The porous media problem: Degassing through a volcanic lava dome

Tara Shreve

Finite Elements Spring 2024

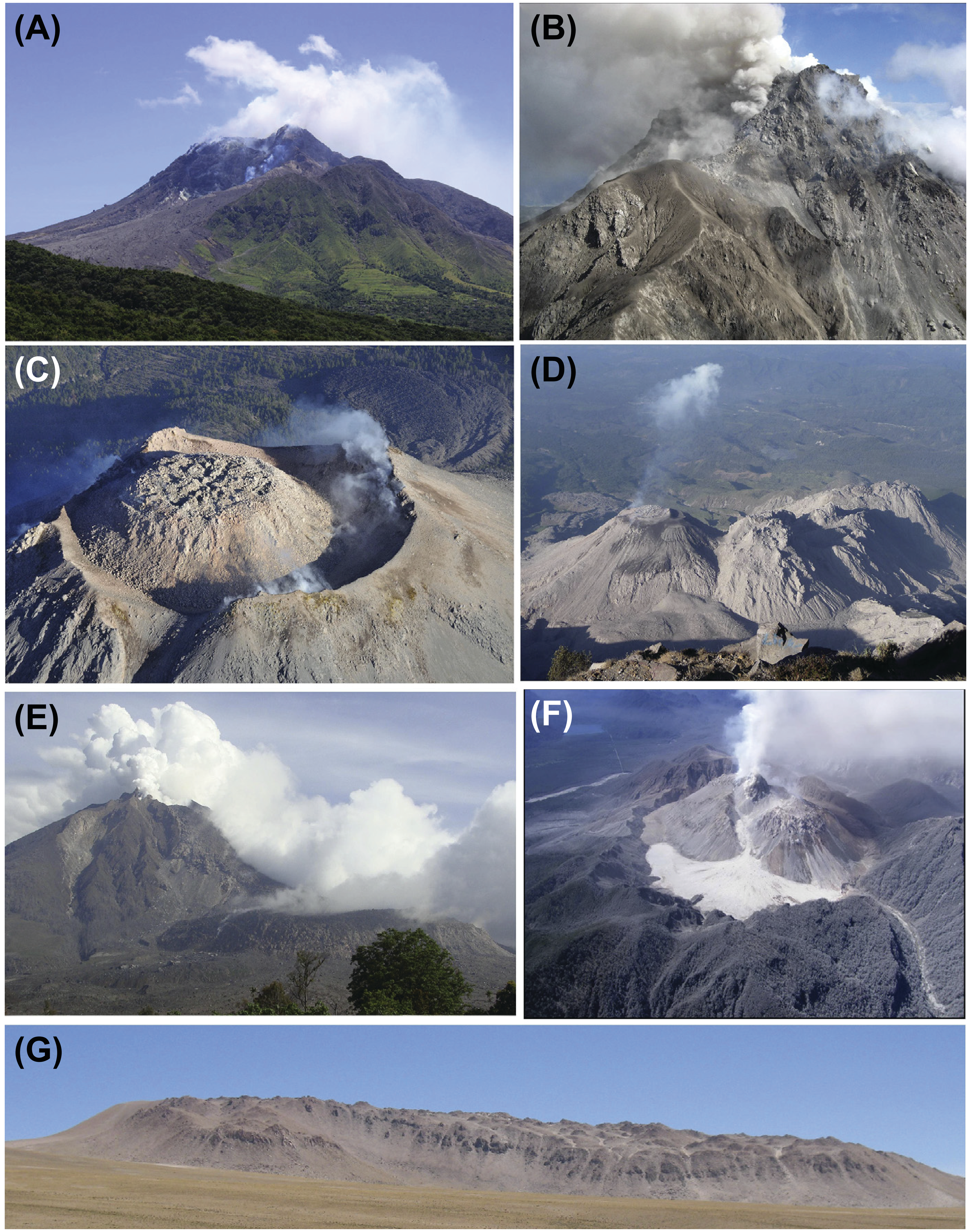
Obsidian dome, California
Photo by John Kupersmith

Outline

1. Geological context
2. Conceptual model
3. Firedrake implementation
4. Run and view code

Geological context

- Volcanic domes are made of lava that builds up around a volcanic vent
- Passive effusion punctuated by explosions or dome collapse
- There have been over 40,000 fatalities in the past 150 years associated with lava dome emplacement and collapse

