The porous media problem: Degassing through a volcanic lava dome

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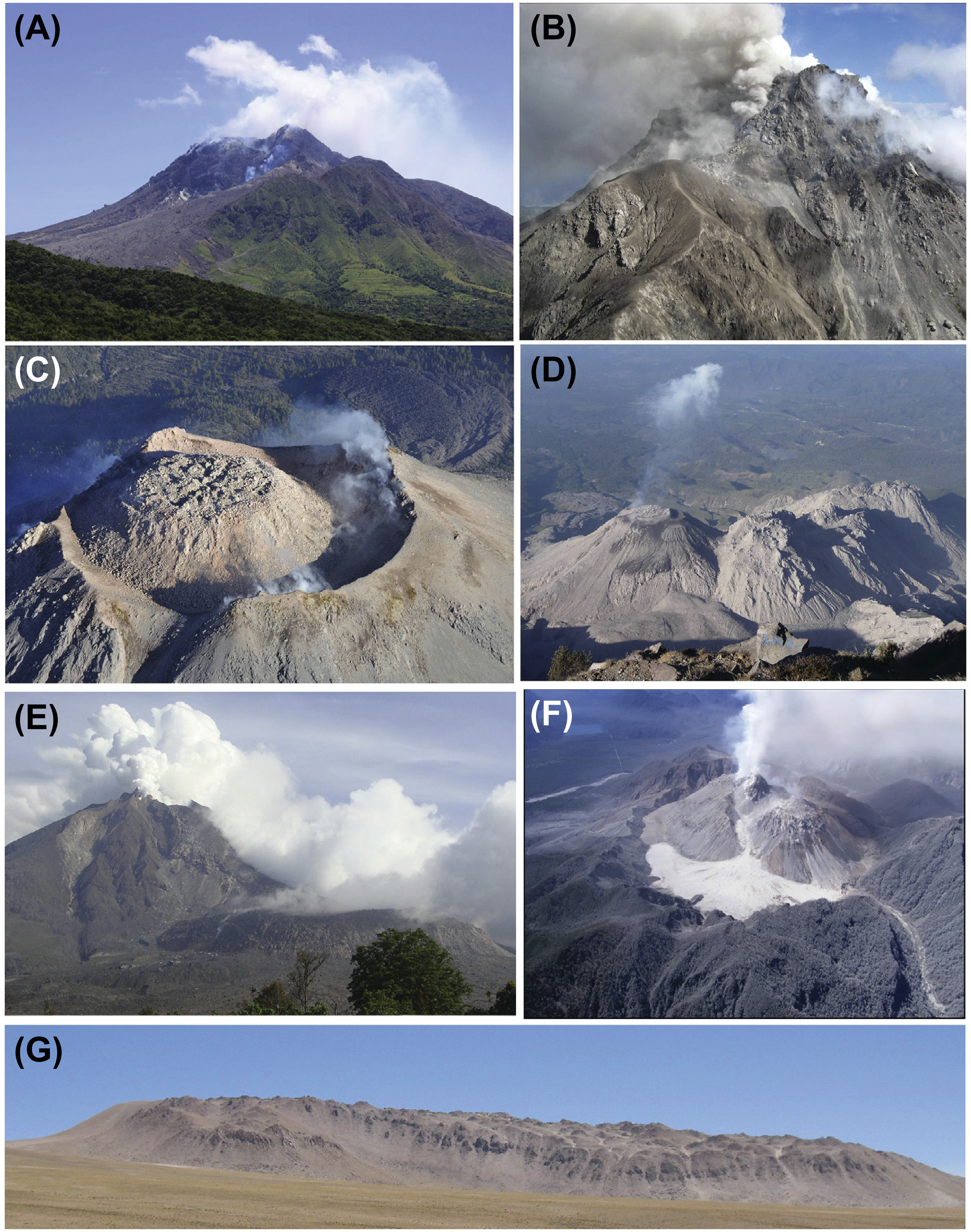
Outline

1. Geological context
2. Conceptual model
3. Firedrake implementation
4. Run and view code

Geological context

- Volcanic domes are made of lava that builds up around a volcanic vent
- Passive effusion punctuated by explosions or dome collapse
- There have been over 40,000 fatalities in the past 150 years associated with lava dome emplacement and collapse



