Geometría y visualización

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Trabajo Fin de Máster
Máster en Matemáticas

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x_1 &= f_1(t_1, \ldots, t_m), \\
&\vdots \\
x_n &= f_n(t_1, \ldots, t_m).
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Este sistema se puede ver como $F : K^m \to K^n$ definido por

$$F(t_1, \dots, t_m) = (f_1(t_1, \dots, t_m), \dots, f_n(t_1, \dots, t_m)).$$

Método de la base de Gröbner

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Resolver el problema de pasar a ecuaciones paramétricas a implícitas equivale a encontrar la variedad mínima que contiene a $F(K^m)$. Véase calculando la base de Gröbner reducida del ideal $\langle x_1 - f_1, \ldots, x_n - f_n \rangle$ y encontrando el elemento que no depende de las variables t_i.

EJEMPLO

$$
\begin{aligned}
x &= r\cos u \cos t + R\cos t \\
y &= r\cos u \sin t + R\sin t \\
z &= r\sin u
\end{aligned}
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EJEMPLO

$$\begin{aligned} x &= r\cos u\cos t + R\cos t \\ y &= r\cos u\sin t + R\sin t \\ z &= r\sin u \end{aligned}$$

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$$\begin{aligned} c_u^2 + s_u^2 - 1 &= 0 \\ c_t^2 + s_t^2 - 1 &= 0 \end{aligned}$$

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$$(x^2 + y^2 + z^2 - r^2 - R^2)^2 = 4R^2(z^2 - r^2)$$

Método de la resultande de Sylvester

$f = a_n x^n + \cdots + a_0$ y $g = b_m x^m + \cdots + b_0$ donde $a_n, b_m \neq 0$

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$$\mathrm{Res}(f,g) = \mathrm{Det}(\mathrm{Syl}(f,g))$$

Donde $\mathrm{Syl}(f,g)$ denota

$$\begin{pmatrix}
a_n & & & & & b_m & & & & \\
a_{n-1} & a_n & & & & b_{m-1} & b_m & & & \\
a_{n-2} & a_{n-1} & a_n & & & b_{m-2} & b_{m-1} & b_m & & \\
\vdots & \vdots & & \ddots & & \vdots & \vdots & \vdots & \ddots & \\
a_1 & \dots & \dots & \dots & a_n & b_1 & \dots & \dots & \dots & b_m \\
a_0 & \dots & \dots & \dots & a_{n-1} & b_0 & \dots & \dots & \dots & b_{m-1} \\
 & a_0 & \dots & \dots & a_{n-2} & & b_0 & \dots & \dots & b_{m-2} \\
 & & \ddots & \vdots & \vdots & & & \ddots & \vdots & \vdots \\
 & & & a_0 & a_1 & & & & b_0 & b_1 \\
 & & & & a_0 & & & & & b_0
\end{pmatrix}$$

EJEMPLO

$$f = x^2y - 1 \qquad\qquad g = x^2 + y^2 + xy - 4$$

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$$\mathrm{Res}(f,g) = \mathrm{Det}\begin{pmatrix} y & 0 & 1 & 0 \\ 0 & y & y & 1 \\ -1 & 0 & y^2-4 & y \\ 0 & -1 & 0 & y^2-4 \end{pmatrix}$$

$$= y^6 - 8y^4 + y^3 + 16y^2 - 8y + 1$$

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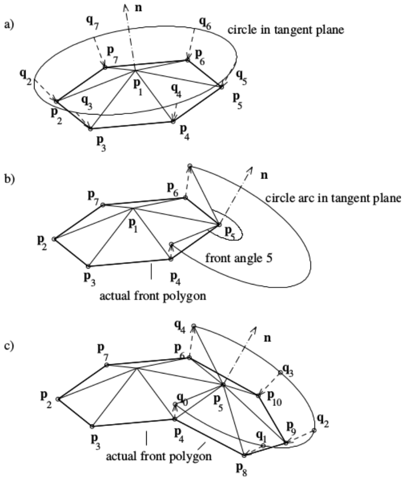
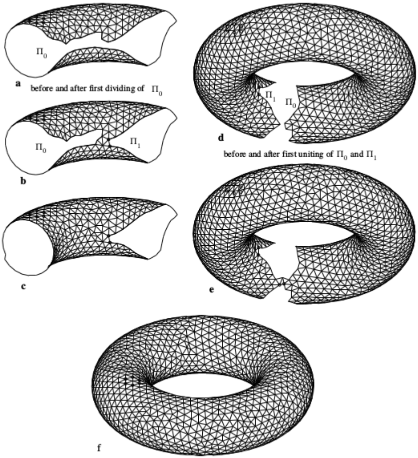
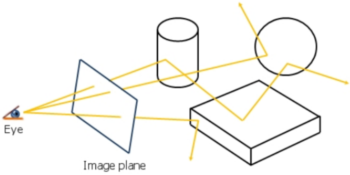
$$\langle x - 4y^5 - y^4 + 32y^3 + 4y^2 - 64y + 16, y^6 - 8y^4 + y^3 + 16y^2 - 8y + 1\rangle$$