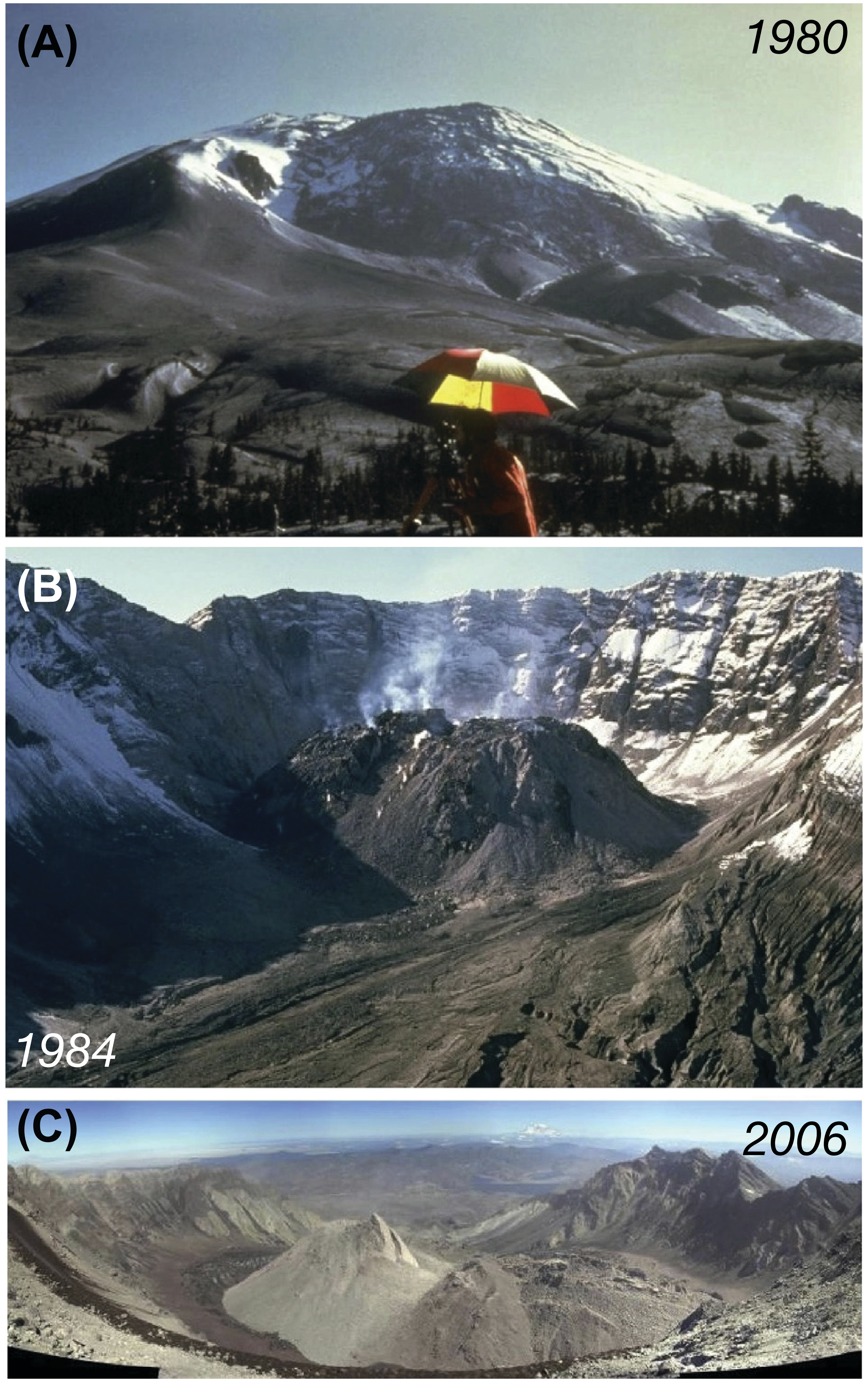


Calder et al. 2015

Geological context

- Observed dome collapses due to:
  - Gravitational loading
  - Internal gas overpressures
  - Topography-controlled
  - Intense rainfall
  - Switch in extrusion direction