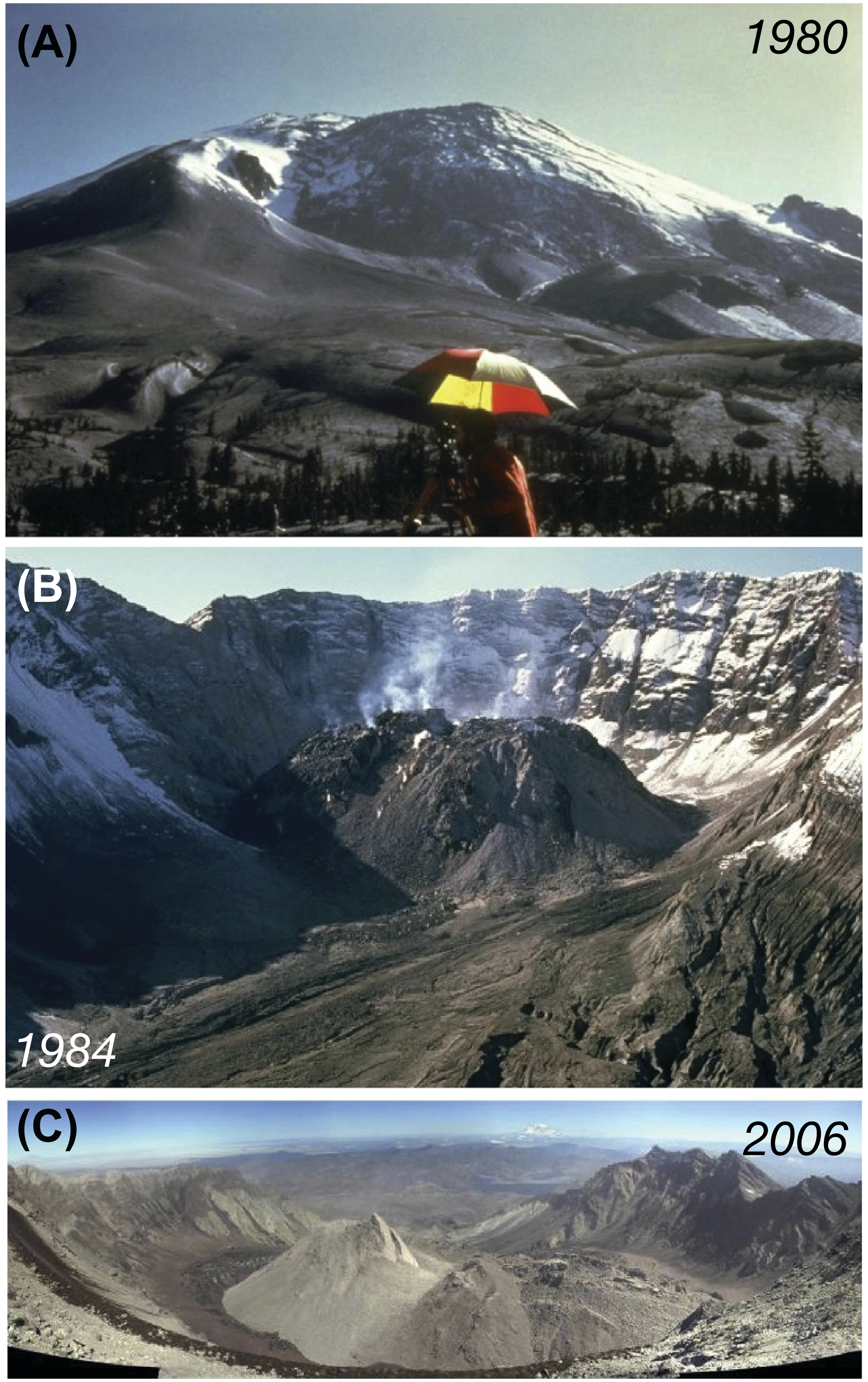
Calder et al. 2015

Geological context

- Observed dome collapses due to:
  - Gravitational loading
  - Internal gas overpressures
  - Topography-controlled
  - Intense rainfall
  - Switch in extrusion direction

https://www.britannica.com/video/82384/awe-geologists-explosion-Mount-Saint-Helens-May-18-1980

Mount St. Helens domes



Calder et al. 2015

Geological context

- Observed dome collapses due to:
    - Gravitational loading
    - **Internal gas overpressures**
    - Topography-controlled
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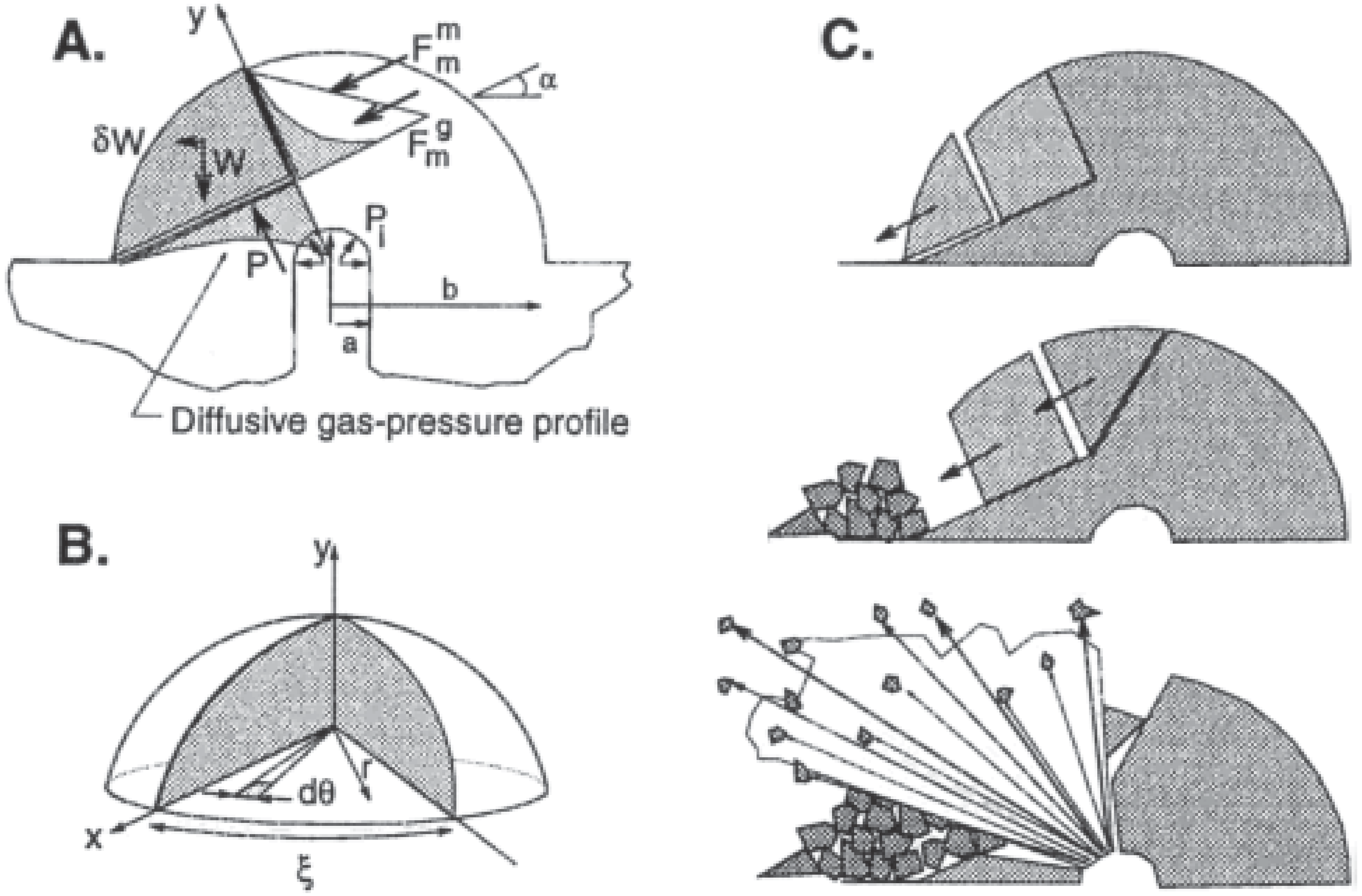
Voight and Elsworth 2000



