SOLUTIONS

Math 615 NADE (Bueler) 8 April 2023 *Nothing to turn in!*

Von Neumann analysis: plug and chug

Section 9.6 of the textbook (R. J. LeVeque, 2007) presents von Neumann analysis without emphasizing how people actually do *the analysis. This worksheet exercises the standard style.*[1]

Example. FTCS on heat equation. It is easiest to explain the idea relative to an example. Suppose we apply the FTCS scheme to the heat equation $u_t = Du_{xx}$ with constant diffusivity $D > 0$:

(1)
$$\frac{U_j^{n+1} - U_j^n}{k} = D\frac{U_{j-1}^n - 2U_j^n + U_{j+1}^n}{h^2}$$

To find what time steps $k > 0$ would be stable for a given spacing $h > 0$, von Neumann substituted

(2)
$$U_j^n = g(\xi)^n e^{ijh\xi}$$

into scheme (1), where $\xi \in \mathbb{R}$ is the *wave number* for the spatial wave $e^{ijh\xi}$, in which $i = \sqrt{-1} \in \mathbb{C}$ (as usual). The scalar function $g(\xi)$ is called the *amplification factor* of the scheme.

The spatial wave is complex, but it really is a wave. In fact, for the interval $0 \le x \le 1$ the grid is $x_j = jh$ and thus

$$e^{ijh\xi} = e^{i\xi x_j} = \cos(\xi x_j) + i\sin(\xi x_j).$$

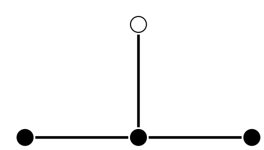
Now we want to find $g(\xi)$. To compute it, substitute form (2) into scheme (1). Indices "$n+1$," "$j-1$," and "$j+1$" will turn into powers. Then use the properties of the exponential. After simplification and trigonometric identities—do the details in Exercise 1—we get

$$g(\xi) = 1 - \frac{4Dk}{h^2}\sin^2\left(\frac{\xi h}{2}\right).$$

Absolute stability $|U_j^{n+1}| \le |U_j^n|$ corresponds to $|g(\xi)| \le 1$ for all $\xi \in \mathbb{R}$. For this scheme[2] we get the condition $k \le \dfrac{h^2}{2D}$, which tells us that the time step k must be very small if the spacing h is small.

Exercise 1. FTCS on heat equation. Label the stencil. Then fill in the above details.

[$u_t = Du_{xx} \Rightarrow (1)$]



put $U_j^n = g(\xi)^n e^{ijh\xi}$ into (1):

$$\frac{g(\xi)^{n+1}e^{ijh\xi} - g(\xi)^n e^{ijh\xi}}{k} = D\,\frac{g(\xi)^n e^{i(j-1)h\xi} - 2g(\xi)^n e^{ijh\xi} + g(\xi)^n e^{i(j+1)h\xi}}{h^2}$$

now cancel $g(\xi)$ & $e^{ijh\xi}$:

$R = \frac{Dk}{h^2}$

$$\frac{g(\xi)-1}{k} = D\,\frac{e^{-ih\xi} - 2 + e^{+ih\xi}}{h^2} \Rightarrow g(\xi) = 1 + R(e^{-ih\xi} + e^{ih\xi} - 2)$$

$$= 1 + R(2\cos(h\xi) - 2)$$

$$= 1 - 2R(1 - \cos(h\xi))$$

$$= 1 - 4R\cdot\tfrac{1}{2}(1 - \cos(h\xi))$$

$$= 1 - 4R\sin^2(h\xi)$$

for all ξ: $|g(\xi)| \le 1 \iff 1 - 4R\sin^2(h\xi) \ge -1$

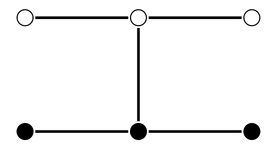
$\iff 4R\sin^2(h\xi) \le 2 \iff \boxed{k \le \frac{h^2}{2D}}$

[1] The textbook eventually says it clearly in section 10.5.

[2] The same condition is derived via MOL in section 9.3, and then by von Neumann analysis in section 9.6.

I did not give you enough space! Only highlights are shown.

Exercise 2. Crank-Nicolson on heat equation. Label the stencil. State the scheme. Do the analysis.



put $U_j^n = g(\xi)^n e^{ijh\xi}$ into CN scheme and cancel:

$$\frac{g(\xi)-1}{k} = \frac{D}{2h^2}\left(g(\xi)e^{-ih\xi} - 2g(\xi) + g(\xi)e^{ih\xi} + e^{-ih\xi} - 2 + e^{ih\xi}\right)$$

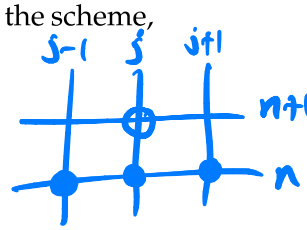
rearrange, use $\cos\theta = \frac{1}{2}(e^{-i\theta}+e^{i\theta})$, use $R = Dk/h^2$:

$$(-R\cos(h\xi)+1)\, g(\xi) = 1 + R\cos(h\xi) - R$$

rearrange use $\sin^2\theta = \frac{1}{2}(1-\cos(2\theta))$:

$$g(\xi) = \frac{-2R\sin^2(h\xi)}{1+2R\sin^2(h\xi)} \implies |g(\xi)| \le 1 \text{ for all } \xi \implies \boxed{\text{stable for any } k>0}$$

Exercise 3. FTCS on the advection equation $u_t + au_x = 0$ with a constant. Write down the scheme, draw and label a stencil, and do the analysis.



scheme: $$\frac{U_j^{n+1} - U_j^n}{k} + a\,\frac{U_{j+1}^n - U_{j-1}^n}{2h} = 0$$

put $U_j^n = g(\xi)^n e^{ijh\xi}$ into scheme, cancel, use $r = ak/h$:

$$g(\xi) - 1 + \tfrac{1}{2} r\left(e^{ih\xi} - e^{-ih\xi}\right) = 0$$

use $\sin\theta = (e^{+i\theta} - e^{-i\theta})/(2i)$, rearrange:

$$g(\xi) = 1 - ir\sin(h\xi)$$

magnitude: $$|g(\xi)| = \sqrt{1 + r^2\sin^2(h\xi)}$$

$|g(\xi)| > 1$ for $k>0$, $\xi \neq 0$

$\boxed{\text{not stable for any } k}$

Exercise 4. CTCS (leapfrog) on advection equation. Again.

scheme: $$\frac{U_j^{n+1} - U_j^{n-1}}{2k} + a\,\frac{U_{j+1}^n - U_{j-1}^n}{2h} = 0$$

put $U_j^n = \ldots$ into scheme, cancel, use $r = ak/h$:

$$g(\xi) - g(\xi)^{-1} + r\left(e^{ih\xi} - e^{-ih\xi}\right) = 0$$

use $\sin\theta = (e^{i\theta} - e^{-i\theta})/2i$, rearrange:

$$g(\xi)^2 + 2ir\sin(h\xi)\, g(\xi) - 1 = 0$$

quad. formula: $$g(\xi) = -ir\sin(h\xi) \pm \sqrt{1 - r^2\sin^2(h\xi)}$$

if $|r| \le 1$ then $|g(\xi)| = 1$

so $\boxed{\text{stable if } \frac{|a|k}{h} \le 1}$