Numerical modelling of glaciers, ice sheets, and ice shelves

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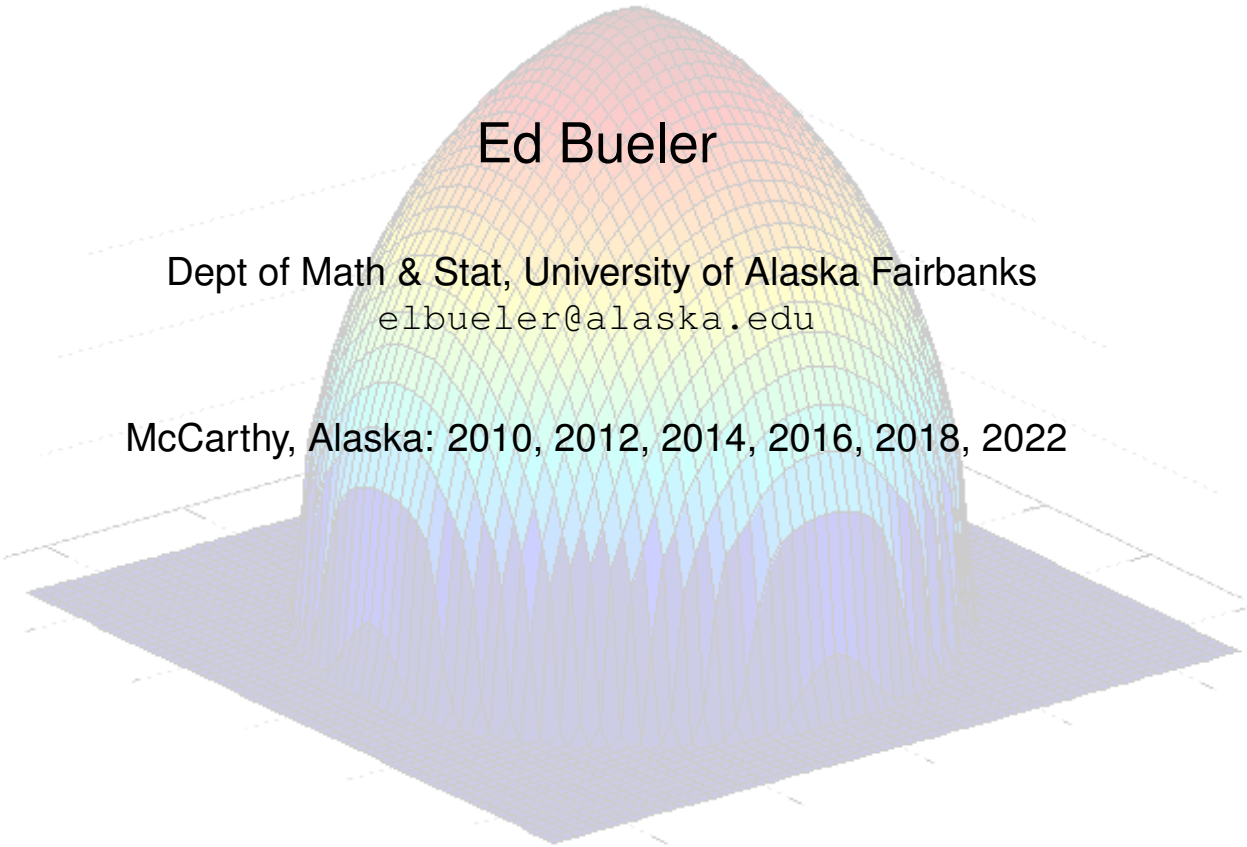
McCarthy, Alaska: 2010, 2012, 2014, 2016, 2018, 2022



slogans & scope

slogans: I will

- approximate ice flow
- show codes that actually work
- always focus on the continuum model

scope:

- models
  - shallow ice approximation (SIA) in 2D
  - shallow shelf approximation (SSA) in 1D
  - mass continuity & surface kinematical equations
- numerical ideas
  - finite difference schemes
  - solving algebraic systems from stress balances
  - verification

notation

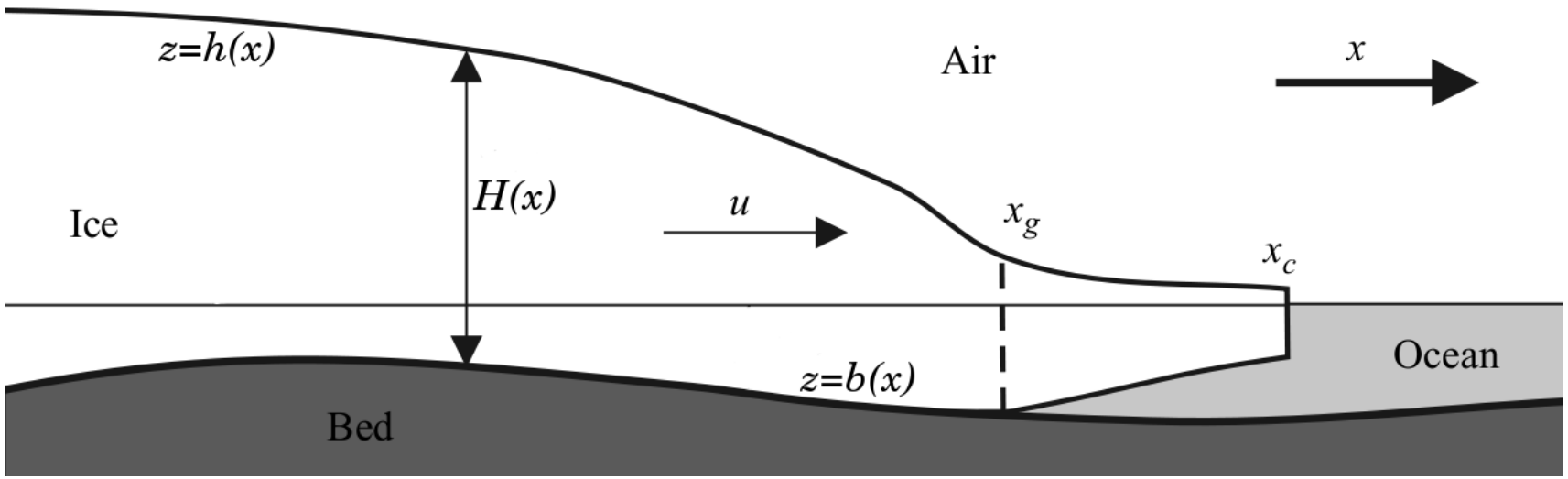


figure modified from Schoof (2007)

- coordinates t, x, y, z (with z vertical, positive upward)
- subscripts for partial derivatives $u_x = \partial u / \partial x$
- $H =$ ice thickness
- $h =$ ice surface elevation
- $b =$ bedrock surface elevation
- $T =$ temperature
- $\mathbf{u} = (u, v, w) =$ ice velocity
- $\rho =$ density of ice
- $\rho_w =$ density of ocean water
- $g =$ acceleration of gravity
- n Glen flow law exponent (usually $= 3$)
- $A = A(T) =$ ice softness in Glen law ($\mathbf{D}_{ij} = A(T)\tau^{n-1}\tau_{ij}$)
- please ask about notation!

Matlab/Octave codes

- lectures and notes are based on 18 Matlab/Octave codes
- . . . a few will appear in these lectures
- each is $\sim$ 1/2 page
- please give them a try! see `mfiles/` directory at

`https://github.com/bueler/mccarthy`

Outline

ice in glaciers is a *fluid*

- what's a fluid?
- is it just a collection of particles?

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- its a mathematical abstraction

ice in glaciers is a *fluid*

- what's a fluid?
- is it just a collection of particles?
- its a mathematical abstraction
- . . . with equations relating these fields:
  - a scalar *density* $\rho(t, x, y, z)$
  - a scalar *pressure* $p(t, x, y, z)$
  - a vector *velocity* $\mathbf{u}(t, x, y, z)$

ice in glaciers is an atypical fluid

- if the ice were
  - faster-moving than it actually is, and
  - linearly-viscous like liquid water

  then it would be a "typical" fluid

- for typical fluids one uses the Navier-Stokes equations:

$$\nabla \cdot \mathbf{u} = 0 \qquad \textit{incompressibility}$$

$$\rho\left(\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u}\right) = -\nabla p + \nabla \cdot \tau_{ij} + \rho \mathbf{g} \qquad \textit{stress balance}$$

$$2\nu D\mathbf{u}_{ij} = \tau_{ij} \qquad \textit{flow law}$$

  - stress balance equation is "$ma = F$"
  - already a subtle model

glaciology as computational fluid dynamics

- yes, numerical ice sheet flow modelling is “computational fluid dynamics”
  - it’s large-scale like atmosphere and ocean
  - . . . but it is a weird one
- consider what makes atmosphere/ocean flow exciting:
  - turbulence
  - convection
  - coriolis force
  - density stratification
- none of the above list is relevant to ice flow
- what could be interesting about the flow of slow, cold, stiff, laminar, inert old ice?
  - *ice dynamics!*

ice is a slow, shear-thinning fluid

- ice fluid is *slow* and *non-Newtonian*
  - "slow" is a technical term:

$$\rho\left(\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u}\right) \approx 0 \qquad \Longleftrightarrow \qquad \begin{pmatrix}\text{forces of inertia}\\ \text{are neglected}\end{pmatrix}$$

  - ice is non-Newtonian in a "shear-thinning" way
    - ★ higher strain rates means lower viscosity
    - ★ viscosity ν is not constant

- thus the standard model is Glen-law Stokes:

$$\begin{aligned} \nabla\cdot\mathbf{u} &= 0 && \textit{incompressibility}\\ 0 &= -\nabla p + \nabla\cdot\tau_{ij} + \rho\,\mathbf{g} && \textit{stress balance}\\ D\mathbf{u}_{ij} &= A\tau^{n-1}\tau_{ij} && \textit{flow law}\end{aligned}$$

“slow” means no memory of velocity/momentum

- note *no time derivatives* in Stokes model:

$$\nabla \cdot \mathbf{u} = 0$$
$$0 = -\nabla p + \nabla \cdot \tau_{ij} + \rho \mathbf{g}$$
$$D\mathbf{u}_{ij} = A\tau^{n-1}\tau_{ij}$$

- thus a time-stepping ice sheet code recomputes the full velocity field at every time step
  - it does not require velocity from the previous time step[1]
- velocity is a “diagnostic” output not needed for starting or restarting the model

[1] you don’t need . . . to know which way the wind blows

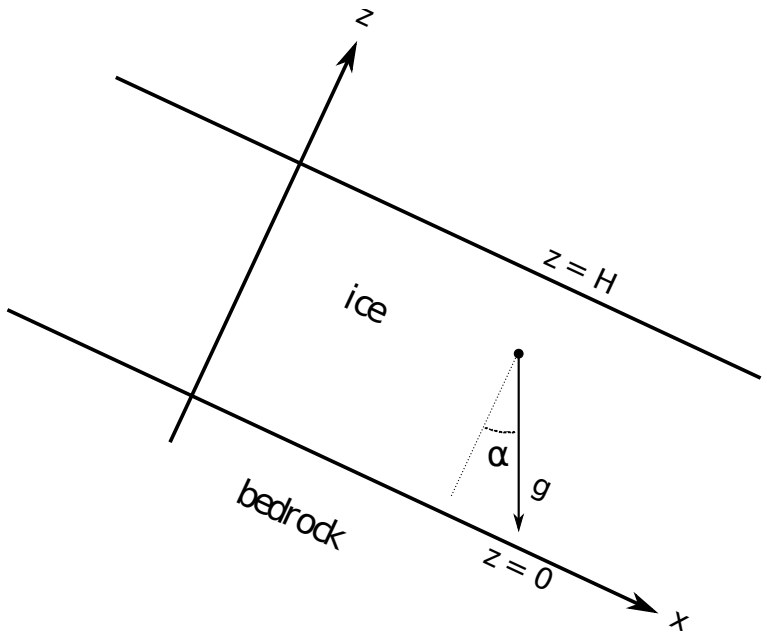
plane flow Stokes

- suppose we work in a x, z plane, such as the centerline of a glacier, or a cross-flow plane
- now the $n = 3$ Stokes equations say

$$
\begin{aligned}
u_x + w_z &= 0 && \textit{incompressibility} \\
p_x &= \tau_{11,x} + \tau_{13,z} && \textit{stress balance } (x) \\
p_z &= \tau_{13,x} - \tau_{11,z} - \rho g && \textit{stress balance } (z) \\
u_x &= A\tau^2\tau_{11} && \textit{flow law (diagonal)} \\
u_z + w_x &= 2A\tau^2\tau_{13} && \textit{flow law (off-diagonal)}
\end{aligned}
$$

  - *notation*: subscripts x, z denote partial derivatives, τ_{13} is the "vertical" shear stress, τ_{11} and $\tau_{33} = -\tau_{11}$ are (deviatoric) longitudinal stresses
- we have five equations in five unknowns $(u, w, p, \tau_{11}, \tau_{13})$
- this is complicated enough . . . what about in a simplified situation?

slab-on-a-slope



- suppose constant thickness
- tilt bedrock by angle α
- rotate the coordinates
- get these new expressions:

$$\mathbf{g} = g \sin\alpha\, \hat{x} - g \cos\alpha\, \hat{z}$$

$$p_x = \tau_{11,x} + \tau_{13,z} + \rho g \sin\alpha$$

$$p_z = \tau_{13,x} - \tau_{11,z} - \rho g \cos\alpha$$

- for this slab-on-a-slope there is *no variation in x*: $\partial/\partial x = 0$
- the equations simplify:

$w_z = 0$	$0 = \tau_{11}$
$\tau_{13,z} = -\rho g \sin\alpha$	$u_z = 2A\tau^2\tau_{13}$
$p_z = -\rho g \cos\alpha$	

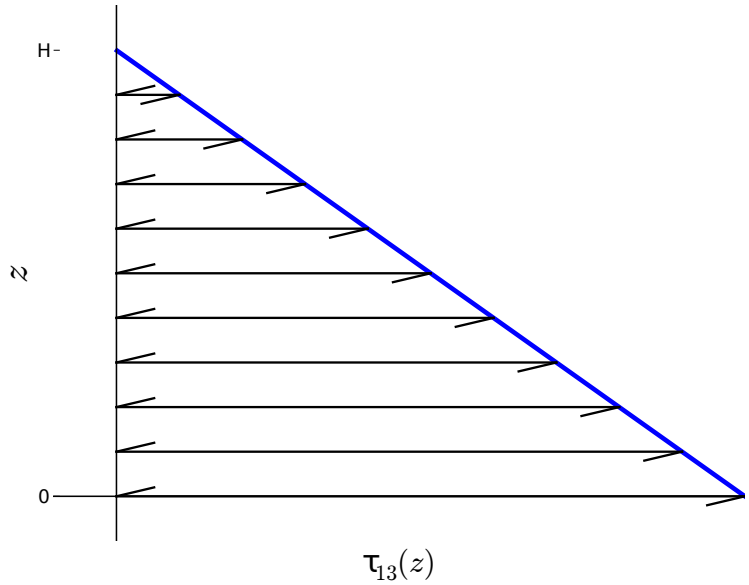
slab-on-a-slope 2

- add boundary conditions:

$$w(\text{base}) = 0, \qquad p(\text{surface}) = 0, \qquad u(\text{base}) = u_0$$

- by integrating vertically, get:

$$w = 0$$
$$p = \rho g \cos\alpha (H - z)$$
$$\tau_{13} = \rho g \sin\alpha (H - z)$$

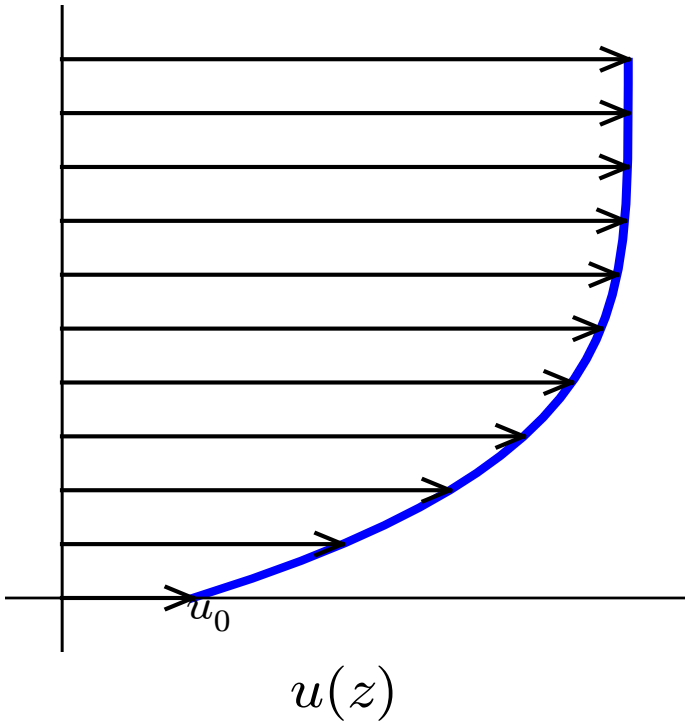


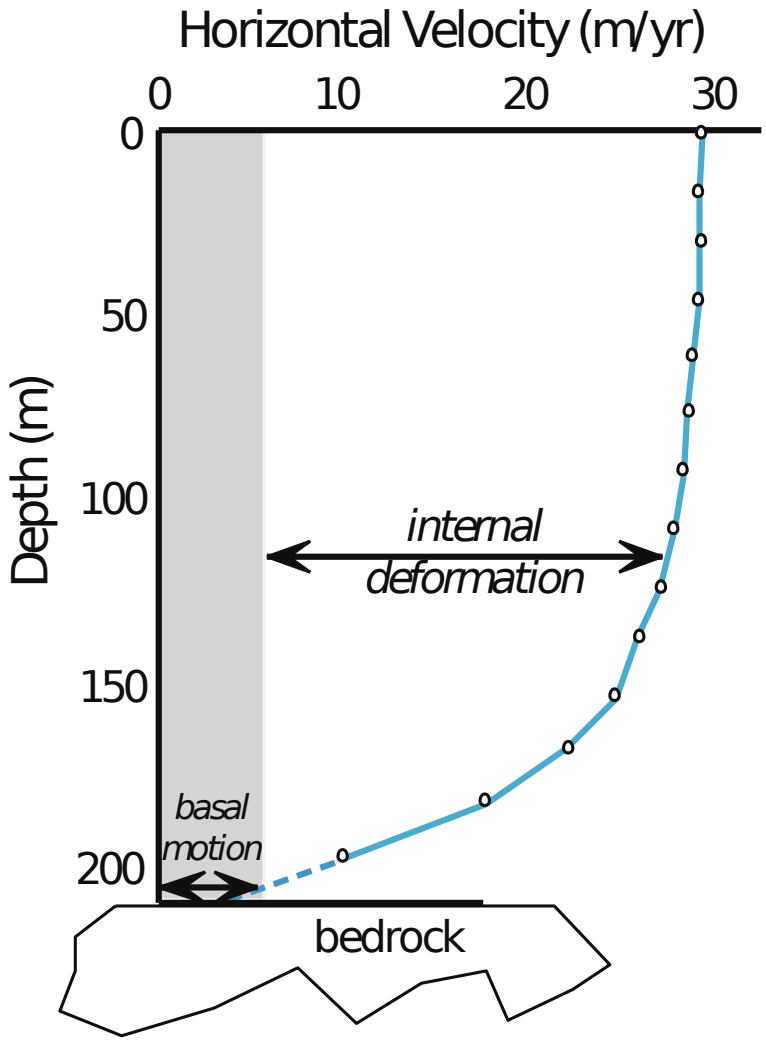
- τ_{13} is linear in depth
- from $u_z = 2A\tau^2\tau_{13}$ get velocity formula

$$u(z) = u_0 + 2A(\rho g \sin\alpha)^3 \int_0^z (H - z')^3 \, dz'$$
$$= u_0 + \frac{1}{2} A(\rho g \sin\alpha)^3 \left(H^4 - (H - z)^4\right)$$

slab-on-a-slope 3

- do we believe these equations?
- velocity formula on last slide gives figure below
- compare to observations at right





Velocity profile of the Athabasca Glacier, Canada, derived from inclinometry (Savage and Paterson, 1963)

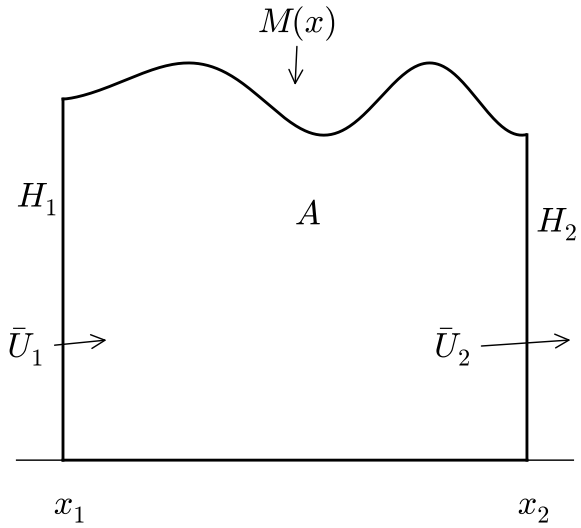
mass conservation

Q: having computed the velocity **u** . . . so what?

