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Multi-variable Calculus

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A Eventually, everything is connected. . . . 33

Chapter 1
Vector and the Geometry of Space

1.1 Three-Dimensional Coordinate Systems

1. Three-dimensional **Rectangular** Coordinate System
   Defined by a triple (x, y, z), can be considered as an orthogonal completed set in three dimension, called the coordinate representation.

   The gradient expression:

$$\boxed{\nabla = \hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}} \tag{1.1}$$

2. Three-dimensional **Cylindrical** Coordinate System
   Defined by a triple (ρ, φ, z).
   Relationship between the (x, y, z) system:

$$\begin{array}{cc} x = \rho\sin\theta & \rho^2 = x^2 + y^2 \\ y = \rho\cos\theta & \tan\theta = \frac{y}{x} \end{array}$$

   The gradient expression:

$$\boxed{\nabla = \hat{\mathbf{r}}\frac{\partial}{\partial r} + \frac{1}{r}\hat{\theta}\frac{\partial}{\partial\theta} + \hat{\mathbf{z}}\frac{\partial}{\partial z}} \tag{1.2}$$

3. Three-dimensional **Spherical** Coordinate System
   Defined by a triple (ρ, θ, φ).
   Relationship between the (x, y, z) system:

$$\begin{array}{cc} x = \rho\sin\varphi\cos\theta & \rho^2 = x^2 + y^2 + z^2 \\ y = \rho\sin\varphi\sin\theta & \tan\theta = \frac{y}{x} \\ z = \rho\cos\varphi & \tan\varphi = \frac{\sqrt{x^2+y^2}}{z} \end{array}$$

   The gradient expression:

$$\boxed{\nabla = \hat{\mathbf{r}}\frac{\partial}{\partial r} + \frac{1}{r}\hat{\theta}\frac{\partial}{\partial\theta} + \hat{\varphi}\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}} \tag{1.3}$$

1.2 Equation for Line and Plane

1.2.1 Line Function

Line function given by the vector function is:

$$\mathbf{r} = \mathbf{r_0} + t\mathbf{v} \text{ where } \mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$$

expressed in the coordinate representation:

$$\begin{cases} x = x_0 + at \\ y = y_0 + bt \\ z = z_0 + ct \end{cases}$$

or can be expressed as following:

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c} = t$$

where the vector $\mathbf{v} = a\hat{\mathbf{i}} + b\hat{\mathbf{j}} + c\hat{\mathbf{k}}$ is the directional vector of this line.
With this, the angle between any two lines can be expressed as following:
Supposing there are two lines l_1 and l_2:

$$\text{line } l_1\text{: } \tfrac{x-x_0}{a_1} = \tfrac{y-y_0}{b_1} = \tfrac{z-z_0}{c_1} = t_1$$
$$\text{line } l_2\text{: } \tfrac{x-x_0}{a_2} = \tfrac{y-y_0}{b_2} = \tfrac{z-z_0}{c_2} = t_2$$

the angle between this two lines equal to the angle between the directional vectors of this two:

$$\cos\theta = \langle \mathbf{v_1}, \mathbf{v_2} \rangle = \frac{\mathbf{v_1} \cdot \mathbf{v_2}}{|\mathbf{v_1}||\mathbf{v_2}|}$$

Therefore, the expression of the angle between two lines are:

$$\boxed{\cos\theta = \frac{|a_1 a_2 + b_1 b_2 + c_1 c_2|}{\sqrt{{a_1}^2 + {b_1}^2 + {c_1}^2}\sqrt{{a_2}^2 + {b_2}^2 + {c_1}^2}}} \tag{1.4}$$

1.2.2 Plane Function

Plane function given by the vector function is:

$$\mathbf{n} \cdot \mathbf{r_0} = 0$$

where $\mathbf{r_0} = x_0\hat{\mathbf{i}} + y_0\hat{\mathbf{j}} + z_0\hat{\mathbf{k}}$ ((x_0, y_0, z_0) is a point in the plane) and the $\mathbf{n} = a\hat{\mathbf{i}} + b\hat{\mathbf{j}} + c\hat{\mathbf{k}}$ are the normal vector of this plane expressed in the coordinate representation:

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

where the vector $\mathbf{n} = a\hat{\mathbf{i}} + b\hat{\mathbf{j}} + c\hat{\mathbf{k}}$ is the normal vector of this plane.

With this, the expression of the angle between line and plane is the following:
Supposing there are one line l and one plane n:

$$\text{line } l\text{: } \tfrac{x-x_0}{a_1} = \tfrac{y-y_0}{b_1} = \tfrac{z-z_0}{c_1} = t$$
$$\text{plane } n\text{: } a_2(x - x_0) + b_2(y - y_0) + c_2(z - z_0) = 0$$

The angle between them are given by the angle between the directional vector of the line and the normal vector of the plane:

$$\boxed{\sin\varphi = \frac{|a_1 a_2 + b_1 b_2 + c_1 c_2|}{\sqrt{{a_1}^2 + {b_1}^2 + {c_1}^2}\sqrt{{a_2}^2 + {b_2}^2 + {c_1}^2}}} \tag{1.5}$$

1.2.3 Line function given by two plane

Notice that the directional vector is perpendicular to two normal vector of two plane $\mathbf{n_1} = a_1\hat{\mathbf{i}} + b_1\hat{\mathbf{j}} + c_1\hat{\mathbf{k}}$ and $\mathbf{n_2} = a_2\hat{\mathbf{i}} + b_2\hat{\mathbf{j}} + c_2\hat{\mathbf{k}}$:

$$\boxed{\mathbf{v} = \mathbf{n_1} \times \mathbf{n_2} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}} \tag{1.6}$$

Moreover the angle between this two plane are:

$$\boxed{\cos\theta = \frac{|a_1 a_2 + b_1 b_2 + c_1 c_2|}{\sqrt{{a_1}^2 + {b_1}^2 + {c_1}^2}\sqrt{{a_2}^2 + {b_2}^2 + {c_1}^2}}} \tag{1.7}$$

1.2.4 Distance

Supposing one point (x_0, y_0, z_0) and a plane with its normal vector $\mathbf{n} = a\hat{\mathbf{i}} + b\hat{\mathbf{j}} + c\hat{\mathbf{k}}$ or $ax + by + cz + d = 0$: the distance between this two can be expressed as the following:

$$\boxed{d = \frac{ax_0 + by_0 + cz_0 + d}{\sqrt{a^2 + b^2 + c^2}}} \tag{1.8}$$

1.3 Cylinders and Quadric Surfaces

Chapter 2
Vector Function

2.1 Vector Function

2.1.1 Definition

Basic form for vector function:

$$\mathbf{v}(t) = \langle f(t), g(t), h(t) \rangle = f(t)\hat{\mathbf{i}} + g(t)\hat{\mathbf{j}} + h(t)\hat{\mathbf{k}}$$

2.1.2 Limit

$$\begin{aligned} \lim_{t \to t_0} \mathbf{v}(t) &= \langle \lim_{t \to t_0} f(t), \lim_{t \to t_0} g(t), \lim_{t \to t_0} h(t) \rangle \\ &= \lim_{t \to t_0} f(t)\hat{\mathbf{i}} + \lim_{t \to t_0} g(t)\hat{\mathbf{j}} + \lim_{t \to t_0} h(t)\hat{\mathbf{k}} \end{aligned}$$

2.1.3 Continuity

$\mathbf{v}(t)$ is continuous at $t = t_0$, iff $\mathbf{v}(t_0)$ is defined.
Criteria for the continuity:

- $\lim_{t \to t_0} \mathbf{v}(t)$ exist
- $\lim_{t \to t_0} \mathbf{v}(t) = \mathbf{v}(t_0)$

2.1.4 Derivative

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{v}(t) &= \mathbf{v}'(t) = \langle f'(t), g'(t), h'(t) \rangle \\ &= \frac{\mathrm{d}}{\mathrm{d}t}f(t)\hat{\mathbf{i}} + \frac{\mathrm{d}}{\mathrm{d}t}g(t)\hat{\mathbf{j}} + \frac{\mathrm{d}}{\mathrm{d}t}h(t)\hat{\mathbf{k}} \end{aligned}$$

