\ x_{n2} \end{pmatrix}, \ldots, \mathbf{X}_n = \begin{pmatrix} x_{1n} \\ x_{2n} \\ \vdots \\ x_{nn} \end{pmatrix}$$

like #17 in §8.1

- *example 12.* determine whether the vectors (solutions) form a fundamental set:

$$\mathbf{X}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-2t}, \quad \mathbf{X}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-6t}$$

solution.

expectations

to learn this material, just listening to a lecture is *not* enough

- *read* section 8.1
- do Homework 8.1
- from now on (§8.2, 8.4) we will focus entirely on *homogeneous* systems