

MA2503
Bachelor Mathematics Lecture Notes
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1 Linear Equation

2 Rectangular System and Echelon Form

GE: Gaussian elimination; G-J: Gauss-Jordan; U : row echelon form; E_A : reduced row echelon form.

Task	Method needed
Determining the rank of A	$A \xrightarrow{GE} U$, $\text{rank}(A) = \text{number of pivot in } U$
Solving linear system $Ax = b$	$[A b] \xrightarrow{GE} [U c]$ or $[A b] \xrightarrow{GJ} [E_A d]$ noted: if $\text{rank}(U) \neq \text{rank}([U c])$, then inconsistent system
Determining the column relation	$A \xrightarrow{GJ} E_A$
Computing the inverse of $A_{n \times n}$	$[A I_n] \xrightarrow{GJ} [I_n A^{-1}]$, noted: if A can not reduce to I_n , then A singular
Testing whether $A \sim B$	$A \xrightarrow{GE} U_A$ $B \xrightarrow{GE} U_B$, then compare $\text{rank}(U_A)$ equal to $\text{rank}(U_B)$ or not
Testing whether $A \overset{row}{\sim} B$	$A \xrightarrow{GJ} E_A$ $B \xrightarrow{GJ} E_B$, then check whether $E_A = E_B$
Testing whether $A \overset{col}{\sim} B$	$A^T \xrightarrow{GJ} E_{A^T}$ $B^T \xrightarrow{GJ} E_{B^T}$, then check whether $E_{A^T} = E_{B^T}$
Testing whether $b \in \text{span}\{v_1, \dots, v_n\}$	$[v_1 \dots v_n b] = [A b] \xrightarrow{GE} [U c]$
Testing whether $\text{span}\{v_1, \dots, v_n\} = \mathbb{R}^m$	$[v_1 \dots v_n] = A_{m \times n} \xrightarrow{GE} U$, then check whether $\text{rank}(A) = m$
Finding the fundamental subspaces of $A_{m \times n}$	$[A I_m] \xrightarrow{GE} [U P]$, then the bases are: $R(A)$: basic columns of A $R(A^T)$: nonzero rows of U $N(A)$: h_i 's in the general solution of $Ax = 0$ $N(A^T)$: last $m - r$ rows of P , where $r = \text{rank}(A)$
Testing whether $R(A^T) = R(B^T)$ $N(A) = N(B)$	$A \xrightarrow{G-J} E_A$, $B \xrightarrow{G-J} E_B$, then check whether $E_A = E_B$ (ie. whether $A \overset{row}{\sim} B$)
Testing whether $R(A) = R(B)$ $N(A^T) = N(B^T)$	$A^T \xrightarrow{G-J} E_{A^T}$, $B^T \xrightarrow{G-J} E_{B^T}$, then check whether $E_{A^T} = E_{B^T}$ (ie. whether $A \overset{col}{\sim} B$)
Testing whether $N(A_{m \times n}) = 0$	$A \xrightarrow{GE} U$, then check whether $\text{rank}(A) = n$
Testing whether $N(A_{m \times n})^T = 0$	$A \xrightarrow{GE} U$, then check whether $\text{rank}(A) = m$
Testing whether $\text{span}\{a_1, \dots, a_n\} = \text{span}\{b_1, \dots, b_n\}$ in \mathbb{R}^n	$(a_1^T \dots a_n^T)^T = A_{r \times n} \xrightarrow{GJ} E_A$, $(b_1^T \dots b_n^T)^T = B_{r \times n} \xrightarrow{GJ} E_B$, then check whether the nonzero rows of E_A and E_B coincide
Testing whether $\{v_1, \dots, v_n\}$ is linearly independent	$[v_1 \dots v_n] = A \xrightarrow{GE} U$, then check whether $\text{rank}(A) = n$
Finding linear relationship among $\{v_1, \dots, v_n\}$	$[v_1 \dots v_n] = A \xrightarrow{G-J} E_A$, then read off the relationships from E_A
Find a basis for $\text{span}\{v_1, \dots, v_n\}$	$[v_1 \dots v_n] = A \xrightarrow{GE} U$, then the basic columns of A form a basis
Extending $\{v_1, \dots, v_n\}$ ($r < n$) to a basis for \mathbb{R}^n	$[v_1 \dots v_r e_1 \dots e_n] = A \xrightarrow{GE} U$, then the basic columns of A form a basis