2.1.5 Operating Rules

- $\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{v}(t) \pm \mathbf{w}(t)) = \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{v}(t) \pm \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{w}(t)$
- $\frac{\mathrm{d}}{\mathrm{d}t}[c(t)\mathbf{v}(t)] = [\frac{\mathrm{d}}{\mathrm{d}t}c(t)]\mathbf{v}(t) + c(t)\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{v}(t)$
- $\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{v}(t) \cdot \mathbf{w}(t)) = \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{v}(t) \cdot \mathbf{w} + \mathbf{v}(t) \cdot \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{w}(t)$
- $\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{v}(t) \times \mathbf{w}(t)) = \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{v}(t) \times \mathbf{w} + \mathbf{v}(t) \times \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{w}(t)$
- $\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{v}(c(t)) = \mathbf{v}'(c(t))c'(t)$

2.1.6 Integration

$$\int_a^b \mathbf{v}(t)dt = \int_a^b f(t)dt\hat{\mathbf{i}} + \int_a^b g(t)dt\hat{\mathbf{j}} + \int_a^b h(t)dt\hat{\mathbf{k}}$$

Also similar to the principle of definite integral:
If $\mathbf{V}'(t) = \mathbf{v}(t)$, then

$$\int_a^b \mathbf{v}(t)dt = \mathbf{V}(t)|_a^b$$

2.1.7 Parameter Representation a tangent line of a space curve

$$\overrightarrow{\tau}(t) = \lim_{\Delta t \to 0} \frac{\overrightarrow{r}(t+\Delta t) - \overrightarrow{r}(t)}{\Delta t}$$

For convenience, we choose to get the unit vector of the tangent vector

$$\hat{\tau}(t) = \frac{\overrightarrow{\tau}(t)}{|\overrightarrow{\tau}(t)|}$$

2.1.8 Arc Length and Curvature

Given a curve $\mathcal{C}$: $x = x(t) \quad y = y(t) \quad z = z(t)$, for small Δt: the length between two points can be approximated as the distance between two points, thus we have the following formula:

$$\begin{aligned}\Delta \mathcal{L} &= \sqrt{|\mathbf{r}(t+\Delta t) - \mathbf{r}(t)|^2} \\ &= \sqrt{(x(t+\Delta t) - x(t))^2 + (y(t+\Delta t) - y(t))^2 + (z(t+\Delta t) - z(t))^2}\end{aligned}$$

$$\begin{aligned}\lim_{\Delta t \to 0} \frac{\Delta \mathcal{L}}{\Delta t} &= \sqrt{|\mathbf{r}(t+\Delta t) - \mathbf{r}(t)|^2} \\ &= \sqrt{\frac{(x(t+\Delta t) - x(t))^2}{\Delta t^2} + \frac{(y(t+\Delta t) - y(t))^2}{\Delta t^2} + \frac{(z(t+\Delta t) - z(t))^2}{\Delta t^2}} \\ \iff \lim_{\Delta t} \frac{\mathcal{L}(t+\Delta t) - \mathcal{L}(t)}{\Delta t} &= \sqrt{x'^2(t) + y'^2(t) + z'^2(t)} = \frac{d\mathcal{L}}{dt}\end{aligned}$$

Therefore, we can express the arc length in the differentiation form:

$$\boxed{d\mathcal{L} = \sqrt{x'^2(t) + y'^2(t) + z'^2(t)}dt} \tag{2.1}$$

If we reconsider the tangent vector $\overrightarrow{\tau}(t)$,

$$\begin{aligned}\overrightarrow{\tau}(t) &= x'(t)\hat{\mathbf{i}} + y'(t)\hat{\mathbf{j}} + z'(t)\hat{\mathbf{k}} = \mathbf{r}'(t) \\ \tfrac{d\mathcal{L}}{dt} &= \sqrt{x'^2(t)\hat{\mathbf{i}}^2 + y'^2(t)\hat{\mathbf{j}}^2 + z'^2(t)\hat{\mathbf{k}}^2} = |\mathbf{r}'(t)|\end{aligned}$$

therefore, we have the formula for the arc length:

$$\boxed{\mathcal{L} = s(t) = \int_{t_0}^{t} |\overrightarrow{\tau}(t)|dt = \int_{t_0}^{t} |\mathbf{r}'(t)|dt} \tag{2.2}$$

In this way, we can see that $\mathcal{L} = s(t)$ is a function of t, suppose that the inverse function exist $t = t(s)$, we can rewrite the curve with respect to s:

$$x = x(t) = x(t(s)) = X(s)$$
$$y = y(t) = y(t(s)) = Y(s)$$
$$z = z(t) = z(t(s)) = Z(s)$$

as that s is the parameter instead of t:

$$\mathcal{L} = s(t) = \int_{t_0}^{t} |\mathbf{r}'(t(s)|dt \qquad \frac{ds}{dt} = |\mathbf{r}'(t)|$$
$$\implies \tfrac{d\mathbf{r}(t(s))}{dt} = \tfrac{d\mathbf{r}}{ds}\tfrac{ds}{dt} = |\mathbf{r}'(t)|\tfrac{d\mathbf{r}}{ds}$$
$$\implies \tfrac{1}{|\mathbf{r}'(t)|}\tfrac{d\mathbf{r}}{dt} = \tfrac{d\mathbf{r}}{ds}$$

$$\boxed{\implies \hat{\tau}(t) = \frac{d\mathbf{r}}{ds}} \tag{2.3}$$

2.1.9 Curvature

Definition:
Supposing that $\hat{\tau}$ is the unit tangent vector of the curve $\mathcal{C}$, then the curvature is:

$$\boxed{\kappa = \left|\frac{d\hat{\tau}}{ds}\right| = \left|\frac{\frac{d\hat{\tau}}{dt}}{\frac{ds}{dt}}\right| = \left|\frac{\hat{\tau}'(t)}{\mathbf{r}'(t)}\right|} \tag{2.4}$$

We want to use $\mathbf{r}(t)$ to represent $\hat{\tau}'(t)$, so consider:

$$\mathbf{r}'(t) = |\mathbf{r}'(t)|\hat{\tau}(t) = \tfrac{ds}{dt}\hat{\tau}(t)$$
$$\mathbf{r}''(t) = \tfrac{d^2s}{dt^2}\hat{\tau}(t) + \tfrac{ds}{dt}\hat{\tau}'(t)$$
$$\mathbf{r}'(t) \times \mathbf{r}''(t) = (\tfrac{ds}{dt})^2(\hat{\tau}(t) \times \hat{\tau}'(t))$$
$$|\mathbf{r}'(t) \times \mathbf{r}''(t)| = |(\tfrac{ds}{dt})^2||(\hat{\tau} \times \hat{\tau}'| = |(\tfrac{ds}{dt})^2||\hat{\tau}||\hat{\tau}'| = |(\tfrac{ds}{dt})^2||\hat{\tau}'|$$

Thus,

$$\boxed{|\hat{\tau}'| = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{(\frac{ds}{dt})^2} = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^2}} \tag{2.5}$$

Therefore:

$$\boxed{\kappa = \frac{|\hat{\tau}'(t)|}{|\mathbf{r}'(t)|} = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}} \tag{2.6}$$

Note that:
for higher dimension general case, this equation always holds true.

For 1-dimensional special case:

$$\boxed{\kappa = \frac{|f''(x)|}{[1 + (f'(x))^2]^{\frac{3}{2}}}} \tag{2.7}$$

2.1.10 The Normal and Binormal Vectors

The principle unit normal vector:

$$\mathbf{N}(t) = \tfrac{\hat{\tau}'(t)}{|\hat{\tau}'(t)|}$$

Osculating plane: the plane that $\hat{\tau}(t)$ and $\mathbf{N}(t)$ span out.
Osculating circle: the approximation of the curve by circle with the radius $\rho = \frac{1}{\kappa}$
The binormal vector:

$$\mathbf{B}(t) = \hat{\tau}(t) \times \mathbf{N}(t)$$

Normal plane: the plane that $\mathbf{B}(t)$ and $\mathbf{N}(t)$ span out.