A: velocity determines changing ice shape through mass conservation

- suppose flowline ice has variable thickness $H(t, x)$
- define the vertical average of velocity:

$$\bar{U}(t, x) = \frac{1}{H} \int_0^H u(t, x, z)\, dz$$



- with climatic mass balance $M(x)$, consider change of area in the figure:

$$\frac{dA}{dt} = \int_{x_1}^{x_2} M(x)\, dx + \bar{U}_1 H_1 - \bar{U}_2 H_2$$

  - area in 2D becomes volume in 3D
- assume width $dx = x_2 - x_1$ is small, so $A \approx dx\, H$, and divide by dx to get the *mass continuity (conservation) equation*

$$H_t = M - \left(\bar{U}H\right)_x$$

combine equations so far

- from slab-on-slope velocity formula in $u_0 = 0$ case, get flux:

$$q = \bar{U}H = \int_0^H \frac{1}{2} A(\rho g \sin\alpha)^3 \left(H^4 - (H-z)^4\right)\, dz$$
$$= \frac{2}{5} A(\rho g \sin\alpha)^3 H^5$$

- note $\sin\alpha \approx \tan\alpha = -h_x$
- combine with mass continuity $H_t = M - \left(\bar{U}H\right)_x$ to get:

$$H_t = M + \left(\frac{2}{5}(\rho g)^5 A H^5 |h_x|^2 h_x\right)_x$$

- this is the "shallow ice approximation" (SIA)
  - a very rough explanation of it!

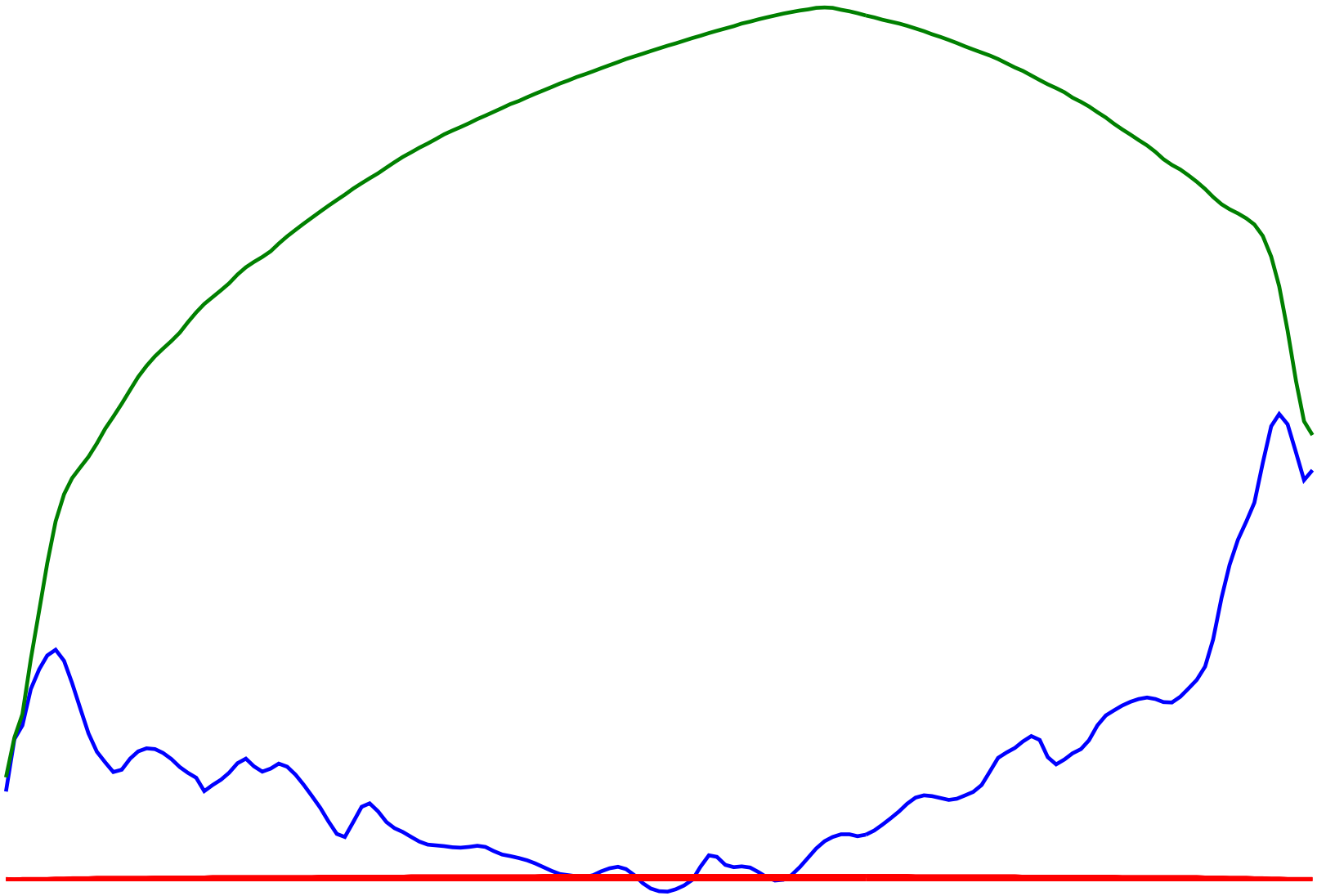
Outline

slow, non-Newtonian, shallow, and sliding

- ice sheets have four outstanding properties *as fluids*:
  1. slow
  2. non-Newtonian
  3. shallow (often)
  4. contact slip (sometimes)

regarding “shallow”

- in red is a Greenland cross section
  - at 71° N: width = 760 km, thickness = 3200 m
- green and blue: standard vertically-exaggerated cross section



flow model I: shallow ice approximation (SIA)

a model which applies to

- small depth-to-width ratio (“shallow”) grounded ice sheets
- for *not* rough/steep bed topography
- when flow is *not* dominated by sliding at the base

“Polaris Glacier,” northwest Greenland, photo 122, Post & LaChapelle (2000)

SIA model equations

- simple slogan: *the SIA uses the formulas from slab-on-a-slope*
  - better explanation: expand the Stokes equations in aspect ratio
- shear stress approximation:

$$(\tau_{13}, \tau_{23}) = -\rho g (h - z) \nabla h$$

- then horizontal velocity $\mathbf{u} = (u, v)$ satisfies this approximation:

$$\begin{aligned} \mathbf{u}_z &= 2A |(\tau_{13}, \tau_{23})|^{n-1} (\tau_{13}, \tau_{23}) \\ &= -2A (\rho g)^n (h - z)^n |\nabla h|^{n-1} \nabla h \end{aligned}$$

- by integrating vertically (non-sliding case):

$$\mathbf{u} = -\frac{2A(\rho g)^n}{n+1} \left[H^{n+1} - (h - z)^{n+1} \right] |\nabla h|^{n-1} \nabla h$$

- integrate again to get $\overline{\mathbf{u}}$
- add mass continuity: $H_t = M - (\overline{\mathbf{u}} H)_x$

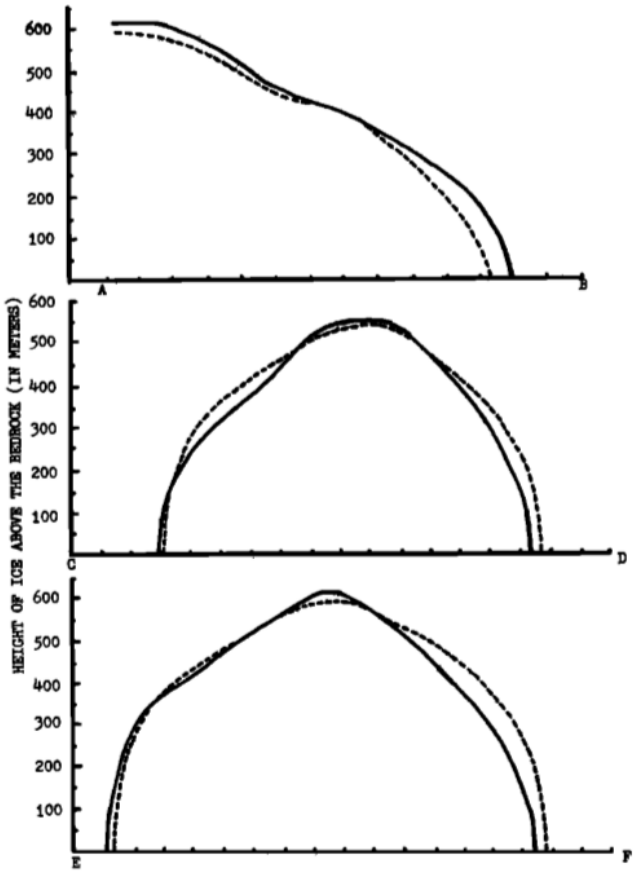
SIA thickness equation

- thus the non-sliding, isothermal SIA for thickness evolution:

$$\boxed{H_t = M + \nabla \cdot \left(\Gamma H^{n+2} |\nabla h|^{n-1} \nabla h \right)} \tag{1}$$

  - H is ice thickness, h is ice surface elevation
  - . . . also b is bed elevation and $h = H + b$
  - M is surface (and basal) mass balance
  - $\Gamma = 2A(\rho g)^n/(n+2)$ is a constant

- numerically solve (1) and you have a usable model for boring ice sheets
  - Barnes ice cap modeled by Mahaffy (1976) $\rightarrow$



good questions:

how to solve (1) numerically?

how to *think* about it?

heat equation

- to understand the SIA, compare to the *heat equation*
- recall Newton's law of cooling

$$\frac{dT}{dt} = -K(T - T_{\text{ambient}})$$

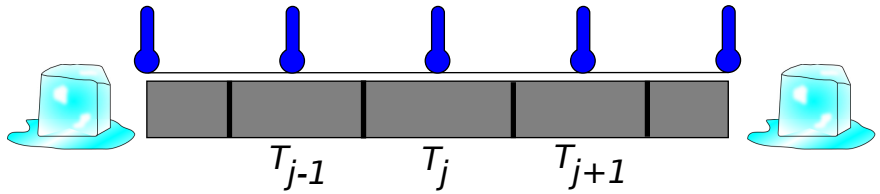
  - T is coffee temperature
  - K from heat conductivity/capacity

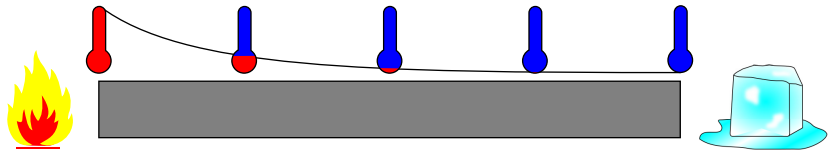
- Newton's law for segments of a rod has T_{ambient} as neighbor average:

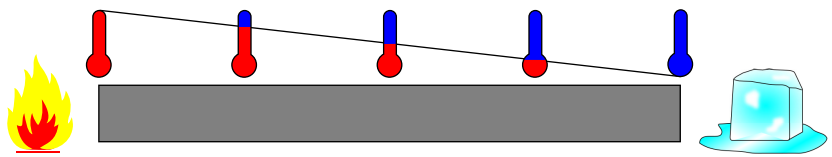
$$\frac{dT_j}{dt} = -K\left(T_j - \frac{1}{2}(T_{j-1} + T_{j+1})\right)$$
$$= \frac{K}{2}(T_{j-1} - 2T_j + T_{j+1})$$

- shrink segments to get *heat equation*:

$$T_t = DT_{xx}$$







major analogy: SIA versus 2D heat equation

- general 2D heat eqn: $T_t = F + \nabla \cdot (D \nabla T)$
- side-by-side comparison:

SIA for ice thickness $H(t,x,y)$	heat eqn for temperature $T(t,x,y)$
$H_t = M + \nabla \cdot \left(\Gamma H^{n+2} \lvert \nabla h \rvert^{n-1} \nabla h \right)$	$T_t = F + \nabla \cdot (D \nabla T)$

- identify the diffusivity in the SIA:

$$D = \Gamma H^{n+2} \lvert \nabla h \rvert^{n-1}$$

- non-sliding shallow ice flow *diffuses* the ice sheet geometry
- considerations when using this analogy:
  - D is not constant; it depends on the solution $H(t,x,y)$
  - $D \to 0$ at margin, where $H \to 0$
  - $D \to 0$ at divides/domes, where $|\nabla h| \to 0$

numerics for heat equation: basic finite differences

- numerical schemes for heat equation are good start for SIA
- only consider *finite difference* (FD) schemes in these slides

- first, for differentiable $f(x)$ *Taylor's theorem* says

$$f(x+h) = f(x) + f'(x)h + \frac{1}{2}f''(x)h^2 + \frac{1}{3!}f'''(x)h^3 + \dots$$

- you can replace "h" by multiples of Δx, e.g.:

$$f(x-\Delta x) = f(x) - f'(x)\Delta x + \frac{1}{2}f''(x)\Delta x^2 - \frac{1}{3!}f'''(x)\Delta x^3 + \dots$$
$$f(x+2\Delta x) = f(x) + 2f'(x)\Delta x + 2f''(x)\Delta x^2 + \frac{4}{3}f'''(x)\Delta x^3 + \dots$$

- *main idea for FD schemes*: combine expressions like these to approximate derivatives from the values of functions on a grid

partial derivatives

- we want FD approximations of partial derivatives
- for example, with any function $u = u(t, x)$:

$$u_t(t,x) = \frac{u(t+\Delta t, x) - u(t,x)}{\Delta t} + O(\Delta t),$$
$$u_t(t,x) = \frac{u(t+\Delta t, x) - u(t-\Delta t, x)}{2\Delta t} + O(\Delta t^2),$$
$$u_x(t,x) = \frac{u(t, x+\Delta x) - u(t, x-\Delta x)}{2\Delta x} + O(\Delta x^2),$$
$$u_{xx}(t,x) = \frac{u(t, x+\Delta x) - 2u(t,x) + u(t, x-\Delta x)}{\Delta x^2} + O(\Delta x^2)$$

- sometimes we want a derivative in-between grid points:

$$u_x(t, x+\tfrac{\Delta x}{2}) = \frac{u(t, x+\Delta x) - u(t,x)}{\Delta x} + O(\Delta x^2)$$

- *note*: "$+O(h^2)$" is better than "$+O(h)$" if h is a small number

explicit scheme for heat equation

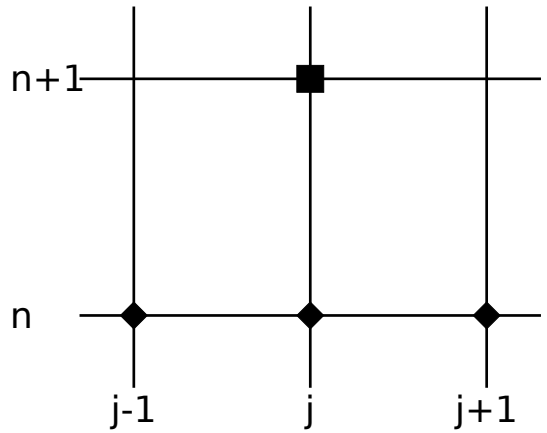
- consider 1D heat equation $T_t = DT_{xx}$
- thus, approximately:

$$\frac{T(t+\Delta t, x) - T(t,x)}{\Delta t} \approx D \frac{T(t, x+\Delta x) - 2T(t,x) + T(t, x-\Delta x)}{\Delta x^2}$$

  - make it equality for a *scheme*
- the difference between $T_t = DT_{xx}$ and the scheme is $O(\Delta t, \Delta x^2)$
- notation:
  - (t_n, x_j) is a point in the time-space grid
  - $T_j^n \approx T(t_n, x_j)$
- let $\mu = D\Delta t/(\Delta x)^2$, so "explicit" scheme is

$$T_j^{n+1} = \mu T_{j+1}^n + (1 - 2\mu) T_j^n + \mu T_{j-1}^n$$

- "stencil" at right $\rightarrow$



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  - make it equality for a *scheme*
- the difference between $T_t = DT_{xx}$ and the scheme is $O(\Delta t, \Delta x^2)$
- notation:
  - (t_n, x_j) is a point in the time-space grid
  - $T_j^n \approx T(t_n, x_j)$ $\leftarrow$ T_j^n from the scheme, $T(t_n, x_j)$ generally unknown
- let $\mu = D\Delta t/(\Delta x)^2$, so "explicit" scheme is

$$T_j^{n+1} = \mu T_{j+1}^n + (1 - 2\mu) T_j^n + \mu T_{j-1}^n$$

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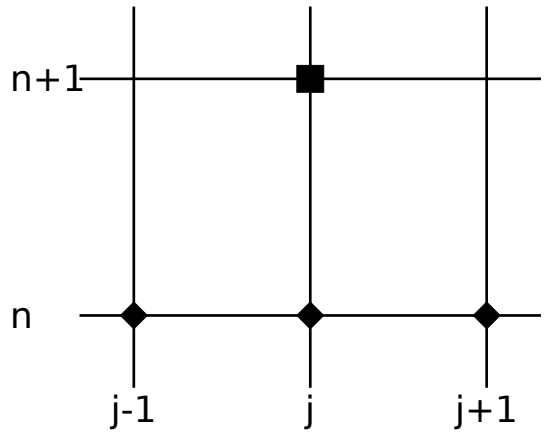


explicit scheme in 2D

- recall heat equation two spatial variables (2D):

$$T_t = D(T_{xx} + T_{yy})$$

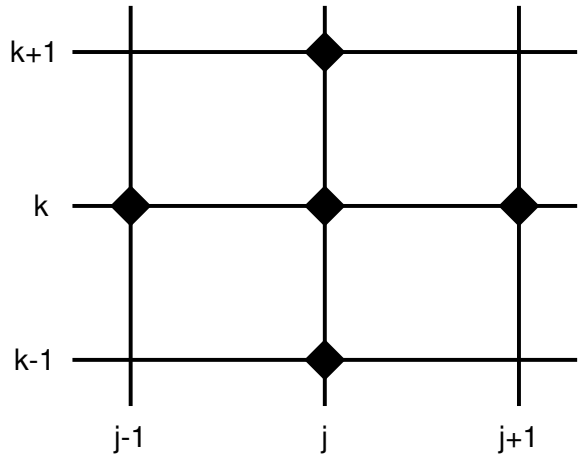
- again: $T^n_{jk} \approx T(t_n, x_j, y_k)$
- the 2D explicit scheme is

$$\frac{T^{n+1}_{jk} - T^n_{jk}}{\Delta t} = D\left(\frac{T^n_{j+1,k} - 2T^n_{jk} + T^n_{j-1,k}}{\Delta x^2} + \frac{T^n_{j,k+1} - 2T^n_{jk} + T^n_{j,k-1}}{\Delta y^2}\right)$$

- which becomes a computable iteration:

$$\begin{aligned} T^{n+1}_{jk} = T^n_{jk} &+ \mu_x(T^n_{j+1,k} - 2T^n_{jk} + T^n_{j-1,k}) \\ &+ \mu_y(T^n_{j,k+1} - 2T^n_{jk} + T^n_{j,k-1}) \end{aligned}$$

where $\mu_x = D\Delta t/\Delta x^2$ and $\mu_y = D\Delta t/\Delta y^2$



implementation

```
function T = heat(D,J,K,dt,N)

dx = 2 / J;     dy = 2 / K;
[x,y] = meshgrid(-1:dx:1, -1:dy:1);
T = exp(-30*(x.*x + y.*y));

mu_x = dt * D / (dx*dx);
mu_y = dt * D / (dy*dy);
for n=1:N
   T(2:J,2:K) = T(2:J,2:K) + ...
       mu_x * ( T(3:J+1,2:K) - 2 * T(2:J,2:K) + T(1:J-1,2:K) ) + ...
       mu_y * ( T(2:J,3:K+1) - 2 * T(2:J,2:K) + T(2:J,1:K-1) );
end

surf(x,y,T),  shading('interp'),  xlabel x,  ylabel y
```

heat.m

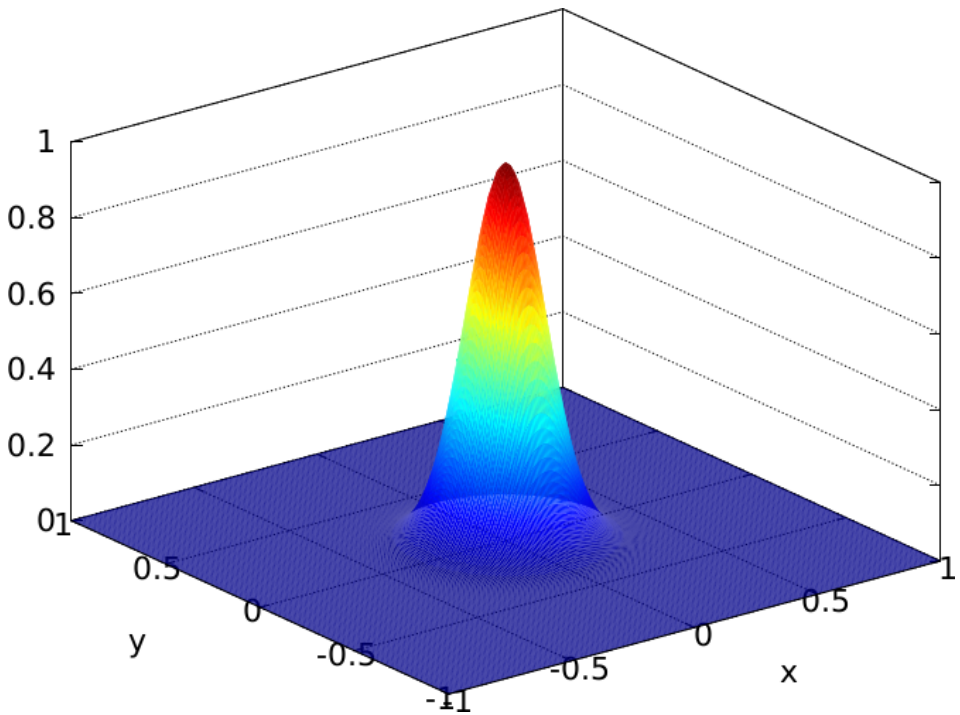
- solves $T_t = D(T_{xx} + T_{yy})$ on the square $-1 < x < 1$, $-1 < y < 1$
- code uses “colon notation” to remove loops (over space)
- example (next slide) uses initial condition $T_0(x, y) = e^{-30r^2}$