Table 1: Summary of applications of Gaussian and Gauss-Jordan elimination

3 Matrix Algebra

3.1 Addition and Transposition

Theorem 1 (Symmetries). *Let A be an $n \times n$ square matrix:*

- *symmetric:* $A^T = A$
- *skew-symmetric:* $A^T = -A$
- *hermitian:* $A^* = A$
- *skew-hermitian:* $A^* = -A$

3.2 Linearity

Linear Function . Suppose that \mathcal{D} and \mathcal{R} are two sets equipped with an addition and a scalar multiplication operation (consider, for example, $\mathcal{D} = \mathbb{C}^n$ and $\mathcal{R} = \mathbb{C}^m$). A function f that maps points in \mathcal{D} to points in \mathcal{R} is said to be a linear function if f satisfies:

$$f(\alpha x + y) = \alpha f(x) + f(y)$$

for all $x, y \in \mathcal{D}$ and all scalars α .

3.3 Matrix Multiplication

General Definition of Matrix Multiplication . If matrices $A_{m \times p}$ and $B_{p \times n}$ are conformable, the matrix product AB is defined to be the $m \times n$ matrix as following:

$$[AB]_{ij} = A_{i*}B_{*j} = \sum_{k=1}^p a_{ik}b_{kj}$$

$$\begin{pmatrix} * & * & \cdots & * \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \vdots & a_{ip} \\ \vdots & \vdots & & \vdots \\ * & * & \cdots & * \end{pmatrix}_{m \times p} \begin{pmatrix} * & \cdots & b_{1j} & \cdots & * \\ * & \cdots & b_{2j} & \cdots & * \\ \vdots & & \vdots & & \vdots \\ * & \cdots & b_{pj} & \cdots & * \end{pmatrix}_{p \times n} = \begin{pmatrix} * & \cdots & * & \cdots & * \\ \vdots & & \vdots & & \vdots \\ * & \cdots & [AB]_{ij} & \cdots & * \\ \vdots & & \vdots & & \vdots \\ * & \cdots & * & \cdots & * \end{pmatrix}_{m \times n}$$

Rows and Columns of a Matrix Product . To express the individual columns and rows of a matrix product:

$$\begin{aligned} [AB]_{*j} &= \begin{pmatrix} [AB]_{1j} \\ [AB]_{2j} \\ \vdots \\ [AB]_{mj} \end{pmatrix} = \begin{pmatrix} a_{11}b_{1j} + a_{12}b_{2j} + \cdots + a_{1p}b_{pj} \\ a_{21}b_{1j} + a_{22}b_{2j} + \cdots + a_{2p}b_{pj} \\ \vdots \\ a_{m1}b_{1j} + a_{m2}b_{2j} + \cdots + a_{mp}b_{pj} \end{pmatrix} = \begin{pmatrix} a_{11}b_{1j} \\ a_{21}b_{1j} \\ \vdots \\ a_{m1}b_{1j} \end{pmatrix} + \cdots + \begin{pmatrix} a_{1p}b_{pj} \\ a_{2p}b_{pj} \\ \vdots \\ a_{mp}b_{pj} \end{pmatrix} \\ &= A_{*1}b_{1j} + A_{*2}b_{2j} + \cdots + A_{*p}b_{pj} \end{aligned}$$

3.4 Matrix Inversion

Matrix Inversion . For a given square matrix $A_{n \times n}$, the matrix $B_{n \times n}$ that satisfied the conditions

$$AB = I_n, \quad BA = I_n$$

is called the inverse of A and is denoted by $B = A^{-1}$. an invertible matrix is said to be nonsingular, and a square matrix with no inverse is called a singular matrix.

Theorem 2 (4.Characterization of nonsingular matrices). For an $n \times n$ matrix A , the following statements are equivalent:

- A^{-1} exists (A is nonsingular);
- $\text{rank}(A) = n$;
- $A \xrightarrow{\text{Gauss-Jordan}} I$;
- $Ax = 0$ has only the trivial solution $x = 0$.

3.5 Elementary Matrices and Equivalence

Elementary Matrix . Matrices of the form $I - uv^T$ where u and v are $n \times 1$ column vectors with $v^T u \neq 1$ are called elementary matrices.