Chapter 3
Partial Derivative

3.1 Multi-variables Function

3.1.1 N-dimensional Space

n-th dimensional space can be considered as the n-th Cartesian product of $\mathbb{R} - \mathbb{R}^n$ which can be expressed as the n-th triple $(x_1, x_2. \cdots, x_n)$, and denote the distance between any point in this space and the origin as the norm: $\|\mathbf{x}\| = \sqrt{{x_1}^2 + {x_2}^2 + \cdots + {x_n}^2}$

3.1.2 Multi-variables Function

A quantity can depend on series variables eq. Temperature T can depend on position x and time t:

$$T = T(x, t)$$

Definition:
Let $\mathcal{D}$ be some region in the xy-plane. If there is a rule that assigns to each point in $\mathcal{D}(x, y)$ a unique real number $\mathbb{Z}$, this rule is called a function of there two variable (x, y), denoted as:

$$z = f(x, y)$$

Associated terminologies:
$\mathcal{D}$ is called the domain of the f
$\mathcal{R}$ is called the range of the f, which is the set of function value
Note that: the definition can be immediately extended to the function of more than three variables say:

$$g(x, y, z)$$
$$h(x, y, z, t)$$

3.1.3 Limit

Definition:
We say that $f(x, y)$ has the limit $\mathcal{L}$ as (x, y) tend to (x_0, y_0) if the following criterion holds:

For every $\varepsilon > 0$, there exist a $\delta > 0$, such that $|f(x, y) - \mathcal{L}| < \varepsilon$, whenever: $0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta$.
In this case, we write:

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = \mathcal{L}$$

If the limit exists, the value will not change along different paths.

Criteria of the nonexistence of the limit at one point:

If along two different paths, the values corresponding to each path are different, therefore, the limit at the point do not exist.
This method can be used to judge existence.

Example: Does the $\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$ exist?

Solution:
Along path one $y = x$

$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4} = \lim_{x\to 0} \frac{x^3}{x^2+x^4} = 0 = \mathcal{L}_1$$

Along path two $y = -x$

$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4} = \lim_{x\to 0} \frac{x^3}{x^2+x^4} = 0 = \mathcal{L}_2$$

Along path three $x = y^2$

$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4} = \lim_{y\to 0} \frac{y^4}{2y^4} = \frac{1}{2} = \mathcal{L}_3$$

Therefore, we see that $\mathcal{L}_1 = \mathcal{L}_2 \neq \mathcal{L}_3$, which indicates that the $\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$ does not exist.

Example: Prove that $\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} = 0.$

Solution: we need to find a $\delta < 0$, such that

$$0 < \sqrt{x^2+y^2} < \delta$$
$$\left|\tfrac{3x^2y}{x^2+y^2} - 0\right| < \varepsilon$$

After the scaling to the LHS of the second equation:

$$\left|\frac{3x^2y}{x^2+y^2}\right| \leq \left|\frac{3x^2y^2}{2xy}\right| \leq \frac{3}{2}|x| \leq \frac{3}{2}\sqrt{x^2+y^2} < \varepsilon$$

Comparing with the first formula, we get $\delta = \frac{2}{3}\varepsilon$
Therefore, the limit exist.

3.1.4 Continuity

The $f(x,y)$ is continuous at (x_0, y_0):

1. $f(x_0, y_0)$ is defined
2. $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$

Example: Determine whether

$$f(x,y) = \begin{cases} \frac{2xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

is continuous at $(0,0)$ or not?

Solution: $f(0,0) = 0$ defined

$$\lim_{(x,y)\to(0,0)} \frac{2xy}{x^2+y^2} = \begin{cases} \lim_{x\to 0} \frac{2x^2}{2x^2} = \frac{2}{3} = \mathcal{L}_1 & \text{path 1 } x = y \\ \lim_{x\to 0} \frac{2x(-x)}{2x^2} = -\frac{2}{3} = \mathcal{L}_2 & \text{path 2 } -x = y \end{cases}$$

$\mathcal{L}_1 \neq \mathcal{L}_2$ indicates that the limits does not exist, therefore, the function is discontinuous at this point.
Note: Continuity and limit can be expanded into higher dimension.

3.2 Definition

Partial derivative defined as following:

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$$

denoted as: $f_x(x, y)$ or $\frac{\partial f(x,y)}{\partial x}$ or z_x.

3.2.1 Calculation of the partial derivatives

To calculate $\frac{\partial f}{\partial x}$, we regard the y as a constant and then take the derivative w.r.t x.

3.2.2 Partial Derivatives Family

For the function of $z = f(x, y)$

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$$
$$\frac{\partial f}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$$

3.2.3 Example

Find the two partial derivatives of the function: $f(x, y) = \sqrt{2xy^2 + 3y}$ and the find their value at $(3, 3)$
Solution:

$$\frac{\partial f}{\partial x} = \frac{1}{2} \frac{\frac{\partial}{\partial x}(2xy^2 + 3y)}{\sqrt{2xy^2 + 3y}} = \frac{2y^2}{2\sqrt{2xy^2 + 3y}}$$
$$\frac{\partial f}{\partial y} = \frac{1}{2} \frac{\frac{\partial}{\partial y}(2xy^2 + 3y)}{\sqrt{2xy^2 + 3y}} = \frac{4xy + 3}{2\sqrt{2xy^2 + 3y}}$$
$$\left.\frac{\partial f}{\partial x}\right|_{(3,3)} = \frac{3}{\sqrt{7}} \qquad \left.\frac{\partial f}{\partial y}\right|_{(3,3)} = \frac{13}{2\sqrt{7}}$$

3.3 Higher Partial Derivatives

Partial derivatives can be easily expanded into higher order and higher dimensions.
For $g(x, y, z)$, we can compute its partial derivatives as the following formula:

$$\frac{\partial g(x,y,z)}{\partial x} = \lim_{\Delta x \to 0} \frac{g(x + \Delta x, y, z) - g(x, y, z)}{\Delta x}$$
$$\frac{\partial g(x,y,z)}{\partial y} = \lim_{\Delta y \to 0} \frac{g(x, y + \Delta y, z) - g(x, y, z)}{\Delta y}$$
$$\frac{\partial g(x,y,z)}{\partial z} = \lim_{\Delta z \to 0} \frac{g(x, y, z + \Delta z) - g(x, y, z)}{\Delta z}$$

For given $z = f(x, y)$, higher order of the partial derivatives are given as the following:

$$\frac{\partial z}{\partial x} = \frac{\partial f(x,y)}{\partial x} = g(x, y)$$
$$\frac{\partial z}{\partial y} = \frac{\partial f(x,y)}{\partial y} = h(x, y)$$

Therefore, for the second partial derivative:

$$\frac{\partial g(x,y)}{\partial x} = \frac{\partial}{\partial x}\frac{\partial f(x,y)}{\partial x} = \frac{\partial^2 f(x,y)}{\partial x^2} = f_{xx}(x, y)$$
$$\frac{\partial h(x,y)}{\partial y} = \frac{\partial}{\partial y}\frac{\partial f(x,y)}{\partial y} = \frac{\partial^2 f(x,y)}{\partial y^2} = f_{yy}(x, y)$$
$$\frac{\partial g(x,y)}{\partial y} = \frac{\partial}{\partial y}\frac{\partial f(x,y)}{\partial x} = \frac{\partial^2 f(x,y)}{\partial x \partial y} = f_{xy}(x, y)$$
$$\frac{\partial h(x,y)}{\partial x} = \frac{\partial}{\partial x}\frac{\partial f(x,y)}{\partial y} = \frac{\partial^2 f(x,y)}{\partial y \partial x} = f_{yx}(x, y)$$

About the last two second order partial derivatives symmetric to each other, their relationship can be determined by the Clairant's Theorem.

3.4 Clairant's Theorem

If $f(x, y)$ is defined in a region that contain (x_0, y_0), then given that f_{xy} and f_{yx} are continuous at (x_0, y_0), by this, the following formula hold true to itself:

$$\boxed{f_{xy}(x_0, y_0) = f_{yx}(x_0, y_0)} \tag{3.1}$$

Noted that: In MA2508 and MA2158, if not states, we always assume the condition in the theorem are satisfied: $f_{xy} = f_{yx}$.

