Bachelor Physics Lecture Notes
AP2212 Measurement & Instrumentation

Peilin WU
City University of Hong Kong

September 6, 2017

Contents

1 Circuit Analysis

1.1 Basic elements in Circuit

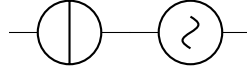### 1.1.1 Sources

- Ideal Current Source

Figure 1: Ideal Current Sources

Ideal Current Sources For the ideal current source, the **internal resistance is regraded as infinity**, which means that a role of **open circuit** in analysis.
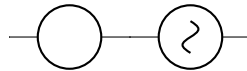
- Voltage Source

Figure 2: Ideal Voltage Sources

Ideal Voltage Sources For the ideal voltage source, the **internal resistance is regraded as zero**, which means that a role of **short circuit** in analysis.

Root-mean-square in Different Phase When the voltage and the current are out of phase, say $V = V_P \sin \omega t$ and $i = i_P \sin(\omega t + \phi)$, then the rms power $P_{rms} = V_{rms} i_{rms} \cos \phi$
Proof:

$$
\begin{aligned}
P_{ave} &= \frac{V_P i_P}{\pi} \int_0^\pi \sin \omega t \sin(\omega t + \phi) d\omega t \\
&= \frac{V_P i_P}{2\pi} \int_0^\pi [\cos \phi - \cos(2\omega t + \phi)] d\omega t \\
&= \frac{V_P i_P}{2\pi} [\cos \phi - \frac{1}{2} \sin(2\omega t + \phi)]|_0^\pi \\
&= \frac{V_P i_P}{2} \cos \phi = \frac{V_P}{\sqrt{2}} \cdot \frac{i_P}{\sqrt{2}} \cos \phi \\
&= V_{rms} i_{rms} \cos \phi
\end{aligned}
$$

1.1.2 Impedance

First, let's give the basic relation between impedance Z, resistance R and reactance X:

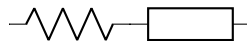$$ Z = R + jX = R + j(\omega L - \frac{1}{\omega C}) $$

- Resistance

Figure 3: Resistance

Resistance   For the resistance, the value of resistance is the **real part** value of its impedance in circuit, which also suggests that **resistor dissipate energy**.

$$Impedance = R$$

Resistances are in **series or parallel** in circuit, the equivalent resistance given by:

$$Series: R_{12} = R_1 + R_2$$

$$Parallel: \frac{1}{R_{12}} = \frac{1}{R_1} + \frac{1}{R_2} \ \ or \ R_{12} = \frac{R_1 R_2}{R_1 + R_2}$$

- Capacitor

Figure 4: Capacitor

Capacitor   For the capacitor, it stores charge $Q = CV$, which means that capacitor in circuit can **store energy**, and current passing through the capacitor is the rate of charge Q with respect to time t:

$$i = \frac{dQ}{dt} = \frac{dV}{dt}$$

$$V = \frac{Q}{C} = \frac{1}{C} \int_T i \, dt$$

and the **reactance of capacitor** in circuit with angular frequency ω is expressed as:

$$\frac{Peak \ value \ of \ voltage}{Peak \ value \ of \ current} = \frac{1}{\omega C}$$

the impedance of the capacitor is given by (the minus sign shows the phase difference):

$$Impedance = -j \cdot Reactance = -j\frac{1}{\omega C}$$

Capacitors are in **series or parallel** in circuit, the equivalent capacitor given by:

$$Series: \frac{1}{C_{12}} = \frac{1}{C_1} + \frac{1}{C_2} \ \ or \ C_{12} = \frac{C_1 C_2}{C_1 + C_2}$$

$$Parallel: C_{12} = C_1 + C_2$$

- Inductive

Figure 5: Inductive

Inductive   For the inductive, it create the magnetic field, which means that inductive in circuit can **store energy**, and since the self-inductance process, the inductance is defined:

$$V = L\frac{di}{dt}$$

$$i = \frac{1}{L} \int_T V \, dt$$

and the **reactance of inductance** in circuit with angular frequency ω is expressed as:

$$\frac{Peak\ value\ of\ voltage}{Peak\ value\ of\ current} = \omega L$$

the impedance of the inductive is given by:

$$Impedance = j \cdot Reactance = j\omega L$$

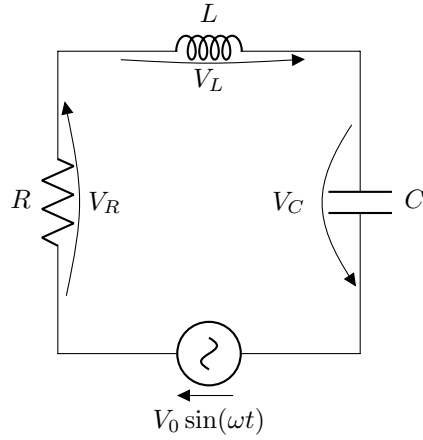Capacitors are in **series or parallel** in circuit, the equivalent capacitor given by:

$$Series:\ L_{12} = L_1 + L_2$$

$$Parallel:\ \frac{1}{L_{12}} = \frac{1}{L_1} + \frac{1}{L_2}\ or\ L_{12} = \frac{L_1 L_2}{L_1 + L_2}$$

1.2   Representation Techniques

For the seek of simplicity, there are three major approaches:
Consider the following RLC series circuit:

1.2.1   Differential Equation

Applying the Kirchhoff's modified loop rule, and also adding the phase constant, we can obtain:

$$V(t) - V_R(t) - V_L(t) - V_C(t) = 0$$

$$L\frac{di}{dt} + iR + \frac{Q}{C} = V_0 \sin(\omega t + \phi)$$

differentiating the equation again:

$$L\frac{d^2 i}{dt^2} + R\frac{di}{dt} + \frac{i}{C} = \omega V_0 \cos(\omega t + \phi)$$

$$\frac{d^2 i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{i}{LC} = \frac{\omega V_0}{L} \cos(\omega t + \phi)$$

the general solution:

$$i(t) \rightarrow \frac{cV_0\omega\left((1 - cL\omega^2)\left(\cos(t\omega)\right) + cR\omega(\sin(t\omega))\right)}{c\omega^2\left(L\left(cL\omega^2 - 2\right) + cR^2\right) + 1} + e^{-\frac{t\left(\frac{\sqrt{cR^2 - 4L}}{\sqrt{c}} + R\right)}{2L}}\left(c_2 e^{\frac{t\sqrt{cR^2 - 4L}}{\sqrt{cL}}} + c_1\right)$$

where c_1 and c_2 are constants
Only consider the particular part:

$$i(t) = \frac{cV_0\omega\left((1 - cL\omega^2)(\cos(t\omega)) + cR\omega(\sin(t\omega))\right)}{c\omega^2\left(L(cL\omega^2 - 2) + cR^2\right) + 1} = \frac{V_0\left(\left(L\omega - \dfrac{1}{c\omega}\right)\cos(t\omega) + R\sin(t\omega)\right)}{\left(\dfrac{1}{c\omega} - L\omega\right)^2 + R^2}$$

$$= \frac{V_0}{\sqrt{\left(L\omega - \dfrac{1}{c\omega}\right)^2 + R^2}}\sin(\omega t - \phi) \qquad where\ \phi = \tan^{-1}\left(\frac{L\omega - \dfrac{1}{c\omega}}{R}\right)$$

1.2.2   Impedance - Complex Representation

Further simplification can be made as we write the triangular form into the complex numbers:

$$Z = R + jX = R + j(\omega L - \frac{1}{\omega C})$$

Note: Please pay attention that: The phase difference of the inductive and the capacitor leads to the positive sign in $j\omega L$ term and the negative sign in $\dfrac{-j}{\omega C}$.

$$Z = \sqrt{X_R^2 + (X_L - X_C)^2} = \sqrt{R^2 + (\omega L - \frac{1}{\omega c})^2}$$

Therefore, we can express the current in RLC circuit above as:

$$I_0 = \frac{V_0}{Z} = \frac{V_0}{\sqrt{R^2 + (\omega L - \dfrac{1}{\omega c})^2}}$$

and the phase constant is:

$$\tan\phi = \left(\frac{X_L - X_C}{X_R}\right) = \frac{1}{R}\left(\omega L - \frac{1}{\omega c}\right)$$

1.2.3   Impedance - Phasor

For the further simplification, we can extract the ωt term as it's invariant in the computation and only left with the phase constant, we have the moivation for phasor representation.
A phasor is a rotating vector having the following properties:

- **Length**: the length corresponding to the amplitude.

- **Angular speed**: the vector rotates counterclockwise with an angular speed ω.

- **Projection**: the projection along the vertical axis corresponding to the value of the alternating quality at time t.

$$X < \phi = Xe^{j\phi}e^{j\omega t} = X\cos(\omega t + \phi) + j\sin(\omega t + \phi)$$

Resistance   The phasor of the resistance can be expressed as:

$$R < 0$$

and we can see that resistance don not change the phase.

Capacitor   The phasor of the capacitor can be expressed as:

$$\frac{1}{\omega C} < -\frac{\pi}{2}$$

and we can see that capacitor will change the phase and we can say that:

$$\boxed{\textit{The current } \textbf{\textit{leads}} \textit{ tthe voltage by } \frac{\pi}{2} \textit{ in a } \textbf{\textit{capavitive circuit}}.}$$

Inductive   The phasor of the inductive can be expressed as:

$$\omega L < \frac{\pi}{2}$$

and we can see that inductive will change the phase and we can say that:

$$\boxed{\textit{The current } \textbf{\textit{lags}} \textit{ tthe voltage by } \frac{\pi}{2} \textit{ in a } \textbf{\textit{inductive circuit}}.}$$

1.3 Analysis

1.3.1 Kirchhoff's Law

Kirchhoff's Law - Current Law at any instant, the algebraic sum of all the currents flowing into any node in a circuit is zero.

$$\sum_n I_n = 0$$

and please pay attention to the orientation you choose about the current: if currents flowing into the node are positive, currents flowing out of the node are negative

Kirchhoff's Law - Voltage Law At any instant the algebraic sum of all the voltages around any loop in a circuit is zero.

$$\sum_n V_n = 0$$

and please pay attention to the orientation you choose about the voltage: if clockwise voltage arrows are positive and then anticlockwise arrows are negative. Better to give each element a voltage flow arrow.

1.3.2 Nodal Analysis

- Determine the reference point as the reference node,

- Label remain nodes with numbers,

- Label any known voltages with flow arrow,

- Apply Kirchhoff's current law to each unknown node,

- Solve simultaneous equation to determine voltage,

- Or the general case, combining the mesh analysis.

1.3.3 Mesh Analysis

- Identify the meshes and assign a clockwise-flowing current to each element,

- Label those elements with numbers,

- Apply Kirchhoff's voltage law to each other,

- Solve the simultaneous equations to determine the current.

- Or the general case, combining the nodal analysis.

1.3.4 Principle of Superposition

Principle of Superposition In a complex linear network of impedance, voltage sources and current sources, each voltage and current in the circuit is equal to the algebraic sum of the voltages or currents that would be present if each source were to be considered separately.

$$V = V_V + V_I \qquad I = I_V + I_I$$

When determining the effects of a single source, the remaining source should be replaced by their internal resistance. This means that the ideal voltage source should be replaced by a short circuit while the ideal current source a open circuit.