Equivalence . whenever B can be derived from A by a combination of elementary row and column operations, we say that A and B are equivalent matrices and write $A \sim B$; in matrix terms,

$$\begin{aligned} A \sim B &\iff PAQ = B && \text{for nonsingular } P \text{ and } Q \\ A \overset{\text{row}}{\sim} B &\iff PAQ = B && \text{for nonsingular } P \\ A \overset{\text{col}}{\sim} B &\iff PAQ = B && \text{for nonsingular } Q \end{aligned}$$

and note that if $A \overset{\text{row}}{\sim} B$, then:

$$B_{*k} = \sum_{j=1}^n \alpha_j B_{*j} \iff A_{*k} = \sum_{j=1}^n \alpha_j A_{*j}$$

Same as the column equivalence, in summary, row equivalence preserves column relationships, and column equivalence preserves row relationships.

Theorem 3 (6.Rank Normal Form). If A is an $m \times n$ matrix such that $\text{rank}(A) = r$, then

$$A \sim N_r = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$$

where N_r is called the rank normal form of A . It is the end product of a complete reduction of A by using both row and column operations.

Theorem 4 (7.Testing for equivalence). For $m \times n$ matrices A and B the following statement are true:

- $A \sim B \iff \text{rank}(A) = \text{rank}(B)$
- $A \overset{\text{row}}{\sim} B \iff E_A = E_B$
- $A \overset{\text{col}}{\sim} B \iff E_{A^T} = E_{B^T}$

Note: in particular, that:

- either $A \overset{\text{row}}{\sim} B$ or $A \overset{\text{col}}{\sim} B$ implies $A \sim B$, but not vice versa
- multiplication by nonsingular matrices doesn't change rank.

3.6 LU Factorization

$$PA = LU$$

Follow the process of Gaussian Elimination.

$$Ax = b : \quad Ly = b \rightarrow y = Ux$$

4 Vector Space

Theorem 5 (3 Characterization of subspaces). *The range of every linear function $f: \mathcal{R}^n \rightarrow \mathcal{R}^m$ is a subspace of \mathcal{R}^m , and every subspace of \mathcal{R}^m is the range of some linear function $g: \mathcal{R}^r \rightarrow \mathcal{R}^m$ ($r \leq m$).*

Theorem 6 (4 Testing for equal ranges). *For $m \times n$ matrices A and B the following statements are true.*

- $R(A^T) = R(B^T)$ if and only if $A \overset{row}{\sim} B$;
- $R(A) = R(B)$ if and only if $A \overset{col}{\sim} B$.

Theorem 7 (5 Testing for equal null spaces). *For $m \times n$ matrices A and B the following statements are true.*

- $N(A) = N(B)$ if and only if $A \overset{row}{\sim} B$;
- $N(A^T) = N(B^T)$ if and only if $A \overset{col}{\sim} B$.

Theorem 8 (7 Linear independence and rank). *If A is $m \times n$, then:*

- the columns of A form a linearly independent set if and only if either of the following holds: (i) $N(A) = \{0\}$, or (ii) $\text{rank}(A) = n$;
- the rows of A form a linearly independent set if and only if either of the following holds: (i) $N(A^T) = \{0\}$, or (ii) $\text{rank}(A) = m$;
- if A is a square matrix, then A is nonsingular if and only if:
 - the columns of A form a linearly independent set, or
 - the rows of A form a linearly independent set.

Theorem 9 (8 Maximal independent subsets). *if A is an $m \times n$ matrix and $\text{rank}(A) = r$, then:*

- any maximal independent subset of columns (rows) from A contains exactly r columns (rows);
- in particular, the r basic columns in A constitute one maximal independent subset of columns from A .

Theorem 10 (9 Basic facts of independence). *For a nonempty set of vectors $\mathcal{S} = \{u_1, u_2, \dots, u_n\}$ in a space \mathcal{V} , the following are true:*

- if \mathcal{S} contains a linearly dependent subset, then \mathcal{S} itself must be linearly dependent; conversely, if \mathcal{S} is linearly independent, then every subset of \mathcal{S} must also be linearly independent;
- if \mathcal{S} is linearly independent and if $v \in \mathcal{V}$, then the extension set $\mathcal{S}_{ext} = \mathcal{S} \cup \{v\}$ is linearly independent if and only if $v \notin \text{span}(\mathcal{S})$;
- if $\mathcal{S} \subseteq \mathcal{R}^m$ and if $n > m$, then \mathcal{S} must be linearly dependent.

Theorem 11 (11 Characterizations of a basis). *Let \mathcal{V} be a subspace of \mathcal{R}^m , and let $\mathcal{B} = \{b_1, b_2, \dots, b_n\} \subseteq \mathcal{V}$. The following statements are equivalent:*

- \mathcal{B} is a basis for \mathcal{V} ;
- \mathcal{B} is a minimal spanning set for \mathcal{V} ;
- \mathcal{B} is a maximal linearly independent subset of \mathcal{V} .