In general, by Clairant's Theorem, higher order partial derivatives can be defined as following:

$$\frac{\partial^{m+n} f(x,y)}{\partial x^m \partial y^n} \text{ or } \frac{\partial^{a+b+c} g(x,y,z)}{\partial x^a \partial y^b \partial z^c} \text{ or even higher.}$$

Noted: For higher derivatives, computing order is important.

Example: Given

$$f(x, y) = (1 + xy)\ln(1 + x^2),$$

Find

$$\frac{\partial^8 f(x,y)}{\partial x^5 \partial y^3}$$

Solution:

$$\begin{cases} \frac{\partial^8 f(x,y)}{\partial x^5 \partial y^3} = \frac{\partial^7 f(x,y)}{\partial x^4 \partial y^3}\frac{\partial f(x,y)}{\partial x} = \frac{\partial^7 f(x,y)}{\partial x^4 \partial y^3}(y\ln(1 + x^3) + (1 + xy)\frac{2y}{1+x^2}) = \cdots \\ \frac{\partial^8 f(x,y)}{\partial x^5 \partial y^3} = \frac{\partial^7 f(x,y)}{\partial x^5 \partial y^2}\frac{\partial f(x,y)}{\partial y} = \frac{\partial^7 f(x,y)}{\partial x^5 \partial y^2}(x\ln(1 + x^2)) = 0 \end{cases}$$

3.5 Geometrical Representation

3.5.1 Tangent Line

The tangent line is given by the tangent vector on the surface $z = f(x, y)$ at the point (x_0, y_0). However, there are infinite number of tangent line at one point that consist a tangent plane. So firstly, let's consider the tangent vector along the fixed y section curve of the surface, the slope of this curve is given by the following:

$$k_y = \frac{\partial f}{\partial x}|_{(x_0,y_0)}$$
$$k_x = \frac{\partial f}{\partial y}|_{(x_0,y_0)}$$

Therefore, we can express the two tangent line of the surface as following;

$$l_y : \quad z - z_0 = \frac{\partial f}{\partial x}|_{(x_0,y_0)}(x - x_0)$$
$$l_x : \quad z - z_0 = \frac{\partial f}{\partial y}|_{(x_0,y_0)}(y - y_0)$$

Furthermore, if we know the two directional tangent vector, we can easily compute the tangent plane.

3.5.2 Tangent Plane

Given the two vectors of the above two directional vector, we can express the tangent plane as following:

For surface $z = f(x, y)$

$$\boxed{z - f(x_0, y_0) = \frac{\partial f}{\partial x}|_{(x_0,y_0)}(x - x_0) + \frac{\partial f}{\partial y}|_{(x_0,y_0)}(y - y_0)} \tag{3.2}$$

For the surface with implicit function $F(x, y, z) = 0$, the tangent plane:

$$\boxed{\frac{\partial F}{\partial x}|_{(x_0,y_0,z_0)}(x - x_0) + \frac{\partial F}{\partial y}|_{(x_0,y_0,z_0)}(y - y_0)\frac{\partial F}{\partial z}|_{(x_0,y_0,z_0)}(z - z_0) = 0} \tag{3.3}$$

Example: Find the equation for the tangent plane of the surface $z = \sin(x + y)$ at the point $(1, -1, 0)$
Solution:

$$\frac{\partial z}{\partial x}\Big|_{(1,-1,0)} = 1 \qquad \frac{\partial z}{\partial y}\Big|_{(1,-1,0)} = 1$$
$$z = (x - 1) + (y + 1) = x + y$$

3.6 Partial differential

3.6.1 Recall

If $y = f(x)$ is differentiable, that is $f(a + dx) = f(a) + \Delta y = f(a) + f'(a)dx + \varepsilon dx$, where dx is the increment with $\varepsilon \to 0$ as $dx \to 0$, then $dy|_a = f'(a)dx$ is called differential.

We can see that:

$$f(a + dx) - f(a) \approx dy|_a \text{ for small } dx$$
$$f(a + dx) \approx f(a) + dy|_a$$

The value at $a + dx$ can be approximately determined.

3.6.2 Definition

$z = f(x, y)$ is said to be differentiable at (x_0, y_0) if:

$$dz|_{(x_0,y_0)} = f(x + dx, y + dy) - f(x, y) = f_x(x_0, y_0)dx + f_y(x_0, y_0) + \varepsilon_1 dx + \varepsilon_2 dy$$

where $(\varepsilon_1, \varepsilon_2) \to (0, 0)$ when $(dx, dy) \to (0, 0)$.

We call $dz|_{(x_0,y_0)} = f(x + dx, y + dy) - f(x, y) = f_x(x_0, y_0)dx + f_y(x_0, y_0)$ to be the differential of f at (x_0, y_0)

3.6.3 Linear Approximation

$$f(x + dx, y + dy) \approx f(x_0, y_0) + f_x(x_0, y_0)dx + f_y(x_0, y_0) = f(x_0, y_0) + dz|_{(x_0,y_0)}$$
$$dz|_{(x_0,y_0)} = \frac{\partial f}{\partial x}\Big|_{(x_0,y_0)} dx + \frac{\partial f}{\partial y}\Big|_{(x_0,y_0)} dy$$

3.6.4 Criteria for Differential

If f_x and f_y exist in a neighborhood of (x_0, y_0) and continuous at (x_0, y_0), then $f(x, y)$ is differentiable at (x_0, y_0).

3.7 The Chain Rule

3.7.1 Recall and Analogy

$y = f(x)$, $x = g(t)$, the derivative of y w.r.t t is following:

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$$

Therefore, for differentiable function $z = f(x, y)$, where $x = g(t)$ and $y = h(t)$, we have the following formula:

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

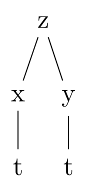
3.7.2 Proof

An increment Δt leads to the increment Δx, Δy and Δz.

$$\begin{aligned}\frac{dz}{dt} = \lim_{\Delta t\to 0}\frac{\Delta z}{\Delta t} &= \lim_{\Delta t\to 0}\frac{1}{\Delta t}(f_x(x,y)dx + f_y(x,y) + \varepsilon_1 dx + \varepsilon_2 dy) \\ &= \lim_{\Delta t\to 0}\frac{\Delta x}{\Delta t}f_x(x,y) + \lim_{\Delta t\to 0}\frac{\Delta y}{\Delta t}f_y(x,y) + \lim_{\Delta t\to 0}\varepsilon_1 \lim_{\Delta t\to 0}\frac{\Delta x}{\Delta t} + \lim_{\Delta t\to 0}\varepsilon_2 \lim_{\Delta t\to 0}\frac{\Delta y}{\Delta t} \\ &= f_x(x,y)\frac{dx}{dt} + f_y(x,y)\frac{dy}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}\end{aligned}$$

3.7.3 Tree diagram

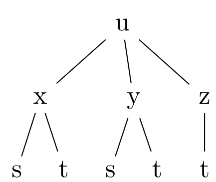
Procedure for writing down the chain formula:
To calculate the derivative of z w.r.t t, we can draw the tree diagram and count how many different path from z to t. For each path, calculate the product of derivative w.r.t the local variable. Finally, add all the product together.