the look of success

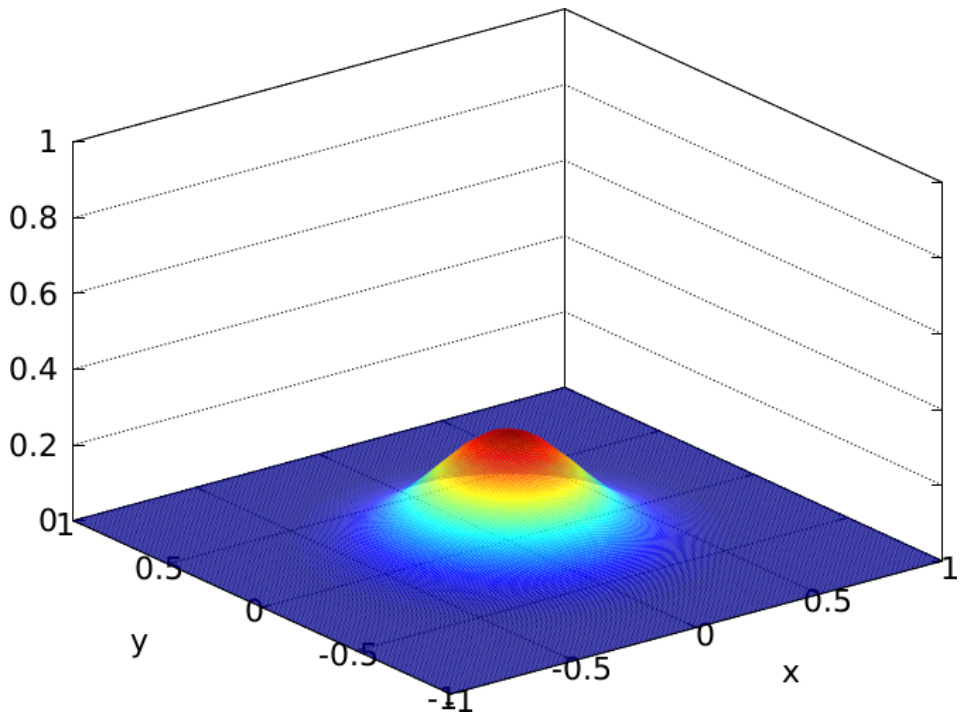
- `» heat(1.0,30,30,0.001,20)`

  solves $T_t = D(T_{xx} + T_{yy})$ for $T(t, x, y)$ with $D = 1$, 30×30 grid, $\Delta t = 0.001$, and $N = 20$ time steps

initial condition $T(0, x, y)$

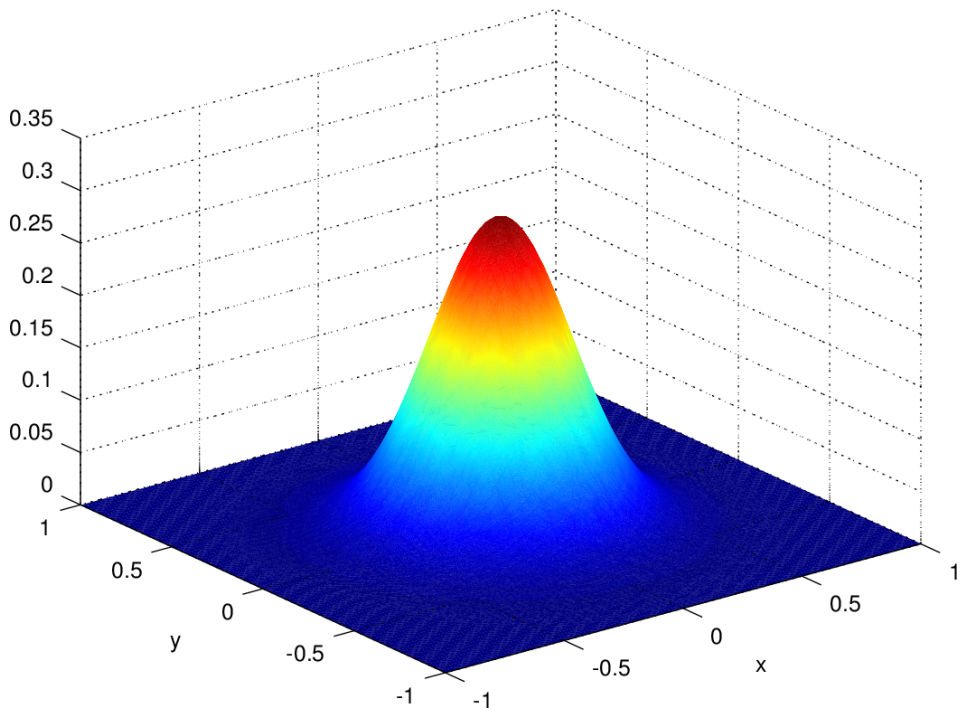


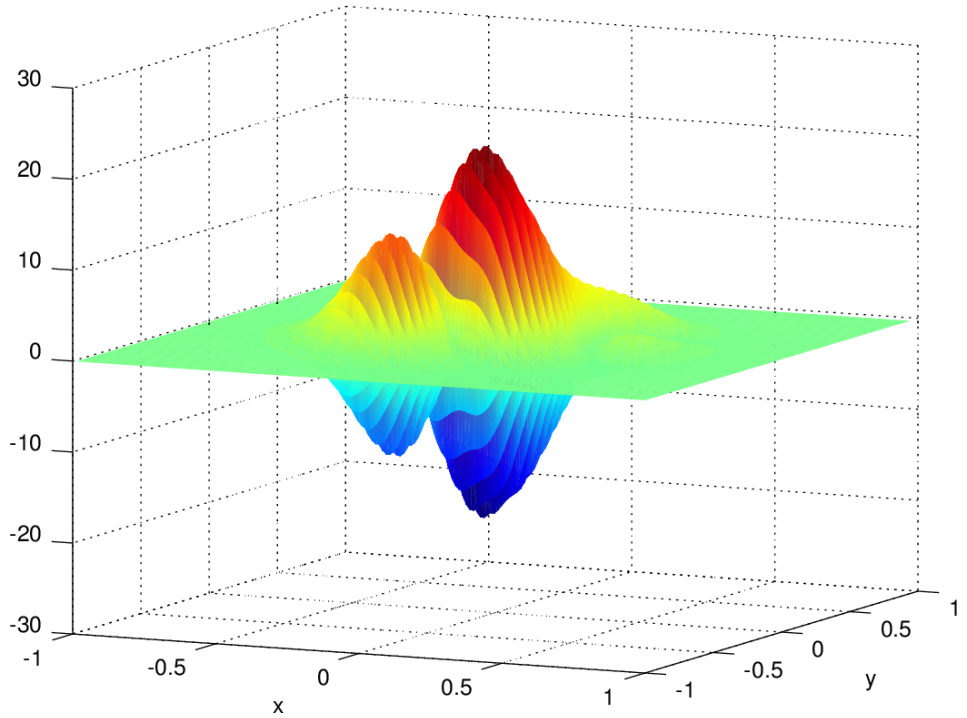
approximate solution $T(0.02, x, y)$



the look of instability

results from solving $T_t = D(T_{xx} + T_{yy})$ on the *same* space grid and at the *same* time, but with slightly-different time steps:





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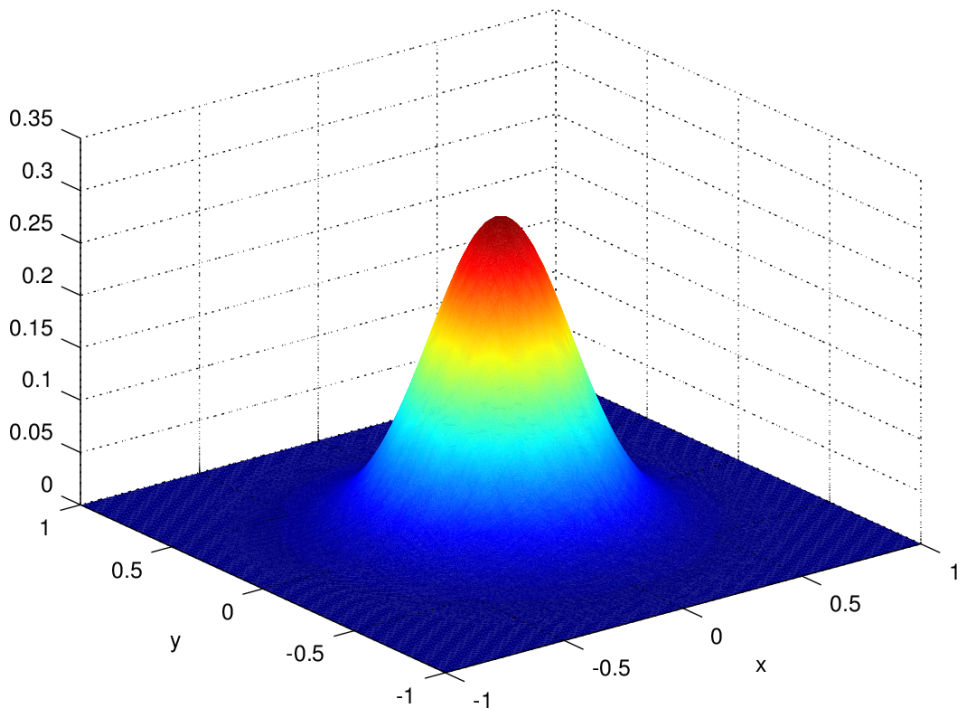
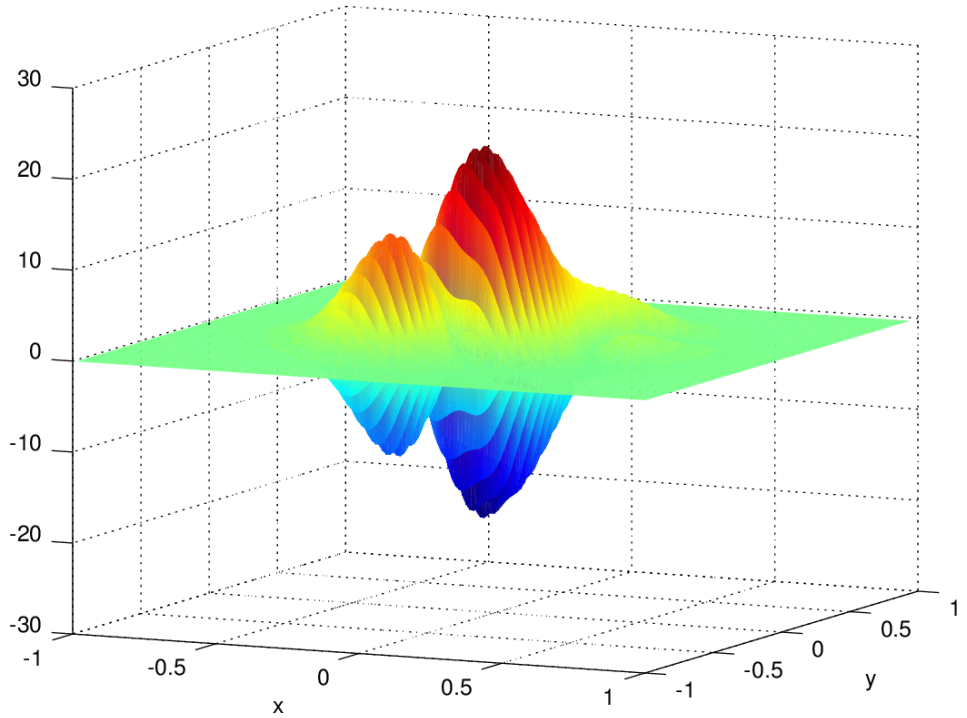




$$\frac{D\Delta t}{\Delta x^2} = 0.2$$

$$\frac{D\Delta t}{\Delta x^2} = 0.4$$

avoid the instability

- recall 1D explicit scheme had the form

$$T_j^{n+1} = \mu T_{j+1}^n + (1 - 2\mu) T_j^n + \mu T_{j-1}^n$$

- thus the new value u_j^{n+1} is an *average* of the old values, *if the middle coefficient is positive*:

$$1 - 2\mu \geq 0 \quad \Longleftrightarrow \quad \frac{D\Delta t}{\Delta x^2} \leq \frac{1}{2} \quad \Longleftrightarrow \quad \Delta t \leq \frac{\Delta x^2}{2D}$$

  - averaging is stable because averaged wiggles are smaller than wiggles
  - this condition is a sufficient *stability criterion*
- instabilities arise because *the time step is too big*

adaptive implementation: guaranteed stability

```
function T = heatadapt(D,L,J,K,tf,Tinitial)

dx = 2 * L / J;     dy = 2 * L / K;
[x,y] = ndgrid(-L:dx:L, -L:dy:L);
if nargin<6, Tinitial = exp(-30*(x.*x + y.*y)); end
T = Tinitial;

t = 0.0;     count = 0;
while t < tf
   dt0 = 0.25 * min(dx,dy)^2 / D;
   dt = min(dt0, tf - t);
   mu_x = dt * D / (dx*dx);     mu_y = dt * D / (dy*dy);
   T(2:J,2:K) = T(2:J,2:K) + ...
       mu_x * ( T(3:J+1,2:K) - 2 * T(2:J,2:K) + T(1:J-1,2:K) ) + ...
       mu_y * ( T(2:J,3:K+1) - 2 * T(2:J,2:K) + T(2:J,1:K-1) );
   t = t + dt;
   count = count + 1;
end

figure(1),  surf(x,y,T),  shading('interp'),  xlabel x,  ylabel y
```

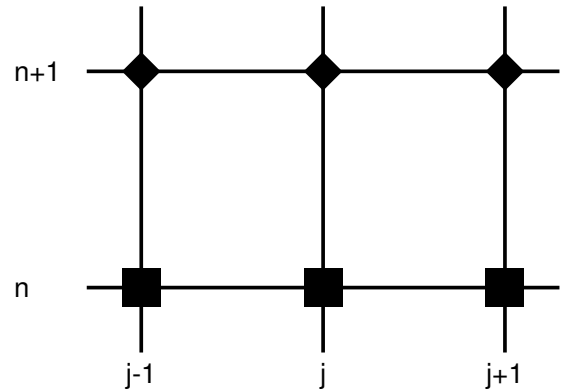
`heatadapt.m`

- same as `heat.m` except:

  choose time step from stability criterion

alternative instability fix: implicitness

- implicit methods can be stable for *any* positive time step Δt
- example is *Crank-Nicolson* scheme $\longrightarrow$



  - has smaller error too: $O(\Delta t^2, \Delta x^2)$
- *but* you must solve linear (or nonlinear) systems of equations to take each time step

- Donald Knuth has advice for ice sheet modelers:

  forget about small efficiencies . . . premature optimization is the root of all evil

  - we will use an explicit adaptive scheme
  - most ice sheet models are explicit
  - Bueler (2016) is an exception; fully-implicit

variable diffusivity and staggered grids

- SIA has diffusivity which varies in space
- consider the generalized diffusion/heat equation for analogy:

$$T_t = F + \nabla \cdot \big(D(x,y)\nabla(T+b)\big)$$

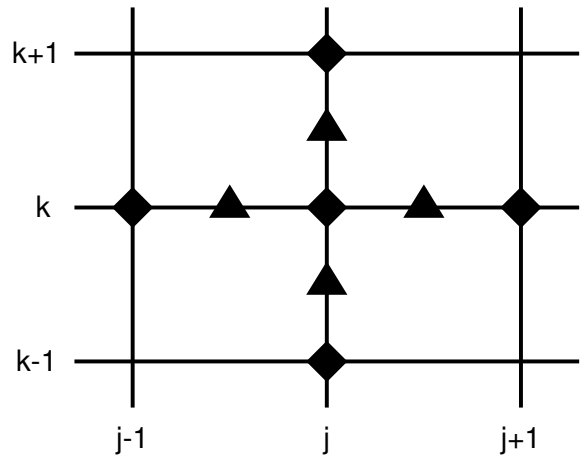
- an explicit method is conditionally stable if we evaluate diffusivity $D(x,y)$ at staggered grid points:

$$\nabla \cdot (D(x,y)\nabla X) \approx \frac{D_{j+1/2,k}(X_{j+1,k} - X_{j,k}) - D_{j-1/2,k}(X_{j,k} - X_{j-1,k})}{\Delta x^2} + \frac{D_{j,k+1/2}(X_{j,k+1} - X_{j,k}) - D_{j,k-1/2}(X_{j,k} - X_{j,k-1})}{\Delta y^2}$$

where $X = T + b$

- in stencil at right $\longrightarrow$

  diamonds: T, b
  triangles: D



general diffusion equation code

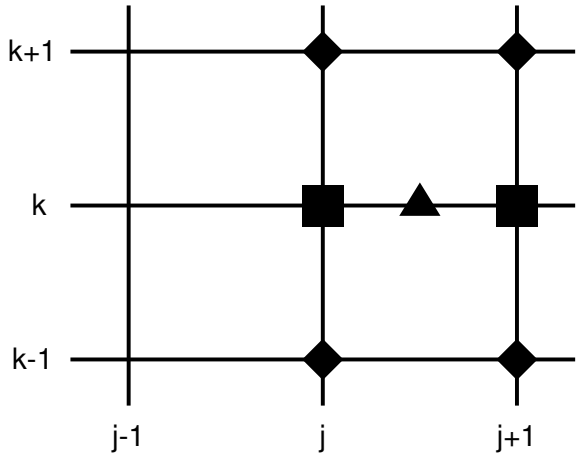
```
function [T,dtav] = diffstag(Lx,Ly,J,K,Dup,Ddown,Dright,Dleft,...
                             T0,tf,F,b)

dx = 2 * Lx / J;     dy = 2 * Ly / K;
[x,y] = ndgrid(-Lx:dx:Lx, -Ly:dy:Ly);
T = T0;
if nargin < 11, F = zeros(size(T0)); end
if nargin < 12, b = zeros(size(T0)); end

t = 0.0;     count = 0;
while t < tf
   maxD = [max(max(Dup))    max(max(Ddown)) ...
           max(max(Dleft)) max(max(Dright))];
   maxD = max(maxD);
   if maxD <= 0.0
     dt = tf - t;
   else
     dt0 = 0.25 * min(dx,dy)^2 / maxD;
     dt = min(dt0, tf - t);
   end
   mu_x = dt / (dx*dx);     mu_y = dt / (dy*dy);
   Tb = T + b;
   T(2:J,2:K) = T(2:J,2:K) + ...
       mu_y * Dup    .* ( Tb(2:J,3:K+1) - Tb(2:J,2:K)    ) - ...
       mu_y * Ddown  .* ( Tb(2:J,2:K)   - Tb(2:J,1:K-1) ) + ...
       mu_x * Dright .* ( Tb(3:J+1,2:K) - Tb(2:J,2:K)    ) - ...
       mu_x * Dleft  .* ( Tb(2:J,2:K)   - Tb(1:J-1,2:K) );
   T = T + F * dt;
   t = t + dt;     count = count + 1;
end
dtav = tf / count;
```

diffstag.m

- solves abstract diffusion equation $T_t = \nabla \cdot (D(x,y)\,\nabla(T+b))$
- user supplies diffusivity on staggered grid

computing diffusivity in SIA

- we need SIA diffusivity $D = \Gamma H^{n+2} |\nabla h|^{n-1}$ on the staggered grid
- various schemes proposed: Mahaffy (1976), Hindmarsh and Payne (1996), Bueler (2016)
- all schemes seek accurate diffusivity by:
  - averaging thickness H
  - differencing surface elevation h
  - in a "balanced" way
- Mahaffy stencil is compact $\longrightarrow$



SIA implementation: flat bed case

```
function [H,dtlist] = siaflat(Lx,Ly,J,K,H0,deltat,tf)

g = 9.81;   rho = 910.0;   secpera = 31556926;
A = 1.0e-16/secpera;   Gamma  = 2 * A * (rho * g)^3 / 5;

N = ceil(tf / deltat);   deltat = tf / N;
dx = 2 * Lx / J;   dy = 2 * Ly / K;   j = 2:J;   k = 2:K;
nk = 3:K+1;   sk = 1:K-1;   ej = 3:J+1;   wj = 1:J-1;

t = 0;   dtlist = [];
H = H0;
for n=1:N
  Hup = 0.5 * ( H(j,nk) + H(j,k) );
  Hdn = 0.5 * ( H(j,k) + H(j,sk) );
  Hrt = 0.5 * ( H(ej,k) + H(j,k) );
  Hlt = 0.5 * ( H(j,k) + H(wj,k) );
  a2up = (H(ej,nk) + H(ej,k) - H(wj,nk) - H(wj,k)).^2 / (4*dx)^2 + ...
         (H(j,nk) - H(j,k)).^2 / dy^2;
  a2dn = (H(ej,k) + H(ej,sk) - H(wj,k) - H(wj,sk)).^2 / (4*dx)^2 + ...
         (H(j,k) - H(j,sk)).^2 / dy^2;
  a2rt = (H(ej,k) - H(j,k)).^2 / dx^2 + ...
         (H(ej,nk) + H(j,nk) - H(ej,sk) - H(j,sk)).^2 / (4*dy)^2;
  a2lt = (H(j,k) - H(wj,k)).^2 / dx^2 + ...
         (H(wj,nk) + H(j,nk) - H(wj,sk) - H(j,sk)).^2 / (4*dy)^2;
  Dup = Gamma * Hup.^5 .* a2up;
  Ddn = Gamma * Hdn.^5 .* a2dn;
  Drt = Gamma * Hrt.^5 .* a2rt;
  Dlt = Gamma * Hlt.^5 .* a2lt;
  [H,dtadapt] = diffstag(Lx,Ly,J,K,Dup,Ddn,Drt,Dlt,H,deltat);
  t = t + deltat;
  dtlist = [dtlist dtadapt];
end
```