Theorem 12 (12 Dimension theorem). *Let \mathcal{V} be a subspace of \mathcal{R}^m . Then any two linearly independent spanning sets (i.e. any two bases) for \mathcal{V} must have the same number of elements.*

Theorem 13 (13 Rank plus nullity theorem). if A is an $m \times n$ matrix, then:

$$\dim R(A) + \dim N(A) = n$$

$$\dim R(A^T) + \dim N(A^T) = m$$

Theorem 14 (14 Dimension of a sum). If \mathcal{X} and \mathcal{Y} are subspaces of a vector space \mathcal{V} , then:

$$\dim(\mathcal{X} + \mathcal{Y}) = \dim \mathcal{X} + \dim \mathcal{Y} - \dim(\mathcal{X} \cap \mathcal{Y})$$

Summary of the rank . if A is an $m \times n$ matrix and $\text{rank}(A) = r$, then:

- $r =$ the number of nonzero rows in any row echelon form of A
- $=$ the number of pivots in any row echelon form of A
- $=$ the number of basic columns in A
- $=$ the size of a maximal independent set of columns from A
- $=$ the size of a maximal independent set of rows from A
- $= \dim \mathcal{R}(A)$
- $= \dim \mathcal{R}(A^T)$
- $= n - \dim \mathcal{N}(A)$
- $= m - \dim \mathcal{N}(A^T)$
- $=$ the size of the largest nonsingular submatrix in A

