

Problem 36. Give a justified example of each, or argue (prove) that it is impossible.

- (a) A sequence that has a subsequence that is bounded, but which contains no subsequence which converges.
- (b) A sequence that does not contain 0 or 1 as a term, but which contains subsequences which converge to each of these values.
- (c) A sequence that contains subsequences converging to every point in the infinite set $\{1, 1/2, 1/3, 1/4, \dots\}$.

Problem 37. Let (a_n) be a bounded sequence. Define the set

$$S = \{x \in \mathbb{R} : x < a_n \text{ for infinitely many terms } a_n\}.$$

Then S is bounded above, and there exists a subsequence (a_{n_k}) which converges to $\sup S$.

Proof. □

Problem 38. Every convergent sequence is a Cauchy sequence.

Proof. □

Problem 39. Give a justified example of each, or argue (prove) that it is impossible.

- (a) A Cauchy sequence that is not monotone.
- (b) A Cauchy sequence containing an unbounded subsequence.
- (c) An unbounded sequence containing a Cauchy subsequence.

Problem 40. Give a justified example of each, or explain (prove) why the request is impossible, by referencing the proper theorem(s).

- (a) Two series $\sum x_n$ and $\sum y_n$ which both diverge, but where $\sum x_n y_n$ converges.
- (b) A convergent series $\sum x_n$ and a bounded sequence (y_n) , such that $\sum x_n y_n$ diverges.
- (c) Two sequences (x_n) and (y_n) where $\sum x_n$ and $\sum (x_n + y_n)$ both converge, but $\sum y_n$ diverges.
- (d) A sequence (x_n) satisfying $0 \leq x_n \leq 1/n$ where $\sum (-1)^n x_n$ diverges.

Problem 41. If $\sum a_n$ converges absolutely then $\sum a_n^2$ converges absolutely.

Proof. □

Problem 42. *Ratio test: For a series $\sum a_n$, if the sequence of terms (a_n) satisfies $a_n \neq 0$ for all n , and if*

$$\lim_{n \rightarrow \infty} \frac{|a_{n+1}|}{|a_n|} = r < 1,$$

then the series converges absolutely.

Proof.

□

Problem 43. *Do the following series converge or diverge? A careful proof is not needed, but a logical and correct justification or explanation is required, possibly using Theorems from Sections 2.1–2.7, or Problems above.*

(a) $\sum_{n=1}^{\infty} \frac{1}{2^n + n}$

(b) $\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2}$

(c) $1 - \frac{3}{4} + \frac{4}{6} - \frac{5}{8} + \frac{6}{10} - \frac{7}{12} + \frac{8}{14} + \dots$

(d) $1 - \frac{1}{2^2} + \frac{1}{3} - \frac{1}{4^2} + \frac{1}{5} - \frac{1}{6^2} + \frac{1}{7} - \frac{1}{8^2} + \frac{1}{9} - \dots$

(e) $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$