

Von Neumann analysis: plug and chug

Section 9.6 of the textbook (R. J. LeVeque, 2007) presents von Neumann analysis without emphasizing how people actually do the analysis. This worksheet exercises the standard style.¹

Example. FTCS on heat equation. It is easiest to explain the idea relative to an example. Suppose we apply the FTCS scheme to the heat equation $u_t = Du_{xx}$ with constant diffusivity $D > 0$:

$$(1) \quad \frac{U_j^{n+1} - U_j^n}{k} = D \frac{U_{j-1}^n - 2U_j^n + U_{j+1}^n}{h^2}$$

To find what time steps $k > 0$ would be stable for a given spacing $h > 0$, von Neumann substituted

$$(2) \quad U_j^n = g(\xi)^n e^{ijh\xi}$$

into scheme (1), where $\xi \in \mathbb{R}$ is the wave number for the spatial wave $e^{ijh\xi}$, in which $i = \sqrt{-1} \in \mathbb{C}$ (as usual). The scalar function $g(\xi)$ is called the amplification factor of the scheme.

The spatial wave is complex, but it really is a wave. In fact, for the interval $0 \leq x \leq 1$ the grid is $x_j = jh$ and thus

$$e^{ijh\xi} = e^{i\xi x_j} = \cos(\xi x_j) + i \sin(\xi x_j).$$

Now we want to find $g(\xi)$. To compute it, substitute form (2) into scheme (1). Indices “ $n+1$,” “ $j-1$,” and “ $j+1$ ” will turn into powers. Then use the properties of the exponential. After simplification and trigonometric identities—do the details in Exercise 1—we get

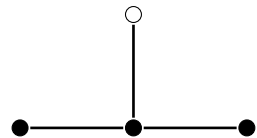
$$g(\xi) = 1 - \frac{4Dk}{h^2} \sin^2\left(\frac{\xi h}{2}\right).$$

Absolute stability $|U_j^{n+1}| \leq |U_j^n|$ corresponds to $|g(\xi)| \leq 1$ for all $\xi \in \mathbb{R}$. For this scheme² we get the condition $k \leq \frac{h^2}{2D}$, which tells us that the time step k must be very small if the spacing h is small.

Exercise 1. FTCS on heat equation. Label the stencil. Then fill in the above details.

$$[u_t = Du_{xx} \Rightarrow (1)]$$

put $U_j^n = g(\xi)^n e^{ijh\xi}$ into (1):



$$\frac{g(\xi)^{n+1} e^{ijh\xi} - g(\xi)^n e^{ijh\xi}}{k} = D \frac{g(\xi)^n e^{i(j-1)h\xi} - 2g(\xi)^n e^{ijh\xi} + g(\xi)^n e^{i(j+1)h\xi}}{h^2}$$

now cancel $g(\xi)$ & $e^{ijh\xi}$:

$$\frac{g(\xi) - 1}{k} = D \frac{e^{-ih\xi} - 2 + e^{ih\xi}}{h^2} \Rightarrow g(\xi) = 1 + R(e^{-ih\xi} + e^{ih\xi} - 2)$$

$$= 1 + R(2\cos(h\xi) - 2)$$

$$= 1 - 2R(1 - \cos(h\xi))$$

¹The textbook eventually says it clearly in section 10.5.

²The same condition is derived via MOL in section 9.3, and then by von Neumann analysis in section 9.6.

$$|g(\xi)| \leq 1 \Leftrightarrow 1 - 4R \sin^2(h\xi/2) \geq -1$$

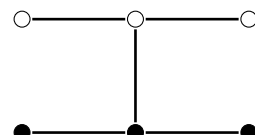
$$\Leftrightarrow 4R \sin^2(h\xi/2) \leq 2 \Leftrightarrow k \leq \frac{h^2}{2D}$$

for all ξ

I did not give you enough space! Only highlights are shown.

Exercise 2. Crank-Nicolson on heat equation. Label the stencil. State the scheme. Do the analysis.

put $U_j^n = g(\xi)^n e^{i j h \xi}$ into CN scheme and cancel:



$$\frac{g(\xi) - 1}{k} = \frac{D}{2h^2} (g(\xi) e^{-i h \xi} - 2g(\xi) + g(\xi) e^{i h \xi} + e^{-i h \xi} - 2 + e^{i h \xi})$$

rearrange, use $\cos \theta = \frac{1}{2}(e^{-i\theta} + e^{i\theta})$, use $R = Dk/h^2$:

$$(-R \cos(h\xi) + 1) g(\xi) = 1 + R \cos(h\xi) - R$$

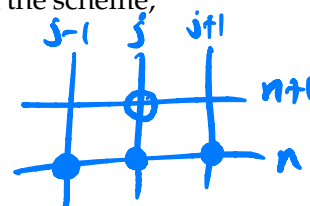
rearrange use $\sin^2 \theta = \frac{1}{2}(1 - \cos(2\theta))$:

$$g(\xi) = \frac{-2R \sin^2(h\xi/2)}{1 + 2R \sin^2(h\xi/2)} \Rightarrow |g(\xi)| \leq 1 \quad \text{for all } \xi$$

stable for any $k > 0$

Exercise 3. FTCS on the advection equation $u_t + au_x = 0$ with a constant. Write down the scheme, draw and label a stencil, and do the analysis.

scheme:
$$\frac{U_j^{n+1} - U_j^n}{k} + a \frac{U_{j+1}^n - U_{j-1}^n}{2h} = 0$$



put $U_j^n = g(\xi)^n e^{i j h \xi}$ into scheme, cancel, use $r = ak/h$:

$$g(\xi) - 1 + \frac{1}{2} r (e^{i h \xi} - e^{-i h \xi}) = 0$$

use $\sin \theta = (e^{i\theta} - e^{-i\theta})/(2i)$, rearrange:

$$g(\xi) = 1 - i r \sin(h\xi)$$

magnitude: $|g(\xi)| = \sqrt{1 + r^2 \sin^2(h\xi)}$

$|g(\xi)| > 1$ for $k > 0, \xi \neq 0$

not stable for any k

Exercise 4. CTCS (leapfrog) on advection equation. Again.

scheme:
$$\frac{U_j^{n+1} - U_j^{n-1}}{2k} + a \frac{U_{j+1}^n - U_{j-1}^n}{2h} = 0$$

put $U_j^n = \dots$ into scheme, cancel, use $r = ak/h$:

$$g(\xi) - g(\xi)^{-1} + r (e^{i h \xi} - e^{-i h \xi}) = 0$$

use $\sin \theta = (e^{i\theta} - e^{-i\theta})/(2i)$, rearrange:

$$g(\xi)^2 + 2i r \sin(h\xi) g(\xi) - 1 = 0$$

quad. formula:

$$g(\xi) = -i r \sin(h\xi) \pm \sqrt{1 - r^2 \sin^2(h\xi)}$$

if $|r| \leq 1$ then $|g(\xi)| = 1$

so

stable if $|a|k/h \leq 1$