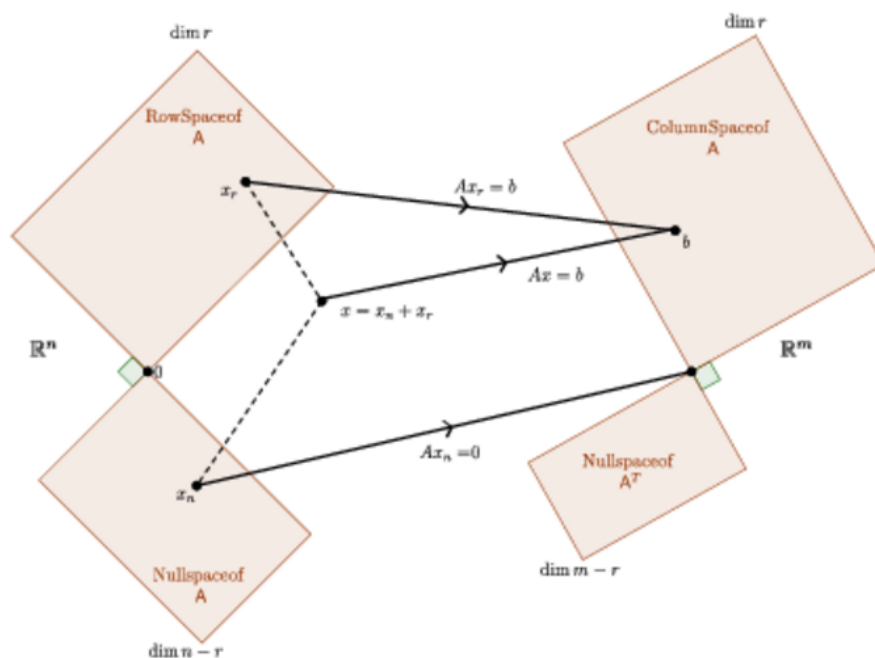


Figure 1: Four Fundamental Subspace of Matrix

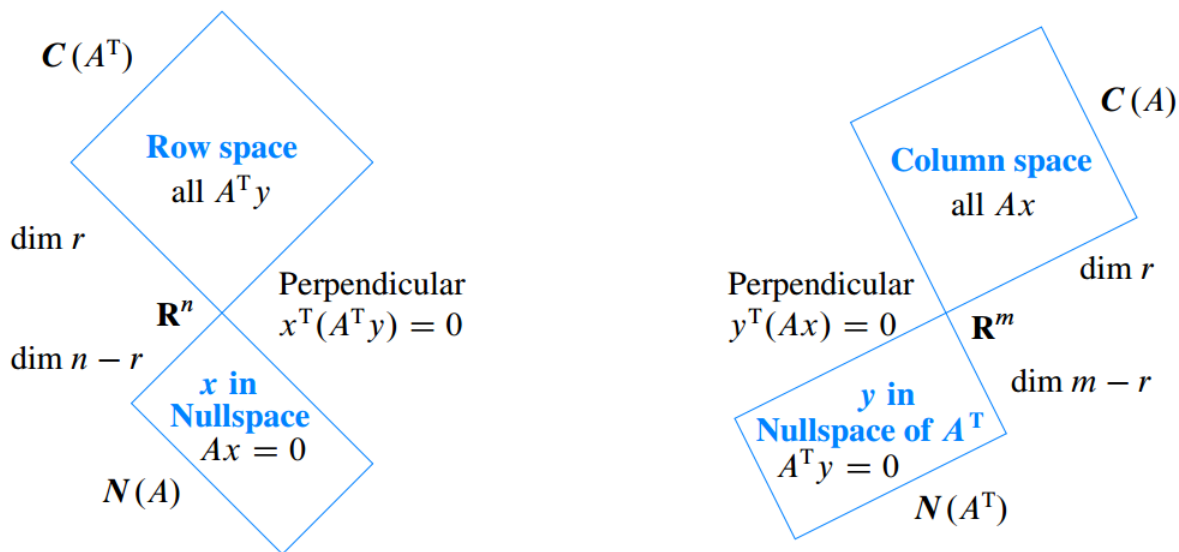


Figure 2: Dimensions and orthogonality for any m by n matrix A of rank r

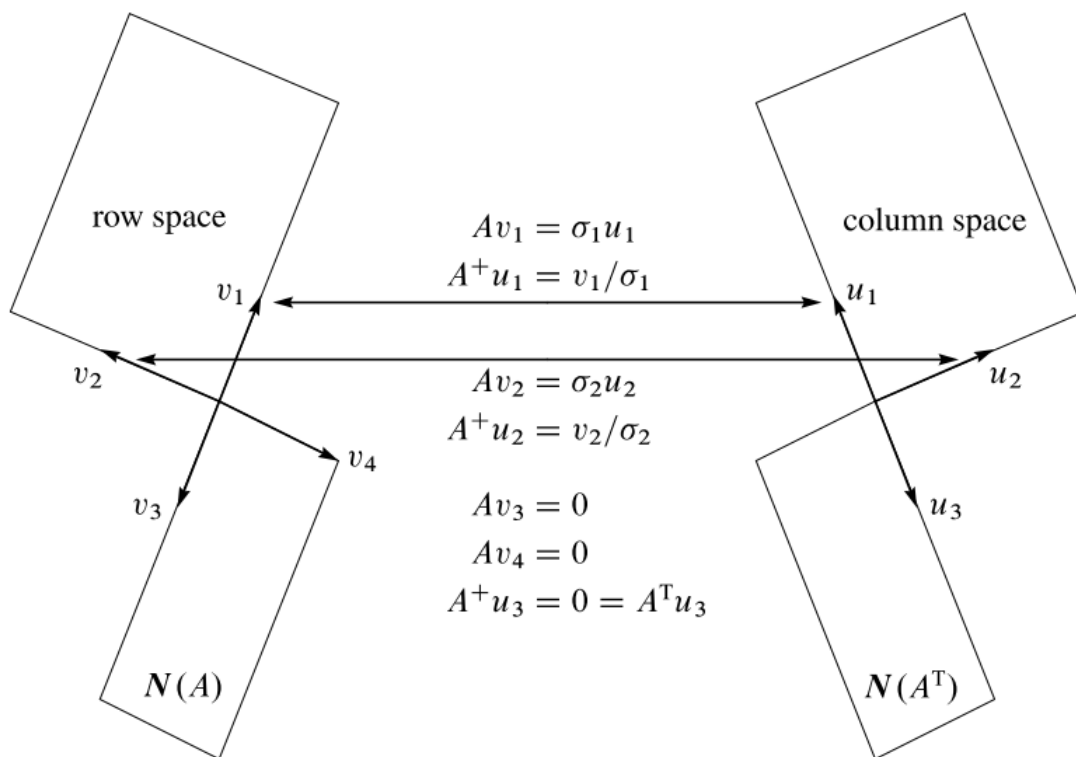


Figure 3: Orthonormal bases that diagonalize A (3 by 4) and A^+ (4 by 3)

