Isothermal sphere

$$\rho(r) = \frac{\sigma^2}{2\pi G r^2} \; ; \; \sigma^2 = \frac{k_B T}{\bar{m}}$$

$$\int \rho(r) = \frac{\sigma^2}{2\pi G} \int r^{-2} dr = A \frac{r^{-2+1}}{-2+1} = -A r^{-1}$$

$$-A \int r^{-1} = -A \ln(r)$$

$$\Phi(r) = -\frac{\sigma^2}{2\pi G} \ln(r)$$

$$F_g = -\frac{d\Phi}{dr} = -1 \cdot \frac{d}{dr} = -\left[1 \cdot \frac{\sigma^2}{2\pi G} \cdot \frac{1}{r} \right]$$

$$F_{net} = \frac{\sigma^2}{2\pi G} \frac{1}{r} = \frac{m v^2}{r} = \underline{\text{Constant}} \; v \; (\text{W.R.T } "r")$$

Get $\rho(r) \Rightarrow \rho(\psi)$

$$\Phi(r) = -\frac{\sigma^2}{2\pi G} \ln(r) \Rightarrow -\frac{2\pi G}{\sigma^2} = \ln(r)$$

$$\Rightarrow r(\Phi) = e^{(-2\pi G \psi / \sigma^2)} \Rightarrow r(\Phi)^{-2} = e^{\left(-\frac{2\pi G \Phi}{\sigma^2}\right) \cdot -2} = e^{\left(\frac{4\pi G \Phi}{\sigma^2}\right)}$$

$$\rho(r) = \frac{\sigma^2}{2\pi G r^2} \Rightarrow \rho(\psi) = \frac{\sigma^2}{2\pi G} \exp\left(\frac{4\pi G}{\sigma^2} \cdot \Phi\right) \propto e^{\Phi}$$

Spherical / isotropic Plummer

$$\rho(r) = \frac{3 M_{TOT}}{4\pi a^3}\left(1 + (r/a)^2\right)^{-5/2}$$

First integral

$$\int \rho(r)\, dr = \frac{3 M a^2}{4\pi} \int \frac{1}{(r^2 + a^2)^{5/2}}\, dr$$

trig sub

$$r = a\tan(u) \;, \; u = \arctan\left(\frac{r}{a}\right)$$

$$\Rightarrow dr = a\sec^2(u)\, du$$

$$\int \frac{1}{(r^2 + a^2)^{5/2}}\, dr = \int \frac{a\sec^2(u)}{(a^2\tan^2(u) + a^2)^{5/2}}$$

$$* \; a^2\tan^2(u) + a^2 = a^2\sec^2(u)$$

$$\int \frac{a\sec^2(u)}{(a^2\tan^2(u) + a^2)^{5/2}} = \int \frac{a\sec^2(u)}{(a^2\sec^2(u))^{5/2}} = \int \frac{a}{a^5}\,\frac{\sec^2(u)}{\sec^5(u)}\, du$$

$$\Rightarrow \int \frac{a\sec^2(u)}{(a^2\tan^2(u) + a^2)^{5/2}}\, du = \frac{1}{a^4}\int \frac{1}{\sec^3(u)}\, du = \frac{1}{a^4}\int \cos^3(u)\, du$$

$$\Rightarrow \frac{1}{a^4}\int \cos^3(u)\, du = \int \cos(u)(1 - \sin^2(u))\, du$$

Substitute

$$v = \sin(u),\; \frac{dv}{du} = \cos(u),\; du = \frac{1}{\cos(u)}\, dv$$

$$\Rightarrow \int (1 - v^2)\, dv$$

$$\Rightarrow \int (1 - v^2)\,dv$$

$$\int (1-v^2)\,dv = \underbrace{\int 1\,dv}_{v} - \int v^2\,dv$$

$$\int v^2\,dv = \frac{v^3}{3}$$

$$\Rightarrow \int 1\,dv - \int v^2\,dv = v - \frac{v^3}{3}$$

plug in $v = \sin(u)$

$$\underbrace{v - \frac{v^3}{3} = \sin(u) - \frac{\sin^3(u)}{3}}_{\text{sol'n to } \int \frac{1}{\sec^3(u)}}$$

$$\Rightarrow \frac{1}{a^4}\int \frac{1}{\sec^3 u} = \frac{\sin(u)}{a^4} - \frac{\sin^3 u}{3a^4}$$

