Optimization and an FEM

a 2D finite element method applied to the p-Laplacian

Ed Bueler
Dept. of Mathematics and Statistics, UAF

12 April 2016

outline for today

Chapter 5 of book:

- the p-Laplacian equation

$$-\nabla \cdot \left(|\nabla u|^{p-2}\nabla u\right) = f$$

  - $p > 1$; eqn is nonlinear if $p \neq 2$
  - arises from minimizing objective $I[u] = \int_\Omega \frac{1}{p}|\nabla u|^p - fu$
  - generalizes Poisson equation ($p = 2$ case) in Chap 2
- introduce Q^1 structured-grid finite element method
- show code `c/ch5/plap.c`:
  - objective-only implementation with `-snes_fd_function`

continuum problem: minimize power of gradient

- $\Omega \subset \mathbb{R}^2$ is (open) domain with nice boundary
  - also given $f \in C(\overline{\Omega})$
- functions with integrable gradient:

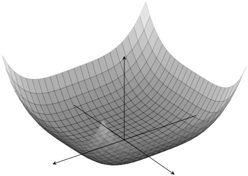
$$W^{1,p}(\Omega) = \left\{ w \,:\, \int_\Omega |w|^p < \infty \;\&\; \int_\Omega |\nabla w|^p < \infty \right\}$$

  - actually want functions with given boundary values, $u\big|_{\partial\Omega} = g$, so consider this subspace:

$$W_g^{1,p}(\Omega) = \left\{ w \in W^{1,p}(\Omega) \;\&\; w\big|_{\partial\Omega} = g \right\}$$

- *goal*: minimize ∞-dimn'l functional

$$I[u] = \int_\Omega \frac{1}{p} |\nabla u|^p - fu$$



- think: minimize a polynomial surface

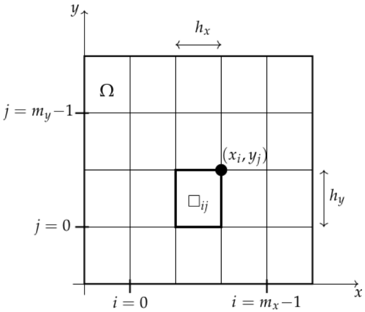
just do it!

- go no further . . . can we just find the minimum now, without doing any more calculus?
- steps:
  1. write code that computes the functional $I[u]$ from a representation $\mathbf{u} = u_{ij}$ of u based on a grid
  2. have PETSc find derivative $\mathbf{F}(\mathbf{u}) \approx I'[u]$ by finite differences
  3. have PETSc solve nonlinear equations $\mathbf{F}(\mathbf{u}) = 0$
     - have PETSc compute Jacobian
     - *also* by finite differences . . . dodgy?
- yes, this sort of works
  - it is a path *toward* a scalable implementation
  - a secondary benefit . . . line search (next week)
- provides concrete transition to finite element method

grid of elements

- domain is unit square
  $\Omega = (0,1) \times (0,1)$
- consider grid (x_i, y_j) with *interior* locations as unknowns u_{ij}
  - boundary values from g
  - $N = m_x m_y$ unknowns
- $(m_x + 1)(m_y + 1)$ *elements* $\square_{ij}$
  - dimensions $h_x \times h_y$
  - indexed by upper-right corner
  - functional can be computed element-by-element:

$$I[u] = \sum_{i=0}^{m_x} \sum_{j=0}^{m_y} \int_{\square_{ij}} \frac{1}{p} |\nabla u|^p - fu$$



Q^1 finite element method (FEM)

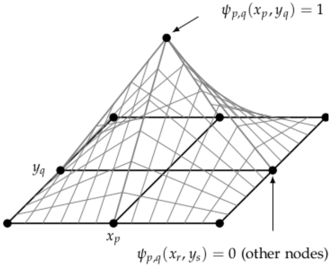
- in a Q^1 FEM, each function is piecewise-bilinear

$$u(x, y) = a + bx + cy + dxy$$

- let S^h be space of *continuous* piecewise-bilinear functions
  - $S^h \subset W^{1,p}(\Omega)$
- also define $S_g^h \subset S^h$ with boundary values g
  - $\dim(S_g^h) = N =$ (number of interior nodes)