1.4   Equivalent Circuit

1.4.1   Thevenin's Theorem

Thevenin's Theorem   In circuit theory terms, the theorem allows any one-port network to be reduced to a single voltage source and a single impedance. Any black box containing resistances only and voltage and current sources can be replaced by a Thevenin equivalent circuit consisting of an equivalent voltage source in series connection with an equivalent resistance.

- Any linear electrical network with voltage and current sources and only resistances can be replaced at terminals A-B by an equivalent voltage source V_{th} in series connection with an equivalent resistance R_{th}.

- The equivalent voltage V_{th} is the voltage obtained at terminals A-B of the network with terminals A-B open circuited.

- The equivalent resistance R_{th} is the resistance that the circuit between terminals A and B would have if all ideal voltage sources in the circuit were replaced by a short circuit and all ideal current sources were replaced by an open circuit.

- If terminals A and B are connected to one another, the current flowing from A to B will be V_{th}/R_{th}. This means that R_{th} could alternatively be calculated as V_{th} divided by the short-circuit current between A and B when they are connected together.
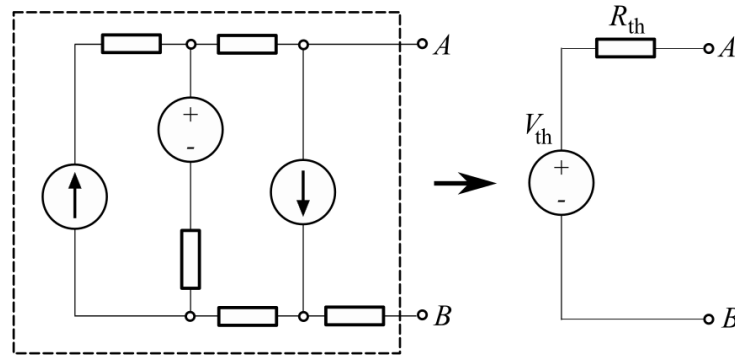

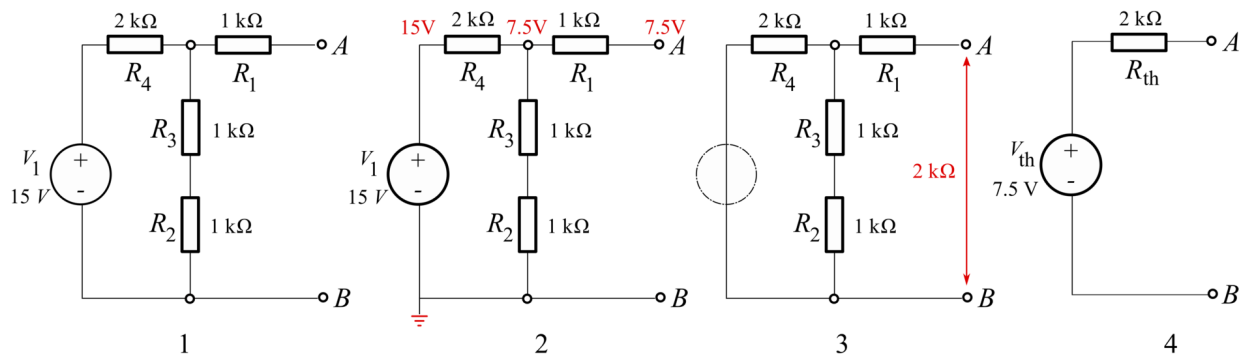Figure 6: Thevenin Equivalent Circuit

For example:

Figure 7: Thevenin's Theorem Example

The evaluation process:

- let A-B be a open circuit and determine the voltages between A-B which is V_{th},

- then replace all sources as its internal resistance and calculate the resistance between A-B which is R_{th}.

1.4.2 Nortons Theorem

Nortons Theorem As the dual of the Thevenin's theorem, Norton's theorem states that any one-port network can be reduced to a single current source and a single impedance. Any black box containing resistances only and voltage and current sources can be replaced by an equivalent circuit consisting of an equivalent current source in parallel connection with an equivalent resistance.

- Any linear electrical network with voltage and current sources and only resistances can be replaced at terminals A-B by an equivalent current source I_{no} in parallel connection with an equivalent resistance R_{no}.

- This equivalent current I_{no} is the current obtained at terminals A-B of the network with terminals A-B short circuited.