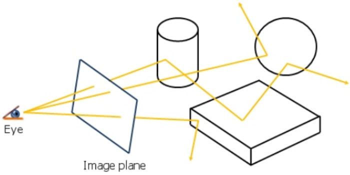

a)
n
q7
q6
circle in tangent plane
p7
p6
q2
q5
p1
q4
q3
p5
p2
p4
p3
b)
n
p6
circle arc in tangent plane
p7
p1
p5
p2
front angle 5
p3
p4
actual front polygon
c)
q4
n
p7
p6
q3
p2
q0
p10
p5
q2
p9
p3
p4
q1
actual front polygon
p8

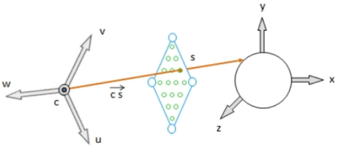


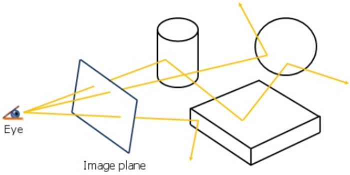
Π0
a
before and after first dividing of Π0
Π0
Π1
b
c
Π1
Π0
d
before and after first uniting of Π0 and Π1
e
f

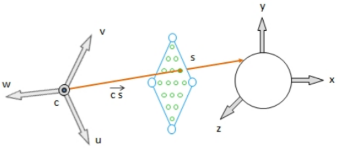
Ray Tracing




Eye
Image plane


v
w
c
u
$\vec{cs}$
s
y
x
z





$$g(t) = f(c_x + t(x_s - c_x), c_y + t(y_s - c_y), c_z + t(z_s - c_z))$$

Una extensión de una función de $\mathbb{R}^n$ a $\mathbb{R}$ dada por $z = f(x_1, \dots, x_n)$ es el intervalo de extensión unida R_f de f. Para el intervalo $X' = (X'_1, \dots, X'_n) \in I(\mathbb{R}^n)$ se define el rango de f-valores en X' como

$$R_f(X'_1, \dots, X'_n) := \{f(x_1, \dots, x_n) : x_i \in X'_i \; \forall i \in \{1, \dots, n\}\}$$

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$$f^*(X) := \left[\min_{x_p \in X'_p} \max_{x_i \in X'_i} f(x_p, x_i), \max_{x_p \in X'_p} \min_{x_i \in X'_i} f(x_p, x_i)\right]$$

$$f^{**}(X) := \left[\max_{x_p \in X'_p} \min_{x_i \in X'_i} f(x_p, x_i), \min_{x_p \in X'_p} \max_{x_i \in X'_i} f(x_p, x_i)\right]$$

```
Evaluate(X,Y,Z):
   If(0 belongs to F(X,Y,Z))
      If(X or Y or Z <= Threshold)
         Add (X,Y,Z) to solution list
      Else
         Subdivide X into X_1 and X_2
         Subdivide Y into Y_1 and Y_2
         Subdivide Z into Z_1 and Z_2
         Evaluate (X_i,Y_j,Z_k) for i,j,k in {1,2}
   Else
      The octant is rejected
```

```
Mitchell(T as [t_1,t_2]):
   If(0 in F(T))
      If(0 not in F'(T))
         If(f(t_1)*f(t_2) <= 0)
            Root refinement over T using Bisection or Newthon method
         Else
            T_1 = [t_1,(t_1 + t_2)/2]
            T_2 = [(t_1 + t_2)/2,t_1]

            If(width(T_1) >= threshold)
               Mitchell(T_1)
            Else
               Root refinement over T_1 using Bisection or Newthon method

            If(width(T_2) >= threshold)
               Mitchell(T_2)
            Else
               Root refinement over T_2 using Bisection or Newthon method
       Else
         Reject T
```

```
Newton(T as [t_1,t_2]):
   If(0 in F(T))
      If(0 not in F'(T))
         t_m = t_1 + midpoint(T)
         NT = t_m - f(t_m)/F'(T)
         NTT = NT intersecting with T

         If(NTT is empty)
            There's no root
         Else
            Newton(NTT)

      Else
         T_1 = [t_1,midpoint(T)]
         T_2 = [midpoint(T), t_2]

         If(width(T_1) >= Threshold)
            Newton(T_1)
         Else
            T_1 is the root

         If(width(T_2) >= Threshold)
            Newton(T_2)
         Else
            T_2 is the root
   Else
      Reject T
```

	Sphere	Drop	Chubs	Crossc.	Gumd. Torus	H. cube	McMul. K3	Kusn.-Schm.
Mitchell	20	36	98	80	240	99	102	75
Newton	28	47	94	96	260	180	123	94
N.+Al.-Han.	27	46	92	99	202	166	110	86
MRFro	48	103	188	105	541	272	276	210
MRFro+Newt.	55	96	174	151	380	283	221	173