Problem 29. *Suppose* $(x_n)_{n=1}^{\infty}$ *converges. Let* $k \in \mathbb{N}$*. The new sequence* $(x_{n+k})_{n=1}^{\infty}$ *also converges, and to the same limit.*

Proof. □

Problem 30. *Give an example of each of the following, or state that such a request is impossible. In the latter case, identify specific theorem(s) that justify your statement.*

(a) *sequences* (x_n) *and* (y_n)*, which both diverge, where the sum* $(x_n + y_n)$ *converges*

(b) *a convergent sequence* (x_n)*, and a divergent sequence* (y_n)*, where* (x_n+y_n) *converges*

(c) *a convergent sequence* (b_n)*, with* $b_n \neq 0$ *for all* n*, such that* $(1/b_n)$ *diverges*

(d) *sequences* (x_n) *and* (y_n)*, where* $(x_n y_n)$ *and* (x_n) *converge but* (y_n) *does not*

Problem 31. *If* $a \geq 0$ *and* $b \geq 0$ *then* $\sqrt{ab} \leq \frac{1}{2}(a+b)$.

Proof. □

Problem 32. *Consider the real sequence generated by setting* $x_1 = 2$ *and then*

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right).$$

(a) *The sequence* (x_n) *is bounded below by* $\sqrt{2}$.

Proof. □

(b) $\lim_{n\to\infty} x_n = \sqrt{2}$.

Proof. □

Problem 33. *The sequence* $\sqrt{2}, \sqrt{2+\sqrt{2}}, \sqrt{2+\sqrt{2+\sqrt{2}}}, \ldots$ *converges to* X.

Proof. □

Problem 34. *For each series, find an explicit formula for the partial sums, and determine if the series converges.*

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

(c) $\displaystyle\sum_{n=1}^{\infty} \log\left(\frac{n+1}{n}\right)$

Problem 35.

(a) *Suppose* $0 \leq a_n \leq b_n$. *If* $\displaystyle\sum_{n=1}^{\infty} a_n$ *diverges then* $\displaystyle\sum_{n=1}^{\infty} b_n$ *diverges.*

Proof. □

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ *diverges.*

Proof. □

Problem 36. *Give an example of each of the following, or argue that it is impossible.*

(a) *A sequence that has a subsequence that is bounded, but which contains no subsequence which converges.*

(b) *A sequence that does not contain* 0 *or* 1 *as a term, but which contains subsequences which converge to each of these values.*

(c) *A sequence that contains subsequences converging to every point in the infinite set* $\{1, 1/2, 1/3, 1/4, \ldots\}$.

Problem 37. *Let* (a_n) *be a bounded sequence. Define the set*

$$S = \{x \in \mathbb{R} : x < a_n \text{ for infinitely many terms } a_n\}.$$

Then S *is bounded above, and there exists a subsequence* (a_{n_k}) *which converges to* $\sup S$.

Proof. □