- This equivalent resistance R_{no} is the resistance obtained at terminals A-B of the network with all its voltage sources short circuited and all its current sources open circuited.
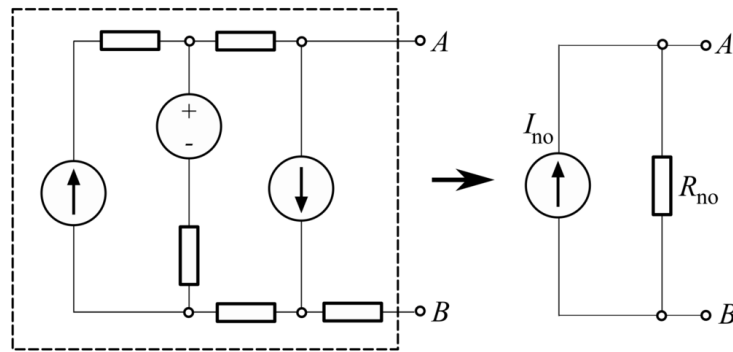
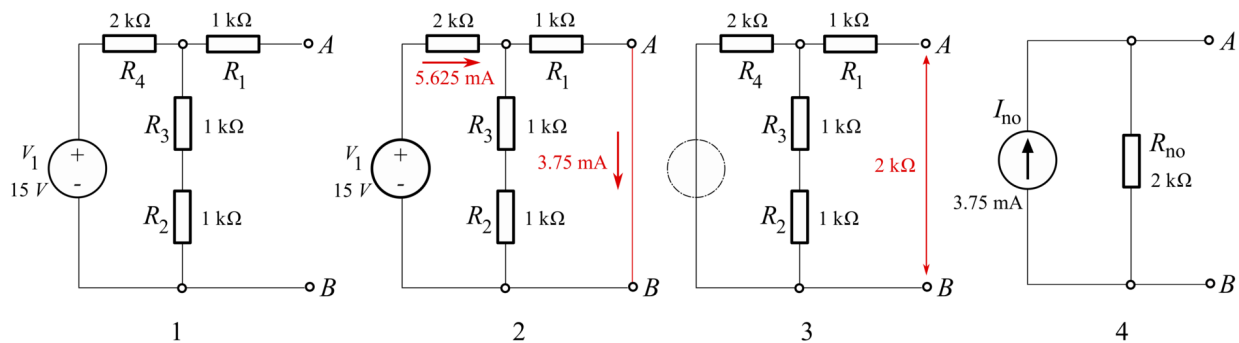Figure 8: Norton Equivalent Circuit

For example:

Figure 9: Norton's Theorem Example

The evaluation process:

- Find the Norton current I_{no}. Calculate the output current, I_{AB}, with a short circuit as the load (meaning 0 resistance between A and B). This is I_{no}.

- Find the Norton resistance R_{no}. When there are no dependent sources (all current and voltage sources are independent), there are two methods of determining the Norton impedance R_{no}.

  1. Calculate the output voltage, V_{AB}, when in open circuit condition (i.e., no load resistor meaning infinite load resistance). R_{no} equals this V_{AB} divided by I_{no}.

  2. Or replace independent voltage sources with short circuits and independent current sources with open circuits. The total resistance across the output port is the Norton impedance R_{no}.

1.4.3 Conversion between both Theorems

Both theorem can be converted into each other:

$$R_{th} = R_{no}$$
$$V_{th} = I_{no}R_{no}$$
$$I_{no} = \frac{V_{th}}{R_{no}}$$
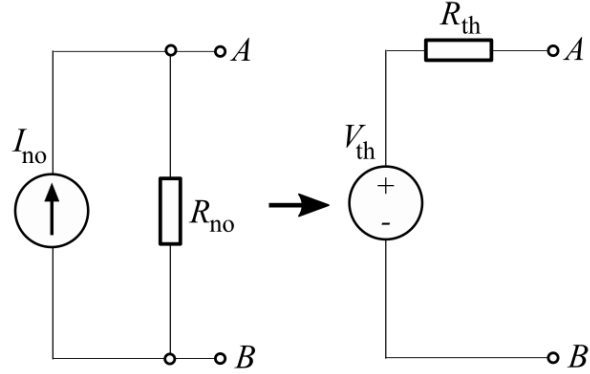
Figure 10: Conversion Demonstration

1.5 RLC Circuit Summaries

1.5.1 R Circuit

1.5.2 C Circuit

1.5.3 L Circuit

1.5.4 RC Circuit

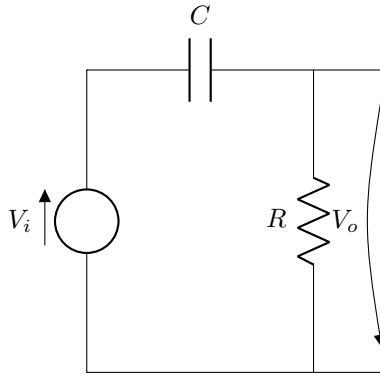RC Circuit can be used as the RC filter - high-pass filter:

Figure 11: RC Filter

In this RC circuit with $V_i = V_P \sin \omega t$, we can first determine the current and then determine the V_{out}
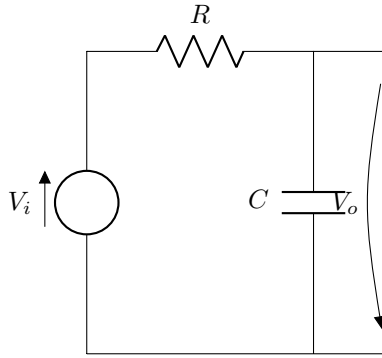
$$i = \frac{V_P}{\sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}} \sin(\omega t + \phi) \qquad where \; \phi = \tan^{-1} \frac{1}{\omega C R}$$

$$V_0 = \frac{V_P R}{\sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}} \sin(\omega t + \phi)$$

$$= \frac{V_P \omega C R}{\sqrt{\omega^2 C^2 R^2 + 1}} \sin(\omega t + \phi)$$

$$= \frac{V_P \omega \tau}{\sqrt{\omega^2 \tau^2 + 1}} \sin(\omega t + \phi)$$

$$\left|\frac{V_o}{V_i}\right| = \sqrt{\frac{\omega^2 \tau^2}{\omega^2 \tau^2 + 1}}$$

From the expression, we can see that as the increase of frequency, the proportion will increase as well. Therefore, this is a high-pass filter.

