

Assignment #8

Due Wednesday, 19 April 2023, at the start of class

Please read textbook¹ Chapters 8, 9, and 10. This assignment is mostly Chapter 9.

Problem P34. Consider the heat equation $u_t = D u_{xx}$ for $D > 0$ constant, $x \in [0, 1]$, and Dirichlet boundary conditions $u(t, 0) = 0$ and $u(t, 1) = 0$. Suppose we have initial condition $u(0, x) = \sin(5\pi x)$.

a) Confirm that

$$u(t, x) = e^{-25\pi^2 Dt} \sin(5\pi x)$$

is an exact solution. (*It is the exact solution, but you do not have to show this.*)

b) Implement the backward Euler (BE) method, as applied to MOL ODE system (9.10), to solve this heat equation problem. Specifically, use diffusivity $D = 1/20$ and final time $t_f = 0.1$. Note that you do not need to use Newton's method to solve the implicit equation, a linear system, but you should use sparse storage and MATLAB's backslash or similar. Feel free to reuse or modify code from **P33** on Assignment #7.

c) Suppose we set $k = h$ for the "refinement path". (*Of course, for BE stability does not constrain our refinement path.*) What do you expect for the convergence rate $O(h^p)$? Then measure it by using the exact solution from **a**), at the final time, and the infinity norm $\|\cdot\|_\infty$, and $h = 0.02, 0.01, 0.005, 0.002, 0.001, 0.0005$. Make a log-log convergence plot of h versus the error.

d) Repeat parts **b**) and **c**) but with the trapezoidal rule, i.e. Crank-Nicolson (CN) equation (9.6). Use the same refinement path. Add the result to the same plot; turn in codes for BE and CN, but only one well-designed log-log convergence plot.

Problem P35. Consider the following scheme, which applies centered differences to both sides of the heat equation $u_t = u_{xx}$:

$$U_j^{n+2} = U_j^n + \frac{2k}{h^2}(U_{j-1}^{n+1} - 2U_j^{n+1} + U_{j+1}^{n+1}).$$

This is called the *Richardson* method. (*L. F. Richardson did many things more important than, and successful than, inventing this scheme!*)

a) Compute the truncation error to determine the order of accuracy of this method, in space and time. The answer will be in form $\tau(t, x) = O(k^p + h^q)$; determine p, q .

b) Derive the method by applying the midpoint ODE method, equation (5.23), to the MOL ODE system (9.10). Also, find the region of absolute stability of the midpoint method (5.23); it is in the textbook. Is the method likely to generate reasonable results? Why or why not?

¹R. J. LeVeque, *Finite Difference Methods for Ordinary and Partial Diff. Eqns.*, SIAM Press 2007

- c) Implement the method on the same problem as in **P34**. Was this successful?
- d) Do a von Neumann stability analysis of this scheme. What do you conclude?

Problem P36. Consider the Jacobi iteration² for the linear system $A\mathbf{u} = \mathbf{b}$ arising from a centered FD approximation of the boundary value problem $u''(x) = f(x)$. Here A is the matrix in equation (2.10); there is *no* need to rederive it. Show that this iteration can be interpreted as forward Euler time-stepping applied to a heat equation MOL system like (9.10), but *with time step* $k = \frac{1}{2}h^2$. Specifically, the MOL equations are those arising from a centered *spatial* FD discretization of $u_t(t, x) = u_{xx}(t, x) - f(x)$.

Comment 1. No implementations or coding is needed for this problem.

Comment 2. The solution of the time-dependent heat equation decays to the steady state solution, that is, to the solution of $u'' = f$. (This assumes steady boundary values.) However, while marching to steady state with an explicit method is one way to solve the steady-state boundary value problem, it is a very inefficient way. (Not recommended!) Instead, just focus on solving the system $A\mathbf{u} = \mathbf{b}$ quickly.

²The Jacobi iteration, equation (4.4) in the textbook, was covered in the “Classical iterative methods” slides bueler.github.io/nade/assets/slides/iterative.pdf