`siaflat.m`

- calls `diffstag.m`

interruption: verification

- how do you make sure your *implemented* numerical ice flow code is correct?
  - *technique* 1 = infallibility: don't make any mistakes
  - *technique* 2 = intercomparison: compare your model with others, and hope that the outliers are the ones with errors
  - *technique* 3 = verification: build-in a comparison to an exact solution, and actually measure the numerical error

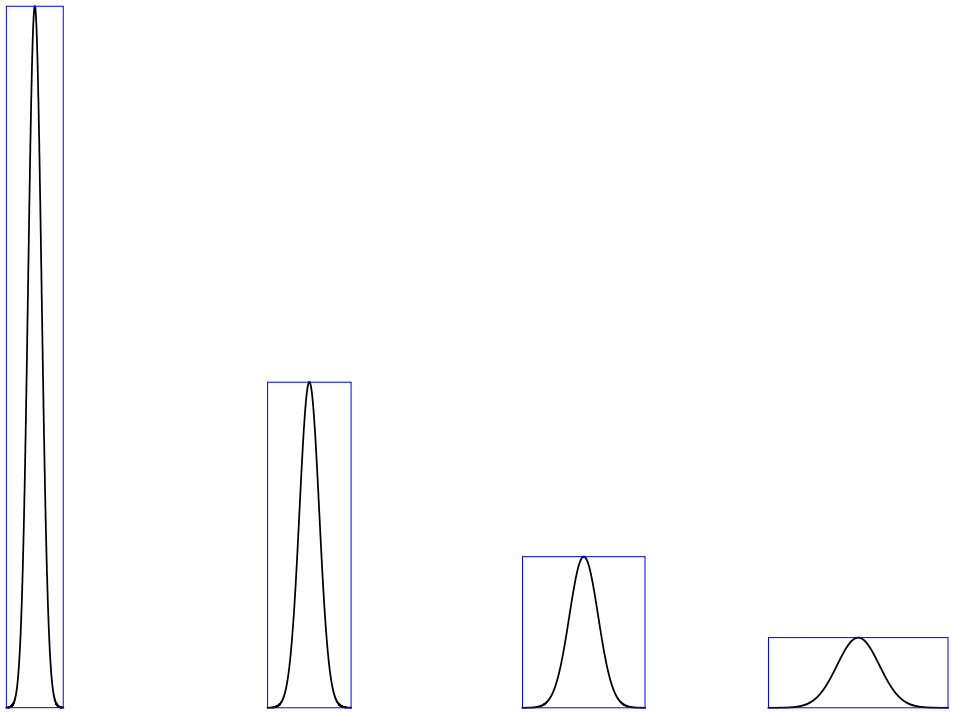
- where to get exact solutions for ice flow models?
  - textbooks: Greve and Blatter (2009), van der Veen (2013)
  - manufactured solns to thermo-coupled SIA (Bueler et al 2007)
  - flowline and cross-flow SSA solns (Bodvardsson, 1955; van der Veen, 1985; Schoof, 2006; Bueler 2014)
  - flowline Blatter solns (Glowinski and Rappaz 2003)
  - constant viscosity flowline Stokes solns (Ladyzhenskaya 1963, Balise and Raymond 1985)
  - manufactured solns to Stokes equations (Sargent and Fastook 2010; Jouvet and Rappaz 2011; Leng et al 2013)

the Green's function of the heat equation

- recall heat equation in 1D with constant diffusivity $D > 0$:

$$T_t = DT_{xx}$$

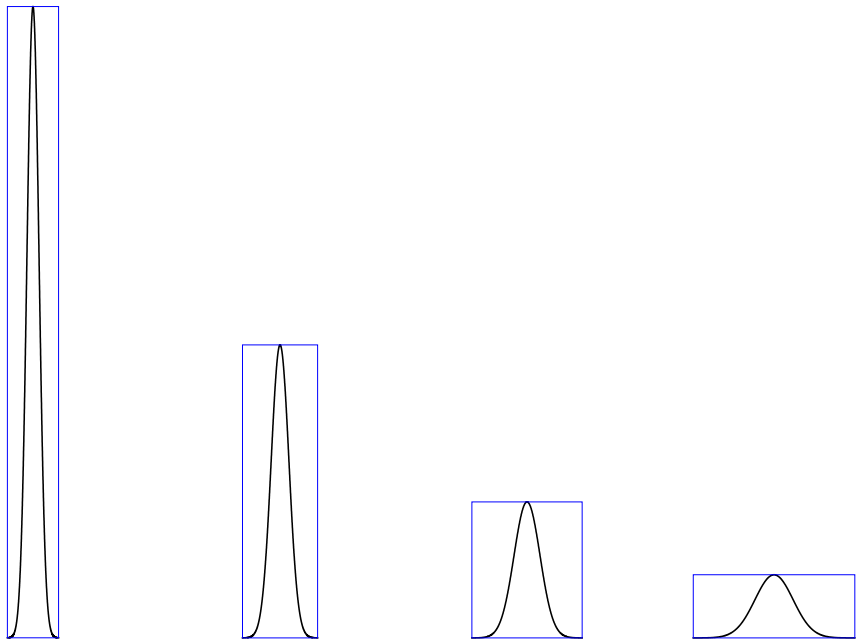
- many exact solutions are known!
- Green's function solution shown below
  - a.k.a. "fundamental solution" or "heat kernel"
  - starts at time $t = 0$ with a "delta function" $T(0, x) = \delta_0(x)$
  - then spreads out over time



increasing time $\longrightarrow$

the Green's function of the heat equation 2

- the Green's function can be found by a method which generalizes to the SIA: the solution is "self-similar" over time
- as time goes it changes shape by
  - shrinking the output (vertical) axis and
  - lengthening the input (horizontal) axis
- . . . but otherwise it is the same shape
- the integral over x is independent of time



similarity solutions

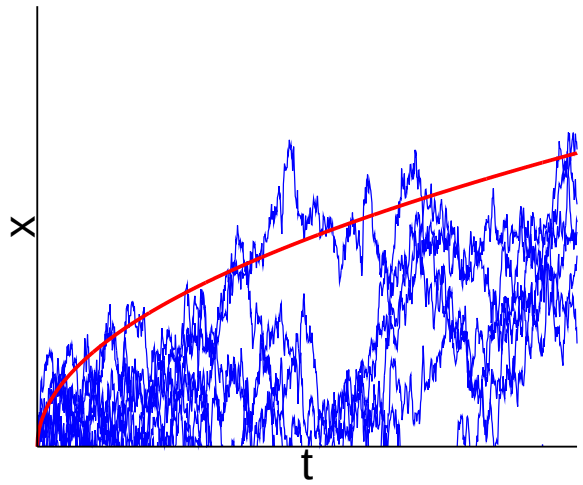
- "similarity" variables for the 1D heat equation are

$$\underset{\text{input scaling}}{s} = t^{-1/2}x, \qquad \underset{\text{output scaling}}{T(t,x)} = t^{-1/2}\phi(s)$$

- derive from ODE: $\phi(s) = C\,e^{-s^2/(4D)}$
- thus the Green's function of heat equation in 1D is

$$T(t,x) = C\,t^{-1/2}e^{-x^2/(4Dt)}$$

- *a tangent?*: Einstein (1905) discovered that the average distance traveled by particles in thermal motion scales like $\sqrt{t}$, so $s = t^{-1/2}x$ is an invariant



similarity solution to SIA

- *1981*: Peter Halfar discovers the similarity solution of the SIA in the case of flat bed and no surface mass balance
  - for 20 years, almost no one cares
- Halfar's 2D solution for $n = 3$ Glen law has scalings

$$H(t, r) = t^{-1/9}\phi(s), \qquad s = t^{-1/18} r$$

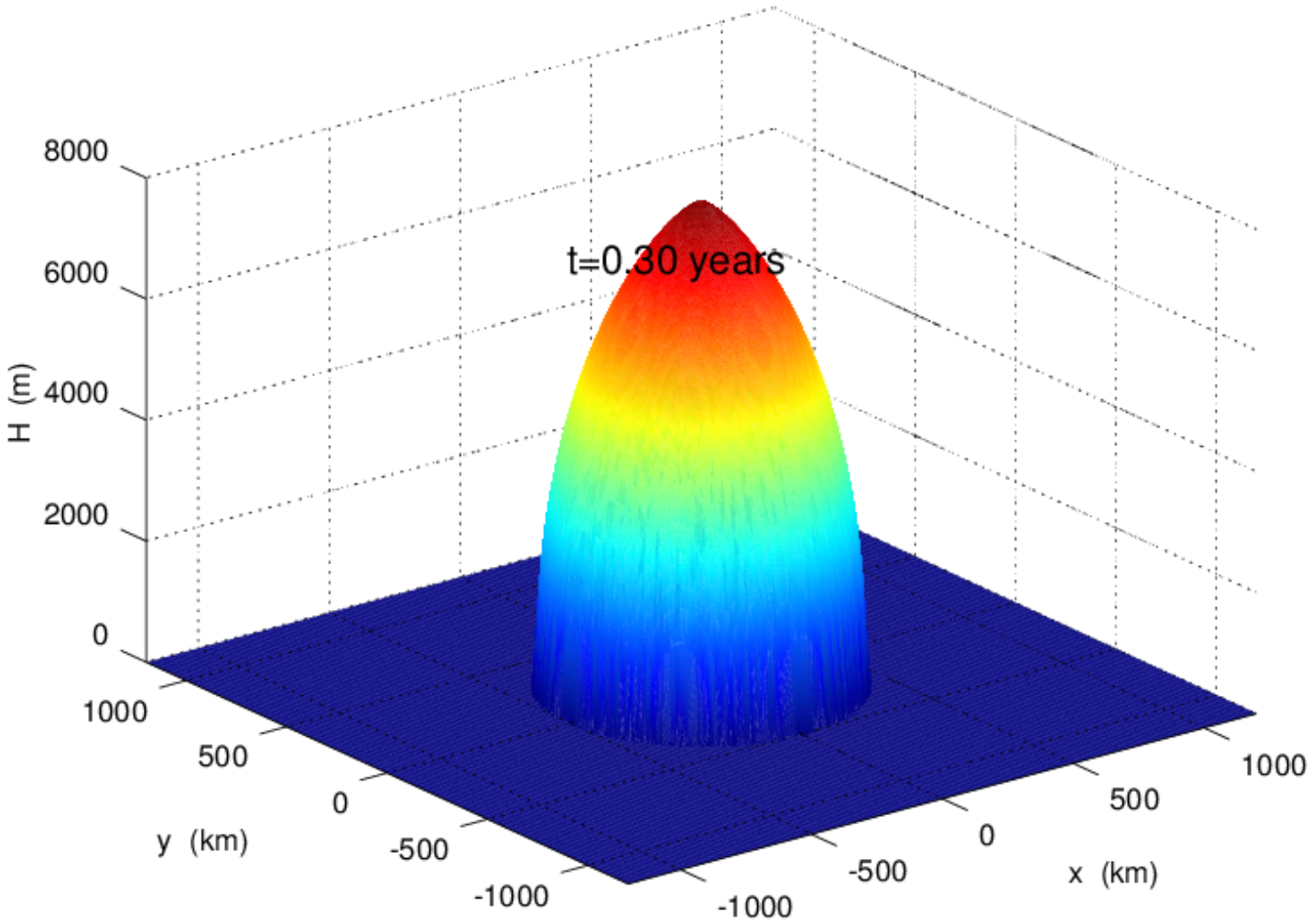
  and a simple invariant shape function $\phi(s)$

- immediate conclusion (movie follows): the diffusion of ice really slows down as the shape flattens out!

Halfar solution: the movie

frames from $t = 0.3$ to $t = 10^6$ years

nearly equally-spaced in *logarithmic* time



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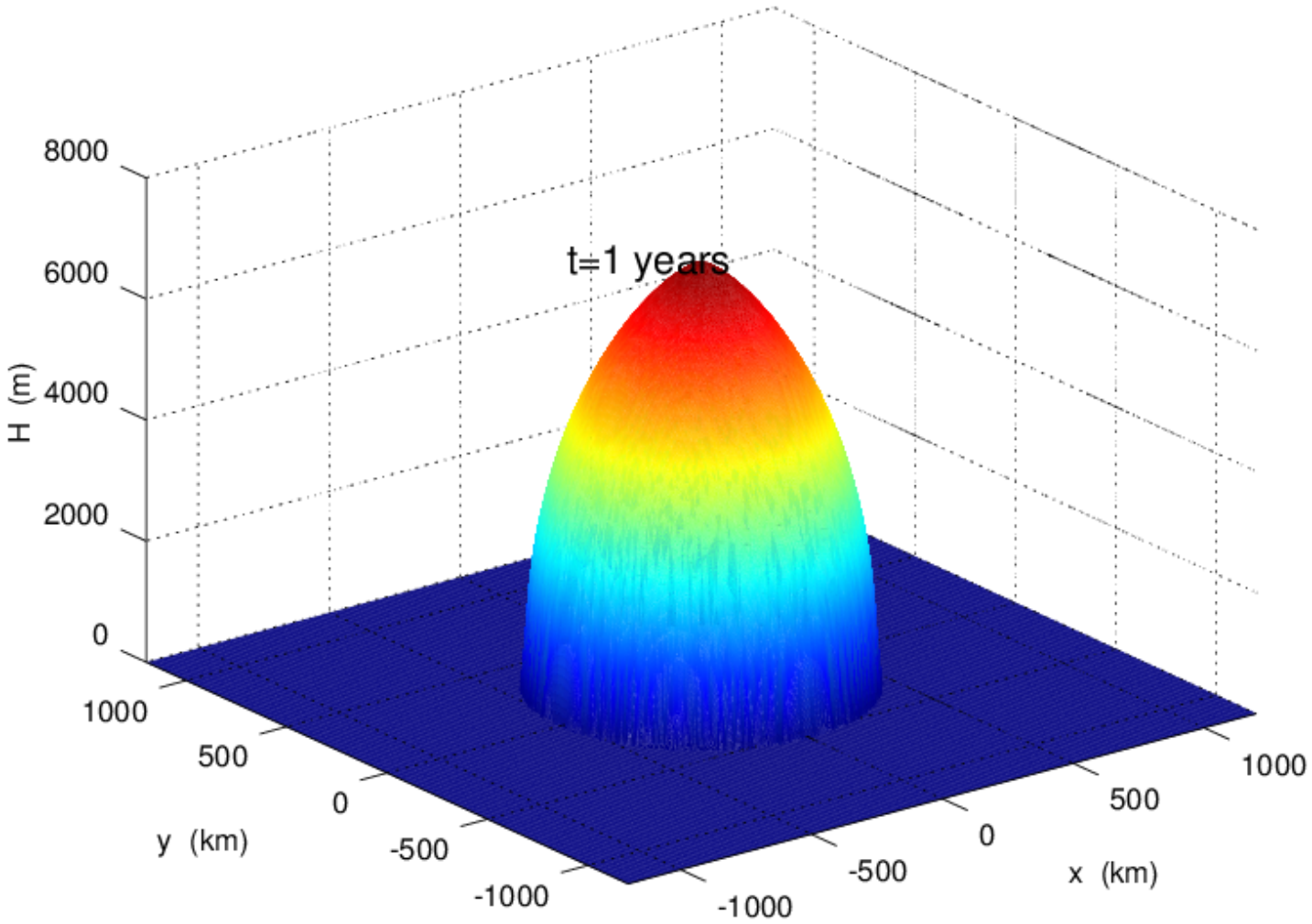

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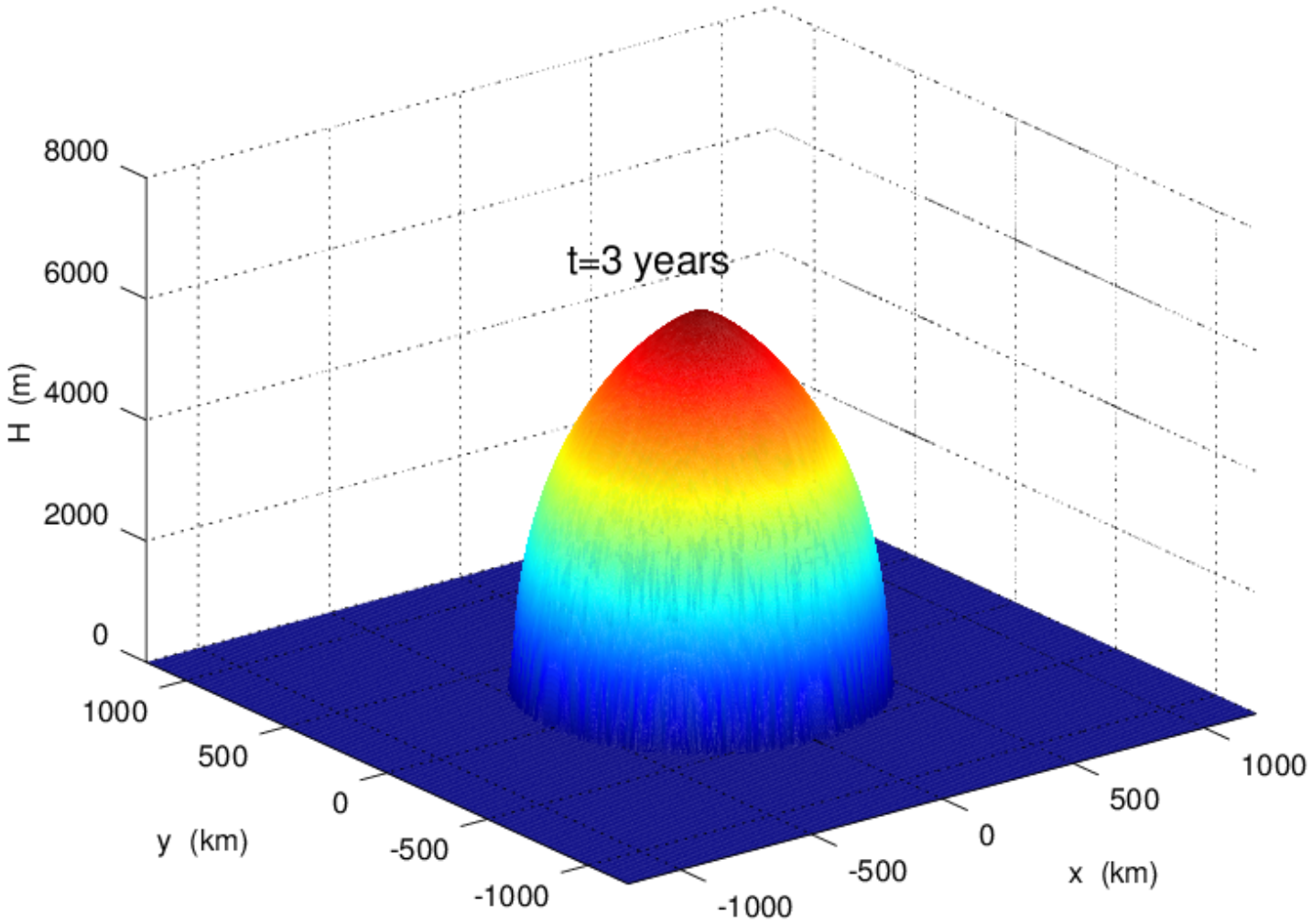

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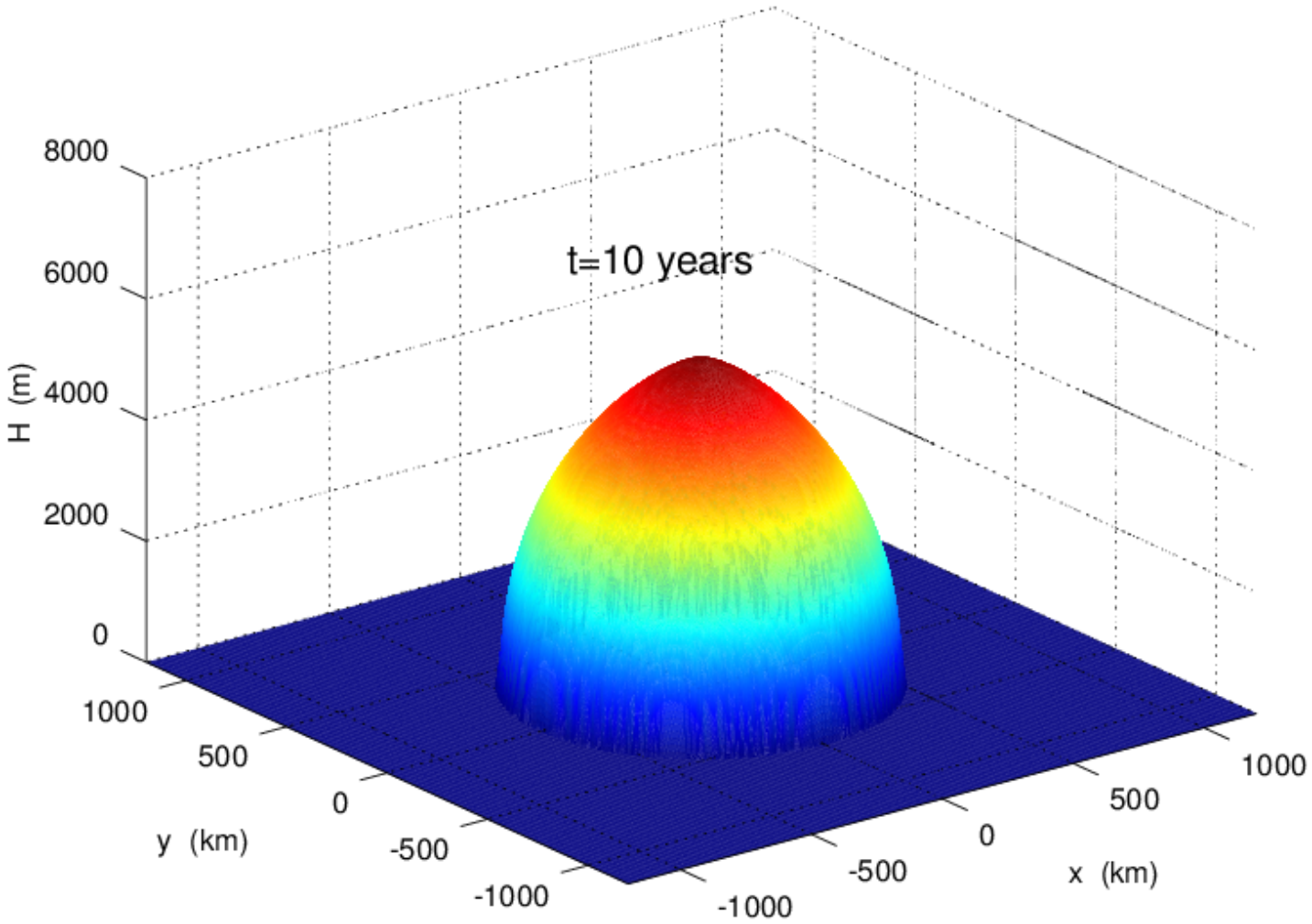