Also, if we take the output voltage as the capacitor, it becomes the low-pass filter:

Figure 12: RC Filter

$$i = \frac{V_P}{\sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}} \sin(\omega t + \phi) \qquad where \; \phi = \tan^{-1} \frac{1}{\omega C R}$$

$$V_0 = \frac{1}{C} \int i \, dt = |V_i \times \frac{Z_C}{Z_C + Z_R}|$$

$$= \frac{V_P}{\omega C \sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}} \cos(\omega t + \phi)$$

$$= \frac{V_P}{\sqrt{\omega^2 C^2 R^2 + 1}} \cos(\omega t + \phi)$$

$$= \frac{V_P}{\sqrt{\omega^2 \tau^2 + 1}} \cos(\omega t + \phi)$$

$$\left|\frac{V_o}{V_i}\right| = \sqrt{\frac{1}{\omega^2 \tau^2 + 1}}$$

1.5.5 RL Circuit

RC Circuit can be used as the RC filter - high-pass filter:
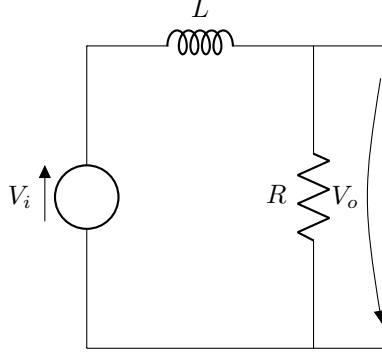
Figure 13: RC Filter

In this RC circuit with $V_i = V_P \sin \omega t$, we can first determine the current and then determine the V_{out}
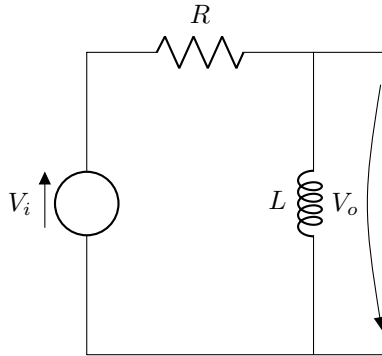
$$i = \frac{V_P}{\sqrt{R^2 + \omega^2 L^2}} \sin(\omega t + \phi) \qquad where\ \phi = \tan^{-1} \frac{\omega L}{R}$$

$$V_0 = \frac{V_P R}{\sqrt{R^2 + \omega^2 L^2}} \sin(\omega t + \phi)$$

$$= \frac{V_P}{\sqrt{1 + \frac{\omega^2 L^2}{R^2}}} \sin(\omega t + \phi)$$

$$= \frac{V_P}{\sqrt{1 + \omega^2 \tau^2}} \sin(\omega t + \phi)$$

$$\left| \frac{V_o}{V_i} \right| = \sqrt{\frac{1}{\omega^2 \tau^2 + 1}}$$

From the expression, we can see that as the decrease of frequency, the proportion will increase as well. Therefore, this is a low-pass filter.

Similar to RC circuit, take the output voltage as the inductive, it becomes a high-pass filter

Figure 14: RC Filter

$$i = \frac{V_P}{\sqrt{R^2 + \omega^2 L^2}} \sin(\omega t + \phi) \qquad where\ \phi = \tan^{-1} \frac{\omega L}{R}$$

$$V_0 = \frac{1}{L}\frac{di}{dt}$$
$$= \frac{V_P \omega L}{\sqrt{R^2 + \omega^2 L^2}} \cos(\omega t + \phi)$$
$$= \frac{V_P \omega \tau}{\sqrt{1 + \omega^2 \tau^2}} \cos(\omega t + \phi)$$
$$\left|\frac{V_o}{V_i}\right| = \sqrt{\frac{\omega^2 \tau^2}{\omega^2 \tau^2 + 1}}$$
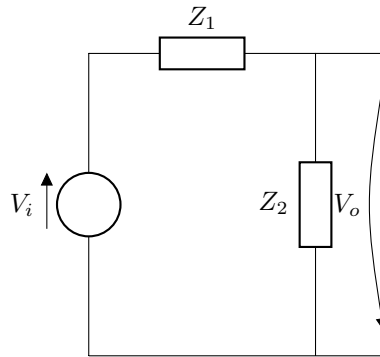
1.5.6 Transfer Function of RC and RL

Figure 15: RC Filter

In this circuit, we can use the voltage divider formula to derive the transfer function:
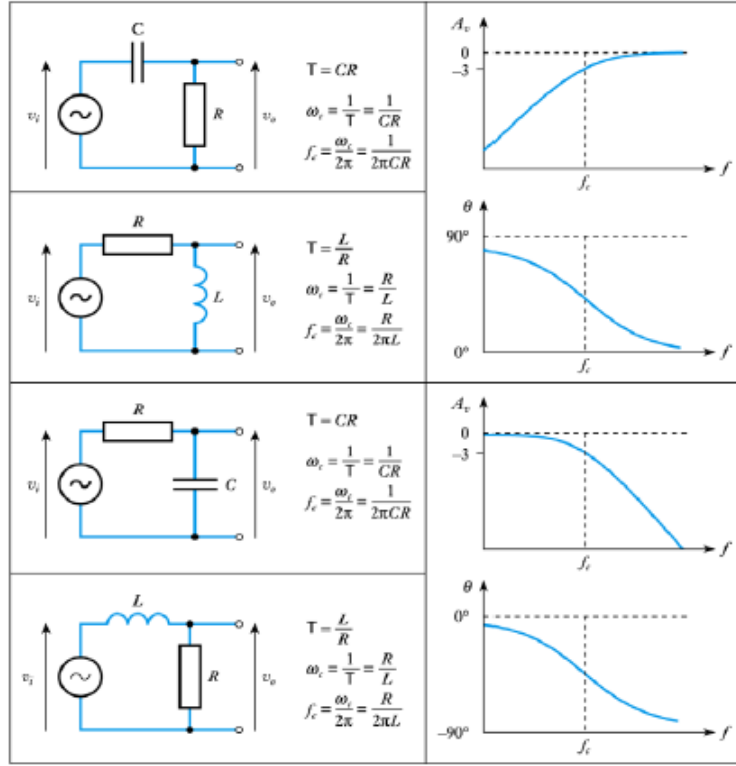
$$\frac{V_o}{V_i} = \frac{Z_2}{Z_1 + Z_2}$$

