Assignment #2

Due Friday, 31 January 2025, at the start of class

Please read sections 1.1–1.4, 2.1–2.10, and Appendix A from the textbook.[1]

Problem P7. Solve, by hand, the ODE boundary value problem

$$y'' + 2y' - 3y = 0, \quad y(0) = \alpha, \quad y(\tau) = \beta,$$

for the solution $y(t)$. Note that α, β, τ are the data of the problem, so the solution will have these parameters in it.

Problem P8. For λ a real number, solve, by hand, the ODE boundary value problem

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y(1) = 0,$$

for the solution $y(t)$. For each λ, find *all* solutions. There will be some values of λ for which there are multiple solutions $y(t)$; identify all of these exceptional λ values.

Problem P9. Suppose this table of data gives samples of a function $Z(h)$:

h	1.0	0.5	0.1	0.05	0.01	0.005
$Z(h)$	56.859	21.694	1.1081	1.1101	0.096909	0.011051

This data may be fitted (regression) by a function $f(h) = Ch^p$ for some values C and p. Find the values of C and p by fitting a straight line to the *logarithms* of the data; in MATLAB you may use `polyfit`. Then graph the data *and* show the fitted line on the same axes, using MATLAB's `loglog` or similar.

Problem P10. *Before doing this exercise, read and understand Example 1.2 in Section 1.2.*

(a) Use the method of undetermined coefficients to set up a 5×5 linear system that determines the fourth-order centered finite difference approximation to $u''(x)$ based on 5 equally-spaced points, namely

$$u''(x) = c_{-2}u(x-2h) + c_{-1}u(x-h) + c_0u(x) + c_1u(x+h) + c_2u(x+2h) + O(h^4).$$

In particular, expand $u(x-2h), u(x-h), u(x+h), u(x+2h)$ in Taylor series. Then collect terms on the right side of the above equation to generate a square linear system $A\,c = g$ in unknowns $c_{-2}, c_{-1}, c_0, c_1, c_2$. This system will have numerical (constant) entries in the matrix A, but the entries of vector g will depend on h.

(b) Use MATLAB/etc. to solve the linear system from part **a)**. A recommended way to do this is to use $h = 1$ in the vector g and solve the system numerically using the "backslash" method. Then write down the answer in a form like (1.11), inserting the correct power of h. Use $h = 0.5$ to confirm that you have captured the correct powers. (*Feel free to use LeVeque's* `fdstencil` *to check your work, but it is not required.*)

[1] R. J. LeVeque, *Finite Difference Methods for Ordinary and Partial Diff. Eqns.*, SIAM Press 2007

Problem P11. *In Section 2.4 the textbook uses finite differences to convert the boundary value problem $u''(x) = f(x)$, $u(0) = \alpha$, $u(1) = \beta$ into matrix equation $AU = F$, with A and F given in (2.10). For any integer $m \geq 1$, this method is based on a grid with $h = 1/(m+1)$ and $x_j = jh$. There are m unknowns $U_1, U_2, \dots, U_m$, located at the interior nodes $x_1, \dots, x_m$. Note that finite difference approximation D^2 from equation (1.13) is used for the u'' term. This problem asks you to generalize this scheme.*

Assume q, x_L, x_R are real numbers with $x_L < x_R$. Similar to the method in Section 2.4, create a finite difference approximation for the problem

$$u''(x) + q\,u(x) = f(x), \quad u(x_L) = \alpha, \quad u(x_R) = \beta.$$

Use the same approximation D^2 for u''. Use the same grid indexing with m unknowns $U_1, \dots, U_m$, and give the new formulas for x_j and the mesh width h. State, in detail, A and F in $AU = F$. (*Note that entries of A will depend on q as well as h.*) Check that, by choosing appropriate constants, you can reproduce formulas (2.10).