ONE PROBLEM, MANY METHODS
a review assignment

Instructions. Please write your solutions neatly on paper, or start a new document on a computer. Include your name in the upper right on each sheet. Clearly label the parts of your solutions following the headings below. Your final document, which should *not* be your first draft, should be at most five pages. Please show your work neatly. Full credit requires evidence of clear understanding *and* clear writing. If you wrote on paper, please scan or photograph your completed document and make the lighting good. Also line things up carefully, and/or use a scanner app. In any case *please generate a single easily-readable PDF file.* Then submit your PDF by uploading using the Google Form link on the "Week 14" page at

`bueler.github.io/math302`

Though many real-world ODEs are nonlinear, a large fraction of the problems this course, and on the Final Exam, are linear ODEs of second order. This Mini-Project reviews our methods for such problems. There is no new content.

Because of how this Mini-Project is structured, the final answer in each part, i.e. the value of $x(2)$, is not worth many points! What you need to do is show understanding and correct application of each method.

One problem. Consider this ODE IVP for an unknown function $x(t)$:

$$x'' + 4x' + 5x = 0, \quad x(0) = 0,\ x'(0) = 10 \tag{1}$$

Parts A–E below ask you to use five different methods to compute, or approximate, $x(2)$. Because this answer is the same in each case, you can check your work.

Also consider this closely-related non-homogeneous variation of the same problem:

$$x'' + 4x' + 5x = 5\sin t, \quad x(0) = 0,\ x'(0) = 10 \tag{2}$$

Parts G–I ask you to compute $x(2)$ for this problem using three different methods.

Many solution methods. The emphasis of your solutions must be on showing clear understanding of the steps of the method, and correctness also. Please consistently write "$x(t)$" for the solution, that is, with independent variable t and dependent variable x.

A. Use *auxiliary equation* methods from §4.3 to solve problem (1). Compute $x(2)$.

B. Use *Laplace transform* methods from Chapter 7 to solve problem (1). Compute $x(2)$.

C. In the slides and video for §5.3/4.10 I show how to use the MATLAB/OCTAVE black-box solver `ode45` to solve second-order ODE IVPs like this one. Use the substitution $z_1(t) = x(t)$ and $z_2(t) = x'(t)$ to write equation (1) as a first-order system. Then use MATLAB/OCTAVE/OctaveOnline to solve the problem *numerically*:

```
>> f = @(t,z) [______; ______];
>> [tt,zz] = ode45(f,[0,2],[______;______]);
>> plot(tt,zz),  xlabel t,  legend('x(t)','dx/dt')
```

You will need to fill in the blanks. Thereby approximate $x(2)$. How accurate is the approximation? (*Compute the absolute error; it is* not *zero;* `ode45` *is* not *magic.*)

D. Use the *power series* methods from §6.2 to solve problem (1). In particular, start your solution with a power series around the basepoint $t_0 = 0$, namely

$$x(t) = \sum_{n=0}^{\infty} c_n t^n.$$

Use known values $x(0)$ and $x'(0)$ to find c_0 and c_1. Find the recurrence relation which generates all of the coefficients c_n. Determine the values of $c_2, \dots, c_6$. Now $x(t)$ is approximated by a polynomial of degree 6, i.e. $x(t) \approx p(t)$. What is the approximation of $x(2)$? (*It will not be close to the result from the other parts.*) Closer to the basepoint, what is the approximation of $x(1)$? (*This should be much better; compare the solution from parts A and/or B.*) On the same axes, plot both the exact solution $x(t)$—from parts A and/or B—and the polynomial $p(t)$ on $0 \le t \le 2$.

E. Write equation (1) as a first order system as in §8.1. That is, use the substitution $y(t) = x'(t)$, and find a 2×2 matrix $\mathbf{A}$ and an initial vector $\mathbf{X}_0$, so that the problem is

$$\mathbf{X}' = \mathbf{A}\mathbf{X}, \quad \mathbf{X}(0) = \mathbf{X}_0$$

for a vector solution $\mathbf{X}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$. The eigenvalues and eigenvectors of this system will be complex. To avoid confusion, you may use the fact that eigenvalues and eigenvectors of $\mathbf{A}$ are

$$\lambda_1 = -2 + i, \quad \lambda_2 = -2 - i, \qquad \mathbf{K}_1 = \begin{pmatrix} -2 - i \\ 5 \end{pmatrix}, \quad \mathbf{K}_2 = \begin{pmatrix} -2 + i \\ 5 \end{pmatrix}$$

Then use the technique in example 6 in section 8.2.3 to write out the general solution as a linear combination of real solutions, and then use the initial values. Confirm that $x(t)$ is the same as from parts A and/or B, and thereby compute $x(2)$.

F. Problem (1) is a *damped mass-spring*, as studied in §5.1: $m = 1$, $\beta = 4$, and $k = 5$. Is it overdamped, critically damped, or underdamped?

G. Use the §4.4 *undetermined coefficients* method to solve problem (2). Compute $x(2)$.

H. Use Chapter 7 *Laplace transform* methods to solve problem (2). Compute $x(2)$.

I. Use the black-box numerical solver `ode45` to solve problem (2). Approximate $x(2)$. How accurate is the approximation?