Conceptual model

- Gas inlet at bottom of dome with perscribed pressure/density P_0
- Gas can escape to the atmosphere with perscribed pressure/density P_{atm}
- Heterogeneous lithology and permeability of dome $k(x,z)$

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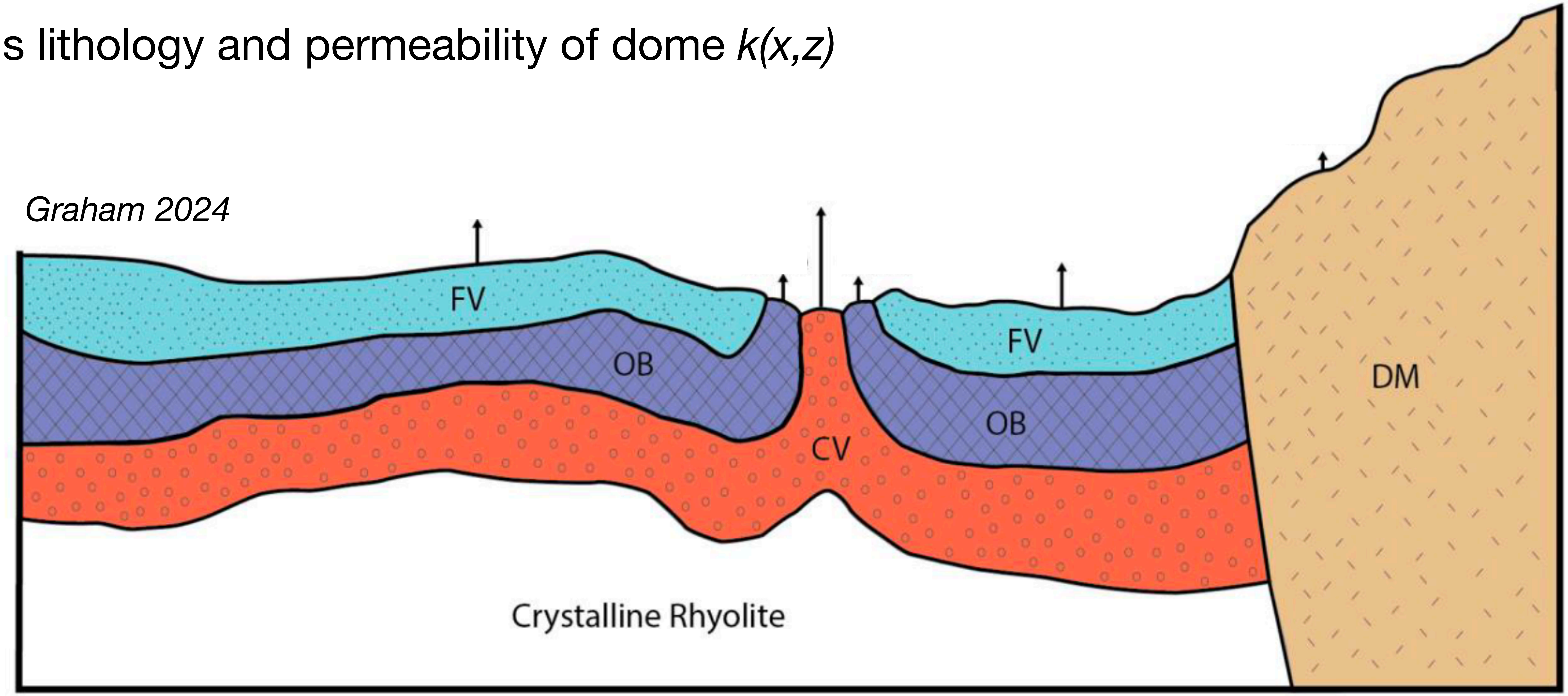

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- **Gas inlet at bottom of dome with perscribed pressure/density P_0**
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Dirichlet boundary conditions

