City University of Hong Kong

MA Courses Review Notes

MA2506

Probability and Statistics

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All LaTeX (`.tex`) files of this document can be accessed from https://github.com/zzw42/review-notes-cityu.

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Preface

Textbook: Probability and Statistics for Engineering and the Sciences, by Jay Devore, 8th Ed., Brooks/Cole Cengage Learning, 2012.

Schedules:

Week	Brief Description
1	Introduction; descriptive statistics
2	Probability; random variables
3	Discrete random variables
4	Continuous random variables
5	Expectation, variance, moments
6	Multivariate random variables
7	Conditional distribution and expectation
8	Correlation coefficient; independence
9	Sampling distribution; point estimation
10	Confidence intervals
11	Hypothesis testing
12	One sample hypothesis tests
13	Two sample hypothesis tests; review

Chapter 1

Overview and Descriptive Statistics

1.1 Populations, Samples, and Processes

Definition 1.1. The **population** is the whole class of individuals which an investigator is interested in.

Definition 1.2. The **sample** is part of population which is examined or observed.

From sample to population is what statistics do: chapter 6-16.
From population to sample is what probability do: chapter 2-5.

Definition 1.3. The **variable** is any characteristic whose value may change from one individual to another in population.

Example 1.4. Household income; Examination score

In statistics, there are two important parts: Estimation and Influence.

- univariable one variable
- bivariable two variables
- multivariable more than two variables

Example 1.5. 77 100 52 78 95 55 86 43 86 73 89 68 57 85 58 79 90 45 95 46 85 77 98 86 100 71 60 24 58 44 64 83 88 95 88 91 86 75 89 77 43 100 88 80 76 0 88 86 69 44 40 84 68 87 86 83

1.2 Pictorial and Tabular Methods in Descriptive Statistics

1.2.1 Stem-and-leaf Plot

Procedure

1. select one or more leading digits as the **stems**. The tailing digits become the **leaves**;
2. List possible **stems** in a verticle column;
3. List the **leaves** for every observation beside the corresponding **stem**;
4. Indicate the unit of **stems** and **leaves** in the plot.

Therefore,

Table 1.1: stem-and-leaf plot, Key: $1|1 = 11$

0	0																		
1																			
2	4																		
3																			
4	0	3	3	4	4	5	6												
5	2	5	7	8	8														
6	0	4	8	8	9														
7	1	3	5	6	7	7	7	8	9										
8	0	0	3	3	4	5	6	6	6	6	6	6	7	8	8	8	8	9	9
9	0	1	5	5	5	8													
10	0	0	0																

1.2.2 Bar Plot

Each observation is repeated as a dot above the corresponding location on a horizontal line with measurement scale.

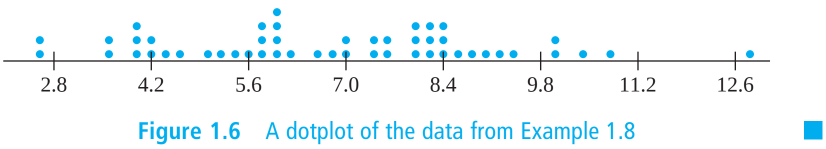


Figure 1.1: A dotplot of the data from Example 1.8

1.2.3 Histogram

Definition 1.6. A variable is **discrete** if its set of possible values either is finite or countable. A variable is **continuous** if its set of possible values consists of an entire interval on the real line.

Discrete cases

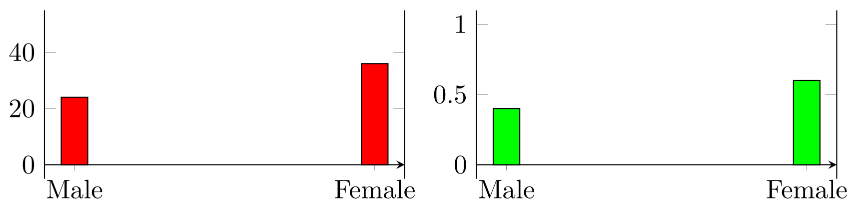
Frequency = number of times a value occur in the dataset.

Relative Frequency $= \frac{\text{number of times the value occurs}}{\text{number of observation in the dataset}}$

Procedure

1. calculate the Frequency and Relative Frequency
2. mark each value on a horizontal scale
3. above each value, draw a rectangular whose height is the frequency or relative frequency of the value.