$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

Example: Given $xy + yz + zx = u$, where $x = st$, $y = \exp st$ and $z = t^2$, find $\frac{dy}{ds}$

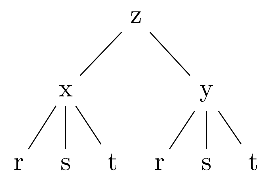
Solution:



$$\frac{du}{ds} = \frac{\partial u}{\partial x}\frac{dx}{ds} + \frac{\partial u}{\partial y}\frac{dy}{ds} = (y+z)t + (x+z)t\exp st$$

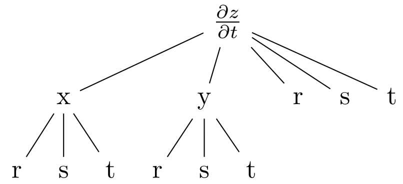
Example: Given: $z = \frac{x}{y}$, $x = r\exp st$, $y = rs\exp t$, find $\frac{\partial^2 z}{\partial t \partial r}$

Solution:



$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = \frac{1}{y}rs\exp st + (-\frac{x}{y^2}rs\exp st)$$

Then draw the second diagram:

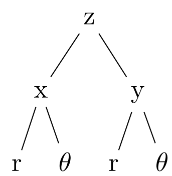


$$\frac{\partial^2 z}{\partial t \partial r} = \frac{\partial}{\partial r}\frac{\partial z}{\partial t} = \frac{\partial}{\partial x}(\frac{\partial z}{\partial t})\frac{\partial x}{\partial r} + \frac{\partial}{\partial y}(\frac{\partial z}{\partial t})\frac{\partial y}{\partial r} + \frac{\partial}{\partial r}(\frac{\partial z}{\partial t}) = \cdots$$

Example: (Important) If $z = f(x,y)$, where $x = r\cos\theta$, $y = r\sin\theta$ show that:

$$\left(\frac{\partial z}{\partial x}\right)^2 + (\frac{\partial z}{\partial x})^2 = \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2$$

Solution:



Path 1: from the R.H.S to the L.H.S

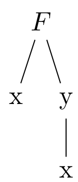
$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} = \cos\theta\frac{\partial z}{\partial x} + \sin\theta\frac{\partial z}{\partial y} \text{ (Eq1)}$$
$$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta} = -r\sin\theta\frac{\partial z}{\partial x} + r\cos\theta\frac{\partial z}{\partial y} \text{ (Eq2)}$$
$$R.H.S = (\frac{\partial z}{\partial r})^2 + \frac{1}{r^2}(\frac{\partial z}{\partial \theta})^2 = (\cos\theta\frac{\partial z}{\partial x} + \sin\theta\frac{\partial z}{\partial y})^2 + \frac{1}{r^2}(-r\sin\theta\frac{\partial z}{\partial x} + r\cos\theta\frac{\partial z}{\partial y})^2 =$$
$$(\frac{\partial z}{\partial x})^2\cos^2\theta + 2\frac{\partial z}{\partial y}\cos\theta\sin\theta + (\frac{\partial z}{\partial x})^2\sin^2\theta + (\frac{\partial z}{\partial x})^2\sin^2\theta - 2\frac{\partial z}{\partial y}\cos\theta\sin\theta + (\frac{\partial z}{\partial x})^2\cos^2\theta = L.H.S$$

Path 2: from the L.H.S to the R.H.S Eq1 $\times r\cos\theta-$ Eq2 $\times r\sin\theta$
Eq1 $\times r\sin\theta+$ Eq2 $\times r\cos\theta$
This yields:

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial r}\cos\theta - \frac{\partial z}{\partial \theta}\frac{\sin\theta}{r}$$
$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial r}\sin\theta + \frac{\partial z}{\partial \theta}\frac{\cos\theta}{r}$$
$$L.H.S = (\frac{\partial z}{\partial x})^2 + (\frac{\partial z}{\partial x})^2 = \cdots = R.H.S$$

3.8 Implicit Differentiation

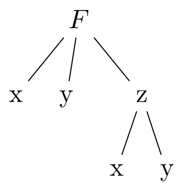
Given: $F(x,y) = 0$, and say y can be regarded as an implicit function of x. Find the $\frac{\mathrm{d}y}{\mathrm{d}x}$



Differentiating w.r.t x:

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0$$
$$\implies \frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}$$

Given: $F(x, y, z) = 0$, and say z can be regarded as an implicit function of x and y. Find the $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial x}$



Differentiating w.r.t x:

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0$$
$$\implies \frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}$$

Similarly, Differentiating w.r.t y:

$$\frac{\partial F}{\partial y} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial y} = 0$$
$$\implies \frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}$$

3.9 Directional Derivatives and Gradient Vector

Consider $z = f(x, y)$:

$$f_x(x_0, y_0) = \lim_{h\to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}$$
$$\text{(net change along the } x\text{-direction)}$$
$$f_x(x_0, y_0) = \lim_{h\to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h}$$
$$\text{(net change along the } y\text{-direction)}$$

Therefore, naturally, what is the net change in any arbitrary direction $\hat{\mathbf{u}} = a\hat{\mathbf{i}} + b\hat{\mathbf{j}}$

$$x_1 = x_0 + h\cos\theta \qquad a = \hat{\mathbf{u}}\cdot\hat{\mathbf{i}}$$
$$y_1 = y_0 + h\sin\theta \qquad b = \hat{\mathbf{u}}\cdot\hat{\mathbf{j}}$$
$$f_u(x_0, y_0) = \lim_{h\to 0} \frac{f(x_0 + ah, y_0 + bh) - f(x_0, y_0)}{h}$$

If the limit exists, we call it to be the directional derivatives of this function $f(x, y)$ at (x_0, y_0) along $\hat{\mathbf{u}}$-direction and denoted it by $D_{\hat{\mathbf{u}}}f(x_0, y_0)$

3.9.1 Theorem

If f is differentiable function of x and y

$$D_{\hat{\mathbf{u}}}f(x_0, y_0) = f_x(x_0, y_0)a + f_y(x_0, y_0)b \qquad \hat{\mathbf{u}} = a\hat{\mathbf{i}} + b\hat{\mathbf{j}}$$

Proof: define $f(x_0 + ha, y_0 + hb) = g(h)$
Then $g(0) = f(x_0, y_0)$, by definition:

$$D_{\hat{\mathbf{u}}}f(x_0, y_0) = \lim_{h\to 0}\frac{g(h) - g(0)}{h} = \left.\frac{dg}{dh}\right|_{h=0}$$

Let $x = x_0 + ah$ and $y = y_0 + bh$ and differentiating w.r.t h:

$$\begin{array}{ccc} & g & \\ / & & \backslash \\ \text{x} & & \text{y} \\ | & & | \\ \text{h} & & \text{h} \end{array}$$

$$\begin{aligned} \frac{dg}{dh}|_{h=0} &= \frac{\partial f}{\partial x}|_{(x_0,y_0)}\frac{dx}{dh}|_{h=0} + \frac{\partial f}{\partial y}|_{(x_0,y_0)}\frac{dy}{dh}|_{h=0} \\ &= f_x(x_0,y_0)a + f_y(x_0,y_0)b \end{aligned}$$

Observation:

We can write the directional derivatives as the dot product of the gradient vector and the directional vector:

$$D_{\hat{\mathbf{u}}} f(x_0, y_0) = \nabla f(x_0, y_0) \cdot \hat{\mathbf{u}}$$
$$\nabla f(x_0, y_0) = \frac{\partial f}{\partial x}|_{(x_0,y_0)}\hat{\mathbf{i}} + \frac{\partial f}{\partial y}|_{(x_0,y_0)}\hat{\mathbf{j}}$$
$$\text{Where the operator: } \nabla = \frac{\partial}{\partial x}\hat{\mathbf{i}} + \frac{\partial}{\partial y}\hat{\mathbf{j}}$$

Note: This gradient can be extended to higher dimension.