Figure 16: RC-RL Filter

1.5.7 RLC Circuit

- RLC in Series
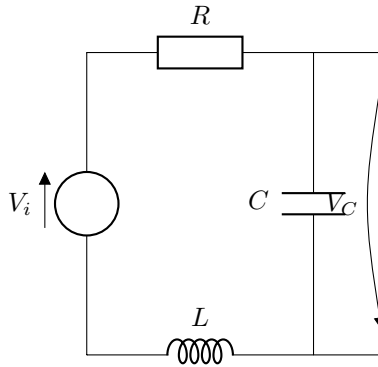
Figure 17: RLC Resonant Circuit

$$Z = Z_R + Z_C + Z_L = R + j\omega L - j\frac{1}{\omega C}$$

When the reactance becomes zero:

$$\omega L = \frac{1}{\omega C} \implies \omega_0 = \frac{1}{\sqrt{LC}}$$

14

Assuming the $V_C = A \sin \omega t$, the current i can be determined:

$$i = C \frac{dV}{dt} = C \frac{d}{dt} A \sin \omega t = A \omega C \cos \omega t$$

therefore, as the definition of the quality factor which can be determined as the following:

$$Q = 2\pi \frac{Max \ of \ the \ energy \ store}{dissipated \ energy \ in \ resonant \ circle}$$

$$= 2\pi \frac{E_S}{E_D} = 2\pi \frac{\frac{1}{2}Li^2 + \frac{1}{2}CV^2}{R < i^2 >}$$

$$= 2\pi \frac{\frac{1}{2}L\omega^2 C^2 A^2 \cos^2 \omega t + \frac{1}{2}CA^2 \sin^2 \omega t}{R \frac{\omega_0^2 C^2 A^2}{2} \frac{2\pi}{\omega_0}}$$

$$= 2\pi \frac{\frac{1}{2}CA^2}{R \frac{\omega_0^2 C^2 A^2}{2} \frac{2\pi}{\omega_0}} = \frac{1}{\omega_0 RC}$$

- RLC in Parallel
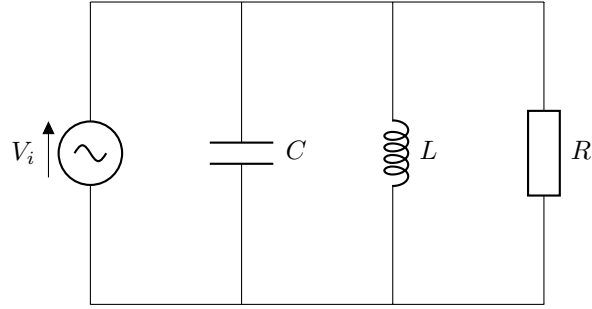
Figure 18: RLC Resonant Circuit

$$\frac{1}{Z} = \frac{1}{Z_C} + \frac{1}{Z_L} + \frac{1}{Z_R}$$

$$Z = \frac{R}{1 + jR(\omega C - 1/\omega L)}$$

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

Assuming that the voltage across the capacitor is $V_C = A \sin \omega t$, then the current across the inductive

is $i = \frac{1}{L}\int V_c dt = \frac{-A}{\omega L}\cos\omega t$ and then we can determine the quality factor:

$$Q = 2\pi\frac{E_S}{E_D} = 2\pi\frac{\frac{1}{2}Li^2 + \frac{1}{2}CV^2}{\frac{<V^2>}{R}}$$

$$= 2\pi\frac{\frac{1}{2}L(\frac{-A}{\omega L}\cos\omega t)^2 + \frac{1}{2}C(A\sin\omega t)^2}{\frac{A^2}{2R}\frac{2\pi}{\omega_0}}$$

$$= 2\pi\frac{\frac{1}{2}CA^2}{\frac{A^2}{2R}\frac{2\pi}{\omega_0}} = \omega_0 RC$$

And for the bandwidth, it can be shown that:

$$Q = \frac{\omega_0}{\Delta\omega}$$

1.6   Transient Response in RL, RC and RLC Circuit

1.6.1   Charging and Discharging Capacitor

- Charging Capacitor
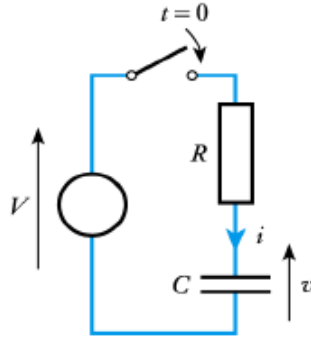
Figure 19: Charging Capacitor

$$V = V(1 - e^{-\frac{t}{RC}}) = V(1 - e^{-\frac{t}{\tau}}) \qquad \tau = \frac{1}{RC}$$

$$i = ie^{-\frac{t}{RC}} = \frac{V}{R}e^{-\frac{t}{\tau}}$$
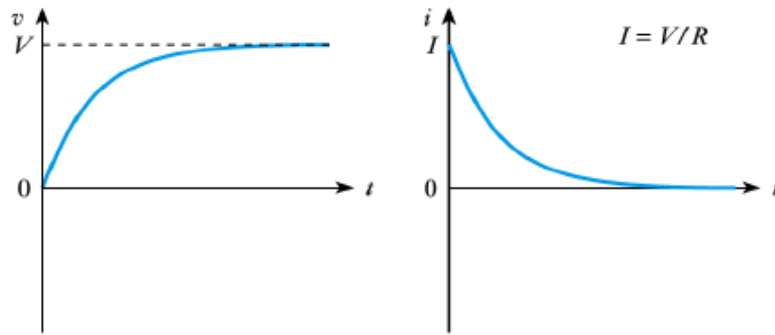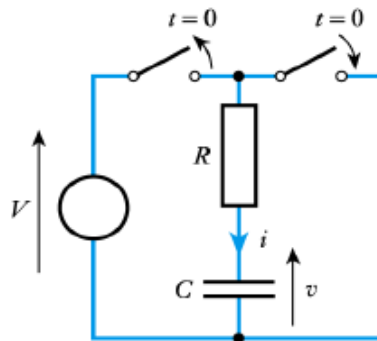
Figure 20: Charging Capacitor