5 Norms. Inner Product, and Orthogonality

Norm . For an $n \times 1$ vector x , the Euclidean norm of x is defined to be:

$$\|x\| = \left(\sum_{i=1}^n x_i^2 \right)^{\frac{1}{2}} = \sqrt{x^T x}$$

whenever $x \in \mathcal{R}^{n \times 1}$, or

$$\|x\| = \left(\sum_{i=1}^n x_i^2 \right)^{\frac{1}{2}} = \sqrt{x^* x}$$

whenever $x \in \mathcal{C}^{n \times 1}$

The p -norm can also defined as:

$$\|x\|_p = \left(\sum_{i=1}^n x_i^p \right)^{\frac{1}{p}} = (x^* x)^{\frac{1}{p}}$$

whenever $x \in \mathcal{C}^{n \times 1}$, $p \geq 1$

Standard Inner Product . The Euclidean vector norm can be viewed as a norm induced by the standard inner product:

$$\langle x, y \rangle = x^T y = \sum_{i=1}^n x_i y_i \quad \text{whenever } x, y \in \mathbb{R}^{n \times 1}$$

$$\langle x, y \rangle = x^* y = \sum_{i=1}^n \bar{x}_i y_i \quad \text{whenever } x, y \in \mathbb{C}^{n \times 1}$$

with the Euclidean norm defined by $\|x\| = \langle x, x \rangle^{1/2}$

Orthogonality . Let \mathcal{V} be either \mathbb{R}^n or \mathbb{C}^n , Two vectors $x, y \in \mathcal{V}$ are said to be orthogonal (to each other) if $\langle x, y \rangle = 0$, and this is denoted by writing $x \perp y$.

And note that:

- For \mathbb{R}^n with the standard product, $x \perp y \implies x^T y = 0$
- For \mathbb{C}^n with the standard product, $x \perp y \implies x^* y = 0$

General Angle . According to the law of cosine:

$$\|u - v\|^2 = \|u\|^2 + \|v\|^2 - 2\|u\|\|v\|\cos \theta$$

in general, it implies that:

$$\cos \theta = \frac{\|u\|^2 + \|v\|^2 - \|u - v\|^2}{2\|u\|\|v\|} = \frac{u^T v}{\|u\|\|v\|}$$

Therefore, the radian measure of the angle between two nonzero vectors $x, y \in \mathbb{R}^n$ is defined to be the number $\theta \in [0, \pi]$ such that:

$$\cos \theta = \frac{\langle x, y \rangle}{\|x\|\|y\|}$$

Orthogonal Set . Let $\mathcal{B} = \{u_1, u_2, \dots, u_n\}$ be a set of vectors in \mathbb{R}^n or \mathbb{C}^n , \mathcal{B} is called an orthogonal set if $\|u_i\| = 1$ for each i and $u_i \perp u_j$ for all $i \neq j$. In other words, \mathcal{B} is orthogonal if:

$$\langle u_i, u_j \rangle = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

where δ_{ij} is the classical Kronecker delta symbol.

Properties of Orthogonal Sets . Let $\mathcal{B} = \{u_1, u_2, \dots, u_r\}$ be an orthogonal set in $\mathbb{R}^n(\mathbb{C}^n)$, then:

- \mathcal{B} is linearly independent;
- if $r = n$, then \mathcal{B} forms an orthogonal basis for the $\mathbb{R}^n(\mathbb{C}^n)$.

Fourier Expansion . For an orthogonal basis $\mathcal{B} = \{u_1, u_2, \dots, u_r\}$:

- the expression:

$$x = \sum_{i=1}^n \langle u_i, x \rangle u_i$$

is called the Fourier expansion of x (with respect to the basis \mathcal{B}), and the scalars $\xi_i = \langle u_i, x \rangle$ are called the Fourier coefficients of x .

- geographically, the Fourier expansion resolves x into n mutually orthogonal vectors $\langle u_i, x \rangle u_i$, each of which represents the orthogonal projection of x onto the space (line) spanned by u_i .

Theorem 15 (Gram-Schmidt Procedure). *The complete Gram-Schmidt procedure proceeds as follows, where $\mathcal{V} = \{v_1, v_2, \dots, v_n\}$, and after the procedure into the orthogonal basis $\mathcal{B} = \{\eta_1, \eta_2, \dots, \eta_n\}$*

$$\begin{array}{ll} \beta_1 = v_1 & \implies \eta_1 = \frac{\beta_1}{\|\beta_1\|} \\ \beta_2 = v_2 - \langle v_2, \eta_1 \rangle \eta_1 & \implies \eta_2 = \frac{\beta_2}{\|\beta_2\|} \\ \beta_3 = v_3 - \langle v_3, \eta_1 \rangle \eta_1 - \langle v_3, \eta_2 \rangle \eta_2 & \implies \eta_3 = \frac{\beta_3}{\|\beta_3\|} \\ \vdots & \vdots \\ \beta_k = v_k - \sum_{i=1}^{k-1} \langle v_k, \eta_i \rangle \eta_i & \implies \eta_k = \frac{\beta_k}{\|\beta_k\|} \\ \vdots & \vdots \\ \beta_n = v_n - \sum_{i=1}^{n-1} \langle v_n, \eta_i \rangle \eta_i & \implies \eta_n = \frac{\beta_n}{\|\beta_n\|} \end{array}$$