Mount St. Helens domes

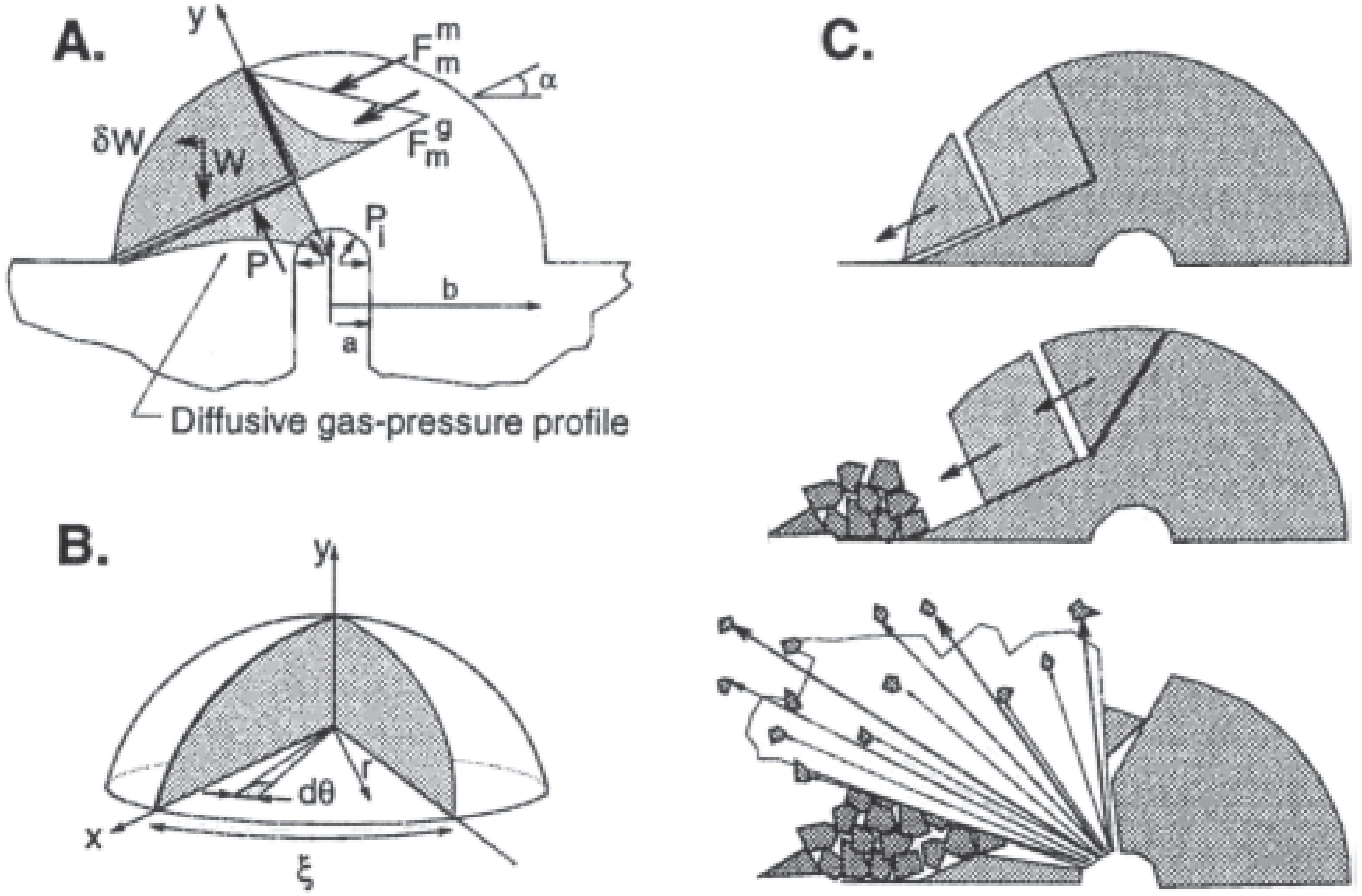


Calder et al. 2015

Geological context

- Observed dome collapses due to:
    - Gravitational loading
    - **Internal gas overpressures**
    - Topography-controlled
    - Intense rainfall
    - Switch in extrusion direction

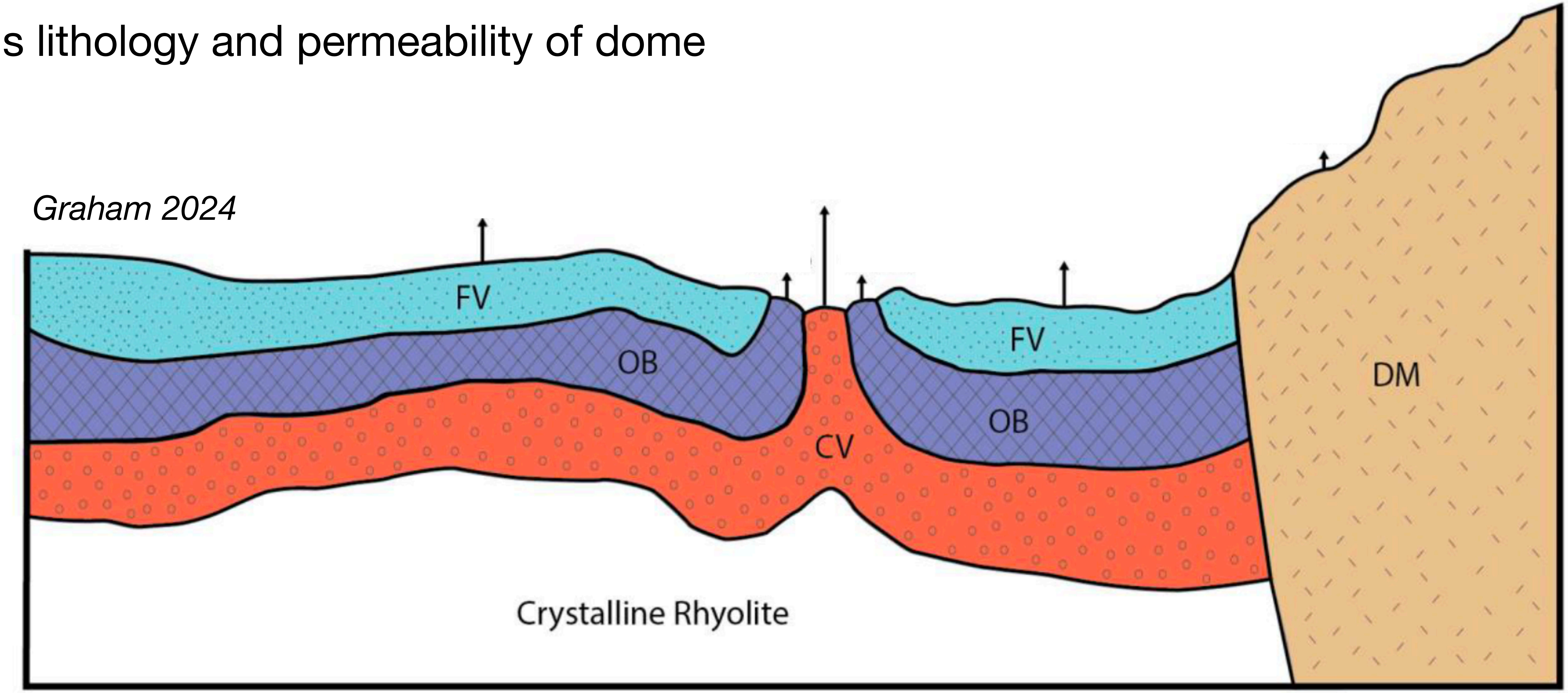
Voight and Elsworth 2000



Conceptual model

- Gas inlet at bottom of dome with perscribed pressure/density
- Gas can escape to the atmosphere with perscribed pressure/density
- Heterogeneous lithology and permeability of dome

