2.4 Exact (first-order differential) Equations

a lesson for MATH F302 Differential Equations

Ed Bueler, Dept. of Mathematics and Statistics, UAF

January 24, 2019

for textbook: D. Zill, *A First Course in Differential Equations with Modeling Applications*, 11th ed.

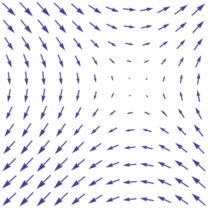
three objects from calculus III

to get started on exact equations we recall these ideas:

1. *vector fields*:

$$\vec{\mathbf{F}} = a(x,y)\hat{\mathbf{i}} + b(x,y)\hat{\mathbf{j}}$$

   - like a slope field
   - ...but *with* magnitudes



$\vec{\mathbf{F}} = \sin y\,\hat{\mathbf{i}} + \sin x\,\hat{\mathbf{j}}$

2. the *gradient* of a function $f(x,y)$:

$$\nabla f(x,y) = \frac{\partial f}{\partial x}\hat{\mathbf{i}} + \frac{\partial f}{\partial y}\hat{\mathbf{j}}$$

   - the gradient *is* a vector field: $a = \frac{\partial f}{\partial x}$, $b = \frac{\partial f}{\partial y}$
   - the gradient points uphill on the surface $z = f(x,y)$

3. the *differential* of f contains the same information as the gradient: $df = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy$

a major idea

- *some vector fields are gradients and some are not:*
  - Example which *is* a gradient:

    $$\vec{\mathbf{F}} = \cos(x+y)\,\hat{\mathbf{\imath}} + (y + \cos(x+y))\,\hat{\mathbf{\jmath}}$$

    supply an f:

  - Example which is *not* a gradient:

    $$\vec{\mathbf{F}} = \cos(x+y)\,\hat{\mathbf{\imath}} + (x + \cos(x+y))\,\hat{\mathbf{\jmath}}$$

    explain why: ...?

- *same idea*: some forms

  $$M(x,y)\,dx + N(x,y)\,dy$$

  are the differentials of an f—they're exact—and some are not
- these ideas are *not obvious*, but true!

recall differentials

- *differentials* were introduced in first-semester calculus as a style for linearizations: $df = f'(x)\,dx$
- now we need differentials for functions of 2 variables:

$$f = f(x, y) \qquad \Longrightarrow \qquad df = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy$$

  - see calculus III
  - differential of $f(x, y)$ describes the tangent plane to the surface $z = f(x, y)$
  - the differential contains the same information as the gradient
  - note: you need to be able to compute partial derivatives!
- Example: find the differential of $f(x, y) = \frac{1}{2}y^2 + \sin(x + y)$

how this relates to DEs

- *definition:* a differential form

$$M(x,y)\,dx + N(x,y)\,dy$$

  is *exact* if there is $f(x,y)$ so that the form is a differential:

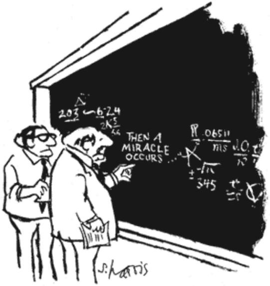
$$M = \frac{\partial f}{\partial x}, \qquad N = \frac{\partial f}{\partial y}$$

- main idea: if we can rewrite an ODE as an exact differential form then we can solve the ODE

example 1

- my first example uses a "miracle" at one step (. . . not sustainable!)
- Example 1: solve

$$y' = \frac{2y}{3y - 2x}$$



"I THINK YOU SHOULD BE MORE EXPLICIT HERE IN STEP TWO."

how to tell if it is exact

- two concerns with above "method":
  1. the differential form has to be exact! how do you tell?
  2. I guessed $f(x, y)$; this is bad—needed miracle
- the following theorem addresses concern 1:

Theorem

The differential form $M(x, y)\,dx + N(x, y)\,dy$ is exact if and only if

$$\frac{\partial M}{\partial y} \overset{*}{=} \frac{\partial N}{\partial x}$$

- ◦ $*$ must be true on simply-connected domain like a rectangle
- ◦ *proof of one direction*: if $M\,dx + N\,dy$ is exact then [fill in]

example 2

- Example 2: use the method of Example 1 to solve

$$y' = \frac{2y}{3y - x^2}$$

- ...trick question? *no*
- not every ODE is solvable by the "exact" method in §2.4
- there is easy test for whether this method will work

example 3

- Example 3: is the equation exact? if so, solve it:

$$(2xy^2 - 3)\,dx + (2x^2y + 4)\,dy = 0$$

example 4

- solve the given initial value problem:

$$(e^x + y)\,dx + (2 + x + ye^y)\,dy = 0, \quad y(0) = 1$$

example 4, cont.

straight from the book

- last two examples are from the book
  - example 3 was #5 in §2.4
  - example 4 was #22 in §2.4
  - expect problems like these on Quizzes and Exam!

- the next example is #46 in §2.4
  - it is *too much computation* for a Quiz or Exam

example 5

- Example 5: the differential equation

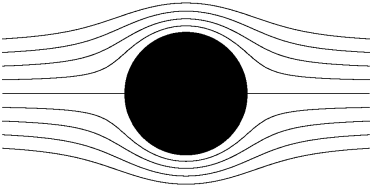
$$\frac{2xy}{(x^2+y^2)^2}\,dx + \left(1 + \frac{y^2-x^2}{(x^2+y^2)^2}\right)\,dy = 0$$

describes a family of curves which are the "streamlines" of an idealized fluid flowing around a circular cylinder

(a) solve the differential equation
  - get a general, but implicit, solution

(b) there is one value of c giving an explicit solution; find it

(c) plot solution curves for $c = 0, \pm 0.2, \pm 0.4, \pm 0.6, \pm 0.8$ using a contour plotting tool



example 5, cont.

(a) solve the differential equation . . . as exact, naturally

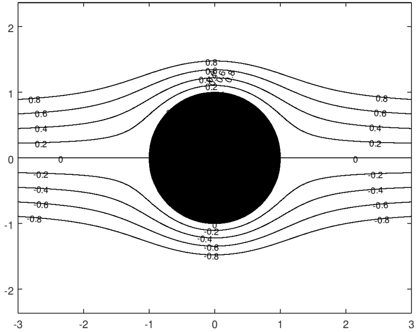
example 5, cont.[2]

example 5, cont.[3]

(b) for what value of c can one find an explicit solution?

(c) contour plot . . . here is Matlab/Octave code:

```
f = @(x,y) y.*(1.0 - 1.0./(x.^2+y.^2));     % define function
x = -3:.1:3;  [xx,yy] = meshgrid(x,x);       % grid of points
c = -0.8:0.2:0.8;                            % contours we want
h = contour(xx,yy,f(xx,yy),c,'k');           % black contours
clabel(h)                                    % ... with labels
axis equal                                   % looks better
```



expectations

to learn this material, just watching this video is *not* enough; also

- *watch* "found online" videos at

  bueler.github.io/math302/week4.html

- *read* section 2.4 in the textbook
- *do* the WebAssign exercises for section 2.4

the biggest issue in §2.4 for most students:

partial derivatives, and integrals, are rusty

- work on fixing this now!
- actually *read* the relevant parts of a calculus book, starting with material on: (i) partial derivatives, (ii) vector fields, (iii) gradients, (iv) differentials
- find a calc. III student and try to help them learn these topics