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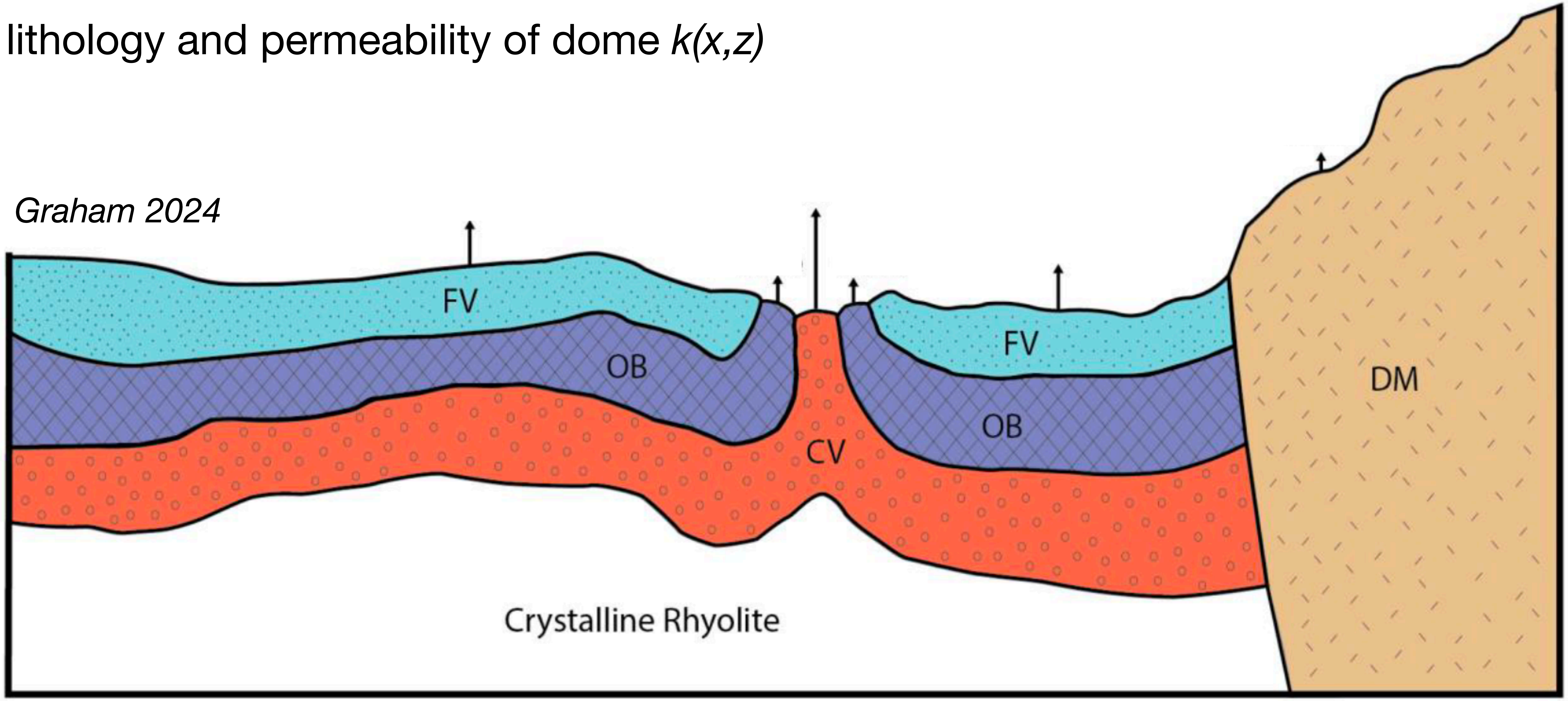

Conceptual model

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Spatially varying material properties