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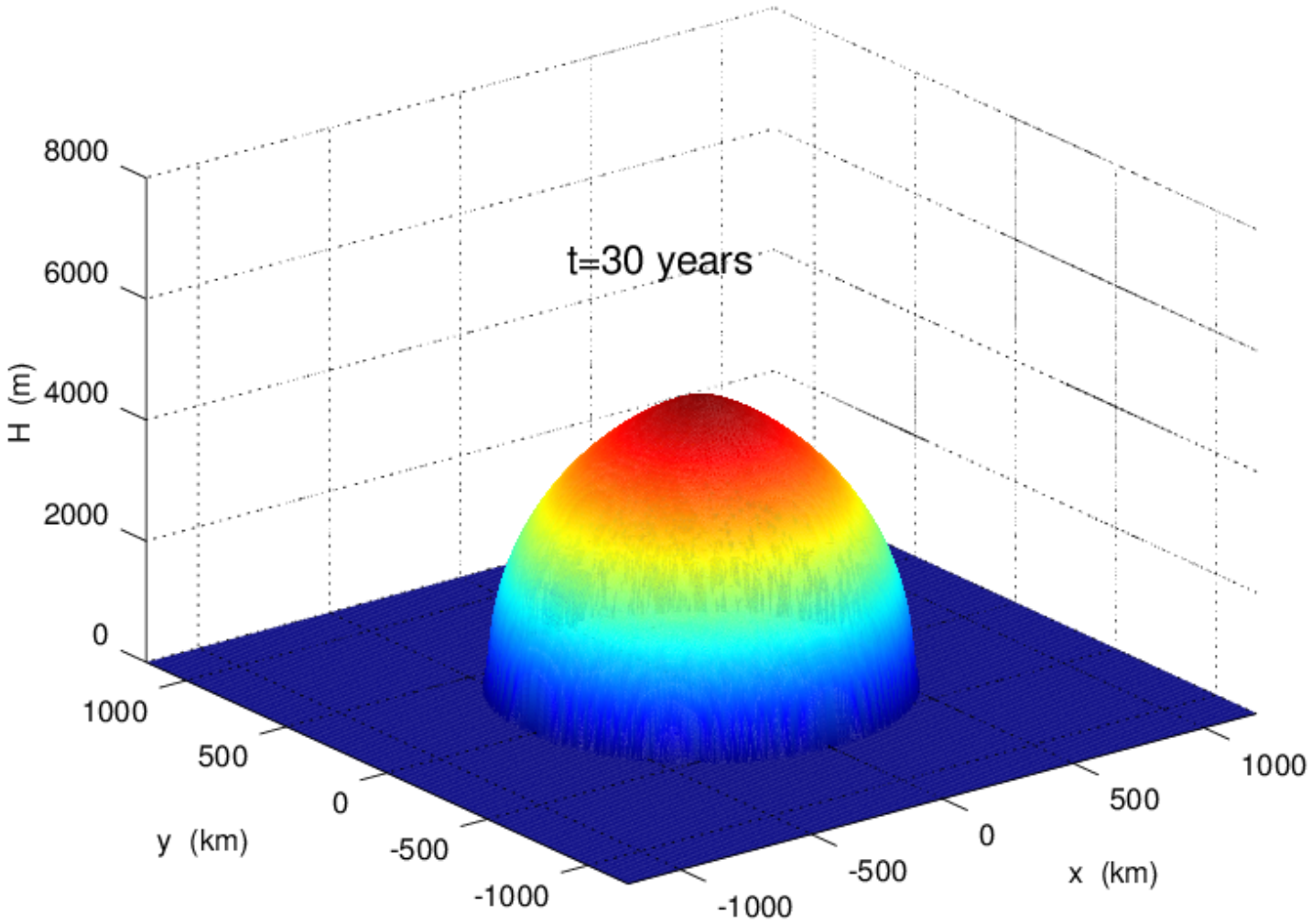

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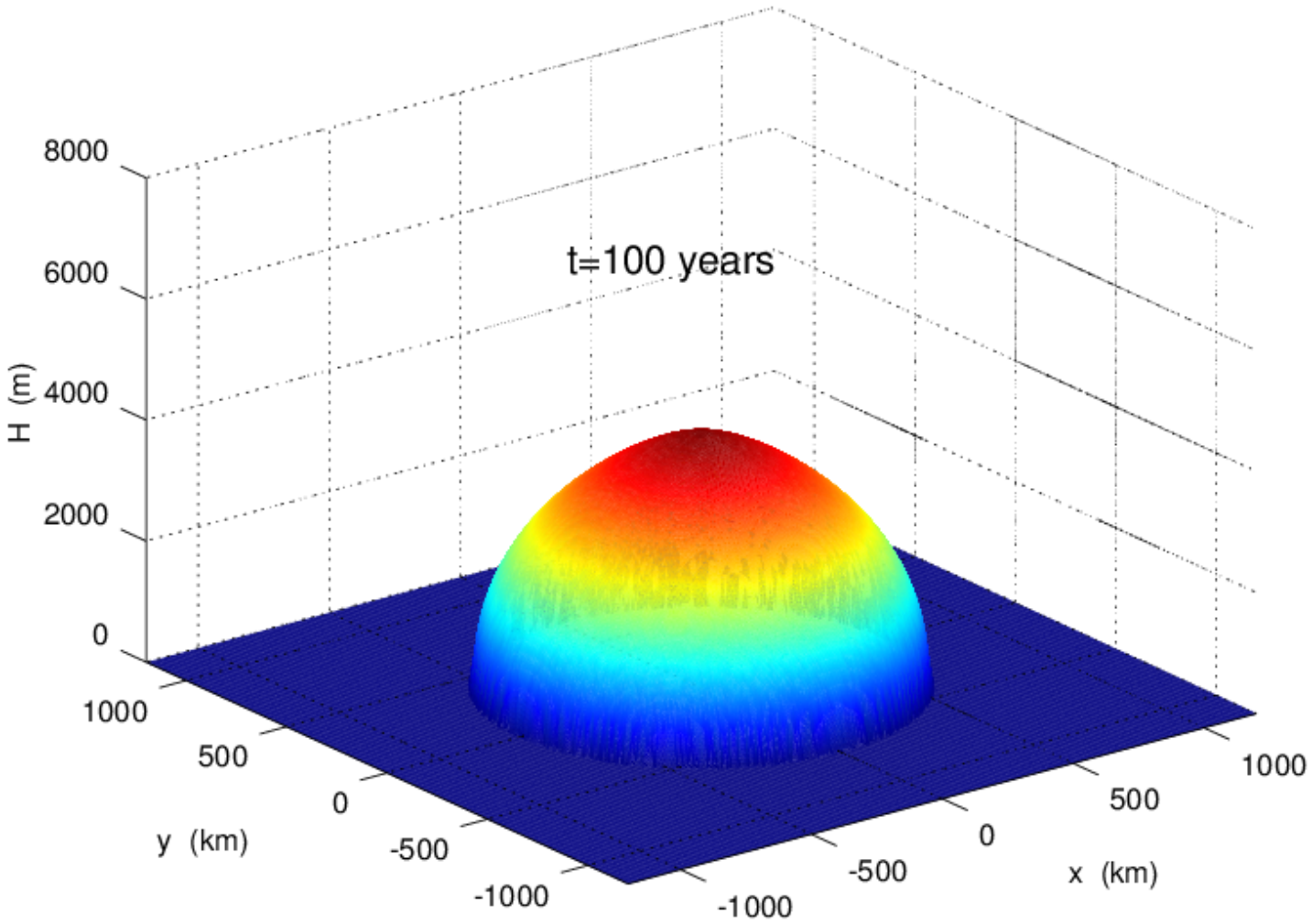

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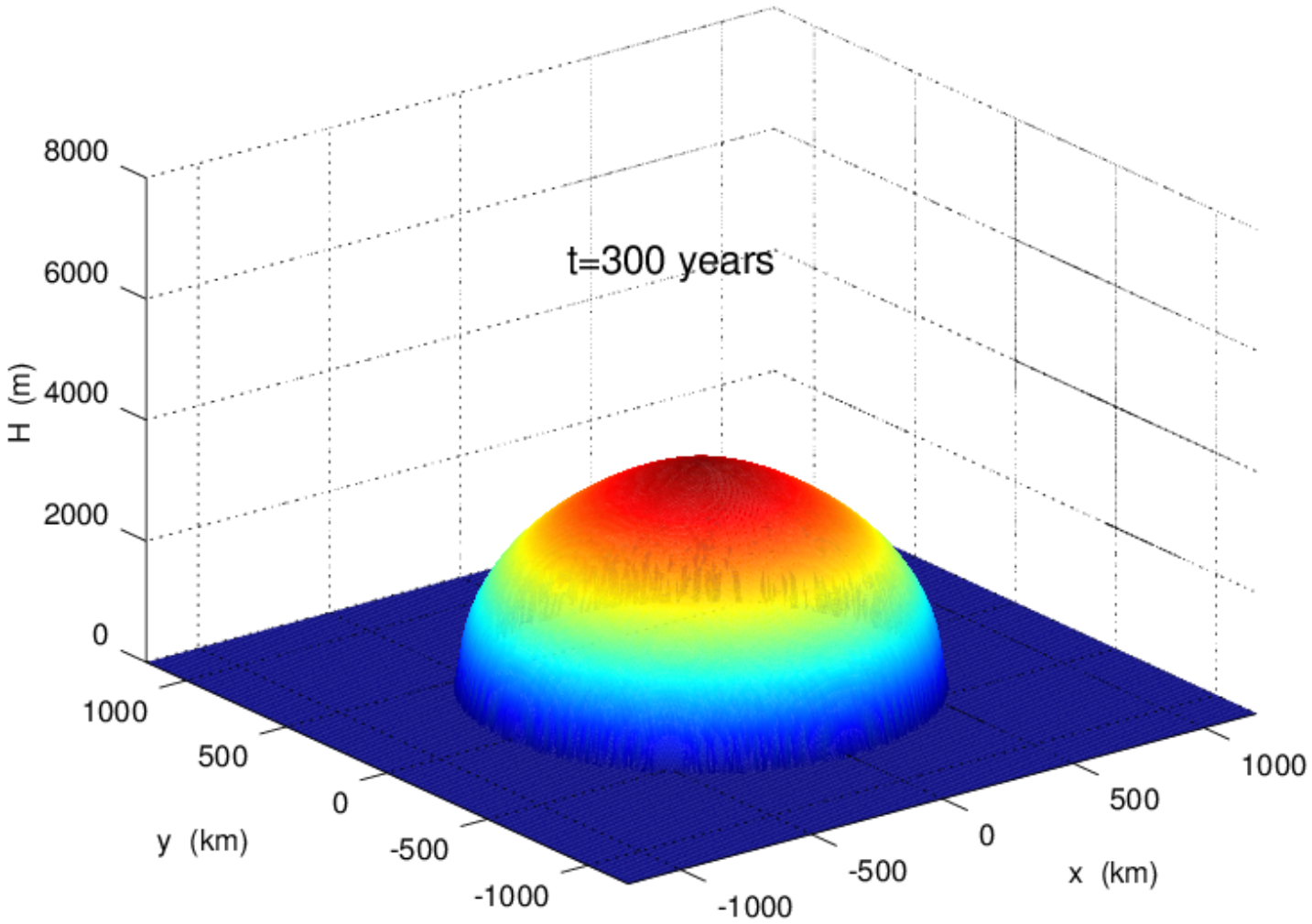

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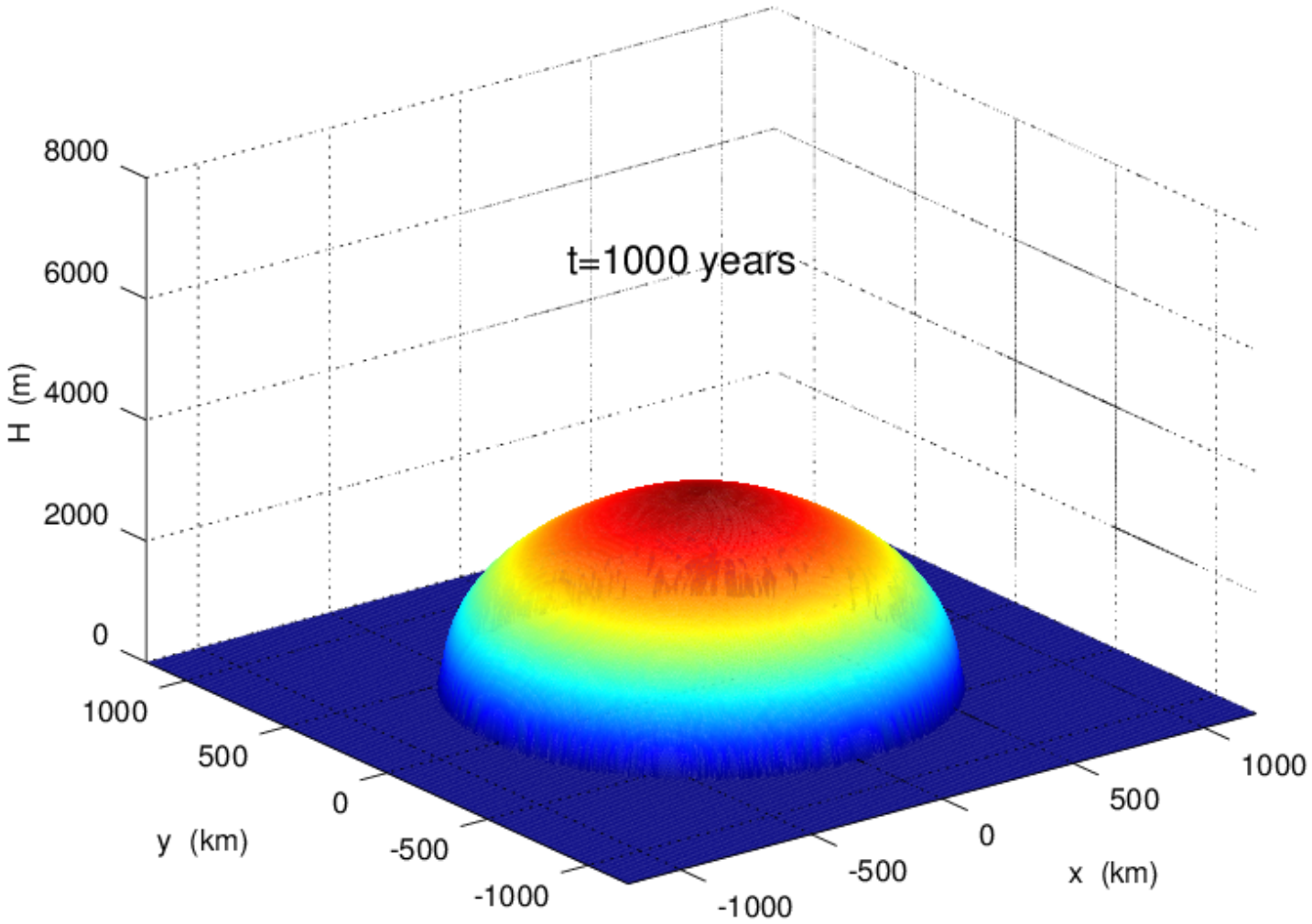

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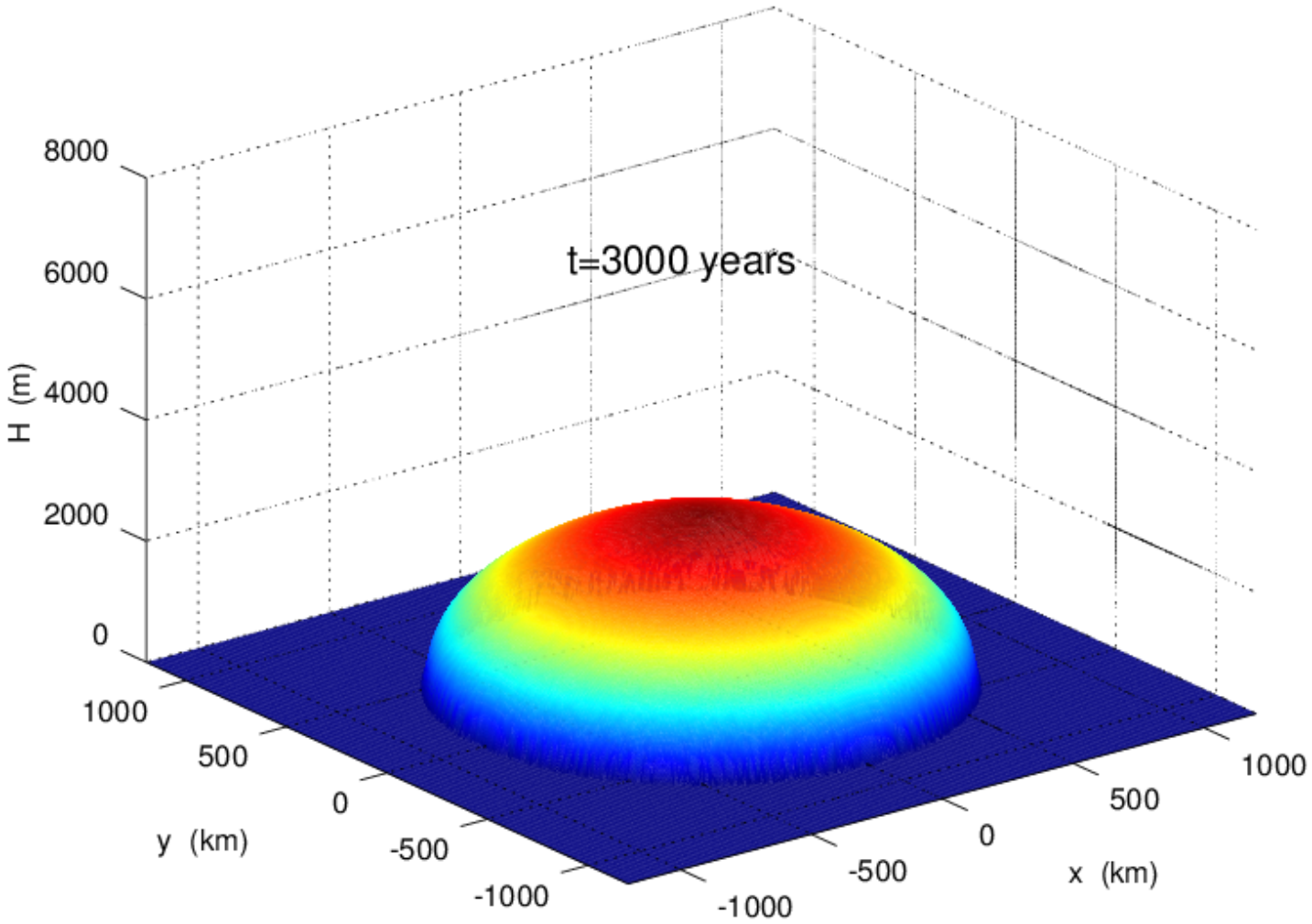

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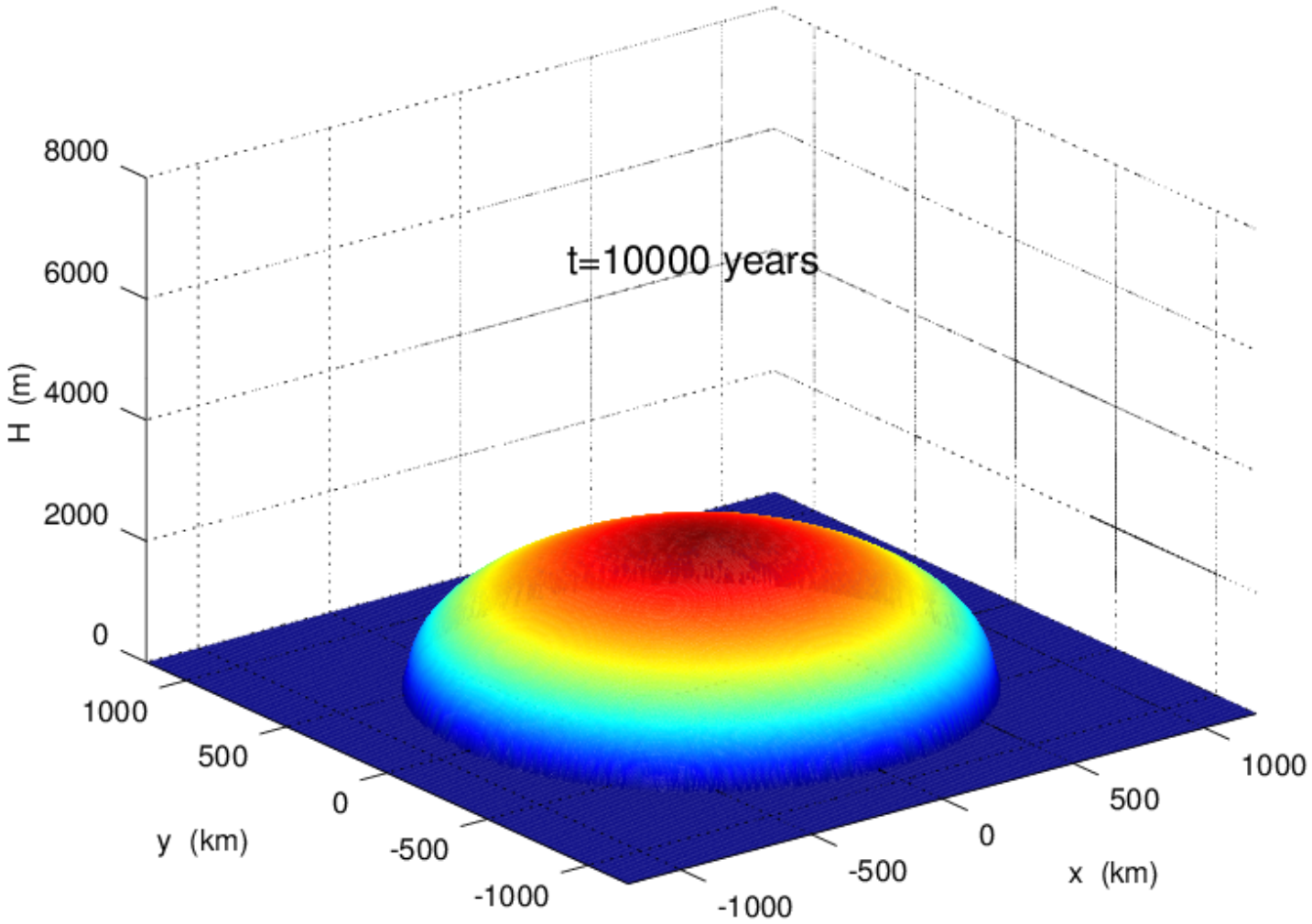