Theorem 16 (Characteristic of Direct Sums). *Let \mathcal{V} be a vector space and \mathcal{X}, \mathcal{Y} be subspaces of \mathcal{V} with respective basis $\mathcal{B}_\mathcal{X}$ and $\mathcal{B}_\mathcal{Y}$. Then the following statements are equivalent:*

- $\mathcal{V} = \mathcal{X} \oplus \mathcal{Y}$
- for each $v \in \mathcal{V}$, there are unique $x \in \mathcal{X}$ and $y \in \mathcal{Y}$, such that $v = x + y$
- $\mathcal{B}_\mathcal{X} \cap \mathcal{B}_\mathcal{Y} = \emptyset$ (empty set) and $\mathcal{B}_\mathcal{X} \cup \mathcal{B}_\mathcal{Y}$ is a basis for \mathcal{V}

Orthogonal Complement . Let \mathcal{V} be either \mathbb{R}^n or \mathbb{C}^n and let \mathcal{M} be a subset of \mathcal{V} . The orthogonal complement \mathcal{M}^\perp of \mathcal{M} is the set of all vectors in \mathcal{V} that are orthogonal to every vector in \mathcal{M} . In other words:

$$\mathcal{M}^\perp = \{y \in \mathcal{V} : \langle x, y \rangle = 0, \forall x \in \mathcal{M}\}$$

Theorem 17 (Orthogonal Complementary Subspaces). *Let \mathcal{V} be either \mathbb{R}^n or \mathbb{C}^n . If \mathcal{M} is a subspace of \mathcal{V} , then:*

$$\mathcal{V} = \mathcal{M} \oplus \mathcal{M}^\perp$$

Furthermore, if \mathcal{N} is a subspace such that $\mathcal{V} = \mathcal{M} \oplus \mathcal{M}^\perp$ and $\mathcal{N} \perp \mathcal{M}$, then:

$$\mathcal{N} = \mathcal{M}^\perp$$

Theorem 18 (Perp Operation). *Let \mathcal{V} be either \mathbb{R}^n or \mathbb{C}^n and let \mathcal{M} be a subset of \mathcal{V} . Then the following statements are true:*

•

$$\dim \mathcal{M}^\perp = n - \dim \mathcal{M}$$

•

$$(\mathcal{M}^\perp)^\perp = \mathcal{M}$$

Theorem 19 (Orthogonal Decomposition Theorem). *For every $A \in \mathbb{R}^{m \times n}$, the following statements are true:*

$$\mathcal{R}(A)^\perp = \mathcal{N}(A^T)$$

$$\mathcal{N}(A)^\perp = \mathcal{R}(A^T)$$

Consequently, every matrix $A \in \mathbb{R}^{m \times n}$ produces an orthogonal decomposition of \mathbb{R}^m and \mathbb{R}^n in the sense that:

$$\mathbb{R}^m = \mathcal{R}(A) \oplus \mathcal{R}(A)^\perp = \mathcal{R}(A) \oplus \mathcal{N}(A^T)$$

$$\mathbb{R}^n = \mathcal{N}(A) \oplus \mathcal{N}(A)^\perp = \mathcal{N}(A) \oplus \mathcal{R}(A^T)$$

6 Determinants

Determinant . Let $A = [a_{ij}]$ be an $n \times n$ matrix. the determinant of A is defined to be the scalar:

$$\det(A) = \sum_p \sigma(p) a_{1p_1} a_{2p_2} \cdots a_{np_n}$$

where the sum is taken over the $n!$ permutations $p = (p_1, p_2, \dots, p_n)$ of $(1, 2, \dots, n)$. The determinant of A can be denoted by $\det(A)$ or $|A|$.