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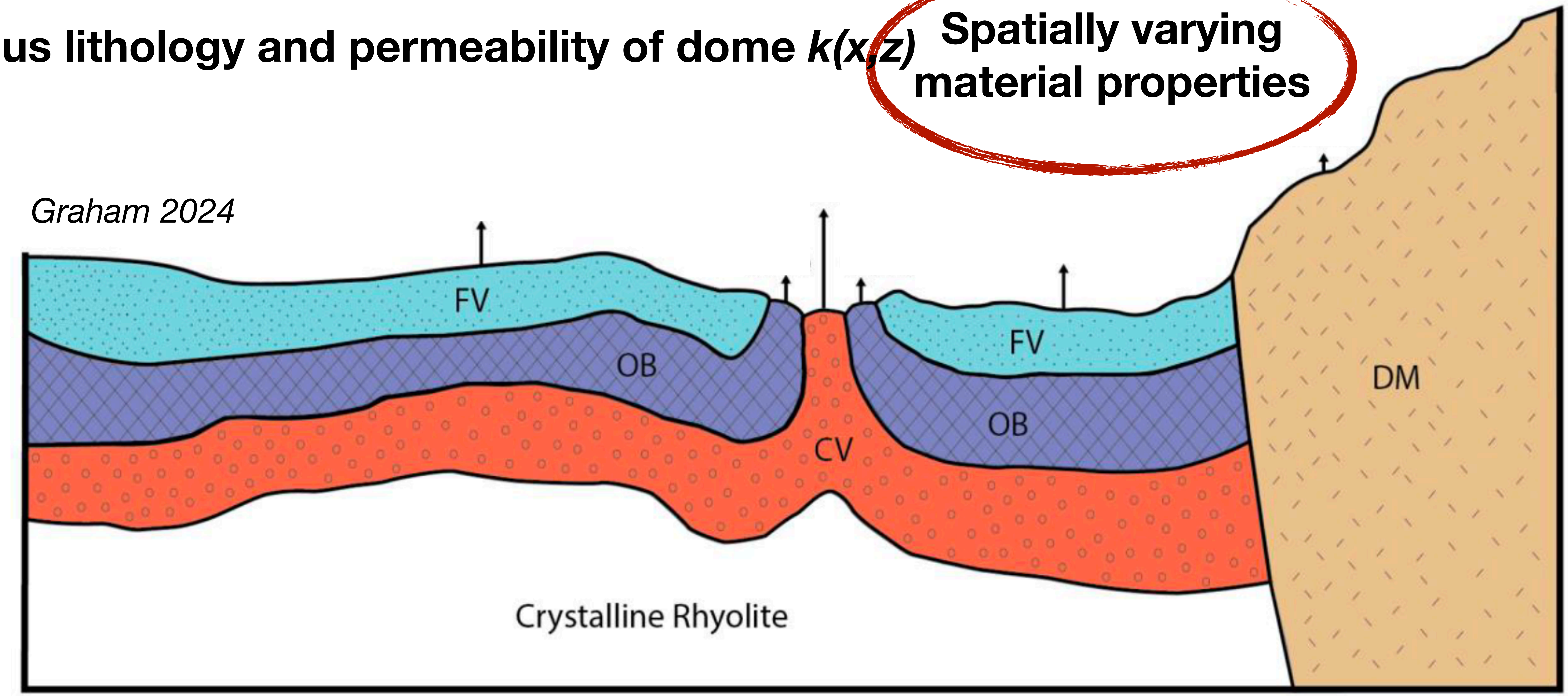
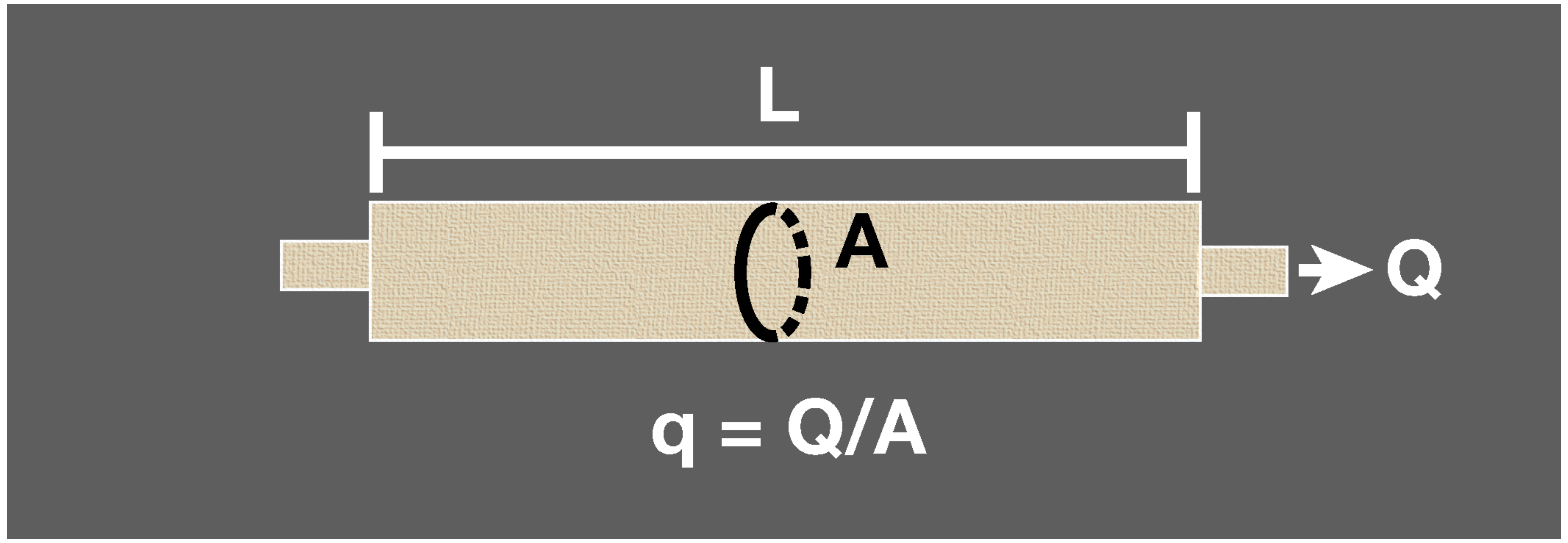


Firedrake implementation

- Darcy's law approximation for **steady-state, laminar *compressible* steam flow** through porous media