- hat function $\psi_{p,q}(x, y) \in S^h$
  - equals one at (x_p, y_q)
  - zero at other nodes
  - $\{\psi_{p,q}\}$ is basis of S^h
- for $v \in S_0^h$:

$$v(x, y) = \sum_{i=0}^{m_x-1} \sum_{j=0}^{m_y-1} v_{i,j}\, \psi_{i,j}(x, y)$$



formulas on one element

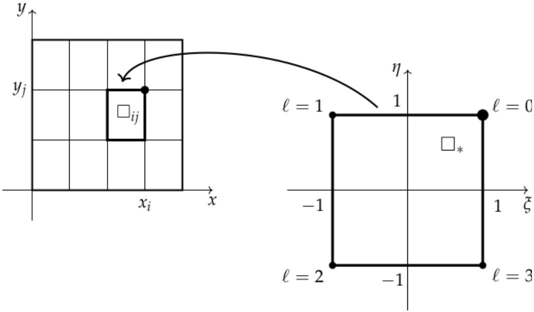
- define *reference element*

$$\square_* = [-1, 1] \times [-1, 1]$$

  - corners labeled $\ell = 0, 1, 2, 3$
- basis for bilinear functions on $\square_*$:

$$\chi_\ell(\xi, \eta) = \frac{1}{4}\left(1 + \xi_\ell \xi\right)\left(1 + \eta_\ell \eta\right)$$

  - monomial basis $\{1, \xi, \eta, \xi\eta\}$ is not convenient



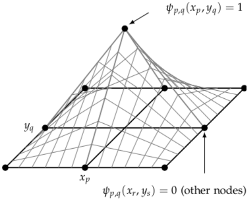
formulas on one element 2

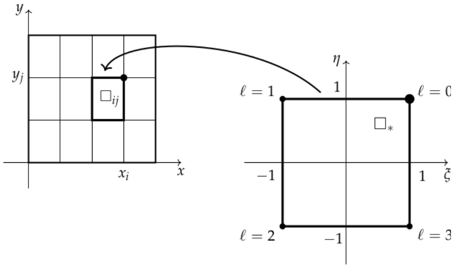
- important formulas on reference element:

$$v(\xi,\eta) = \sum_{\ell=0}^{3} v_\ell \chi_\ell(\xi,\eta)$$

$$\psi_{p,q}(x(\xi,\eta), y(\xi,\eta)) = \chi_\ell(\xi,\eta)$$

$$\det \frac{\partial(x,y)}{\partial(\xi,\eta)} = \det \begin{bmatrix} \frac{h_x}{2} & 0 \\ 0 & \frac{h_y}{2} \end{bmatrix} = \frac{h_x h_y}{4}$$





quadrature on quadrilaterals

- n point Gauss-Legendre quadrature on interval $[-1,1]$:

$$\int_{-1}^{1} f(z)\,dz \approx \sum_{q=0}^{n-1} w_q f(z_q)$$

- on reference element $\square_*$:

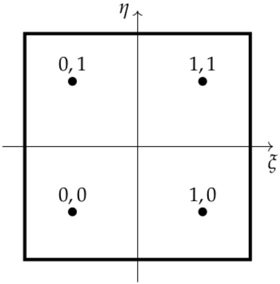
$$\int_{\square_*} v(\xi,\eta)\,d\xi\,d\eta \approx \sum_{r=0}^{n-1}\sum_{s=0}^{n-1} w_r w_s v(z_r, z_s)$$

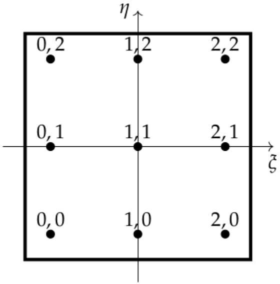
- we can go back to x, y element $\square_{ij}$ by change of variables:

$$\int_{\square_{ij}} v(x,y)\,dx\,dy = \frac{h_x h_y}{4}\int_{\square_*} v(\xi,\eta)\,d\xi\,d\eta$$

$n = 1, 2, 3$ Gauss-Legendre quadrature

n	nodes z_q	weights w_q
1	0	2
2	$-\frac{1}{\sqrt{3}}, +\frac{1}{\sqrt{3}}$	$1, 1$
3	$-\sqrt{\frac{3}{5}}, 0, +\sqrt{\frac{3}{5}}$	$\frac{5}{9}, \frac{8}{9}, \frac{5}{9}$





approximation to functional

- let

$$G_{ij}(\xi, \eta) = \left[\frac{1}{p}|\nabla u|^p - fu\right]_{\square_*}$$

  - details in Exercise 5.7 in book
- putting it all together, we have a computable approximation to the functional which we want to minimize
- this is our *objective*:

$$I^h[u] = \frac{h_x h_y}{4} \underbrace{\sum_{i=0}^{m_x} \sum_{j=0}^{m_y}}_{\substack{\text{sum} \\ \text{over} \\ \text{elements}}} \underbrace{\sum_{r=0}^{n-1} \sum_{s=0}^{n-1}}_{\substack{\text{sum} \\ \text{over} \\ \text{quadrature points}}} w_r w_s G_{ij}(z_r, z_s)$$

structure of `c/ch5/plap.c`

- dashed lines denote un-implemented code parts
- diagram like this appears for all remaining book examples