3.9.2 Example

Find the directional derivative of $g(x, y) = \exp x \cos y$ at $(1, \frac{\pi}{6})$ in the directional vector $\mathbf{v} = \hat{\mathbf{i}} - \hat{\mathbf{j}}$:

Solution:

$$\hat{\mathbf{v}} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\hat{\mathbf{i}}-\hat{\mathbf{j}}}{\sqrt{1^2+1^2}} = \frac{\hat{\mathbf{i}}}{\sqrt{2}} - \frac{\hat{\mathbf{j}}}{\sqrt{2}}$$
$$\nabla g(x, y) = \frac{\partial g}{\partial x}\hat{\mathbf{i}} + \frac{\partial g}{\partial y}\hat{\mathbf{j}} = \exp x(\cos y\hat{\mathbf{i}} - \sin y\hat{\mathbf{j}})$$
$$\nabla g(1, \tfrac{\pi}{6}) = e(\frac{\sqrt{3}\hat{\mathbf{i}}}{2} - \frac{\hat{\mathbf{j}}}{2})$$

Then, after the dot product

$$\nabla g(1, \tfrac{\pi}{6}) \cdot \hat{\mathbf{v}} = \frac{\sqrt{3}+1}{2\sqrt{2}}e$$

3.9.3 Extension

To function of three variables (dimension)

$$\hat{u} = a\hat{i} + b\hat{j} + c\hat{k} \qquad w = f(x, y, z)$$
$$\implies D_{\hat{u}} f(x, y, z) = \lim_{h\to 0} = \frac{f(x + ha, y + hb, z+, hc) - f(x, y, z)}{h}$$
$$\nabla f(x, y, z) = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$
$$\boxed{D_{\hat{u}} f(x, y, z) = \nabla f(x, y, z) \cdot \hat{u}}$$

3.9.4 Fastest Changing Direction

Theorem: The maximum value of the $D_{\hat{u}} f(\mathbf{v})$ is the $|\nabla f(\mathbf{r})|$

Proof:

$$\begin{aligned} D_{\hat{u}} f(\mathbf{r}) &= \nabla f(\mathbf{r}) \cdot \hat{u} \\ &= |\nabla f(\mathbf{r})| \cdot |\hat{u}| \cos\theta \le |\nabla f(\mathbf{r})| \end{aligned}$$

This suggests: along the gradient direction, the function changes fastest.

3.9.5 Geometrical Meaning of the Gradient Vector

Consider a surface $\mathcal{S}$ given by:

$$F(x, y, z) = 0$$

Then, we consider an arbitrary curve $\mathcal{C}$ on $\mathcal{S}$ surface that suppose that $\mathcal{C}$: $x = x(t), y = y(t), z = z(t)$

$$F(x(t), y(t), z(t)) = 0$$

Differentiating this equation w.r.t t

$$\tfrac{\partial F}{\partial x}\tfrac{dx}{dt} + \tfrac{\partial F}{\partial y}\tfrac{dy}{dt} + \tfrac{\partial F}{\partial z}\tfrac{dz}{dt} = 0$$

Then, we denote:

$$\begin{aligned}\mathbf{r}'(t) &= \frac{dx}{dt}\hat{\mathbf{i}} + \frac{dy}{dt}\hat{\mathbf{j}} + \frac{dz}{dt}\hat{\mathbf{k}} \\ \nabla F(x, y, z) &= \frac{\partial F}{\partial x}\hat{\mathbf{i}} + \frac{\partial F}{\partial y}\hat{\mathbf{j}} + \frac{\partial F}{\partial z}\hat{\mathbf{k}}\end{aligned}$$

After this, rewrite the differentiating equation:

$$\boxed{\nabla F(x, y, z) \cdot \mathbf{r}'(t) = 0} \tag{3.4}$$

This means that: $\nabla F(x, y, z)$ is the normal vector of the tangent plane since $\nabla F(x, y, z)$ is perpendicular to any tangent vector along an arbitrary curve.

Given surface $F(x, y, z) = 0$:

The normal vector gives by the following equation:

$$\boxed{\nabla F(x, y, z) = \frac{\partial F}{\partial x}\hat{\mathbf{i}} + \frac{\partial F}{\partial y}\hat{\mathbf{j}} + \frac{\partial F}{\partial z}\hat{\mathbf{k}}} \tag{3.5}$$

substitute (x_0, y_0, z_0) to determine the normal vector at point (x_0, y_0, z_0).

3.9.6 Example

Find the tangent plane of the surface at $(1, 0, 5)$:

$$xe^{yz} = 1$$

Solution:
Follow thee equation:
the tangent vector can be obtained:

$$\begin{aligned}\nabla F(x, y, z) &= \frac{\partial F}{\partial x}\hat{\mathbf{i}} + \frac{\partial F}{\partial y}\hat{\mathbf{j}} + \frac{\partial F}{\partial z}\hat{\mathbf{k}} \\ &= e^{yz}\hat{\mathbf{i}} + xze^{yz}\hat{\mathbf{j}} + xye^{yz}\hat{\mathbf{k}} \\ &= \hat{\mathbf{i}} + 5\hat{\mathbf{j}} + 0\hat{\mathbf{k}}\end{aligned}$$

Therefore, the equation of tangent plane:

$$(x - 1) + 5(y - 0) = 0$$

3.10 Maximum and Minimum Values

Definition: (For) z=f(x,y) defined in the a region $\mathcal{D}$ in the xy-plane)
$f(x,y)$ has local maximum (minimum) at (a,b) if:

$$f(x,y) \leq f(a,b)$$

for all points (x,y) in the neighbourhood of (a,b).

If this relation holds true to all points in the region $\mathcal{D}$, then $f(x,y)$ has an absolute maximum (minimum) at (a,b).

3.10.1 Theorem

If f has a total extremum at (a,b), the first order partial derivatives exist at (a,b), then:

$$f_x(a,b) = f_y(a,b) = 0$$

3.10.2 Critical Point for Extremum

- point that satisfy both $f_x = 0$ and $f_y = 0$.
- one or both partial derivatives do not exist.

local extremums must occur at those point, which one called critical points.

3.10.3 Example

Find all the critical point of this function:

$$f(x,y) = y\sqrt{x} - y^2 - x + 6y \qquad (x \geq 0)$$

Solution:

$$\begin{aligned} f_x &= y\frac{1}{2\sqrt{x}} - 1 \\ f_y &= \sqrt{x} - 2y + 6 \end{aligned}$$

From those formula, we can see that:
f_x doesn't exist at $x = 0$,thus ant point lay on the y-axis is critical point.
Then:

$$\begin{aligned} f_x &= y\frac{1}{2\sqrt{x}} - 1 = 0 \\ f_y &= \sqrt{x} - 2y + 6 = 0 \end{aligned}$$

this gives another critical point $(4,4)$

3.11 Second Derivative Test

Suppose that the second derivatives of $f(x,y)$ are continuous in a disc centered at (a,b), $f_x(a,b) = f_y(a,b) = 0$

let:

$$\boxed{\mathcal{D} = \begin{vmatrix} f_{xx}(a,b) & f_{xy}(a,b) \\ f_{xy}(a,b) & f_{yy}(a,b) \end{vmatrix}}$$

Then the second derivative test gives:

- $\mathcal{D} > 0$ & $f_{xx}(a,b) > 0 \rightarrow$ Local minimum
- $\mathcal{D} > 0$ & $f_{xx}(a,b) < 0 \rightarrow$ Local maximum
- $\mathcal{D} < 0 \rightarrow$ Sudden point
- $\mathcal{D} = 0 \rightarrow$ Failure

3.11.1 Absolute Test

For the absolute extremum, we have not only carry out the second derivative test but also the boundary test.
For $y = f(x,y)$ defined in a region $\mathcal{D}$, then $f(x,y)$ has the absolute maximum or absolute minimum at either critical point or some point on the boundary of $\mathcal{D}$

Chapter 4
Multiple Integral

4.1 Double Integral

4.1.1 Definition

Recall in two dimensional case:

The integral is defined as following:

$$\int_a^b f(x) = \lim_{\Delta x \to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

Then, move on to the higher dimension case, say three dimension case:

Consider $f(x,y)$ defined in a region $\mathcal{R}$ in the xy-plane.

$$\int_c^d \int_a^b f(x,y)dxdy = \lim_{\substack{\Delta x \to 0 \\ \Delta x \to 0}} \sum_{j=1}^{m} \sum_{i=1}^{n} f(x_i^*, y_j^*)\Delta A_{ij}$$

If the limit exists,we called it the double integral for this function within the region $\mathcal{R}$ and denoted by:

$$\boxed{\iint_{\mathcal{R}} f(x,y)dA = \iint_{\mathcal{R}} f(x,y)dxdy} \tag{4.1}$$

In order to evaluate the double integral by the definite integral (For the single variables integral), we introduce by double iterated integral directly from definite integrals.