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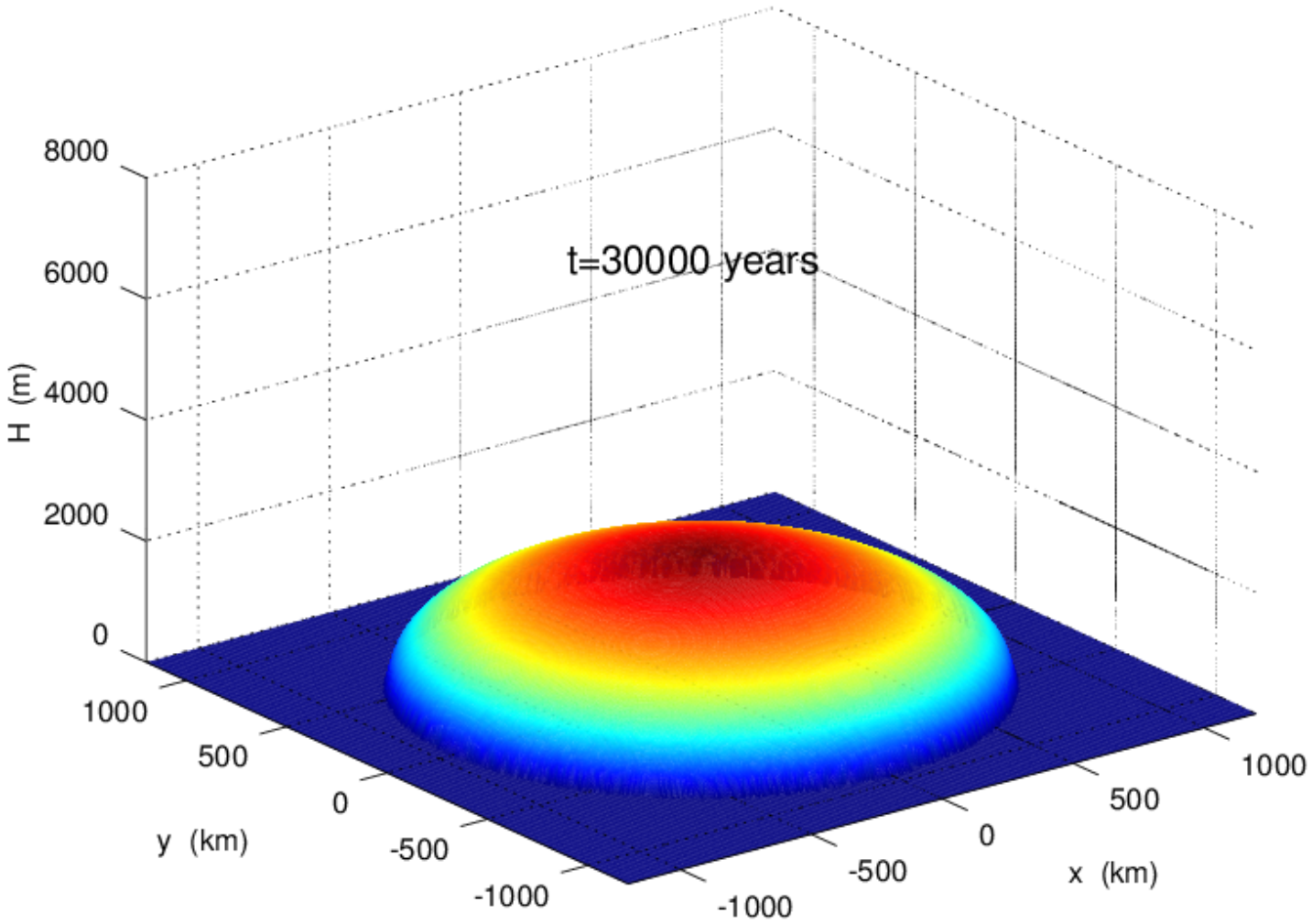

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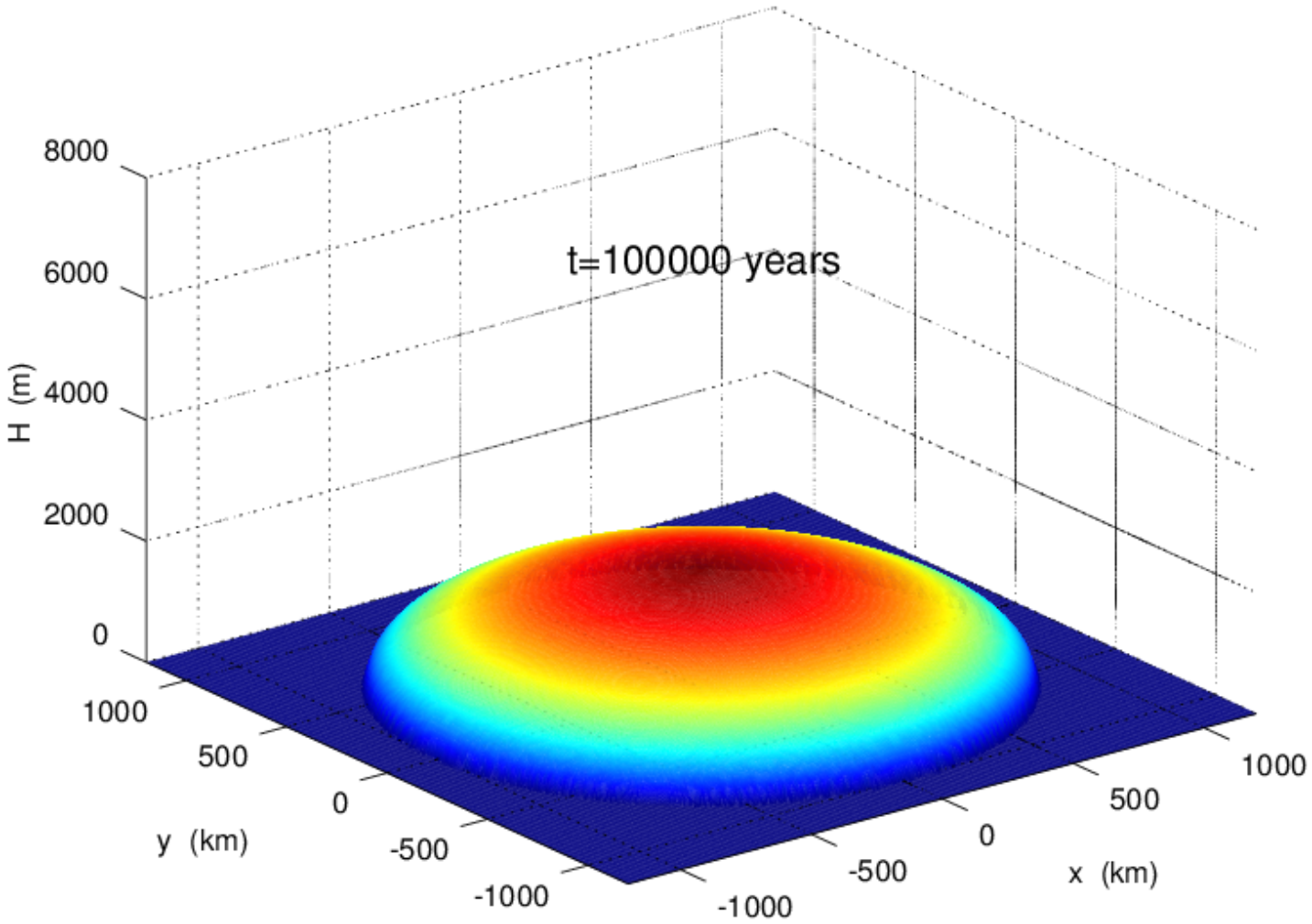

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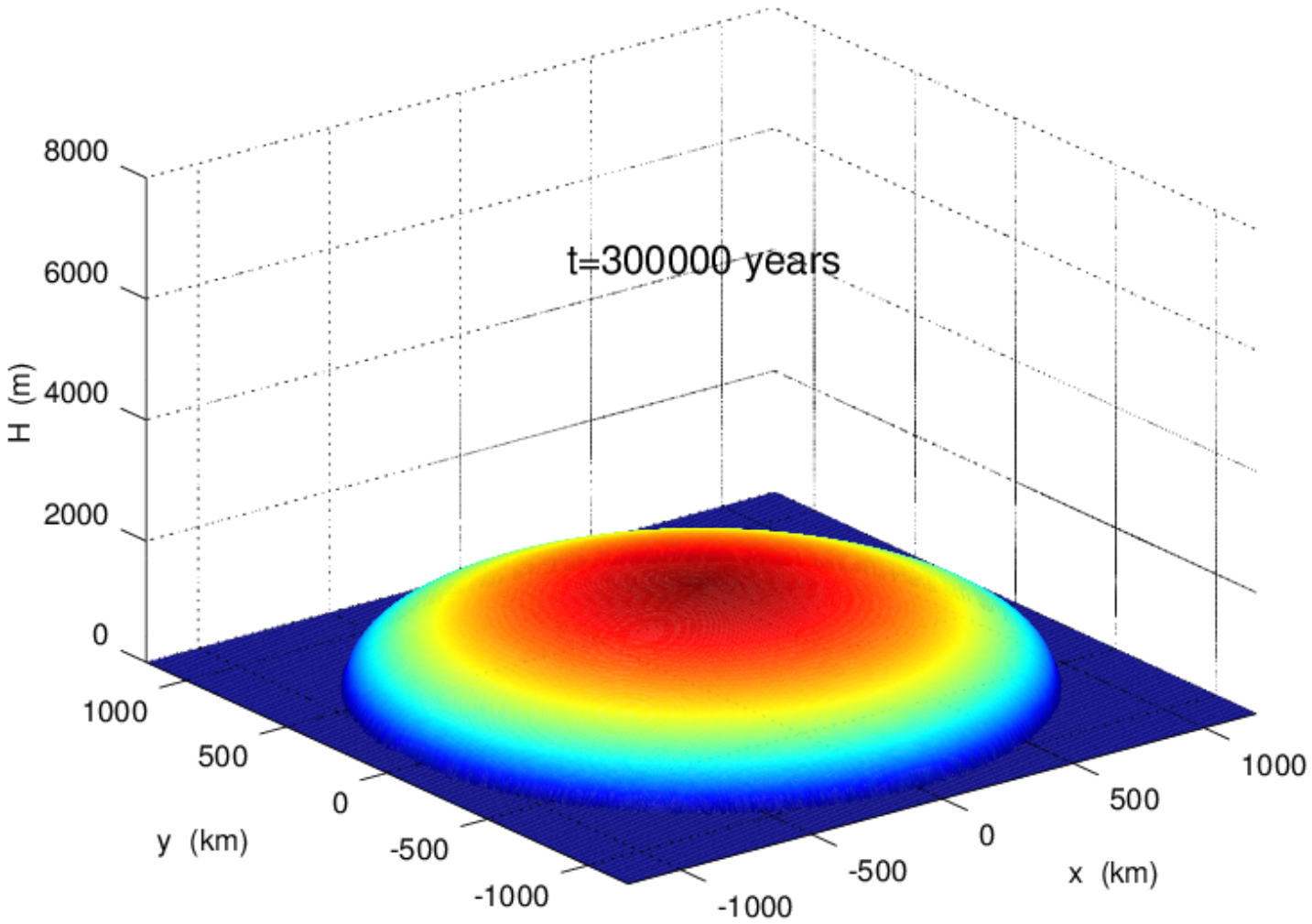

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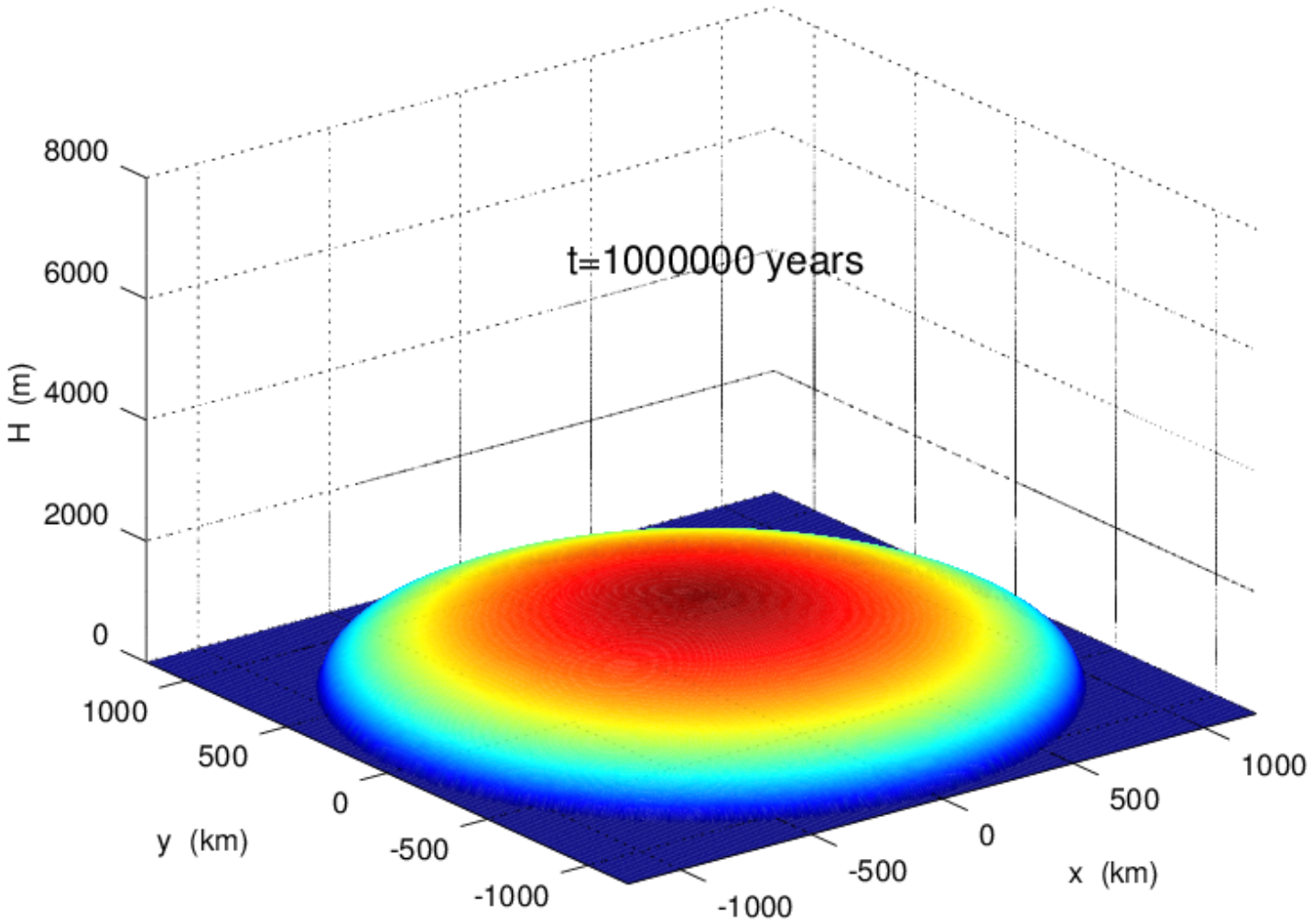


Halfar solution: the formula

- for $n = 3$ the solution formula is:

$$H(t,r) = H_0 \left(\frac{t_0}{t}\right)^{1/9} \left[1 - \left(\left(\frac{t_0}{t}\right)^{1/18} \frac{r}{R_0}\right)^{4/3}\right]^{3/7}$$

  if H_0, R_0 are central height and ice cap radius
- the "characteristic time"

$$t_0 = \frac{1}{18\Gamma}\left(\frac{7}{4}\right)^3 \frac{R_0^4}{H_0^7}$$

- it is a simple formula to use for verification!
  - it finally appears in a textbook: van der Veen (2013)

is the Halfar solution useful for modelling?

- John Nye and others (2000) compared different flow laws for the South Polar Cap on Mars
- they evaluated CO_2 ice and H_2O ice softness parameters by comparing the long-time behavior of the corresponding Halfar solutions
- conclusions:

  . . . none of the three possible [CO_2] flow laws will allow a 3000-m cap, the thickness suggested by stereogrammetry, to survive for 10^7 *years, indicating that the south polar ice cap is probably not composed of pure CO_2 ice . . . the south polar cap probably consists of water ice, with an unknown admixture of dust*

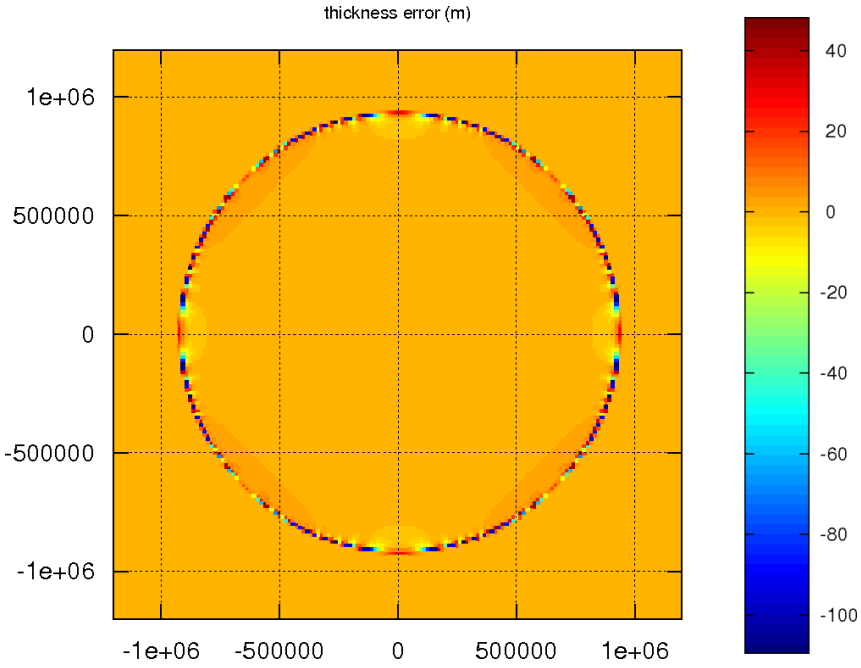
- direct observation by landers and orbiters: yes indeed!

verifying SIA code vs Halfar

- run `siaflat.m` using Halfar solution for verification:

```
>> verifysia(20)
average thickness error  = 22.310
maximum thickness error  = 227.845
>> verifysia(40)
average thickness error  = 9.459
maximum thickness error  = 240.941
>> verifysia(80)
average thickness error  = 2.771
maximum thickness error  = 153.845
>> verifysia(160)
average thickness error  = 1.085
maximum thickness error  = 104.605
```

- how much does effort increase when halving the grid spacing?



