Assignment #1

Due Wednesday, 22 January 2025, at the start of class

Please read sections 1.1–1.4, 2.1–2.4, and Appendix A from the textbook.[1] The Problems on this assignment are designed to encourage review of important prerequisite topics. In fact, please find three prerequisite textbooks, or their online equivalents:

- Find a **calculus** book.
- Find an introductory textbook on **ordinary differential equations** (ODEs).
- Find an introductory textbook on **linear algebra**.

You will need these references throughout the semester.

For this Assignment, please review these mathematical topics:

- Taylor's theorem with the remainder formula. This may be best explained by an introductory numerical analysis textbook.
- Solution of linear, homogeneous, and constant-coefficient ODEs.
- Euler's method for approximately solving first-order systems of ODEs, from an initial value.

This Assignment also requires that you get started in the programming language of your choice. Recommended: MATLAB, OCTAVE, PYTHON, or JULIA.

Problem P1. Calculate $129^{1/7}$ to within 10^{-5} of the exact value *without* any computing machinery except a pencil or pen. Prove that your answer has this accuracy. (*Hint: Taylor on $f(x) = x^{1/7}$, with a carefully-chosen base point. Note 10^{-5} is the maximum absolute error. Feel free to use a computer to* check *your by-hand value, but otherwise this is not a computer question.*)

Problem P2. Assume f' is continuous. Derive the remainder formula

$$\int_0^a f(x)\,dx = af(0) + \frac{1}{2}a^2 f'(\nu) \tag{1}$$

for some (unknown) ν between zero and a. (*Hint: Start by showing $f(x) = f(0) + f'(\xi)x$ where $\xi = \xi(x)$ is some number between 0 and x. Then integrate.*) Use at least two sentences to explain the meaning of (1) as an approximation to the integral. That is, answer the question "What properties of $f(x)$ or a make the left-endpoint rule $\int_0^a f(x)\,dx \approx af(0)$ more or less accurate?"

Problem P3. Work at the command line, in the programming language of your choice, to compute a finite sum approximation to

$$\sum_{n=1}^{\infty} \frac{\arctan(\cos n)}{n^3+1}.$$

[1] R. J. LeVeque, *Finite Difference Methods for Ordinary and Partial Diff. Eqns.*, SIAM Press 2007

Compute the partial sums for $N = 10$ and $N = 100$ terms. Turn your command-line work into a function `mysum(N)`, defined in a file `mysum.m` (or similar), and check that it yields the same numbers. Turn in both the command line session and the code. (*These can be very brief.*) How close do you think the $N = 100$ partial sum is to the infinite sum?

Problem P4. *Please do* not *waste paper by turning in tables of numbers unless specifically asked! Here, check that you have the same numbers as in Table 1.1, but don't turn in a table.*

Reproduce Figure 1.2 on page 6 of the textbook. In particular, write a code which generates the data in Table 1.1 by doing the calculations described by Example 1.1, with $u(x) = \sin x$ and $\bar{x} = 1$. Then generate the Figure, which has logarithmic scaling on both axes. Make sure to label the axes as shown, and also put in the labels "D_0" etc. in approximately the right locations. (*Use* `text` *in Matlab.*) The data should be shown as markers, but the lines between can be generated however is convenient. Turn in both the code and the figure you generate.

Problem P5. *Observe that you are making a* prediction *of $y(t)$ at $t = 4$, given initial data and a precise "law" about how $y(t)$ evolves in time, namely the differential equation.*

Solve, by hand, the ODE initial value problem

$$y'' + 4y' - 5y = 0, \quad y(2) = 0, \quad y'(2) = -1, \tag{2}$$

for the solution $y(t)$. Then find $y(4)$. Give a reasonable by-hand sketch on t, y axes which shows the initial values, the solution, and the value $y(4)$.

Problem P6. Using Euler's method for approximately solving ODEs, write your own program to solve initial value problem (2) to find $y(4)$. A first step is to convert the second-order ODE into a system of two first-order ODEs. Use a few different step sizes, decreasing as needed, so that you get apparent three-digit accuracy. (*You may use a black-box ODE solver as a 3rd method to* check *your work, but don't turn this in.*)