# Assignment #9

## Due Friday, 25 April 2025, at the start of class

Please read Chapters 9 and 10 of the textbook (R. J. LeVeque, *Finite Difference Methods for Ordinary and Partial Diff. Eqns.*, SIAM Press 2007). We will not get to Chapter 11 in regular lecture or homework, but it could be beneficial to browse it for insight into your project.

**Problem P36.** Consider the advection equation $u_t + au_x = 0$ for $u(t,x)$. Suppose $a > 0$, and consider (10.21), the *forward time, first-order upwinding* scheme:[1]

$$U_j^{n+1} = U_j^n - \frac{ak}{h}(U_j^n - U_{j-1}^n).$$

**(a)** Derive the scheme by applying first-order finite differences to the advection equation. Draw and label the stencil.

**(b)** Implement this method for $x \in [0,1]$ and $a = 1$. Assume as a boundary condition that $u(t,0) = 0$; no other boundary condition will be needed. For the initial condition, let

$$u(x,0) = \begin{cases} 1, & 0.1 < x < 0.3 \\ 0, & \text{otherwise} \end{cases}$$

The scheme is only conditionally stable, subject to the CFL condition $|a|k/h \le 1$. Therefore, given user input of $m$ or $h$, choose the time step from this condition. *However*, please do not choose the very special case $k = h/|a|$, which will cause the hard-to-interpret weird behavior of getting the exact solution! Instead, use 80%, or so, of the maximum allowed time step. (*The solution to this part is the code itself.*)

**(c)** The exact solution of the problem in **(b)** is easily found by thinking about how advection works. Sketch the exact solution at $t = 0.5$ and $t = 1$. Then show, for $h = 0.1$, the numerical solution from your code in **(b)** at the same times. Now add the result from $h = 0.01$. In a couple of sentences, describe the character of the convergence you are (*or should be*) seeing.

**(d)** Describe briefly, showing an easy code modification if you wish, how you would modify the code from **(b)** to handle the more general advection equation $u_t + a(t,x)u_x = 0$, where the continuous function $a(t,x)$ is real-valued but can have any sign and magnitude.

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[1]Commonly just "first-order upwinding".

**Problem P37.** Consider (10.13) for solving the advection equation $u_t + au_x = 0$, for $u(t, x)$, where $a \in \mathbb{R}$ is a constant of either sign:

$$U_j^{n+1} = U_j^{n-1} - \frac{ak}{h}(U_{j+1}^n - U_{j-1}^n).$$

This applies centered differences to all derivatives; it is the *leapfrog* method.[2]

**(a)** Compute the local truncation error $\tau(t, x)$ of this method, and find its order of accuracy; that is, determine $p, q$ in $\tau(t, x) = O(k^p + h^q)$.

**(b)** Apply a von Neumann analysis. (*Hint. See the worksheet for Wednesday 4/23.*)

**(c)** State the MOL ODE system $U(t)' = AU(t)$ from which the above method comes, assuming periodic boundary conditions on the interval $x \in [0, 1]$, giving details for $A$. What are the eigenvalues of $A$? Derive the method by applying the midpoint ODE method to it. From the stability region of the midpoint method, explain what is understood about the stability of this PDE method. (*Hint. Examine Chapter 10. You may extract most of this answer from the book.*)

**(d)** Implement this leapfrog method on the following problem: $x \in [0, 1]$, $a = -0.5$, $u(0, x) = \sin(8\pi x)$. Assume periodic boundary conditions. To make the implementation work you will have to compute the first step by some other scheme; describe and justify what you do. (*The solution to this part is the code itself.*)

**(e)** Suppose the final time is $t_f = 10$. Then the exact solution in part **(d)** is $u(t_f, x) = \sin(8\pi x)$; please explain why. Then use $h = 0.1, 0.05, 0.02, 0.01, 0.005, 0.002$ and $k = h$ and show a log-log convergence plot using the infinity norm for the error. What $O(h^p)$ do you expect for the rate of convergence, and what do you measure, and what is the right data to measure?

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[2]Note that Lax-Wendroff is generally recommended as a better scheme for advection than leapfrog, supposing one wants a simple and linear scheme formula. However if you are willing to use serious and modern methods then the story changes. What you actually do is go back to first-order upwinding and modify it with a higher-order, nonlinear flux-limiting or slope-limiting scheme, often from a finite-volume frame of mind.