7.4 Laplace Transforms: convolutions

a lesson for MATH F302 Differential Equations

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April 5, 2019

for textbook: D. Zill, *A First Course in Differential Equations with Modeling Applications*, 11th ed.

Laplace transform a mass-spring system

- consider a damped mass-spring system with driving force $g(t)$:

$$mx'' + \beta x' + kx = g(t), \qquad x(0) = 0,\, x'(0) = 0$$

- consider Laplace transform of the solution: $X(s) = \mathcal{L}\{x(t)\}$
- *exercise.* compute $X(s)$

$$X(s) = \boxed{}$$

the transfer function

$$mx'' + \beta x' + kx = g(t), \qquad x(0) = 0,\ x'(0) = 0 \qquad (*)$$

- applying $\mathcal{L}$ we get, after simplifying,

$$X(s) = \frac{1}{ms^2 + \beta s + k}\, G(s)$$

- the function

$$F(s) = \frac{1}{ms^2 + \beta s + k}$$

  is the *transfer function* of $(*)$

- *so*: the Laplace transform of $x(t)$ is simply the product of the transfer function and the transformed driving force:

$$X(s) = F(s)G(s)$$

remember the table?

TABLE OF LAPLACE TRANSFORMS:

$$\mathcal{L}\{1\} = \frac{1}{s}$$

$$\mathcal{L}\{t\} = \frac{1}{s^2}$$

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$$

$$\mathcal{L}\left\{t^{-1/2}\right\} = \frac{\sqrt{\pi}}{s^{1/2}}$$

$$\mathcal{L}\left\{t^{1/2}\right\} = \frac{\sqrt{\pi}}{2s^{3/2}}$$

$$\mathcal{L}\{t^\alpha\} = \frac{\Gamma(\alpha+1)}{s^{\alpha+1}}$$

$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$$

$$\mathcal{L}\{\sin(kt)\} = \frac{k}{s^2+k^2}$$

$$\mathcal{L}\{\cos(kt)\} = \frac{s}{s^2+k^2}$$

$$\mathcal{L}\{\sinh(kt)\} = \frac{k}{s^2-k^2}$$

$$\mathcal{L}\{\cosh(kt)\} = \frac{s}{s^2-k^2}$$

$$\mathcal{L}\{te^{at}\} = \frac{1}{(s-a)^2}$$

$$\mathcal{L}\{t^n e^{at}\} = \frac{n!}{(s-a)^{n+1}}$$

$$\mathcal{L}\{e^{at}\sin(kt)\} = \frac{k}{(s-a)^2+k^2}$$

$$\mathcal{L}\{e^{at}\cos(kt)\} = \frac{s-a}{(s-a)^2+k^2}$$

$$\mathcal{L}\{t\sin(kt)\} = \frac{2ks}{(s^2+k^2)^2}$$

$$\mathcal{L}\{t\cos(kt)\} = \frac{s^2-k^2}{(s^2+k^2)^2}$$

$$\mathcal{L}\{e^{at}f(t)\} = F(s-a)$$

$$\mathcal{L}\{\mathcal{U}(t-a)\} = \frac{e^{-as}}{s}$$

$$\mathcal{L}\{f(t-a)\mathcal{U}(t-a)\} = e^{-as}F(s)$$

$$\mathcal{L}\{g(t)\mathcal{U}(t-a)\} = e^{-as}\mathcal{L}\{g(t+a)\}$$

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}F(s)$$

$$\mathcal{L}\left\{f^{(n)}(t)\right\} = s^n F(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$$

note last two entries $\longrightarrow$

$$(f*g)(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau$$

$$\mathcal{L}\{f*g\} = F(s)G(s)$$

convolution

the last two entries say:

- *definition.* given functions $f(t)$ and $g(t)$ defined on $(0, \infty)$, the function

$$(f * g)(t) = \int_0^t f(\tau)g(t - \tau)\, d\tau$$

  is called the *convolution* of f and g

- *the convolution theorem.* if $F(s) = \mathcal{L}\{f(t)\}$ and $G(s) = \mathcal{L}\{g(t)\}$ then

$$\mathcal{L}\{f * g\} = F(s)G(s)$$

why the convolution matters

$$mx'' + \beta x' + kx = g(t), \qquad x(0) = 0,\ x'(0) = 0 \qquad (*)$$

$$X(s) = F(s)G(s) \quad \text{where} \quad F(s) = \frac{1}{ms^2 + \beta s + k}$$

- let $f(t) = \mathcal{L}^{-1}\{F(s)\}$
  - the *impulse response* or *weight function* of problem $(*)$
- by the convolution theorem,

$$x(t) = (f * g)(t)$$

  - the solution comes from *convolving* the impulse response and the driving force

warning: convolution vs. multiplication

- the Laplace transform of a product of function is *NOT* the product of Laplace transforms

$$\mathcal{L}\left\{f(t)\,g(t)\right\} \neq F(s)G(s)$$

  - why not?