Graham 2024

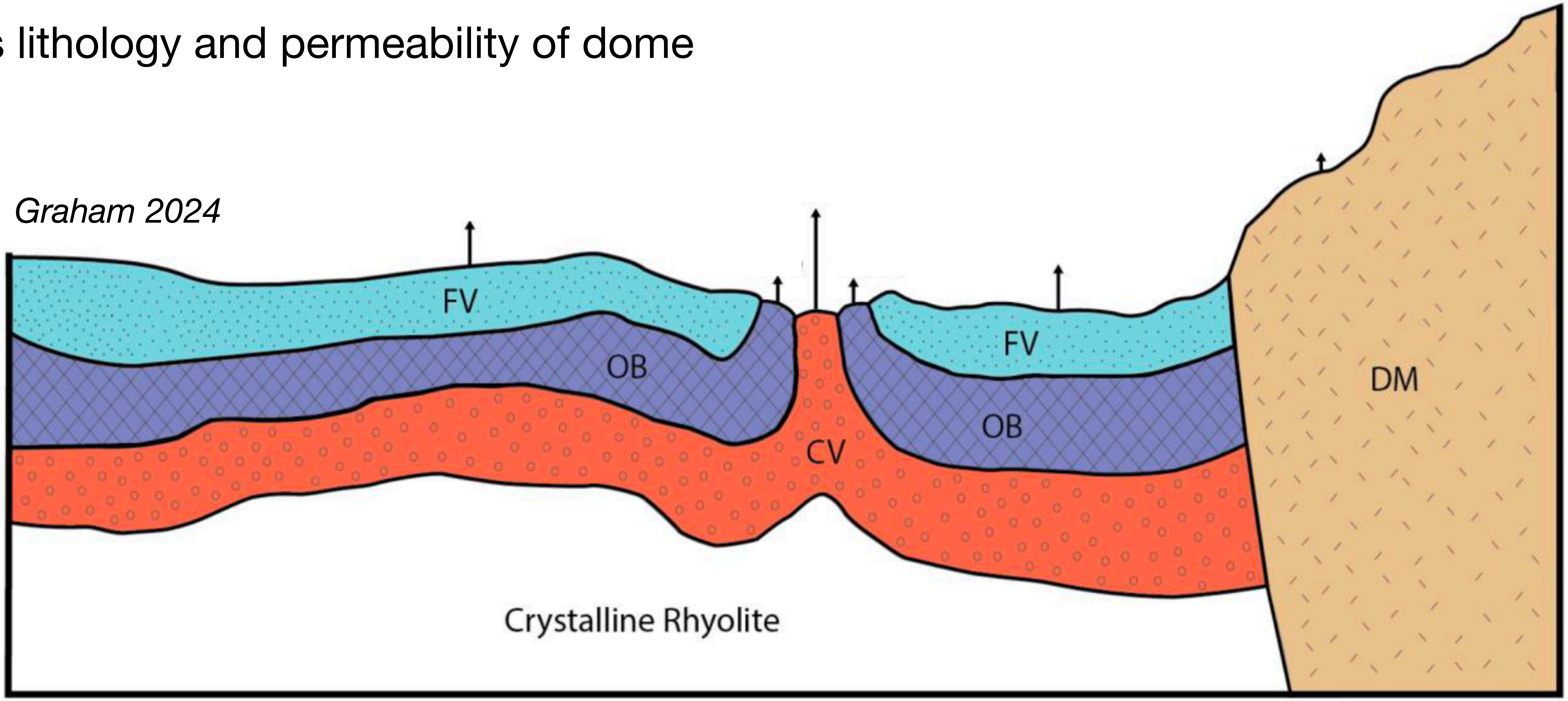


Conceptual model

- **Gas inlet at bottom of dome with perscribed pressure/density**
- **Gas can escape to the atmosphere with perscribed pressure/density**
- Heterogeneous lithology and permeability of dome