consider $f(x,y)$. if we hold y as a constant, then we may regard $f(x,y)$ as a function of single variables x, then:

$$\int_a^b f(x,y)dx = A(y)$$
$$\int_c^d A(y)dy = \int_c^d \left(\int_a^b f(x,y)dx\right) dy$$
$$\int_c^d f(x,y)dy = B(x)$$
$$\int_a^b B(x)dx = \int_a^b \left(\int_c^d f(x,y)dy\right) dx$$

Generalized result in two different evaluation process corresponding to the simplification:

$$\boxed{\int_c^d \left(\int_{h_1(y)}^{h_2(y)} f(x,y)dx \right) dy \quad \& \quad \int_a^b \left(\int_{g_1(x)}^{g_2(x)} f(x,y)dy \right) dx}$$

4.1.2 Evaluation in Different Cases

- Rectangular Case:

$$\begin{aligned}
\iint_{\mathcal{R}} f(x,y)dA &= \lim_{\substack{\Delta x\to 0\\ \Delta x\to 0}} \sum_{j=1}^{m}\sum_{i=1}^{n} f(x_i^*,y_j^*)\Delta x_i \Delta y_j \\
&= \lim_{\Delta y\to 0} \sum_{j=1}^{m} \left(\lim_{\Delta x\to 0} \sum_{i=1}^{n} f(x_i^*,y_j^*)\Delta x_i \right) \Delta y_j \\
&= \lim_{\Delta y\to 0} \sum_{j=1}^{m} \left(\int_a^b f(x,y)dx \right) \Delta y_j \\
&= \int_c^d \int_a^b f(x,y)dxdy
\end{aligned}$$

Therefore, in rectangular case, we can convert the double integral to the double iterated integral:

$$\begin{aligned}
\iint_{\mathcal{R}} f(x,y)dA = \quad \cdots \quad &= \int_c^d \int_a^b f(x,y)dxdy \\
\iint_{\mathcal{R}} f(x,y)dA = \quad \cdots \quad &= \int_a^b \int_c^d f(x,y)dydx
\end{aligned}$$

These two relationship hold true only for the rectangular region.
Furthermore, Fubini's Theorem is based on the rectangular region

$$\mathcal{R} = [a,b]\times[c,d]$$

then the double integral can be evaluated in a simple way:

$$\iint_{\mathcal{R}} f(x)g(y)dA = \int_a^b f(x)dx \int_c^d g(y)dy$$

- $y = h_i(x)$ bounded region case:
In this case, region $\mathcal{R} = (x,y)|a\le x\le b, h_1(x)\le y\le h_2(x)$

$$\begin{aligned}
\iint_{\mathcal{R}} f(x,y)dA &= \lim_{\substack{\Delta x\to 0\\ \Delta x\to 0}} \sum_{i=1}^{n}\sum_{j=1}^{m} f(x_i^*,y_j^*)\Delta y_j \Delta x_i \\
&= \lim_{\Delta x\to 0} \sum_{i=1}^{n} \left(\lim_{\Delta y\to 0} \sum_{j=1}^{m} f(x_i^*,y_j^*)\Delta y_j \right) \Delta x_i \\
&= \lim_{\Delta x\to 0} \sum_{i=1}^{n} \left(\int_{h_1(x)}^{h_2(x)} f(x,y)dy \right) \Delta x_i \\
&= \int_a^b \left(\int_{h_1(x)}^{h_2(x)} f(x,y)dy \right) dx
\end{aligned}$$

- $x = g_i(y)$ bounded region case:
In this case, region $\mathcal{R} = (x,y)|g_1(x)\le x\le g_2(x), c\le y\le d$

$$\iint_{\mathcal{R}} f(x,y)dA = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \sum_{i=1}^{n} \sum_{j=1}^{m} f(x_i^*, y_j^*) \Delta x_i \Delta y_j$$
$$= \int_c^d \left(\int_{g_1(y)}^{g_2(y)} f(x,y)dx \right) dy$$

- Two well-defined boundary region case (two point closed loop):
  In this case, region $\mathcal{R} = (x,y)|g_1(x) \leq x \leq g_2(x), c \leq y \leq d$or
  $\mathcal{R} = (x,y)|a \leq x \leq b, h_1(x) \leq y \leq h_2(x)$

$$\iint_{\mathcal{R}} f(x,y)dA = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \sum_{i=1}^{n} \sum_{j=1}^{m} f(x_i^*, y_j^*) \Delta y_j \Delta x_i$$
$$= \int_a^b \left(\int_{h_1(x)}^{h_2(x)} f(x,y)dy \right) dx$$
$$= \int_c^d \left(\int_{g_1(y)}^{g_2(y)} f(x,y)dx \right) dy$$

Note:

- For an irregular regions, we divided it into regular regions.

$$\iint_{\mathcal{R}} f(x,y)dA = \iint_{\mathcal{R}_1} f(x,y)dA + \iint_{\mathcal{R}_2} f(x,y)dA$$

- A double integral (DI) can be evaluated ny a double iterated integral (DII) while DII can be evaluated by some DI

4.1.3 Properties of DI

The equations:

1. $\iint_{\mathcal{R}} dA = \text{area of } \mathcal{R}$
2. if $\mathcal{R} = \mathcal{R}_1 + \mathcal{R}_2$, then:

$$\iint_{\mathcal{R}} f(x,y)dA = \iint_{\mathcal{R}_1} f(x,y)dA + \iint_{\mathcal{R}_2} f(x,y)dA$$

3. $\iint_{\mathcal{R}} [f(x,y) \pm g(x,y)]dA = \iint_{\mathcal{R}} f(x,y)dA \pm \iint_{\mathcal{R}} g(x,y)dA$
4. $\iint_{\mathcal{R}} cf(x,y)dA = c \iint_{\mathcal{R}} f(x,y)dA$

The inequality:

1. if $m \leq f(x,y) \leq M$, for all (x,y) in $\mathcal{R}$:

$$mA(\mathcal{R}) \leq \iint_{\mathcal{R}} f(x,y)dA \leq MA(\mathcal{R})$$

4.2 Double Integral in Polar Coordinates

Consider the following function:

$$\iint_{\mathcal{R}} f(x,y)dA$$

take an arbitrary point (x_{ij}^*, y_{ij}^*), say $(x_{ij}^*, y_{ij}^*) = (r_i \cos\theta_j, r_i \sin\theta_j)$, then the integral can be rewritten as the following:

$$\begin{aligned}\iint_{\mathcal{R}} f(x,y)dA &= \lim_{\substack{\Delta r \to 0 \\ \Delta\theta \to 0}} \sum_{i=1}^{n} \sum_{j=1}^{m} f(r_i \cos\theta_j, r_i \sin\theta_j) \Delta A_{ij} \\ &= \cdots = f(r_i \cos\theta_j, r_i \sin\theta_j)\end{aligned}$$

Generalization: As we can use the Jacobian Matrix to generalize the gradient of the scalar field. The Jacobian can also be thought of as describing the amount of "stretching", "rotating" or "transforming" that a transformation imposes locally.

Therefore, in the polar coordinates case:

$$\begin{aligned}dx\,dy &= \det \mathcal{J}(r,\theta) dr d\theta \\ &= \det \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} dr d\theta \\ &= \det \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} dr d\theta \\ &= r dr d\theta\end{aligned}$$

Thus we obtain:

$$dx\,dy = r\,dr\,d\theta$$

Therefore, after substitute into the definition of the double integral in the polar coordinates:

- $\iint_{\mathcal{R}} f(x,y)dA = \int_{\alpha_1}^{\alpha_2} \int_{\rho_1}^{\rho_2} f(r_i \cos\theta_j, r_i \sin\theta_j) r dr d\theta$
- $\iint_{\mathcal{R}} f(x,y)dA = \int_{\alpha_1}^{\alpha_2} \int_{h_1(\theta)}^{h_2(\theta)} f(r_i \cos\theta_j, r_i \sin\theta_j) r dr d\theta$
- $\iint_{\mathcal{R}} f(x,y)dA = \int_{\rho_1}^{\rho_2} \int_{g_1(r)}^{g_1(r)} f(r_i \cos\theta_j, r_i \sin\theta_j) r d\theta dr$

Note that: Usually, if the region involves a circle or part of a circle, we use the polar coordinate to the evaluation, otherwise we use the rectangular coordinates.