- *convolution* on the t side becomes multiplication on the s side:

$$\mathcal{L}\left\{f(t) * g(t)\right\} = F(s)G(s)$$

- i.e.

$$\mathcal{L}\left\{f * g\right\} = \mathcal{L}\left\{f\right\}\mathcal{L}\left\{g\right\}$$

computing a convolution: example 1

- definition: $(f * g)(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau$
- *example 1.* compute $f * g$:

$$f(t) = 1 - \mathcal{U}(t-2) = \begin{cases} 1, & 0 \le t < 2 \\ 0, & t \ge 2 \end{cases}, \qquad g(t) = e^{-t}$$

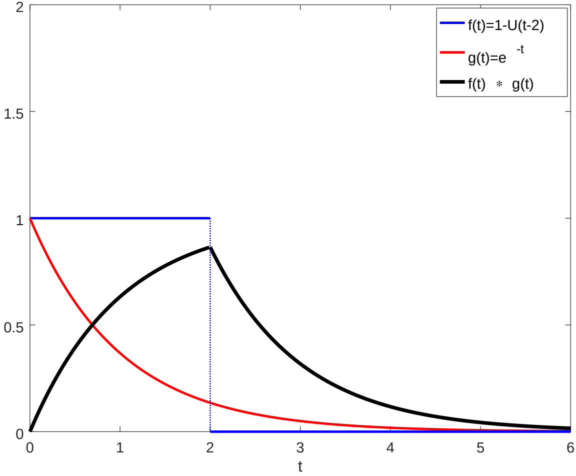
computing a convolution: example 2

- *example 2.* compute $f * g$:

$$f(t) = \sin(t), \qquad g(t) = e^{-t}$$

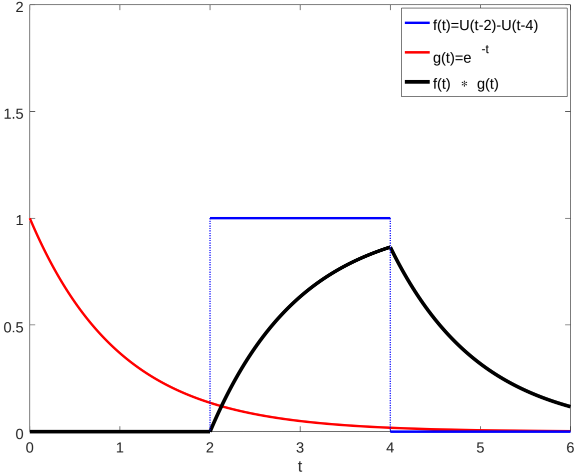
but what does a convolution *look* like?

ex. 1: $f(t) = 1 - \mathcal{U}(t-2)$ and $g(t) = e^{-t}$



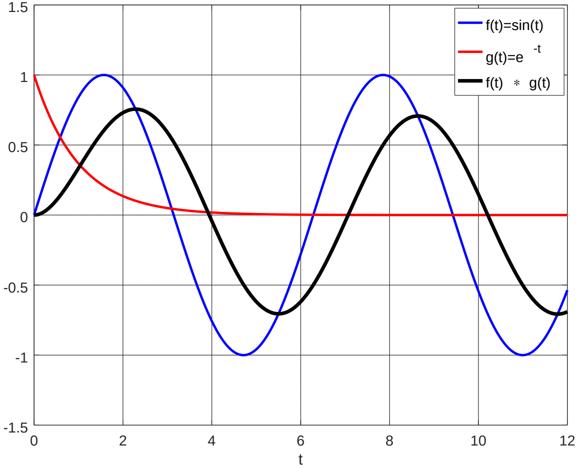
but what does a convolution *look* like?

- $f(t) = \mathcal{U}(t-2) - \mathcal{U}(t-4)$ and $g(t) = e^{-t}$



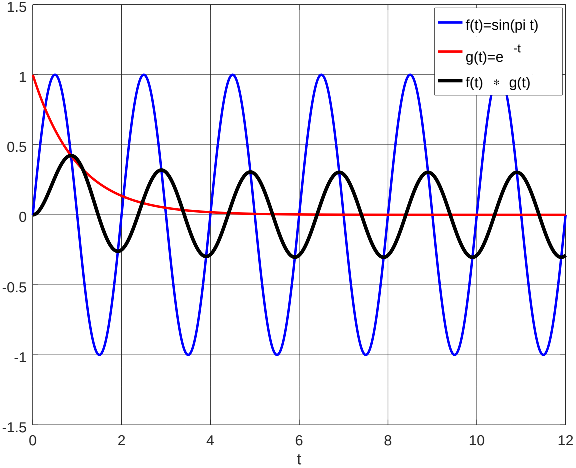
but what does a convolution *look* like?

ex. 2: $f(t) = \sin(t)$ and $g(t) = e^{-t}$



but what does a convolution *look* like?