Dirichlet boundary conditions

Graham 2024

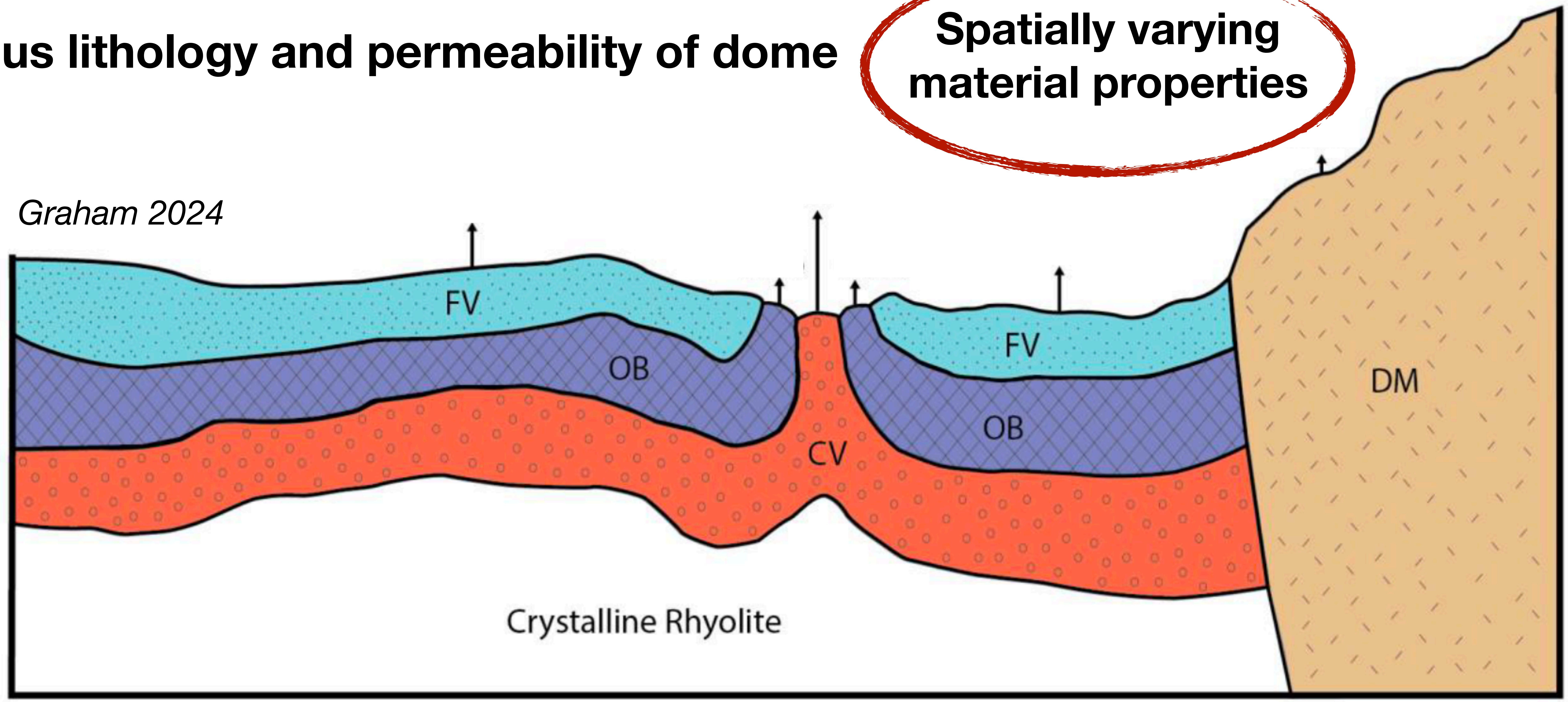


Conceptual model

- Gas inlet at bottom of dome with perscribed pressure/density
- Gas can escape to the atmosphere with perscribed pressure/density
- **Heterogeneous lithology and permeability of dome**

Spatially varying material properties

Graham 2024



Firedrake implementation

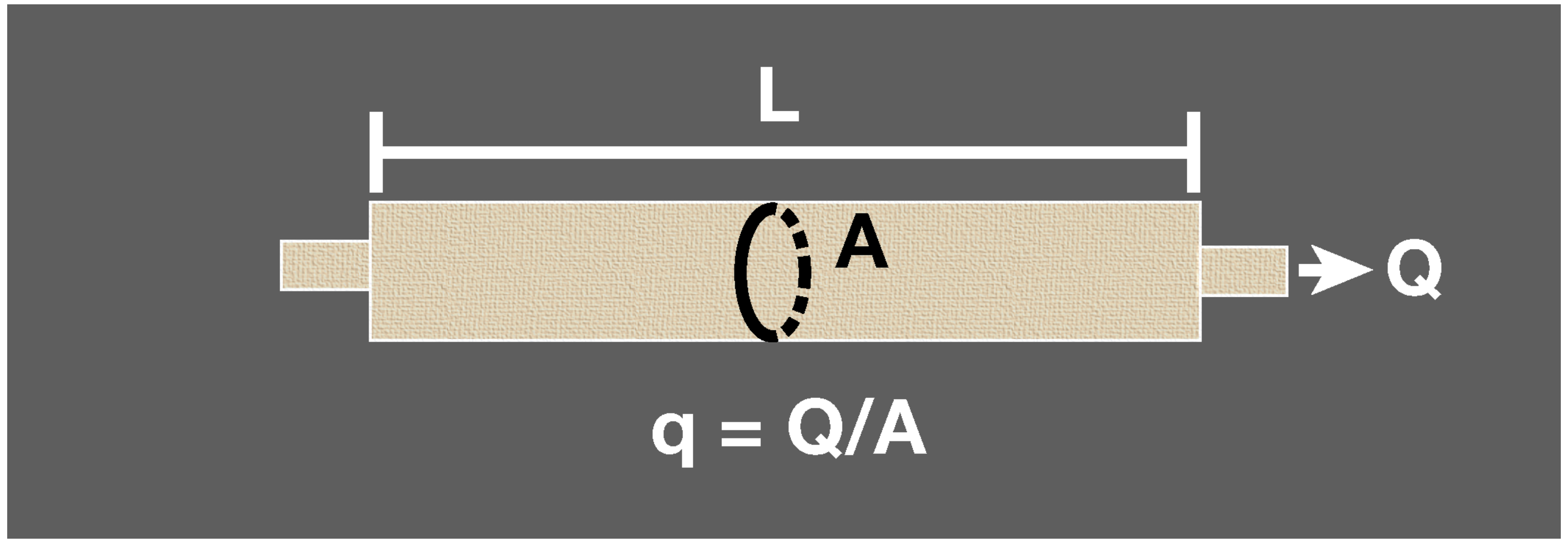
- Darcy's law approximation for **steady-state, laminar *compressible* steam flow** through porous media