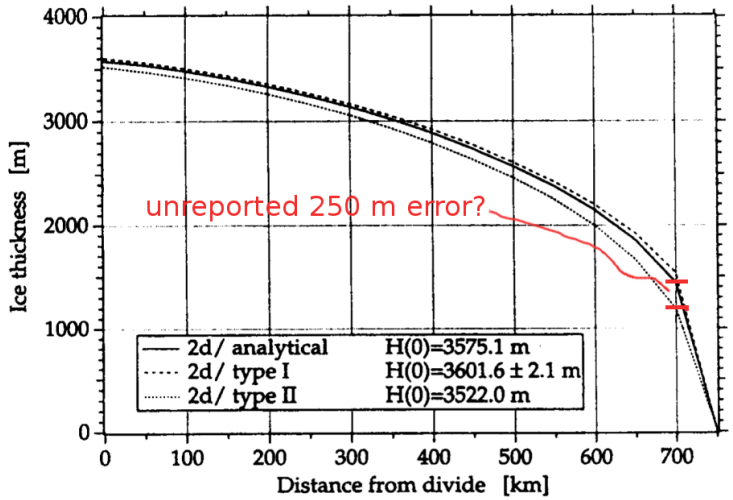
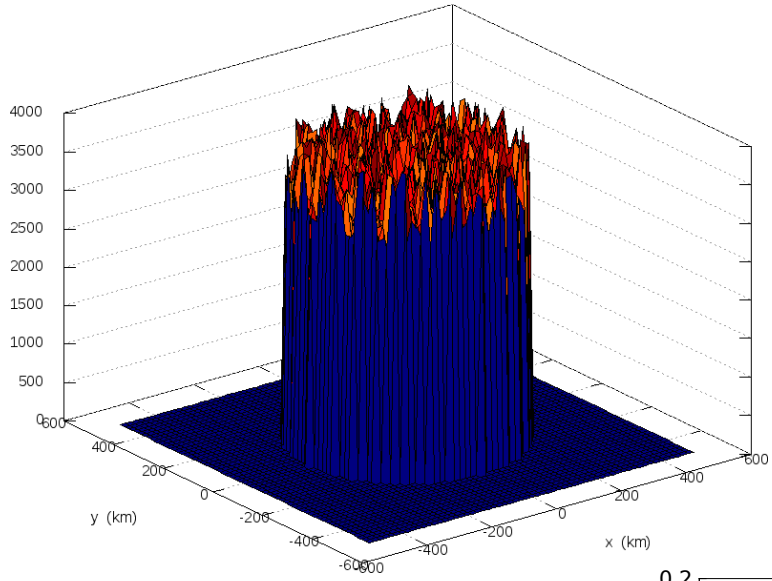
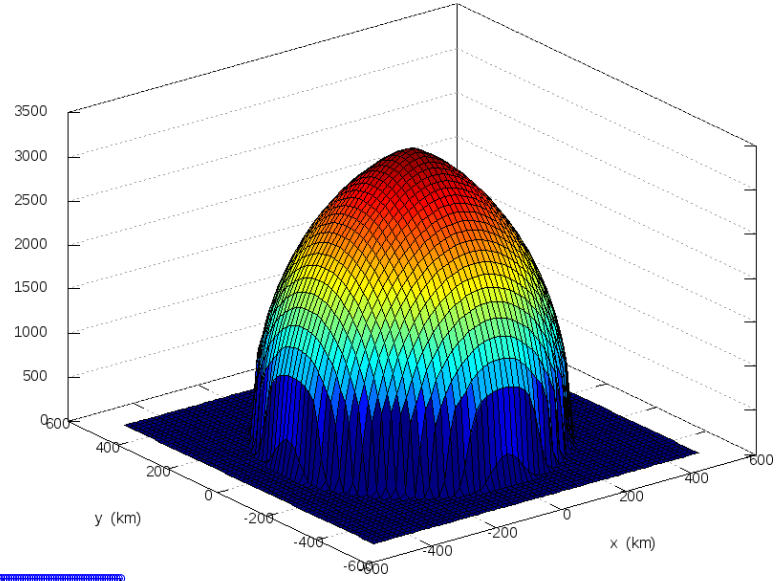


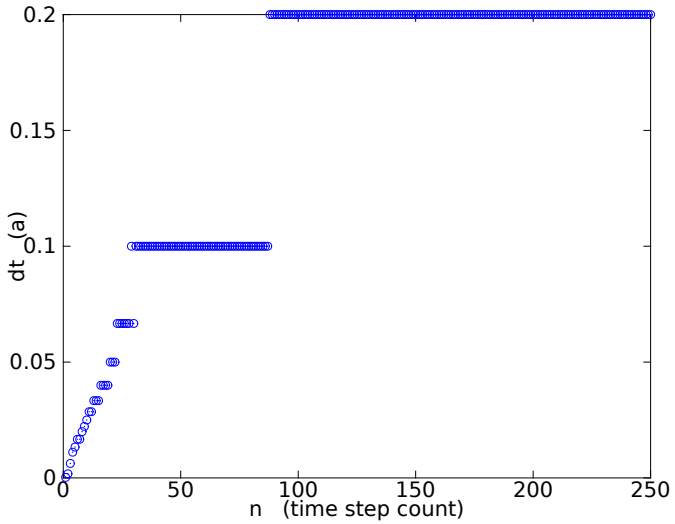
figure 2 in Huybrechts et al. (1996)

demonstrate robustness

- `roughice.m` sets-up the nasty initial state at left
- calls `siaflat.m` to evolve it for 50 years . . . adaptive time steps!
- get familiar-looking dome at right

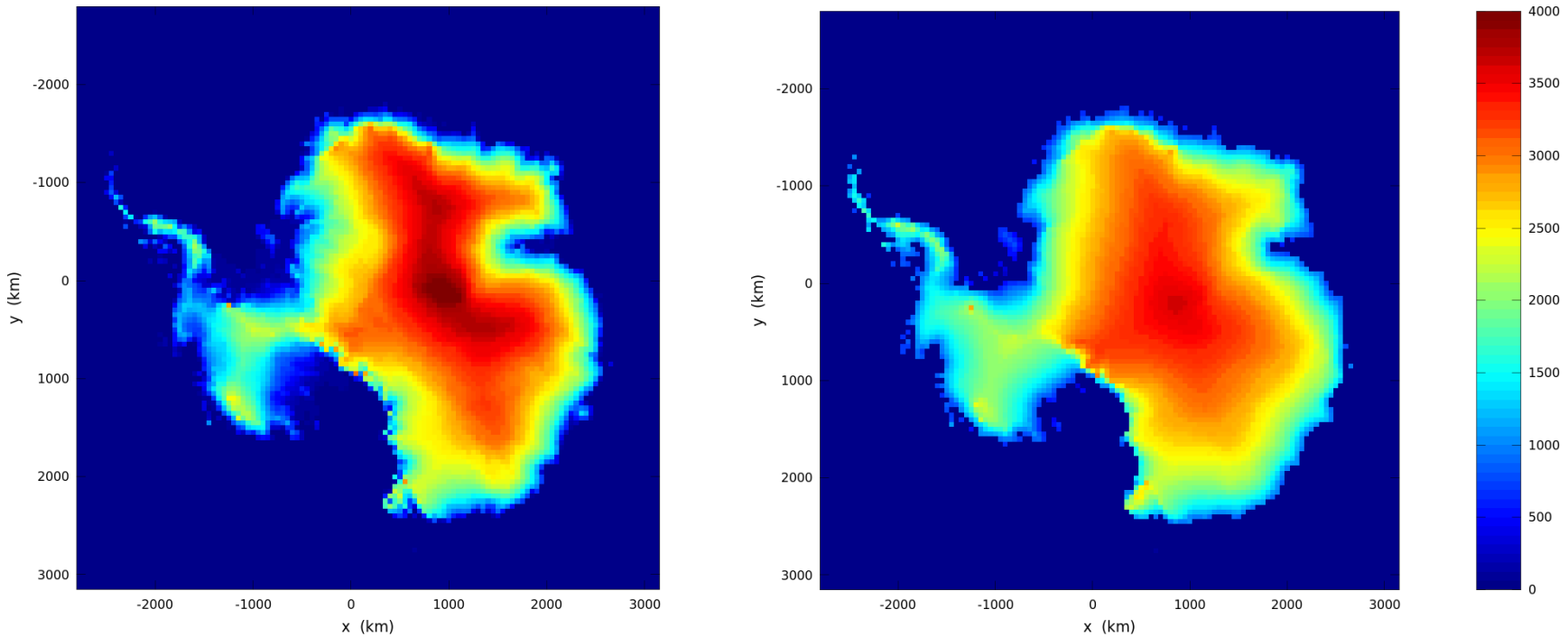






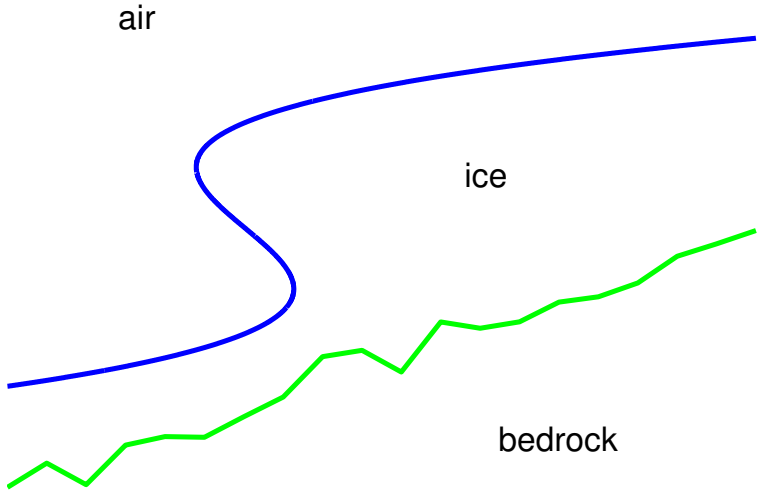
model the Antarctic ice sheet

- modify `siaflat.m` into `siageneral.m`:
  - observed accumulation as surface mass balance,
  - allow non-flat bed (so $H \neq h$),
  - calve any ice that is floating
- makes a good exercise . . . only add 10 new lines of code
- results from this *toy* Antarctic flow model, a 2000 model year run on a $\Delta x = 50$ km grid; runtime a few seconds



Outline

the most-basic shallow assumption



not shallow!

- there are many shallow theories: SIA, SSA, hybrids, Blatter, ...
- *all* make one assumption not required in Stokes:

  the surface and base of the ice are given by functions

  - namely $z = h(t, x, y)$ and $z = b(t, x, y)$
  - surface overhang is not allowed
  - most Stokes models make this assumption too

not shallow! not a fluid!

three equations for geometry change

- let a be the climatic (surface) mass balance function
  - $a > 0$ is accumulation and $a < 0$ is ablation
- let s be the basal melt rate (mass balance) function
  - $s > 0$ is basal melting and $s < 0$ is freeze-on
- let $M = a - s$
  - "climatic-basal mass balance function" in glossary
- define the map-plane flux of ice,

$$\mathbf{q} = \int_b^h (u, v)\, dz = \bar{\mathbf{u}}\, H$$

- the three equations for glacier geometry change:

surface kinematical	$h_t = a - u\big\|_h h_x - v\big\|_h h_y + w\big\|_h$
base kinematical	$b_t = s - u\big\|_b b_x - v\big\|_b b_y + w\big\|_b$
mass continuity	$H_t = M - \nabla \cdot \mathbf{q}$

kinematic and mass continuity equations

Q: what does the "most-basic shallow assumption" get you?

A: any two equations imply the third:

surface kinematical	$h_t = a - u\big\vert_h h_x - v\big\vert_h h_y + w\big\vert_h$
base kinematical	$b_t = s - u\big\vert_b b_x - v\big\vert_b b_y + w\big\vert_b$
mass continuity	$H_t = M - \nabla \cdot \mathbf{q}$

- to show the equivalences:
  - recall the incompressibility of ice

$$u_x + v_y + w_z = 0$$

  - use the Leibniz rule for differentiating integrals

$$\frac{d}{dx}\left(\int_{g(x)}^{f(x)} h(x,y)\,dy\right) = f'(x)h(x,f(x)) - g'(x)h(x,g(x)) + \int_{g(x)}^{f(x)} h_x(x,y)\,dy$$

  - it's an exercise

the mass continuity equation: a summary

- most ice sheet models use the mass continuity equation and the base kinematical equation

- regarding how to think about the *mass continuity equation*

$$H_t = M - \nabla \cdot (\overline{\mathbf{u}} H),$$

  a summary:
  - its character depends on the stress balance
  - it *is* a transport equation
  - . . . but it is a diffusion for frozen bed, large scale flows (SIA)
    - ⋆ if your fancy Stokes model is not diffusive in this case . . . *it's wrong*
  - it is not very diffusive for membrane stresses and low basal resistance (e.g. SSA for ice shelves)
  - additional mass continuity equations are needed for liquid water on surface and base . . . with much shorter time scales . . . which gets complicated

- there is *not* much helpful theory on the transport problems in glaciology . . . maybe you will help find this theory!

standard recipe for ice sheet models

1. use stress balance to compute velocity
   - often: get (u, v), then compute w from incompressibility
2. somewhere in here do other processes: thermodynamics, basal melt, calving, . . .
3. decide on time-step Δt from diffusivity D (or: *fixed* Δt, sadly)
4. from velocity, surface balance, and base balance do time-step of mass continuity equation to get H_t
5. update surface elevation: $h \leftarrow h + H_t \Delta t$
6. repeat at 1.

- this paradigm is explicit time stepping of the whole model
  - it will always be low-performance

Outline

flow model II: shallow shelf approximation (SSA)

SSA model applies very well to ice shelves

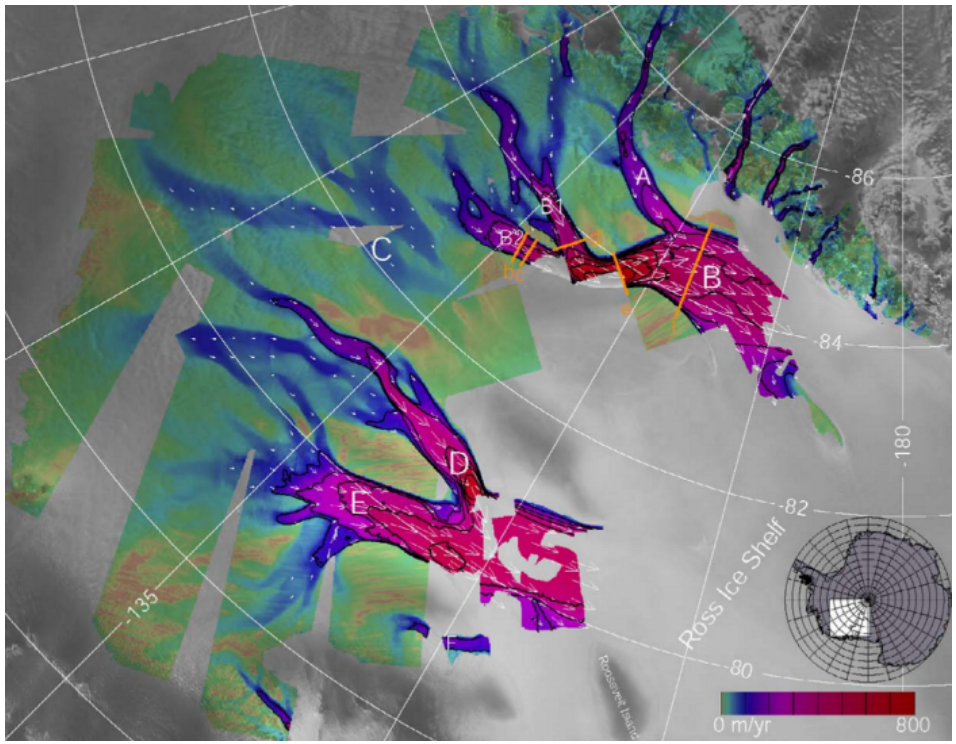
- . . . for parts away from grounding lines
- . . . and away from calving fronts

Ekström ice shelf (Hans Grobe)

application of SSA to ice streams

SSA also applies reasonably well to ice streams

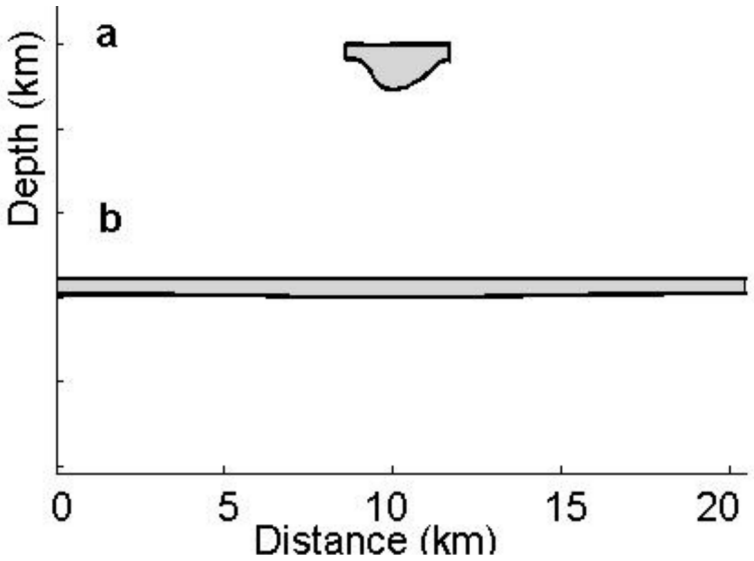
- . . . with modest bed topography
- . . . and weak bed strength[2]
- limited applicability near shear margins and grounding lines



surface velocity for Siple Coast ice streams, Antarctica

[2]energy conservation (esp. ice temperature and basal melt rate) and subglacial hydrology (esp. subglacial water pressure) are major aspects of ice stream flow . . . but not addressed here!

but what is, *and is not*, an ice stream?



Jakobshavns Isbrae (**a**) and Whillans Ice Stream (**b**); plotted without vertical exaggeration (Truffer and Echelmeyer 2003))

a. outlet glaciers:
   - fast surface speed (up to 10 km a^{-1})
   - uncertain how much is sliding
   - vertical shear in much of ice column
   - *not* flat bed topography
   - soft temperate ice may play big role
   - shallow simplifications are dubious

b. ice streams:
   - fast surface speed (up to 1 km a^{-1})
   - very concentrated vertical shear in thin layer near base (“sliding”)
   - cold ice nearly down to bed
   - SSA is accurate

SSA stress balance equation

- only plane flow case ("flow line") shown here
- the stress balance equation, ice stream case:

$$\boxed{\left(2A^{-1/n}H|u_x|^{1/n-1}u_x\right)_x - C|u|^{m-1}u = \rho g H h_x} \qquad (2)$$

- the red term inside parentheses is the vertically-integrated "longitudinal" or "membrane" stress
- the blue term is basal resistance; $C = 0$ in ice shelf
- the green term is driving stress

- this equation determines velocity u
- derived originally by Morland (1987), MacAyeal (1989)

once again, good questions:

how to solve (2) numerically?

how to *think* about it?

flotation criterion and grounding line

- the inequality "$\rho H < -\rho_w b$" is the flotation criterion
  - assumes $z = 0$ is sea level
  - grounding line $x = x_g$ is where $\rho H = -\rho_w b$
  - H, u, u_x are all continuous at $x = x_g$
- surface elevation and driving stress according to flotation:

grounded:

$$\rho H > -\rho_w b$$
$$h = H + b$$
$$\rho g H h_x = \rho g H (H_x + b_x)$$

floating:

$$\rho H < -\rho_w b$$
$$h = (1 - \rho/\rho_w) H$$
$$\rho g H h_x = \rho (1 - \rho/\rho_w) g H H_x$$

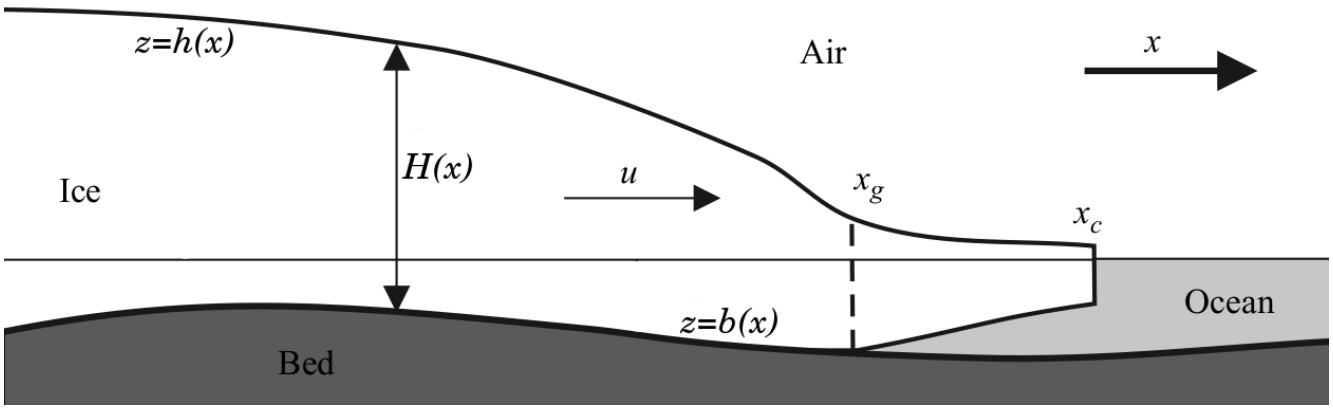
flow line model: from stream to shelf

$$u = u_0 \qquad \text{at } x = 0$$

$$\left.\begin{aligned} \left(2A^{-1/n}H|u_x|^{1/n-1}u_x\right)_x - C|u|^{m-1}u &= \rho g H h_x \\ h &= H + b \end{aligned}\right\} \qquad \text{on } 0 < x < x_g$$

$$\left.\begin{aligned} \left(2A^{-1/n}H|u_x|^{1/n-1}u_x\right)_x &= \rho g H h_x \\ h &= (1 - \rho/\rho_w)H \end{aligned}\right\} \qquad \text{on } x_g < x < x_c$$

$$2A^{-1/n}H|u_x|^{1/n-1}u_x = \frac{1}{2}\rho(1-\rho/\rho_w)gH^2 \qquad \text{at } x = x_c$$