4.2.1 Example

Evaluate:

$$\iint_{\mathcal{R}} x\,dA$$

where the region $\mathcal{R}$ is bounded by $x^2 + y^2 = 2x$

Solution: First we find the polar coordinate representation of $\mathcal{R}$:

$$r^2 = 2r\cos\theta$$

then, we start to evaluate the integral:

$$\begin{aligned}
\iint_{\mathcal{R}} x dA &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{0}^{2\cos\theta} r\cos\theta r dr d\theta \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{\cos\theta}{3} r^3\right) |_0^{2\cos\theta} d\theta \\
&= \frac{8}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^4\theta d\theta \\
&= \frac{8}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{1+\cos 2\theta}{2}\right)^2 d\theta \\
&= \frac{2}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(1+2\cos 2\theta+\cos^2 2\theta\right)^2 d\theta = \cdots
\end{aligned}$$

4.3 Applications of Double Integral

4.3.1 Area

$$\iint_{\mathcal{R}} dA = A(\mathcal{R})$$

4.3.2 Mass of a Plate

Consider a plate with density varies in the density function $\rho(x, y)$, find the mass of this plate:

$$\begin{aligned}
M &= \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \sum_{j=1}^{m}\sum_{i=1}^{n} \rho(x_{ij}^*, y_{ij}^*)\Delta A_{ij} \\
&= \iint_{\mathcal{R}} \rho(x, y) dA
\end{aligned}$$

4.3.3 Moment of a Plate

After finding the mass of a plate with density function $\rho(x, y)$, we can further find the moment of a plate:

1. Moment about the x-direction
$$\iint_{\mathcal{R}} y\rho(x, y) dA$$
2. Moment about the x-direction
$$\iint_{\mathcal{R}} x\rho(x, y) dA$$
3. Center of mass (x, y)
$$m\overline{x} = M_y \implies \overline{x} = \frac{M_y}{m} = \frac{\iint_{\mathcal{R}} x\rho(x, y) dA}{\iint_{\mathcal{R}} \rho(x, y) dA}$$
$$m\overline{y} = M_x \implies \overline{y} = \frac{M_x}{m} = \frac{\iint_{\mathcal{R}} y\rho(x, y) dA}{\iint_{\mathcal{R}} \rho(x, y) dA}$$

4.3.4 Moment of Inertia

1. moment of inertia about x-axis
$$\iint_{\mathcal{R}} y^2 \rho(x,y) dA$$
2. moment of inertia about y-axis
$$\iint_{\mathcal{R}} x^2 \rho(x,y) dA$$
3. moment of inertia about origin
$$\iint_{\mathcal{R}} (x^2 + y^2) \rho(x,y) dA$$

4.3.5 Radii of Gyration

$$\bar{\bar{x}}^2 = \frac{\iint_{\mathcal{R}} x^2 \rho(x,y) dA}{\iint_{\mathcal{R}} \rho(x,y) dA}$$
$$\bar{\bar{y}}^2 = \frac{\iint_{\mathcal{R}} y^2 \rho(x,y) dA}{\iint_{\mathcal{R}} \rho(x,y) dA}$$

4.4 Surface Area

Given a surface $z = f(x,y)$, the following equation gives the surface area:

$$\mathcal{S} = \iint_{\mathcal{R}} \sqrt{1 + |\nabla f(x,y)|^2} dA$$

$$\mathcal{S} = \iint_{\mathcal{R}} \sqrt{1 + \frac{\partial f}{\partial x}^2 + \frac{\partial f}{\partial y}^2} dA$$

4.5 Triple Integral

Consider a three dimensional function $f(x,y,z)$ which is defined in a region $\mathcal{D}$ on the space bounded by surface $\mathcal{S}$. We divided $\mathcal{D}$ into many small cube in a similar manner to the double integral, then wecan define the triple integral:

$$\boxed{\iiint_{\mathcal{D}} f(x,y,z) dv = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0 \\ \Delta z \to 0}} \sum_{k=1}^{q} \sum_{j=1}^{m} \sum_{i=1}^{n} f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk}) \Delta x_i \Delta y_j \Delta z_k}$$

Generalization:
By considering the boundary condition, the triple integral can be evaluated in three manners:
Note: Usually, the three manners is chosen regard to which the projection region is easy to find.

- $\iiint_{\mathcal{D}} f(x,y,z) dv = \iint_{\mathcal{R}} \left(\int_{\phi_1(x,y)}^{\phi_2(x,y)} f(x,y,z) dz \right) dA$

- $\iiint_{\mathcal{D}} f(x,y,z)dv = \iint_{\mathcal{R}} \left(\int_{\psi_1(x,z)}^{\psi_2(x,z)} f(x,y,z)dy \right) dA$
- $\iiint_{\mathcal{D}} f(x,y,z)dv = \iint_{\mathcal{R}} \left(\int_{g_1(y,z)}^{g_2(y,z)} f(x,y,z)dx \right) dA$

4.6 Triple Integral in Cylindrical Spherical Coordinates

Similar to the transformation in the double integral, the volume unit is given by the Jacobian matrix determinant.

$$dx\,dy\,dz = \det \mathbf{J}(r,\theta,z)\,dr\,d\theta\,dz = r\,dr\,d\theta\,dz$$
$$dx\,dy\,dz = \det \mathbf{J}(\rho,\theta,\phi)\,d\rho\,d\theta\,d\phi = \rho^2 \sin\phi\,d\rho\,d\theta\,d\phi$$

Therefore, substitute into the original definition, we can evaluate the triple integral in both cylindrical and spherical coordinates:

$$\begin{aligned}\iiint_{\mathcal{R}} f(x,y,z)dv &= \iiint_{\mathcal{R}} f(r\cos\theta, r\sin\theta, z) r dr d\theta dz \\ &= \iiint_{\mathcal{R}} f(r\sin\phi\cos\theta, r\sin\phi\sin\theta, r\cos\phi)\rho^2 \sin\phi d\rho d\theta d\phi\end{aligned}$$

4.7 Application of Triple Integral

4.7.1 Volume

$$\iiint_{\mathcal{D}} dv = V(\mathcal{D})$$

4.7.1.1 Mass

In a region $\mathcal{D}$, with the density function $\rho(x,y,z)$, the mass can be given by the following equation:

$$\iiint_{\mathcal{D}} \rho(x,y,z)dv = M(\mathcal{D})$$

4.7.2 Mass Center and Moment

$$\bar{x} = \frac{m_{yz}}{M} = \frac{\iiint_{\mathcal{D}} x\rho(x,y,z)}{\iiint_{\mathcal{D}} \rho(x,y,z)dv}$$

$$\bar{y} = \frac{m_{xz}}{M} = \frac{\iiint_{\mathcal{D}} y\rho(x,y,z)}{\iiint_{\mathcal{D}} \rho(x,y,z)dv}$$

$$\bar{z} = \frac{m_{xy}}{M} = \frac{\iiint_{\mathcal{D}} z\rho(x,y,z)}{\iiint_{\mathcal{D}} \rho(x,y,z)dv}$$

4.7.3 Radii of Gyration

$$\bar{\bar{x}}^2 = \frac{I_{yz}}{M} = \frac{\iiint_{\mathcal{D}} (y^2+z^2)\rho(x,y,z)}{\iiint_{\mathcal{D}} \rho(x,y,z)dv}$$

$$\bar{\bar{y}}^2 = \frac{I_{xz}}{M} = \frac{\iiint_{\mathcal{D}} (x^2+z^2)\rho(x,y,z)}{\iiint_{\mathcal{D}} \rho(x,y,z)dv}$$

$$\bar{\bar{z}}^2 = \frac{I_{xy}}{M} = \frac{\iiint_{\mathcal{D}} (x^2+y^2)\rho(x,y,z)}{\iiint_{\mathcal{D}} \rho(x,y,z)dv}$$

Please refer to the link:
http://math.ucsd.edu/~lni/math20e/schedule.html
https://web.math.rochester.edu/people/faculty/edummit/handouts.html

Chapter 5
Vector Calculus

Chapter 6
Second Order Differential Equation

Appendix A
Eventually, everything is connected.