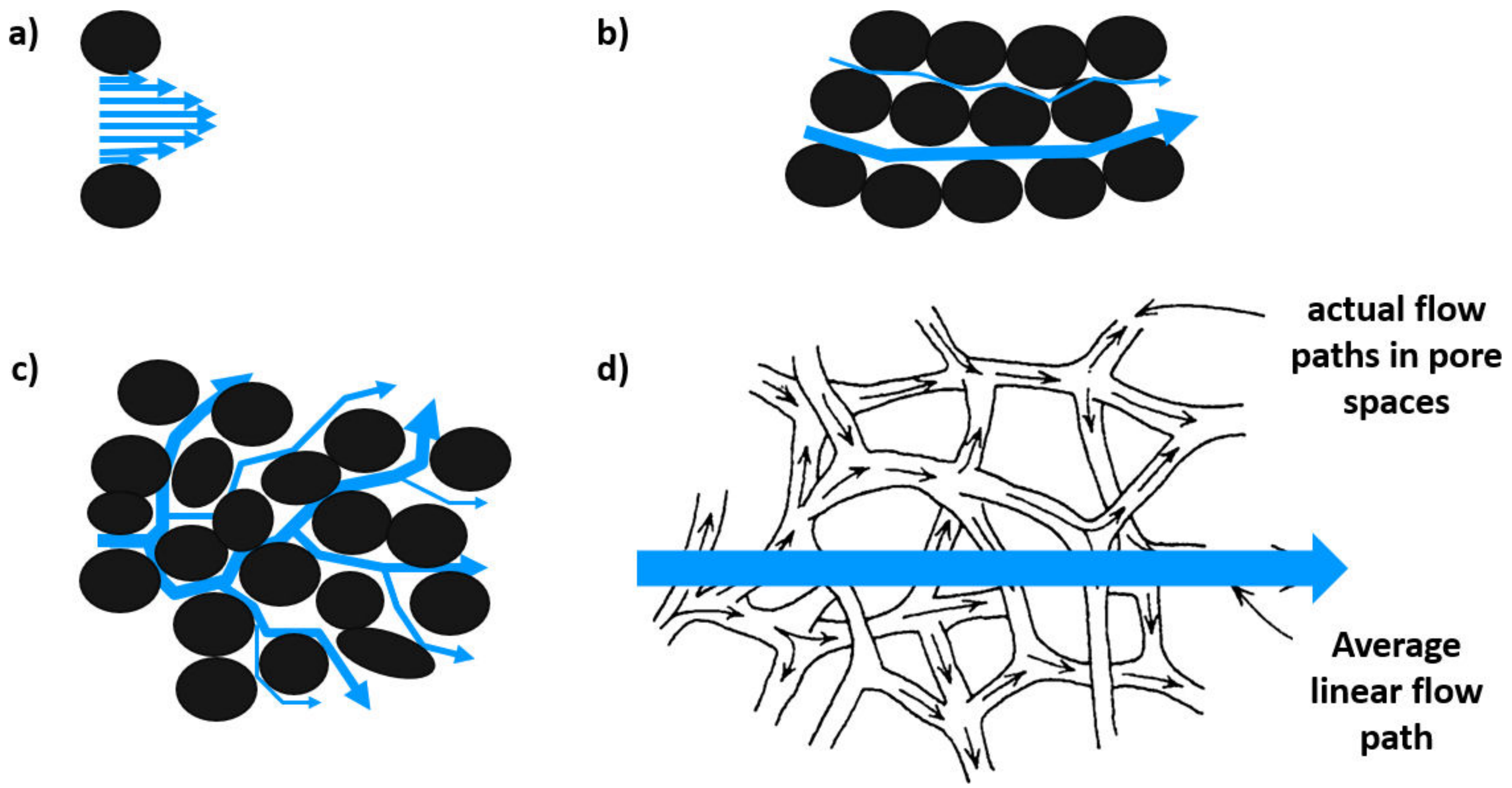


- Strong form:

$$\mathbf{q} = -\frac{k}{\mu}(\nabla P + \rho g z)$$

$$\phi\frac{d\rho}{dt} + \nabla\cdot(\rho\mathbf{q}) = 0$$

$$\rho = \frac{PM}{RT}$$



After Freeze and Cherry, 1979

Firedrake implementation

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- Strong form:

$$\mathbf{q} = -\frac{k}{\mu}(\nabla P + \rho g z)$$

$$\phi\frac{d\rho}{dt} + \nabla\cdot(\rho\mathbf{q}) = 0 \xrightarrow{} \nabla\cdot\left(-\frac{k}{\mu}\rho\nabla(c\rho + \rho g z)\right) = 0$$

Steady-state solution

$$\rho = \frac{PM}{RT}$$

Firedrake implementation

- Darcy's law approximation for **steady-state, laminar *compressible* steam flow** through porous media

- Strong form:

$$\mathbf{q} = -\frac{k}{\mu}(\nabla P + \rho g z)$$

$$\boldsymbol{\sigma} = \rho\frac{\mathbf{q}}{\phi} = -\frac{k}{\mu\phi}\rho\nabla(c\rho + \rho g z)$$

$$c = \frac{RT}{M}$$

$$\rho > 0$$

$$\phi\frac{d\rho}{dt} + \nabla\cdot(\rho\mathbf{q}) = 0 \longrightarrow \nabla\cdot\left(-\frac{k}{\mu}\rho\nabla(c\rho + \rho g z)\right) = 0$$

Steady-state solution

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Firedrake implementation

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Steady-state solution

$$\rho = \frac{PM}{RT}$$

$$\nabla\cdot(\nabla u) = 0$$

Firedrake implementation

- Apply a conservative mixed finite element method

- Weak form derived from:

$$\int_\Omega \boldsymbol{\sigma} \cdot \boldsymbol{\omega} \mathrm{dx} + \int_\Omega \frac{k}{\mu} \rho \nabla (c\rho + \rho g z) \cdot \boldsymbol{\omega} \mathrm{dx} = 0 \qquad \forall \boldsymbol{\omega} \in S_P^0(\Omega)(??)$$

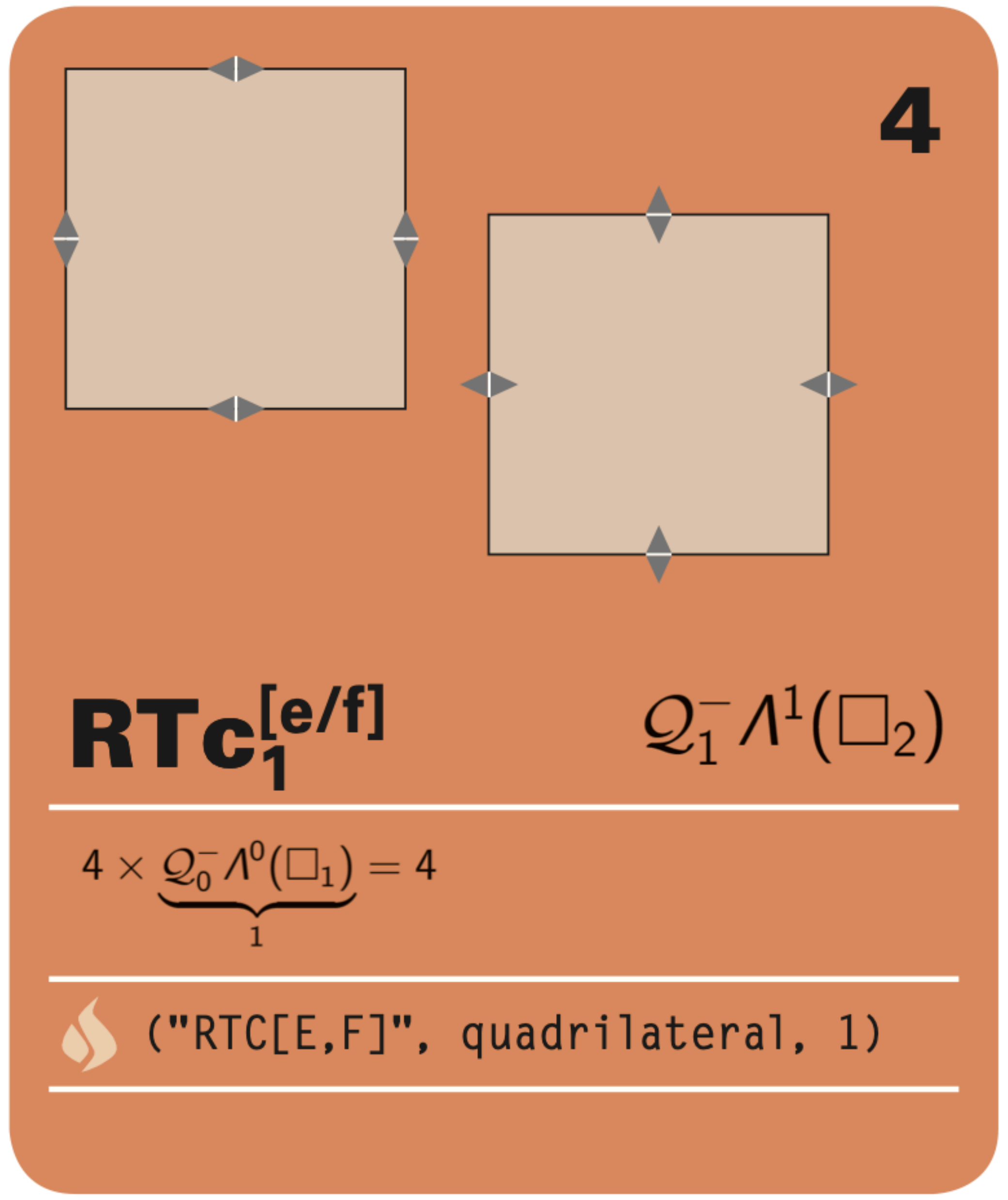
$$\int_\Omega (\nabla \cdot \boldsymbol{\sigma}) v \mathrm{dx} = 0$$