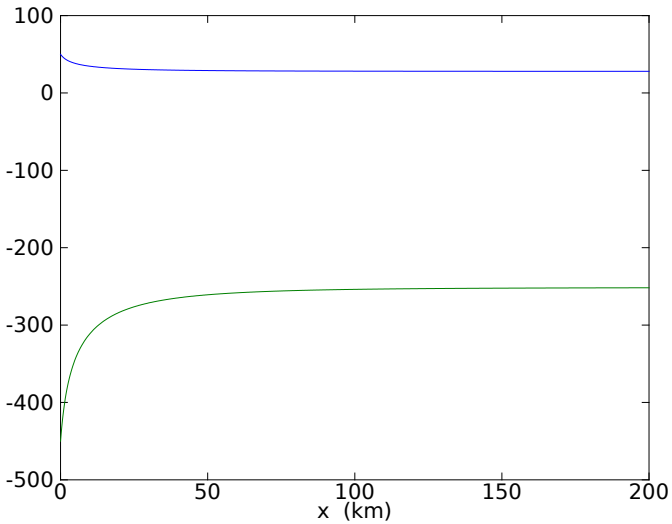


steady ice shelf

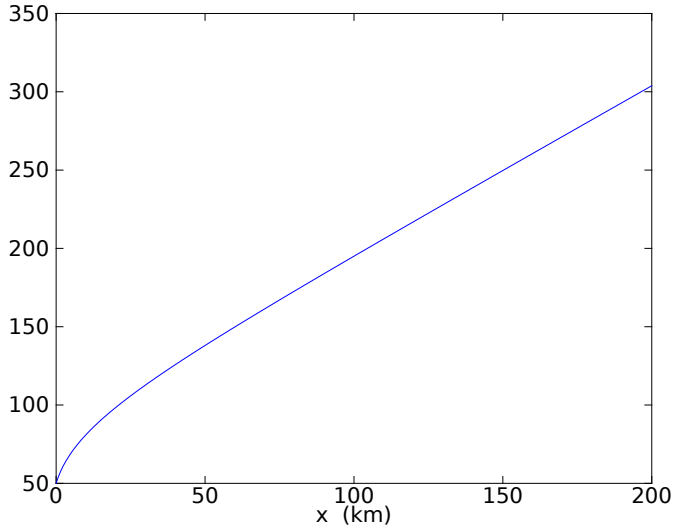
- I have limited goal here:

  describe a steady state, 1D ice shelf

- *by-hand* result (van der Veen 1983): steady SSA ice shelf found from thickness H_g and velocity u_g at the grounding line

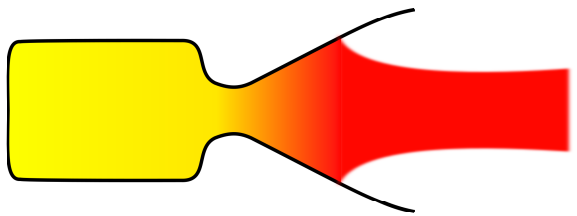


surface/base elevation



velocity

- we will use this to
  - understand the SSA better
  - verify a numerical SSA code



- an ice shelf is a kind of fluid *jet*

numerical SSA stress balance example

- fix the ice thickness $H(x)$ from the exact van der Veen formula
- find the velocity numerically
- verify using the exact van der Veen formula for velocity

numerics of SSA stress balance

- the stress balance is a nonlinear equation in the velocity:

$$\left(2A^{-1/n}H|u_x|^{1/n-1}u_x\right)_x - C|u|^{m-1}u = \rho g H h_x$$

  - thus iteration is needed

- coefficient $\bar{\nu} = A^{-1/n}|u_x|^{1/n-1}$ is the "effective viscosity":

$$(2\,\bar{\nu}\,Hu_x)_x - C|u|^{m-1}u = \rho g H h_x$$

- *simplest iteration idea*: use old $\bar{\nu}$ to get new velocity solution, and repeat until things stop changing
  - this is "Picard" iteration (versus Newton iteration, generally superior)
  - from last iterate $u^{(k-1)}$ define

$$W^{(k-1)} = 2\bar{\nu}H = 2A^{-1/n}|u_x^{(k-1)}|^{1/n-1}H$$

  - solve for current iterate $u^{(k)}$:

$$\left(W^{(k-1)}u_x^{(k)}\right)_x - C|u^{(k-1)}|^{m-1}u^{(k)} = \rho g H h_x$$

where do you get an initial guess $u^{(0)}$?

- *for floating ice*, assume a uniform strain rate:

$$u^{(0)}(x) = u_g + \gamma(x - x_g)$$

- *for grounded ice*, assume ice is held by basal resistance only:

$$u^{(0)}(x) = \left(-C^{-1}\rho g H h_x\right)^{1/m}$$

abstract the “inner” linear problem

- abstract problem:
$$(W(x)\,u_x)_x - \alpha(x)\,u = \beta(x)$$
  - on $0 < x < L$
  - at grounding line $x_g = 0$: $u(0) = u_g$
  - at calving front $x_c = L$: $u_x(L) = \gamma$
- an *elliptic* boundary value problem
- $W(x)$, $\alpha(x)$, $\beta(x)$ are known functions in the SSA context:
  - both $W(x)$ and $\alpha(x)$ come from previous iteration
  - $\beta(x)$ is driving stress

numerics of the "inner" linear problem

- suppose $j = 1, 2, \ldots, J+1$, where $x_1 = x_g$ and $x_{J+1} = x_c$ are endpoints
- $W(x)$ is needed on the staggered grid; the approximation is:

$$\frac{W_{j+1/2}(u_{j+1} - u_j) - W_{j-1/2}(u_j - u_{j-1})}{\Delta x^2} - \alpha_j u_j \stackrel{*}{=} \beta_j$$

- left-hand boundary condition: $u_1 = u_g$ given
- right-hand boundary condition:
  - introduce notional point x_{J+2}
  - approximate "$u_x(L) = \gamma$" by

$$\frac{u_{J+2} - u_J}{2\Delta x} = \gamma$$

  - use $j = J+1$ case of $*$ to eliminate u_{J+2} (by-hand)

numerics of the "inner" linear problem 2

- thus iterate of SSA stress balance has form $A\mathbf{x} = \mathbf{b}$:

$$\begin{bmatrix} 1 & & & & & \\ W_{3/2} & A_{22} & W_{5/2} & & & \\ & W_{5/2} & A_{33} & & & \\ & & \ddots & \ddots & & \\ & & W_{J-1/2} & A_{JJ} & W_{J+1/2} \\ & & & A_{J+1,J} & A_{J+1,J+1} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_J \\ u_{J+1} \end{bmatrix} = \begin{bmatrix} u_g \\ \beta_2 \Delta x^2 \\ \beta_3 \Delta x^2 \\ \vdots \\ \beta_J \Delta x^2 \\ b_{J+1} \end{bmatrix}$$

- diagonal entries

$$A_{22} = -(W_{3/2} + W_{5/2} + \alpha_1 \Delta x^2), \quad A_{33} = -(W_{5/2} + W_{7/2} + \alpha_2 \Delta x^2), \quad \dots$$

- special cases in final equation:

$$A_{J+1,J} = 2W_{J+1/2}$$
$$A_{J+1,J+1} = -(2W_{J+1/2} + \alpha_{J+1} \Delta x^2)$$
$$b_{J+1} = -2\gamma \Delta x W_{J+3/2} + \beta_{J+1} \Delta x^2$$

- this is a *tridiagonal* system

numerics of the “inner” linear problem 3

```
function u = flowline(L,J,gamma,W,alpha,beta,ug)

dx = L / J;
b = dx^2 * beta(:);
b(1) = ug;  b(J+1) = b(J+1) - 2 * gamma * dx * W(J+1);

A = sparse(J+1,J+1);
A(1,1) = 1.0;
for j=2:J
  A(j,j-1:j+1) = [ W(j-1), -(W(j-1) + W(j) + alpha(j) * dx^2), W(j) ];
end
A(J+1,J) = W(J) + W(J+1);
A(J+1,J+1) = - (W(J) + W(J+1) + alpha(J+1) * dx^2);

scale = full(max(abs(A),[],2));
for j=1:J+1,  A(j,:) = A(j,:) ./ scale(j);  end
b = b ./ scale;

u = A \ b;
```

flowline.m

testing the abstracted linear code

- first we test the linear code `flowline.m` for the abstract problem
- test by "manufacturing" solutions
  - see `testflowline.m`
  - result: converges at optimal rate $O(\Delta x^2)$
- then write Picard iteration loop to create SSA nonlinear solver
  - `ssaflowline.m` on next slide

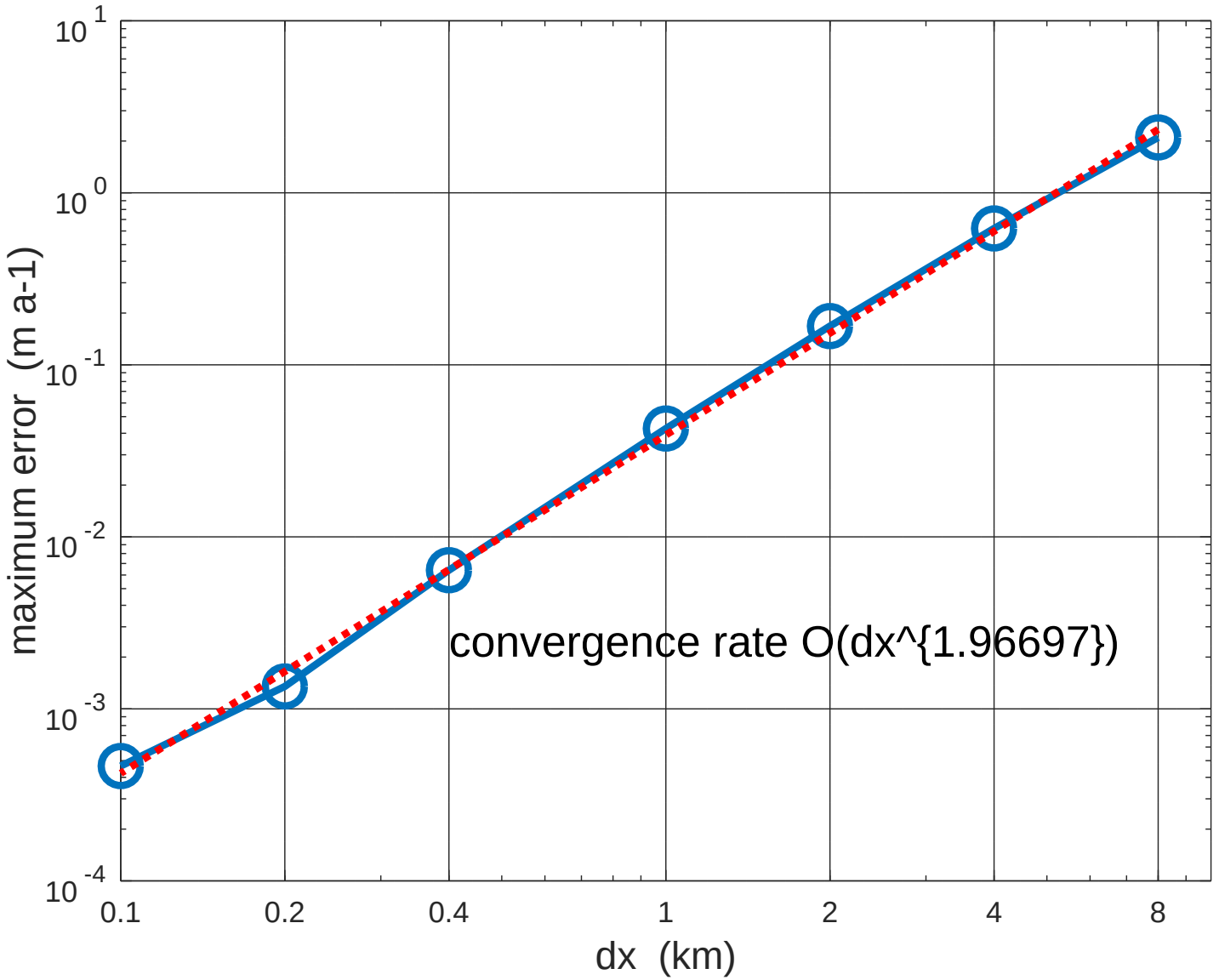
numerical SSA implementation

```
function [u,u0] = ssaflowline(p,J,H,b,ug,initchoice)

if nargin ~= 6, error('exactly 6 input arguments required'), end

dx = p.L / J;  x = (0:dx:p.L)';
xstag = (dx/2:dx:p.L+dx/2)';

alpha = p.C * ones(size(x));
h = H + b;
hx = [(h(2)-h(1))/dx; (h(3:J+1)-h(1:J-1))/(2*dx); (h(J+1)-h(J))/dx];

beta = p.rho * p.g * H .* hx;
gamma = ( 0.25 * p.A^(1/p.n) * (1 - p.rho/p.rhow) *...
          p.rho * p.g * H(end) )^p.n;

u0 = ssainit(p,x,beta,gamma,initchoice);  u = u0;

Hstag = 0.5 * (H(1:end-1) + H(2:end));
tol = 1.0e-14;
eps_reg = (1.0 / p.secpera) / p.L;
maxdiff = Inf;  W = zeros(J+1,1);  iter = 0;
while maxdiff > tol
  uxstag = (u(2:end) - u(1:end-1)) / dx;
  sqr_ux_reg = uxstag.^2 + eps_reg^2;
  W(1:J) = 2 * p.A^(-1/p.n) * Hstag .* sqr_ux_reg.^(((1/p.n)-1)/2.0);
  W(J+1) = W(J);

  unew = flowline(p.L,J,gamma,W,alpha,beta,ug);
  maxdiff = max(abs(unew-u));
  u = unew;
  iter = iter + 1;
end
```

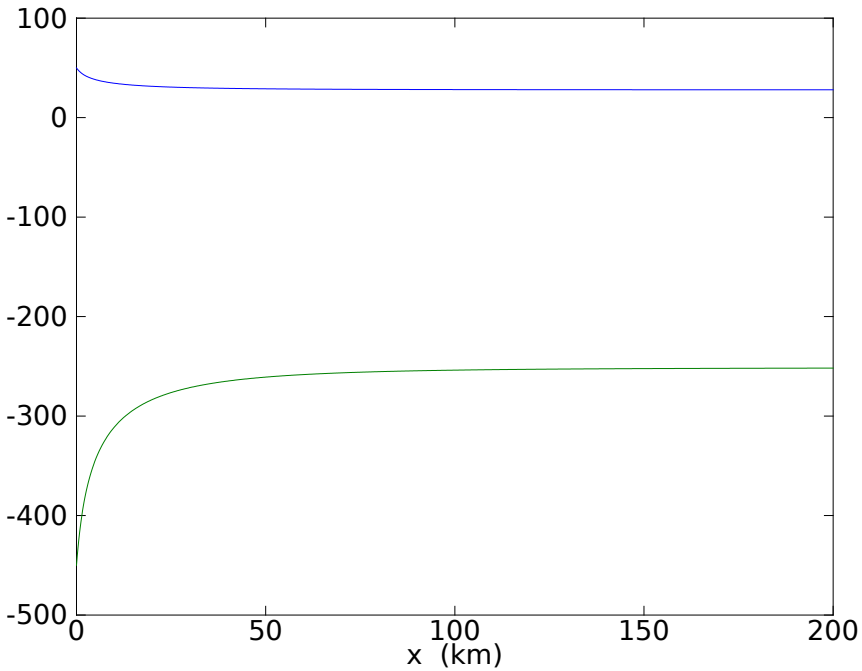
ssaflowline.m

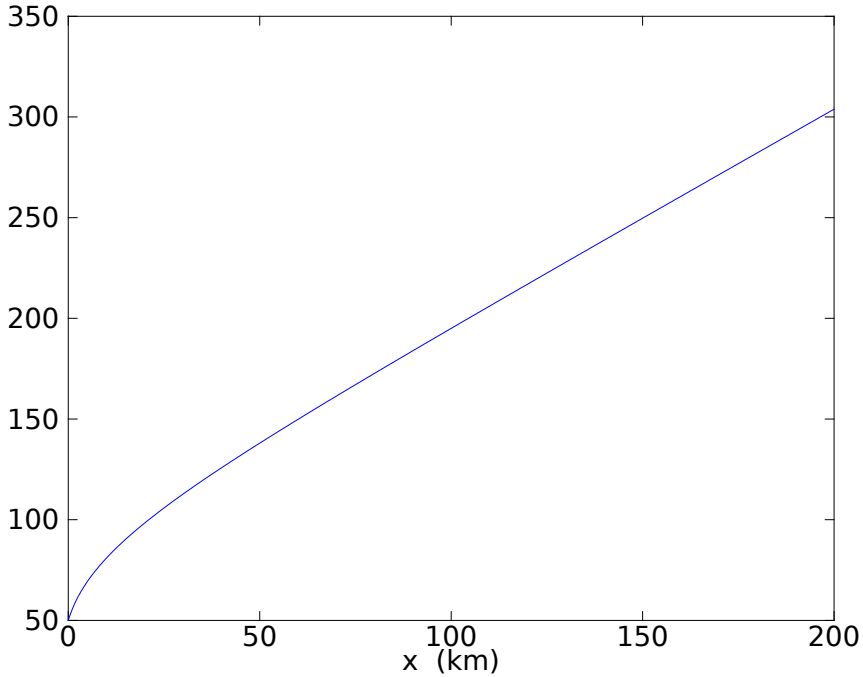
numerical convergence

convergence analysis of `ssaflowline.m`:



SSA numerical model output

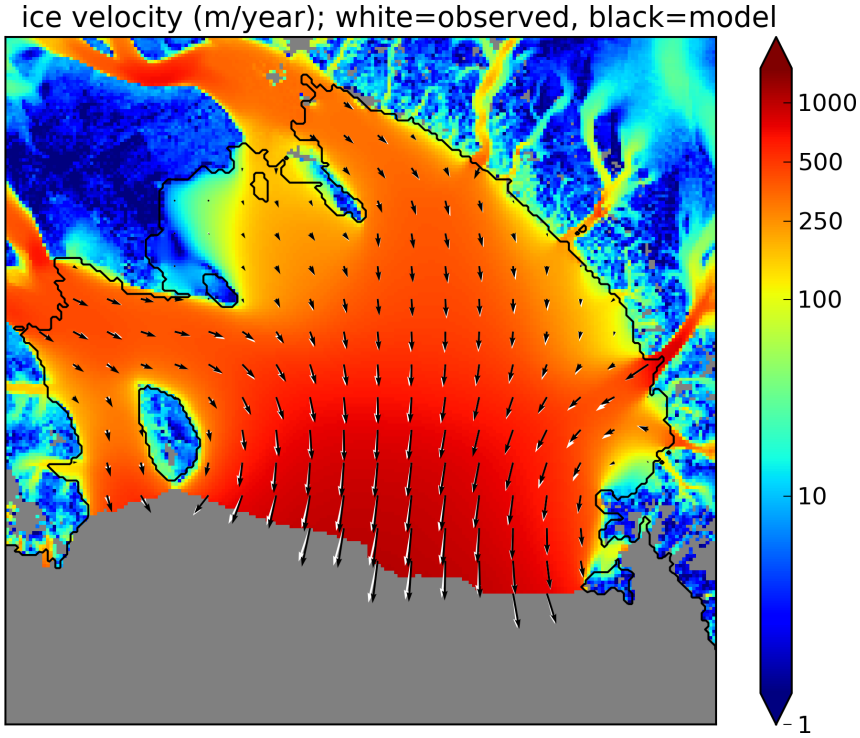


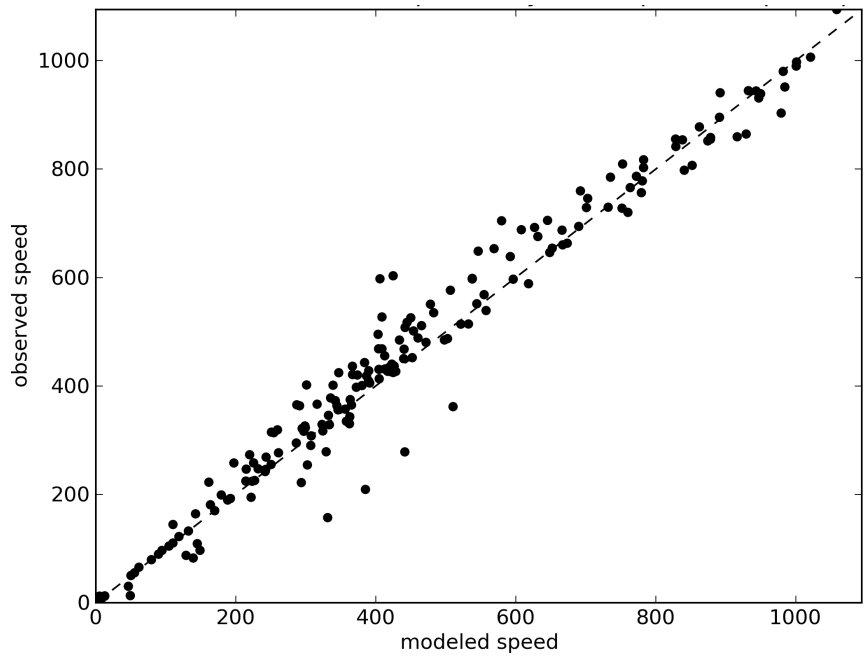


- *this looks suspiciously like figures for the exact solution . . .*
- yes

numerical ice shelf modeling in 2D

- "diagnostic" (static geometry) ice shelf modeling in 2D has been quite successful
- observed surface velocities validate SSA stress balance model
  - e.g. Ross ice shelf example below using PISM
  - . . . many models can do this





numerical solution of stress balances: a summary

- stress balance equations (e.g. SSA or Stokes) determine velocity from geometry and boundary conditions
- for ice the equations are nonlinear so iteration is necessary
  - at each iteration a sparse matrix "inner" problem is solved
  - give it to a linear algebra package; don't write it yourself
- when geometry and boundary stresses are *known*, many modern stress balance solvers do well
  - e.g. ice shelves
  - model shallowness often is *not* the issue
  - see: Aschwanden et al. (2016), *Complex Greenland outlet glacier flow captured*

Outline

best practices for numerical modeling

- learn a version-control system `git`
- put your project on a public host from the beginning `github.com`
  - ignore your secretive advisor
- modularize your codes
- test the parts
  - make testing easily repeatable `make test`
- if you are stuck:
  - write documentation for what you have `pdflatex`
  - go find exact solutions and literature about submodels `google`

- this thing

does not know what you *intend*, namely your science/modelling goals

  - it does “know” your discrete equations, the program you wrote

- you *can* aspire for your computer to “know” your continuum model, the one that you write as PDEs
  - achieve this through verification with exact solutions