Plug in $u = \arctan\left(\frac{r}{a}\right)$ + use $\sin\left(\arctan\left(\frac{r}{a}\right)\right) = \frac{r}{a\sqrt{\frac{r^2}{a^2}+1}}$

$$\frac{\sin(u)}{a^4} - \frac{\sin^3 u}{3a^4} = \frac{r}{a^5\left(1+\frac{r^2}{a^2}\right)^{1/2}} - \frac{r^3}{3a^7\left(1+\frac{r^2}{a^2}\right)^{3/2}} = \int \frac{1}{(r^2+a^2)^{5/2}}$$

Plug into $\frac{3Ma^2}{4\pi}\int \frac{1}{(r^2+a^2)^{5/2}}$

$$\frac{3Ma^2}{4\pi}\int \frac{1}{(r^2+a^2)^{5/2}} = \frac{3Mr}{4\pi a^3\left(1+(r/a)^2\right)^{1/2}} - \frac{Mr^3}{4\pi^5\left(1+(r/a)^2\right)^{3/2}} = \frac{Mr(2r^2+3a^2)}{4\pi a^5\left(1+\left(\frac{r}{a}\right)^2\right)^{3/2}}$$

$$\longrightarrow \int \rho\,dr = \frac{Mr(2r^2+3a^2)}{4\pi a^5\left(1+\left(\frac{r}{a}\right)^2\right)^{3/2}} \propto \frac{1}{\left(1+\left(\frac{r}{a}\right)^2\right)^{3/2}}$$

(ignoring constants)

2nd Int

$$\iint \rho \, dr \simeq \int \frac{1}{\left(1+\left(\frac{r}{a}\right)^2\right)^{3/2}} = a^3 \int \frac{1}{(r^2+a^2)^{3/2}}\, dr$$

$$\int \frac{1}{(r^2+a^2)^{3/2}}\, dr$$

Trig Sub

$$\begin{cases} r = a\tan(u) \\ u = \arctan(r/a) \\ dr = a\sec^2(u)\,du \end{cases}$$

$$\to \int \frac{1}{(r^2+a^2)^{3/2}}\, dr = \int \frac{a\sec^2(u)\,du}{(a^2\tan^2(u)+a^2)^{3/2}} \; ; \; a^2\tan^2(u)+a^2 = a^2\sec^2(u)$$

$$\to \int \frac{a\sec^2(u)}{(a^2\sec^2(u))^{3/2}}\, du = \int \frac{a\sec^2(u)}{a^3\sec^3(u)} = \frac{1}{a^2}\int \frac{1}{\sec(u)}\, du$$

$$\to \frac{1}{a^2}\int \frac{1}{\sec(u)}\, du = \frac{1}{a^2}\int \cos(u)\, du = \frac{1}{a^2}\sin(u)$$

plug back in

$$u = \arctan\left(\frac{r}{a}\right)$$

$$\sin\left(\arctan\left(\frac{r}{a}\right)\right) = \frac{r}{a\left(1+\left(\frac{r}{a}\right)^2\right)^{1/2}}$$

$$\Rightarrow \frac{r}{a^3\sqrt{1+(r/a)^2}} = \frac{\sin(u)}{a^2}$$

$$\Rightarrow a^3\int \frac{1}{(r^2+a^2)^{3/2}}\, dr = a^3 \cdot \frac{r}{a^3\sqrt{1+(r/a)^2}}$$

$$= \iint \rho\, dr \simeq \frac{r}{\left(1+\left(\frac{r}{a}\right)^2\right)^{1/2}} \simeq -\Phi \Rightarrow \Phi \propto -\left(1+(r/a)^2\right)^{-1/2}$$

looks about right

Find $\rho(\psi)$ & $V_{circ}(r)$

$$\Phi \propto \frac{1}{\sqrt{1+r^2/a^2}} \rightarrow (1+r^2/a^2) = \Phi^{-2}$$

$$\Rightarrow r^2 = a^2(\Phi^{-2} - 1)$$

$$\Rightarrow r = a(\Phi^{-2} - 1)^{1/2}$$

$$r(\psi) \rightarrow \rho(\psi)$$

$$\rho(r) \propto (1 + r^2/a^2)^{-5/2}$$

$$r(\psi)^2 = a^2(\Phi^{-2} - 1)$$

$$\rho(\psi) = \left(1 + \frac{a^2(\Phi^{-2}-1)}{a^2}\right)^{-5/2}$$

$$\rho(\psi) = (\Phi^{-2} - 1 + 1)^{-5/2}$$

$$\rho(\psi) = (\Phi^{-2})^{-5/2}$$

$$\Rightarrow \rho(\psi) = \Phi^5$$

$$V_c^2 = r\left|\frac{d\Phi}{dr}\right| \simeq r^2(1 + r^2/a^2)^{-3/2}$$

$$V_c = r(1 + r^2/a^2)^{-3/4}$$