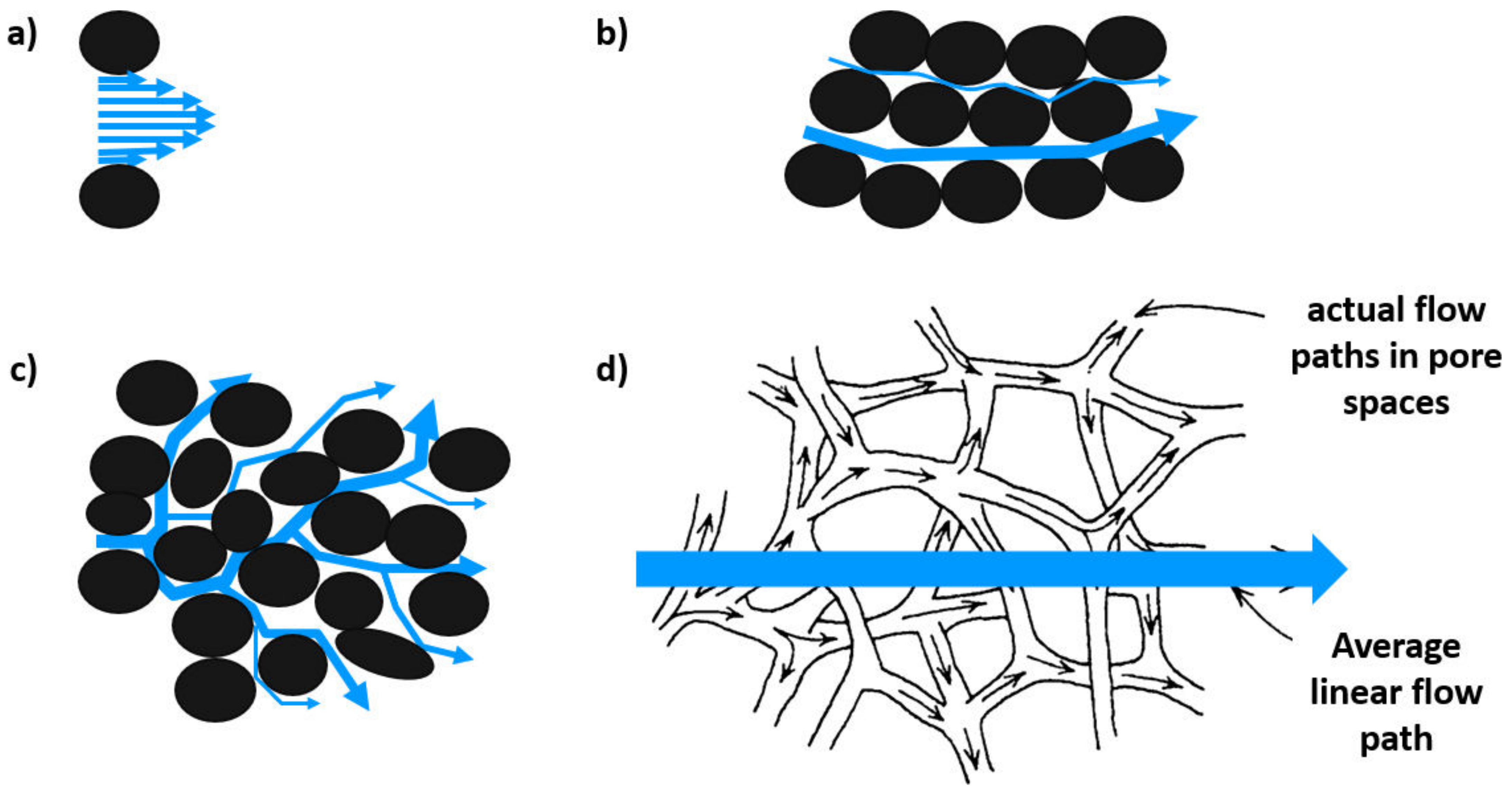
- Strong form:

$$\mathbf{q} = -\frac{\kappa}{\mu}(\nabla P + \rho g z)$$

$$\phi\frac{d\rho}{dt} + \nabla\cdot(\rho\mathbf{q}) = 0$$

$$\rho = \frac{PM}{RT}$$





After Freeze and Cherry, 1979

Firedrake implementation

- Darcy's law approximation for **steady-state, laminar *compressible* steam flow** through porous media

- Strong form:

$$\mathbf{q} = -\frac{\kappa}{\mu}(\nabla P + \rho g z)$$

$$\phi\frac{d\rho}{dt} + \nabla \cdot (\rho\mathbf{q}) = 0 \xrightarrow{\hspace{3em}} \nabla \cdot \left(-\frac{k}{\mu}\rho\nabla(c\rho + \rho g z)\right) = 0$$

Steady-state solution

$$\rho = \frac{PM}{RT}$$

Firedrake implementation

- Darcy's law approximation for **steady-state, laminar *compressible* steam flow** through porous media

- Strong form:

$$\mathbf{q} = -\frac{\kappa}{\mu}(\nabla P + \rho g z)$$

Note that we have an issue if $\rho \to 0$!

$$\phi\frac{d\rho}{dt} + \nabla\cdot(\rho\mathbf{q}) = 0 \xrightarrow{} \nabla\cdot\left(-\frac{k}{\mu}\rho\nabla(c\rho + \rho g z)\right) = 0$$

Steady-state solution

$$\rho = \frac{PM}{RT}$$

Firedrake implementation

- Darcy's law approximation for **steady-state, laminar *compressible* steam flow** through porous media

- Strong form:

$$\mathbf{q} = -\frac{\kappa}{\mu}(\nabla P + \rho g z)$$

$$\boldsymbol{\sigma} = \rho\frac{\mathbf{q}}{\phi} = -\frac{k}{\mu\phi}\rho\nabla(c\rho + \rho g z)$$

$$c = \frac{RT}{M}$$

$$\rho > 0$$

$$\phi\frac{d\rho}{dt} + \nabla\cdot(\rho\mathbf{q}) = 0 \xrightarrow{} \nabla\cdot\left(-\frac{k}{\mu}\rho\nabla(c\rho + \rho g z)\right) = 0$$

Steady-state solution

$$\rho = \frac{PM}{RT}$$

Firedrake implementation

- Darcy's law approximation for **steady-state, laminar *compressible* steam flow** through porous media