- Apply the divergence theorem and we obtain:

$$\int_\Omega \boldsymbol{\sigma} \cdot \boldsymbol{\omega} \mathrm{dx} - \int_\Omega \frac{1}{\mu}(c\rho + \rho g z) \nabla \cdot (k \rho \boldsymbol{\omega}) \mathrm{dx} + \int_{\partial\Omega} \frac{k}{\mu} \rho (c\rho + \rho g z) \boldsymbol{\omega} \cdot \mathbf{n} \mathrm{ds} = 0$$

Firedrake implementation

- We need appropriate function spaces for $\boldsymbol{\sigma}$ and ρ
  - Normal vectors continuous across element boundaries for $\boldsymbol{\sigma}$
  - Can be discontinuous for ρ
    - Raviart-Thomas and discontinuous Galerkin spaces with quadrilateral elements



- Apply appropriate Dirichlet boundary conditions by defining the boundary integral for known conditions

- **Final weak form:**

$$\int_\Omega \boldsymbol{\sigma} \cdot \boldsymbol{\omega} \mathrm{dx} - \int_\Omega \frac{1}{\mu}(c\rho + \rho g z) \nabla \cdot (k\rho\boldsymbol{\omega})\mathrm{dx} = - \int_{\Gamma_0} \frac{k}{\mu}\rho_0(c\rho_0 + \rho_0 g z)\boldsymbol{\omega} \cdot \mathbf{n}\mathrm{ds}$$

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Dirichlet boundary conditions

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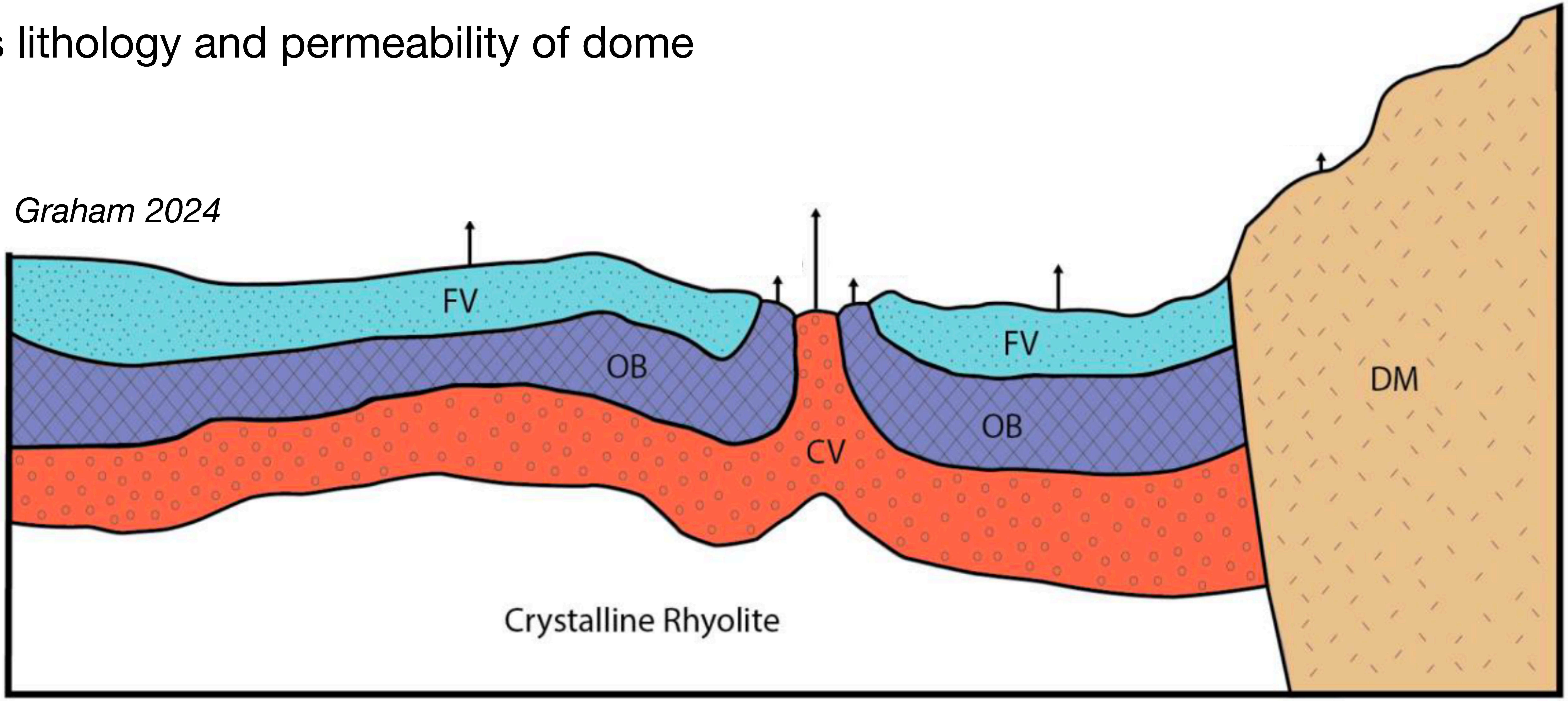