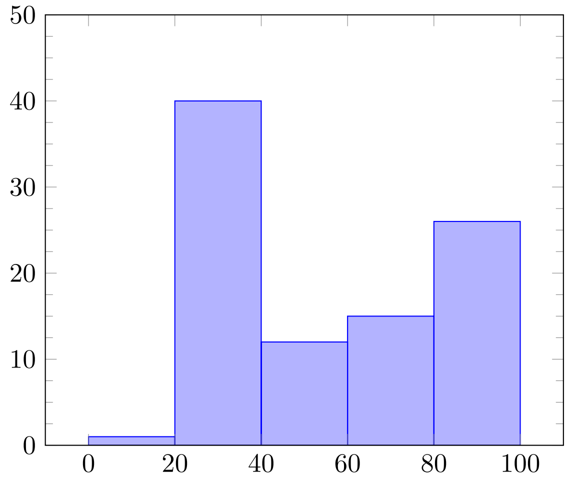
(a) frequency (b) relative frequency



Continuous cases

Need to determine the size of each class.

(a) Equal class similar to discrete case.



You can also use the relative frequency.

(b) The unequal class

Example 1.7. 0-10K, 10K-20K, 20-30K, ..., 500K-510K, 510K-520K, ...(a waste of space!)
$\Rightarrow$ 0-10K, 10K-20K, 20K-30K, 30K-40K, 40K-50K, 50K-100K, 100K-200K

For the unequal class, frequency or relative frequency may mislead some people because of a wide range. Therefore, we use density.

$$\text{Density} = \frac{\text{relative frequency of the class}}{\text{class width}}$$

Use density as height to draw histogram within unequal class.

Shape of histogram

- Mode: unimodal, bimodal, multimodal
- Symmetry: symmetric, positive skewed, negative skewed

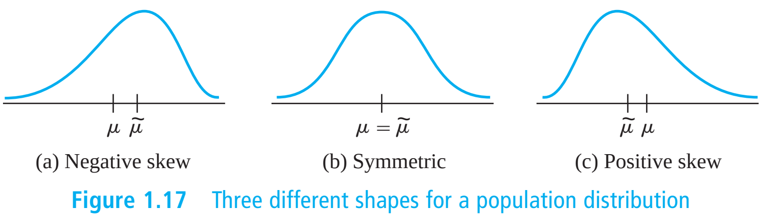


Figure 1.2: Three different shapes for a population distribution

1.3 Measures of Location and Variability

1.3.1 Location

Observations:

$$x_1, x_2 \dots x_n$$

Sample mean:

$$\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n}$$

Sample median:

$$\tilde{x} = \begin{cases} \frac{n+1}{2}\text{th ordered value,} & \text{if } n \text{ is odd} \\ \frac{n}{2} \text{ or } \frac{n+2}{2}\text{th ordered value.} & \text{if } n \text{ is even} \end{cases}$$

- symmetric: $x \approx \tilde{x}$
- positive skewed: $x > \tilde{x}$
- negative skewed: $x < \tilde{x}$

If you want your mean closer to your sample median. You can use **truncated mean**.

1.3.2 Variability

Example 1.8. Two dataset

- Dataset 1 1,100 $\bar{x} = 50.5$ $\tilde{x} = 50.5$
- Dataset 2 50,51 $\bar{x} = 50.5$ $\tilde{x} = 50.5$

Sample Variance:

$$S^2 = \frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{n-1}$$

Sample Standard deviation (s.d): $S = \sqrt{S^2}$
Short-cut formula:

$$S^2 = \frac{\sum_{i=1}^{n} x_i^2 - n\bar{x}^2}{n-1}$$

Proof.

$$\sum_{i=1}^{n}(x_i - \bar{x})^2 = \sum_{i=i}^{n}(x_i^2 + 2x_i\bar{x} + \bar{x}^2) = \sum_{i=i}^{n} x_i^2 - 2\bar{x}\sum_{i=1}^{n} x_i + \sum_{i=i}^{n} \bar{x}^2$$

Since

$$\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n}, \qquad n\bar{x} = \sum_{i=1}^{n} x_i$$

Substitute this, and the proof is done. ■

Proposition 1.9. *Let $x_1 \dots x_n$ be a sample, and c be any nonzero constant.*

1. *Let $y_1 = x_1 + c, y_2 = x_2 + c, \dots, y_n = x_n + c$, then*

$$\bar{y} = \bar{x} + c, S_y^2 = S_x^2$$

2. *Let $z_1 = cx_1, z_2 = cx_2, \dots, z_n = cx_n$, then*

$$\bar{z} = c\bar{x}, S_z^2 = c^2 S_z^2$$

1.3.3 Boxplot

The simplest boxplot is based on the following five-number summary:
smallest x_i, lower fourth, median, upper fourth, largest x_i

Definition 1.10. Any observation farther than $1.5f_s$ from the closest fourth is an **outlier**. An outlier is **extreme** if it is more than $3f_s$ from the nearest fourth, and it is **mild** otherwise.