- Strong form:

$$\mathbf{q} = -\frac{\kappa}{\mu}(\nabla P + \rho g z)$$

$$\boldsymbol{\sigma} = \rho\frac{\mathbf{q}}{\phi} = -\frac{k}{\mu\phi}\rho\nabla(c\rho + \rho g z)$$

$$c = \frac{RT}{M}$$

$$\rho > 0$$

$$\phi\frac{d\rho}{dt} + \nabla\cdot(\rho\mathbf{q}) = 0 \longrightarrow \nabla\cdot\left(-\frac{k}{\mu}\rho\nabla(c\rho + \rho g z)\right) = 0$$

Steady-state solution

$$\rho = \frac{PM}{RT}$$

$$\nabla\cdot(\nabla u) = 0$$

Firedrake implementation

- Apply a conservative mixed finite element method

- Weak form derived from:

$$\int_\Omega \boldsymbol{\sigma} \cdot \boldsymbol{\omega} \mathrm{dx} + \int_\Omega \frac{k}{\mu} \rho \nabla (c\rho + \rho g z) \cdot \boldsymbol{\omega} \mathrm{dx} = 0$$

$$\forall v \in H^1_D(\Omega)$$

$$\int_\Omega (\nabla \cdot \boldsymbol{\sigma}) v \mathrm{dx} = 0$$

$$\boldsymbol{\omega} \in ??$$

- Apply the divergence theorem and we obtain:

$$\int_\Omega \boldsymbol{\sigma} \cdot \boldsymbol{\omega} \mathrm{dx} - \int_\Omega \frac{1}{\mu} (c\rho + \rho g z) \nabla \cdot (k\rho \boldsymbol{\omega}) \mathrm{dx} + \int_{\partial\Omega} \frac{k}{\mu} \rho (c\rho + \rho g z) \boldsymbol{\omega} \cdot \mathbf{n} \mathrm{ds} = 0$$

Firedrake implementation

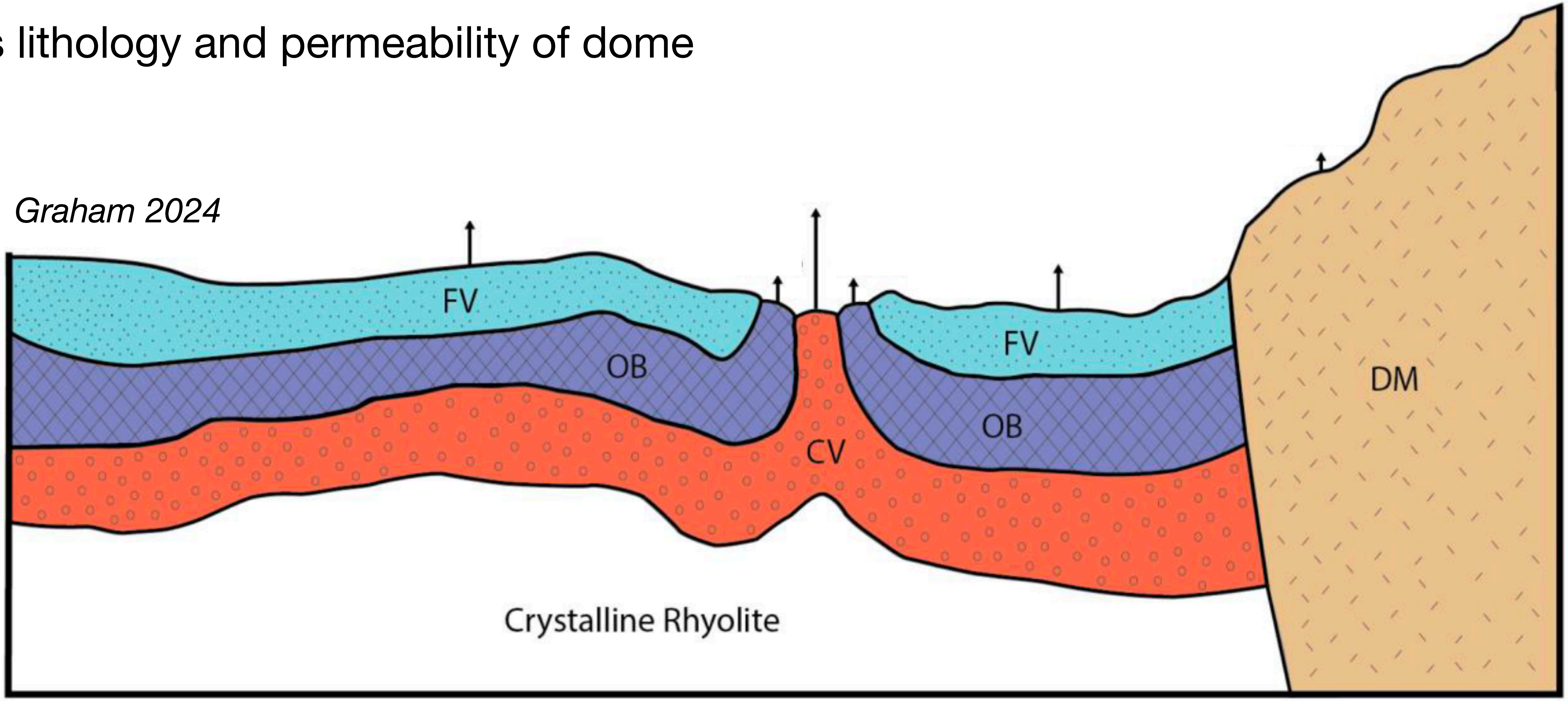
- We need appropriate function spaces for $\boldsymbol{\sigma}$ and ρ
  - Normal vectors continuous across element boundaries, discontinuous
    - Raviart-Thomas and DG spaces with quadrilateral elements
- Apply appropriate Dirichlet boundary conditions by defining the boundary integral for known conditions
- **Final weak form:**

$$\int_{\Omega} \boldsymbol{\sigma} \cdot \boldsymbol{\omega} \mathrm{dx} - \int_{\Omega} \frac{1}{\mu}(c\rho + \rho g z) \nabla \cdot (k\rho\boldsymbol{\omega}) \mathrm{dx} = - \int_{\Gamma_0} \frac{k}{\mu} \rho_0 (c\rho_0 + \rho_0 g z) \boldsymbol{\omega} \cdot \mathbf{n} \mathrm{ds}$$