Triangular Determinants . For a triangular matrix, its determinant is equal to the product of its diagonal entries:

$$\det(T) = \begin{vmatrix} t_{11} & t_{12} & \cdots & t_{1n} \\ 0 & t_{22} & \cdots & t_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & t_{nn} \end{vmatrix} = t_{11} t_{22} \cdots t_{nn}$$

Theorem 20 (2.Effects of row operations). Let B be the matrix obtained from $A_{n \times n}$ by one of the three elementary row operations:

- type I: interchange rows i and j
- type II: multiply row i by $\alpha \neq 0$
- type III: add α times row i to row j

Then, the determinant $\det(B)$ is given by:

- $\det(B) = -\det(A)$ for the type I operations
- $\det(B) = \alpha \det(A)$ for the type II operations
- $\det(B) = \det(A)$ for the type III operations

Theorem 21 (3.Invertibility and Determinants). For every $n \times n$ matrix A , the following statements are true:

- A is non-singular if and only if $\det(A) \neq 0$, or equivalently
- A is singular if and only if $\det(A) = 0$

Theorem 22 (4.Product Rule). For all $n \times n$ matrices A and B ,

$$\det(AB) = \det(A)\det(B)$$

Cofactor Expansion . Let A be an $n \times n$ matrix, with $n \geq 2$. The (i, j) -minor of A , M_{ij} , is defined to be the determinant of the $(n-1) \times (n-1)$ submatrix of A obtained by deleting the i -th row and the j -th column of A . The (i, j) -cofactor of A , A_{ij} , is defined to be $(-1)^{i+j} M_{ij}$. And follow the definition, we can express the determinant as the following:

$$\det(A) = a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in}, \quad (\text{about the row } i)$$

$$\det(A) = a_{1j}A_{1j} + a_{2j}A_{2j} + \cdots + a_{nj}A_{nj}, \quad (\text{about the column } j)$$

Characteration of Nonsingular Matrices . If A is a $n \times n$ matrix, the the following statement are equivalent:

- A^{-1} exists (A is nonsingular)
- $\text{rank}(A) = n$

- $A \xrightarrow{G-J} I$
- $Ax = 0$ has only the trivial solution $x = 0$
- A is the product of elementary matrices of type I, II or III
- the columns of A forms a linearly independent set
- the rows of A forms a linearly independent set
- $\det(A) \neq 0$

7 Eigenvalue and Eigenvectors

Eigenvalue and Eigenvector . For an $n \times n$ matrix A , scalar λ and vectors $x_{n \times 1} \neq 0$ satisfying $Ax = \lambda x$ are called the eigenvalue and eigenvectors of A , and any pair, (λ, x) , is called an eigenpair for A . The set of distinct eigenvalues, denoted by $\sigma(A)$, is called the spectrum of A .

Similarity Transformation . Two $n \times n$ matrices A and B are said to be similar if there exists a non-singular matrix P such that $P^{-1}AP = B$. The product $P^{-1}AP$ is called a similarity transformation of A .

Diagonalizability . Let A be an $n \times n$ matrix

- A is said to be diagonalizable if A is similar to a diagonal matrix.
- A is said to have a complete set of eigenvectors if A has a set of n linearly independent eigenvectors, if A fails to possess a complete set of eigenvectors, then A is called deficient or defective.

Theorem 23 (4.Schurs triangularization theorem). *Every square matrix is unitarily similar to an upper-triangular matrix. That is, for each $n \times n$ matrix A , there exists a unitary matrix U (not unique) and an upper-triangular matrix T (not unique) such that $U^*AU = T$, and the diagonal entries of T are the eigenvalues of A .*

Theorem 24 (5.The Cayley-Hamilton Theorem). *Every square matrix satisfies its own characteristic equation $p(\lambda) = 0$*

Multiplicities . Let $\lambda \in \sigma(A) = \{\lambda_1, \lambda_2, \dots, \lambda_s\}$

- The algebraic multiplicity of λ , denoted by $\text{alg mult}_A(\lambda)$, is the number of times it is repeated as a root of the characteristic polynomial.
- when $\text{alg mult}_A(\lambda) = 1$, λ is called a simple eigenvalue.
- the geometric multiplicity of λ , denoted by $\text{geo mult}_A(\lambda)$, is the dimension of the eigenspace $\mathcal{N}(A - \lambda I)$.
- Eigenvalues such that $\text{alg mult}_A(\lambda) = \text{geo mult}_A(\lambda)$ are called semisimple eigenvalues of A .

Theorem 25 (7.Diagonalizability and multiplicities). *An $n \times n$ matrix A is diagonalizable if and only if:*

$$\text{geo mult}_A(\lambda) = \text{alg mult}_A(\lambda)$$

for each $\lambda \in \sigma(A)$, i.e. if and only if every eigenvalue is semisimple.