- Discharging Capacitor

Figure 21: Discharging Capacitor

$$V = Ve^{-\frac{t}{RC}} = Ve^{-\frac{t}{\tau}}$$
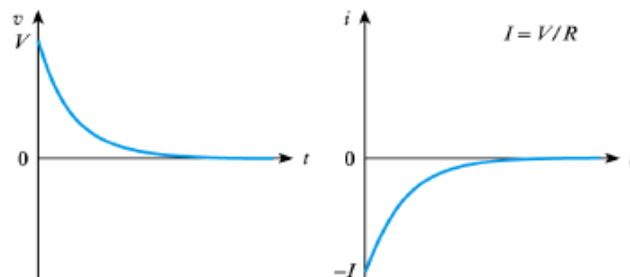$$i = -ie^{-\frac{t}{RC}} = -\frac{V}{R}e^{-\frac{t}{\tau}}$$

Figure 22: Discharging Capacitor

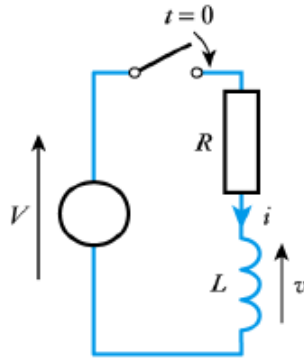### 1.6.2 Energizing and De-energizing Inductor

- Energizing Inductor

Figure 23: Energizing Inductor

$$V = Ve^{-\frac{Rt}{L}} = Ve^{-\frac{t}{\tau}} \qquad \tau = \frac{L}{R}$$

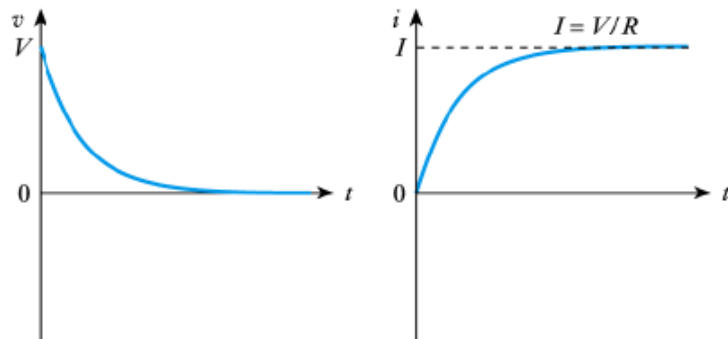$$i = \frac{V}{R}(1 - e^{-\frac{Rt}{L}}) = \frac{V}{R}(1 - e^{-\frac{t}{\tau}})$$

Figure 24: Energizing Inductor

- De-energizing Inductor
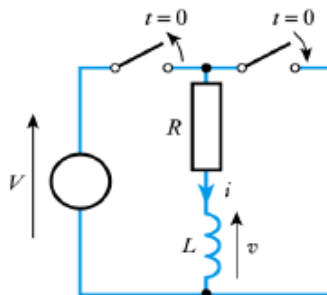

Figure 25: De-energizing Inductor

18

$$V = -Ve^{-\frac{Rt}{L}} = -Ve^{-\frac{t}{\tau}}$$

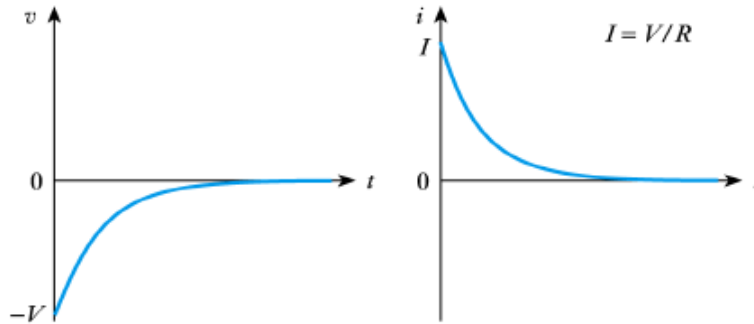$$i = \frac{V}{R}e^{-\frac{Rt}{L}} = \frac{V}{R}e^{-\frac{t}{\tau}}$$

Figure 26: De-energizing Inductor

1.6.3 Response for the First Order System

Initial and final value formulae:

$$V = V_f + (V_i - V_f)e^{-\frac{t}{\tau}}$$

$$i = i_f + (i_i - i_f)e^{-\frac{t}{\tau}}$$

The first term in each case is the **steady-state response**
The second term represents the **transient response**
The combination gives the **total response** of the arrangement

1.6.4 Response for the Second Order System

$$\frac{1}{\omega_n^2}\frac{d^2y}{dt^2} + \frac{2\zeta}{\omega_n}\frac{dy}{dt} + y = x$$

For this second order differential equation, we have:

ω_n is the **undamped natural frequency** *(rad/s)*

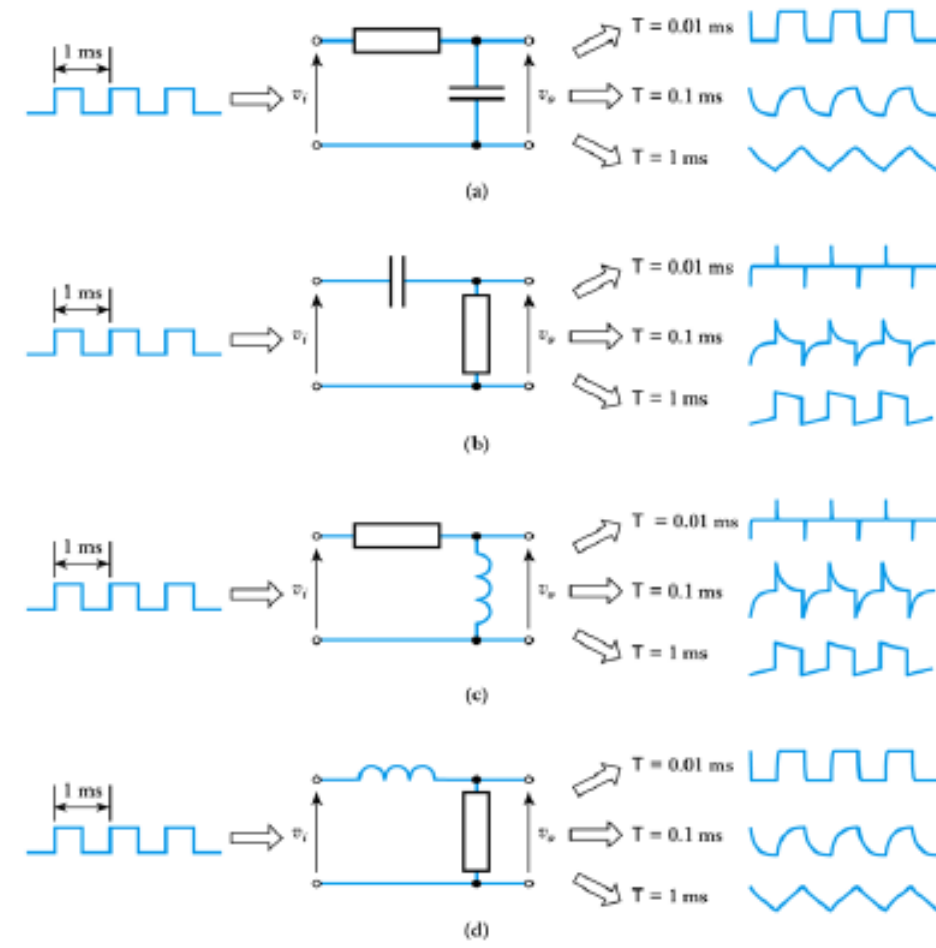ζ is the **damping factor**

Figure 27: Output of first-order systems to a square waves - for different time response
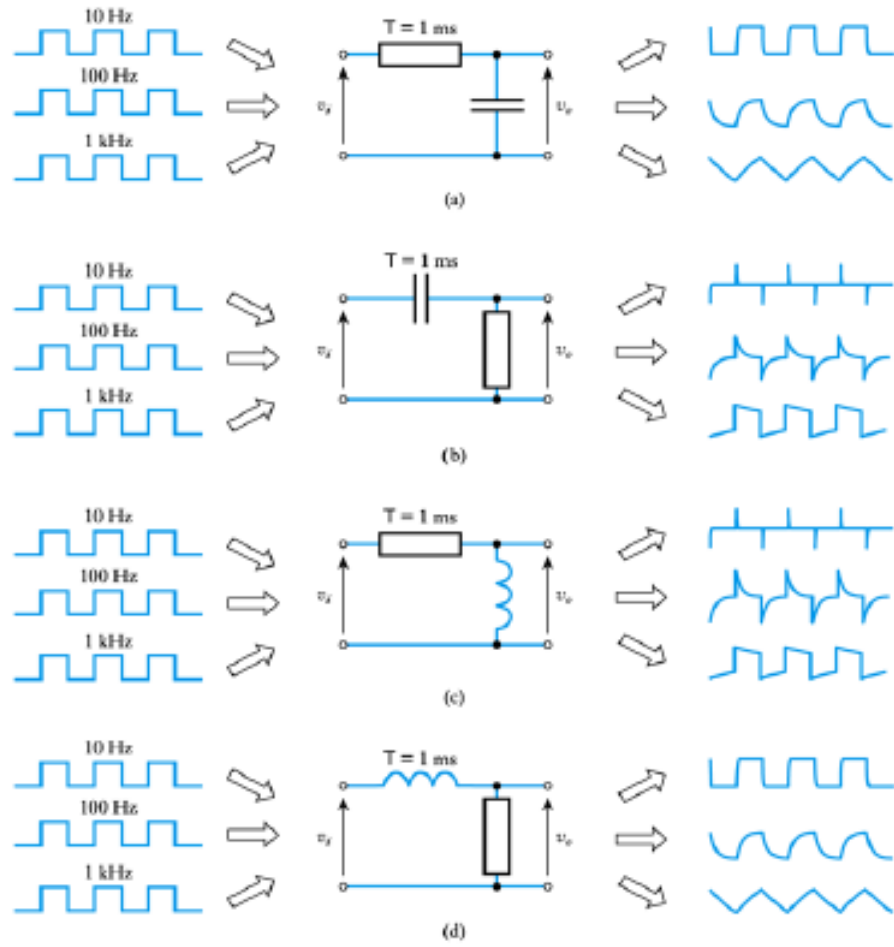
Figure 28: Output of first-order systems to a square waves - for different frequencies