- **Gas inlet at bottom of dome with perscribed pressure/density**
- **Gas can escape to the atmosphere with perscribed pressure/density**
- Heterogeneous lithology and permeability of dome

Dirichlet boundary conditions

Graham 2024



Firedrake implementation

- Use Newton’s method from the SNES component of PETSc
- Back-tracking type line search option
- Linear Newton step equation solved by sparse direct matrix method

Heterogeneous permeability solution

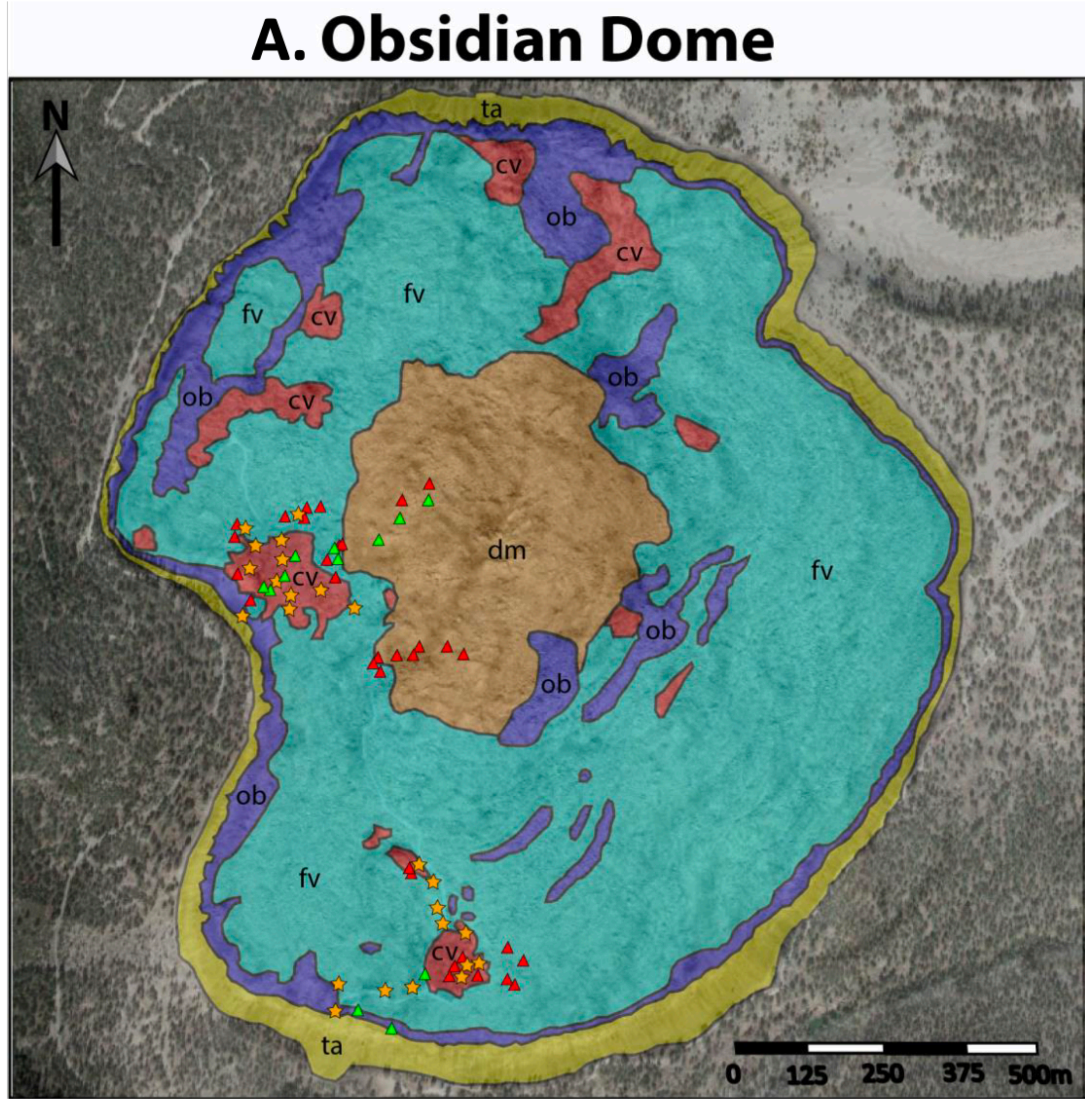
- Run and view code!

Firedrake implementation

- Compare with CG1 implementation to ensure correct implementation
- Final weak form:

$$\int_\Omega k\rho \nabla(c\rho + \rho g z) \cdot \nabla \omega = 0 \qquad \forall \omega \in H^1_D(\Omega)$$

Heterogeneous 3D solution



Graham 2024

Map Key

Finely Porphyritic
- dm Dense Microcrystalline
- cv Coarsely Vesicular
- fv Finely Vesicular
- ob Obsidian

Coarsely Porphyritic
- cp Coarsely Porphyritic

Talus
- ta Talus

Station Symbols
- Oriented Samples
- Field Stations
- Observation Stations