- $f(t) = \sin(\pi t)$ and $g(t) = e^{-t}$



convolutions are a big deal

instead of trying to show you here, go to:

`en.wikipedia.org/wiki/Convolution`

- images and movies of 1D convolutions
- discrete 1D convolution is a *filter* in signal processing
- discrete 2D convolution is a *filter* in image processing

exercise §7.4 #19

- *example 3.* find the convolution $f * g$ of the functions, and then find the Laplace transform of the result:

$$f(t) = 4t, \qquad g(t) = 3t^2$$

exercise §7.4 #26

- convolution theorem: $\mathcal{L}\{f * g\} = F(s)G(s)$
- *example 4.* find the Laplace transform of $f * g$ using the convolution theorem; do not evaluate the convolution integral before transforming:

$$f(t) = e^{2t}, \qquad g(t) = \sin(t)$$

exercise §7.4 #28

- the convolution of $f(t)$ with the constant 1 is just the integral:

$$f(t) * 1 = \int_0^t f(\tau) 1 \, d\tau = \int_0^t f(\tau) \, d\tau$$

- so

$$\mathcal{L}\left\{\int_0^t f(\tau)\, d\tau\right\} = \mathcal{L}\left\{f(t) * 1\right\} = \frac{F(s)}{s}$$

- *example 5.* compute the Laplace transform:

$$\mathcal{L}\left\{\int_0^t \cos\tau \, d\tau\right\} =$$

like exercise §7.4 #35

- *example 6.* compute the inverse Laplace transform:

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s-3)}\right\} =$$

what was the old way?

the main idea (*a summary*)

for a damped mass-spring system with driving force $g(t)$,

$$mx'' + \beta x' + kx = g(t), \qquad x(0) = 0,\, x'(0) = 0$$

1. the *transfer function*

$$F(s) = \frac{1}{ms^2 + \beta s + k}$$

   is multiplied by the transformed driving force to give the Laplace transform of the solution $x(t)$:

$$X(s) = F(s)G(s)$$

2. $f(t) = \mathcal{L}^{-1}\{F(s)\}$ is *impulse response* of mass-spring system
3. solution is convolution of impulse response and driving force:

$$x(t) = (f * g)(t)$$

a definition from §7.5

- *definition.* the *Dirac delta* $\delta(t)$ is a "function" which equals zero when $t \neq 0$ and yet has area one:

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1$$

  - this is impossible for a real function, but it is a useful concept
  - ultimately made into honest math by generalizing "function"
  - the next slide relates to this definition ...
  - but you are not responsible for it (*no WebAssign/quiz/exam*)
- because of the above property,

$$\mathcal{L}\{\delta(t)\} = 1$$

- $\delta(t)$ is an "impulse" at $t = 0$

why “impulse response”?

$$mx'' + \beta x' + kx = g(t), \qquad x(0) = 0,\ x'(0) = 0 \qquad (*)$$

- we call $F(s) = \frac{1}{ms^2+\beta s+k}$ the *transfer function*
- ...and $f(t) = \mathcal{L}^{-1}\{F(s)\}$ the *impulse response* ...why?
- because if $g(t) = \delta(t)$ then for $(*)$ we have

$$X(s) = F(s)\,G(s) = \frac{1}{ms^2 + \beta s + k}\,1 = \frac{1}{ms^2 + \beta s + k}$$

  - i.e. $g(t) = \delta(t)$, a sharp impulse at $t = 0$, generates $X(s)$
  - a standard whack
  - the response of the system is $x(t) = \mathcal{L}^{-1}\{X(s)\}$

the Γ function (and $n!$)

- you are not responsible (*no WebAssign/quiz/exam*) for this
  ... but I want to explain one remaining item in the table
- compare

$$\mathcal{L}\left\{t^n\right\} = \frac{n!}{s^{n+1}} \qquad \text{versus} \qquad \mathcal{L}\left\{t^\alpha\right\} = \frac{\Gamma(\alpha+1)}{s^{\alpha+1}}$$

- what is $\mathcal{L}\left\{t^{1.7}\right\}$? or any other non-integer power?
- *definition.*

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt$$

- *facts.*

$$\Gamma(n+1) = n!$$

$$\Gamma(\tfrac{1}{2}) = \sqrt{\pi}$$

the table: is it all clear?

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$$\mathcal{L}\{f*g\} = F(s)G(s)$$

expectations

- just watching this video is *not* enough!
  - see "found online" videos and stuff at
    `bueler.github.io/math302/week12.html`
  - *read* "7.4.2 Transforms of Integrals" in §7.4
    - and read nearby stuff if you are interested
  - *do* the WebAssign exercises for section 7.4