

7.1 Laplace Transforms

(starting from the definition)

a lesson for MATH F302 Differential Equations

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March 21, 2019

for textbook: D. Zill, *A First Course in Differential Equations with Modeling Applications*, 11th ed.

the definition

- the Laplace transform of a function $f(t)$ defined on $(0, \infty)$ is

$$\mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t) dt$$

- this is the book's notation
- the result of applying the Laplace transform is a function of s
 - so slightly better notation would be

$$\mathcal{L}\{f\}(s) = \int_0^{\infty} e^{-st} f(t) dt$$

- a common (and good) way to write it is

$$F(s) = \int_0^{\infty} e^{-st} f(t) dt$$

why do we use \mathcal{L} in differential equations?

- why do we use

$$\mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t) dt \quad ?$$

- because the Laplace transform **converts a linear DE** (in t) **into an algebra problem** (in s)
 - this is especially useful for solving *nonhomogeneous* DEs
 - ... it's how many engineers think about nonhomogeneous DEs
 - the Laplace transform *linear*, that is,

$$\begin{aligned}\mathcal{L}\{f(t) + g(t)\} &= \mathcal{L}\{f(t)\} + \mathcal{L}\{g(t)\}, \text{ and} \\ \mathcal{L}\{\alpha f(t)\} &= \alpha \mathcal{L}\{f(t)\}\end{aligned}$$

- the Laplace transform is basically limited to linear DEs

practice with integrals on $[a, \infty)$

- a Laplace transform is an integral $\int_0^\infty \dots$
- we need practice!
- *practice 1.* compute

$$\int_2^\infty e^{-3t} dt =$$

$$= \frac{1}{3}e^{-6}$$

- *practice 2.* compute and sketch

$$\int_1^\infty \frac{1}{t} dt =$$

$$= +\infty$$

practice integrals, cont.

- *practice 3.* compute and sketch

$$\int_0^{\infty} te^{-t} dt =$$

$$= +1$$

Laplace transforms, from the definition

- the technique in Chapter 7 requires pre-computing the Laplace transforms of some familiar functions, and then using these to solve DEs
- *example 1.* compute $\mathcal{L}\{e^{kt}\}$
- *example 2.* compute $\mathcal{L}\{1\}$

from the definition, cont.

- *example 3.* compute $\mathcal{L}\{t\}$

- *example 4.* compute $\mathcal{L}\{\cos(kt)\}$

from the definition, cont.²

- *example 5.* compute $\mathcal{L}\{t^n\}$

$$\mathcal{L}\{t^n\} = \frac{n}{s} \mathcal{L}\{t^{n-1}\}$$

$$\implies \mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$$

Theorem 7.1.1 Transforms of Some Basic Functions

$$(a) \quad \mathcal{L}\{1\} = \frac{1}{s}$$

$$(b) \quad \mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}, \quad n = 1, 2, 3, \dots$$

$$(c) \quad \mathcal{L}\{e^{at}\} = \frac{1}{s-a}$$

$$(d) \quad \mathcal{L}\{\sin kt\} = \frac{k}{s^2 + k^2}$$

$$(e) \quad \mathcal{L}\{\cos kt\} = \frac{s}{s^2 + k^2}$$

$$(f) \quad \mathcal{L}\{\sinh kt\} = \frac{k}{s^2 - k^2}$$

$$(g) \quad \mathcal{L}\{\cosh kt\} = \frac{s}{s^2 - k^2}$$

- you will have a table like this on quizzes and exams
- *and* it is a fair question to ask you to show any one from the definition

another example from the definition

- *example 6.* compute $\mathcal{L}\{f\}$ if

$$f(t) = \begin{cases} 0, & 0 \leq t < a \\ 1, & t > a \end{cases}$$

$$\mathcal{L}\{f\} = \frac{e^{-as}}{s}$$

- several WebAssign problems are piecewise like this

key fact from §7.2

- *example 7.* let $Y(s) = \mathcal{L}\{y(t)\}$. use the definition to show

$$\mathcal{L}\{y'(t)\} = sY(s) - y(0)$$

an *actual* example

- so far, examples just compute $\mathcal{L}\{f(t)\}$ for particular $f(t)$
- ... they *don't* show how \mathcal{L} is actually used!
- *example 8.* solve by using \mathcal{L} :

$$y' + 5y = t, y(0) = 0$$

$$y(t) = \frac{1}{25}(e^{-5t} - 1) + \frac{t}{5}$$

the old way, to check

- *example 8'*. solve by using Chapter 2 methods:

$$y' + 5y = t, y(0) = 0$$

can you always compute \mathcal{L} ?

- your function $f(t)$ has to be defined on the interval $[0, \infty)$ so you can do the integral $\int_0^\infty e^{-st} f(t) dt$
- even then, the function has to *not blow-up too fast*
 - *bad example.* try to compute

$$\mathcal{L}\{e^{t^2}\} =$$

- the result may not be defined for all s
 - *example.* explain why this result only makes sense for $s > 7$:

$$\mathcal{L}\{e^{7t}\} = \frac{1}{s-7}$$

can you always compute \mathcal{L} ? cont.

- *definition.* a function $f(t)$, defined on $[0, \infty)$, is of *exponential order c* if there are constants M and c so that

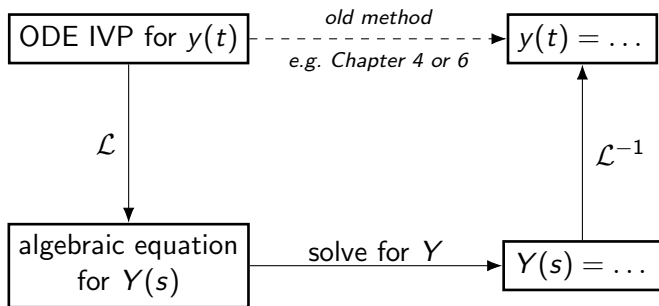
$$|f(t)| \leq Me^{ct}$$

for all t in $[0, \infty)$

Theorem

if $f(t)$ is piecewise continuous on $[0, \infty)$ and of exponential order c then $\mathcal{L}\{f(t)\}$ is defined for $s > c$

the Laplace transform strategy



- example 8 used this strategy
- we get serious about this strategy in §7.2 and §7.3

beating a dead DE ...

- by the end of this Chapter we will have a *third* good way of solving linear, constant-coefficient DEs:

Chapter 4 use auxiliary equation and undetermined coefficients

Chapter 6 use power series

Chapter 7 use Laplace transform

- both homogenous and nonhomogeneous
- all these methods use linearity ... they are *not* suited to nonlinear DEs
- only Chapter 6 methods are well-suited to variable coefficients
- in Chapter 8 we will get one more method

expectations

- just watching this video is *not* enough!
 - see “found online” videos and stuff at bueler.github.io/math302/week11.html
 - *read* section 7.1 in the textbook
 - *do* the WebAssign exercises for section 7.1