Firedrake implementation

- Use Newton's method from the SNES component of PETSc
- Back-tracking type line search option
- Linear Newton step equation solved by sparse direct matrix method

Heterogeneous permeability solution

- Run and view code!

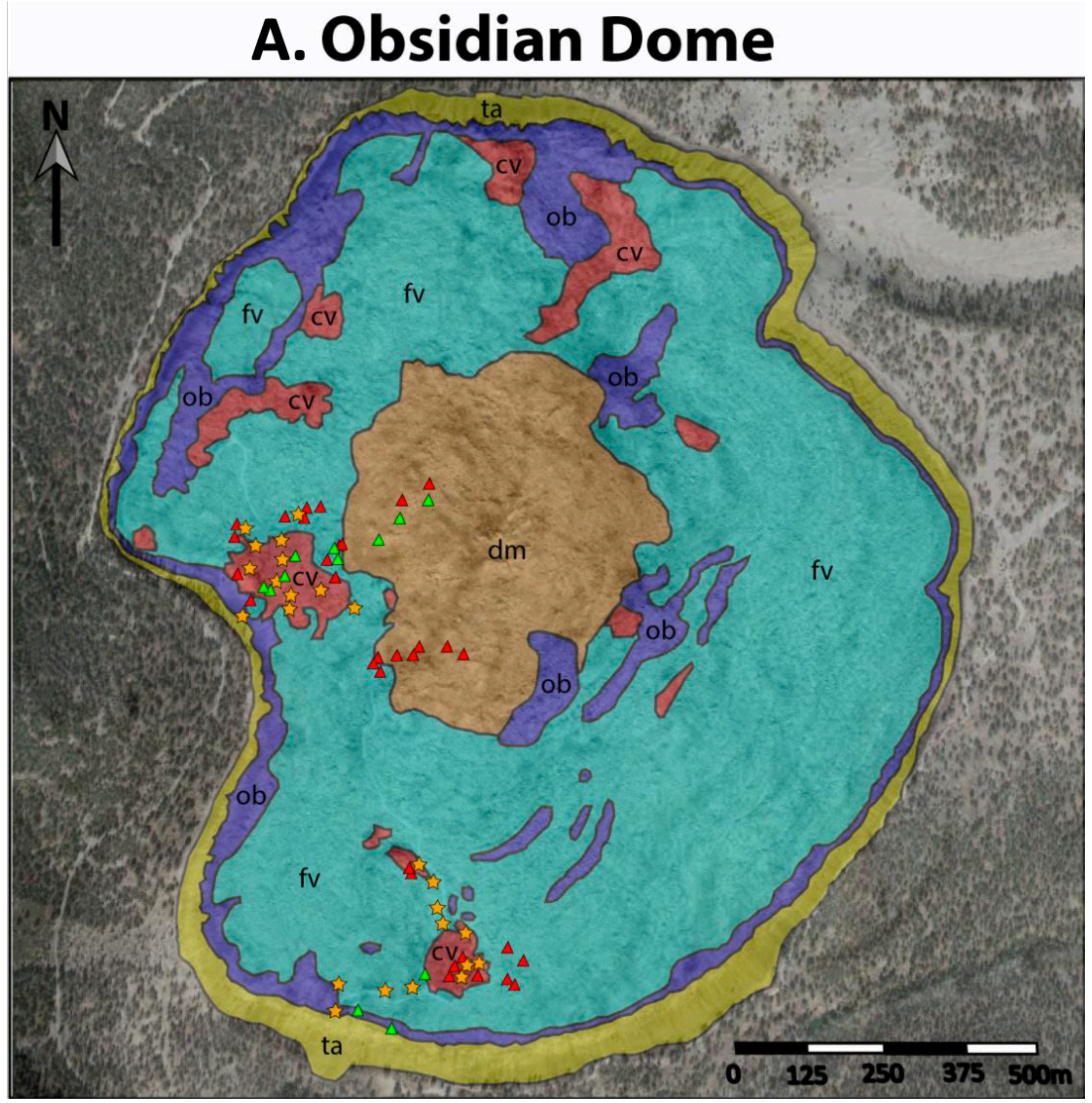
Firedrake implementation

- Compare with the Continuous Galerkin (CG1) implementation to ensure correct implementation

- Final weak form:

$$\int_\Omega k\rho \nabla(c\rho + \rho g z) \cdot \nabla \omega = 0 \qquad \forall \omega \in H_D^1(\Omega)$$

Heterogeneous 3D solution



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