Each mild outlier is represented by a closed circle and each extreme outlier by an open circle.

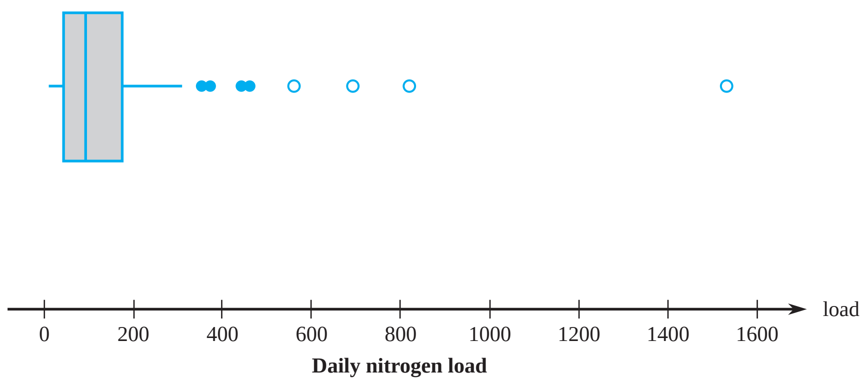


Figure 1.22 A boxplot of the nitrogen load data showing mild and extreme outliers

Figure 1.3: Boxplots That Show Outliers

Chapter 2

Probability

2.1 Sample Spaces and Events

Definition 2.1. An **experiment** is any action or process that generates observation.

Example 2.2. Flip a coin once, observe either H or T.

Example 2.3. Roll a dice, observe one one spot, two spot ...six spot.

Example 2.4. Choose a card from a well-shuttled deck, observe a deck of cards.[1]

2.1.1 The Sample Space of an Experiment

Definition 2.5. Sample space of an experiment, denoted by $\mathcal{S}$, is the set of all possible outcomes of the experiment.

Example 2.6. $\mathcal{S} = \{\mathrm{H}, \mathrm{T}\}$

Example 2.7. $\mathcal{S} = \{1, 2, 3, 4, 5, 6\}$

Example 2.8. $\mathcal{S} = \{A\spadesuit, 2\spadesuit, \ldots, K\heartsuit\}$

Example 2.9. Flip a coin twice, $\mathcal{S} = \{\mathrm{HH}, \mathrm{HT}, \mathrm{TH}, \mathrm{HH}\}$

2.1.2 Events

Definition 2.10. An **event** is a collection of outcomes of the sample space, denoted by E.

Example 2.11. $E = \{H\}$

Example 2.12. $E = \{4, 5, 6\}$

Example 2.13. $E = \{A\clubsuit, 2\clubsuit, \ldots, K\clubsuit\}$

Example 2.14. $\mathcal{E} = \{HH, TT\}$

2.1.3 Some Relations from Set Theory

Definition 2.15. The **union** of two events A and B is the event consisting of all outcomes that are either in A or in B. Notation:$A \cup B$

Definition 2.16. The **intersection** of two events A and B is the event consisting of all outcomes that are in **both** A or in B. Notation: $A \cap B$

Definition 2.17. The **complement** of an event A is the event consisting of all outcome in $\mathcal{S}$ but not in A. Notation: A'

[1] Four suits: $\spadesuit$spade; $\heartsuit$heart; $\diamondsuit$diamond; $\clubsuit$club. 13 cards in each suit: A,2,3,...,10,J, Q,K.

Example 2.18. Roll a dice, $\mathcal{S} = \{1, 2, 3, 4, 5, 6\}$
Let $A = \{1, 2, 3\}$, $B = \{1, 3, 5\}$
$A \cup B = \{1, 2, 3, 5\}$, $A \cap B = \{1, 3\}$
$A' = \{4, 5, 6\}$, $B' = \{2, 4, 6\}$

Definition 2.19. If A and B have no outcome in common, then they are **mutually exclusive** or **disjoint** $\Rightarrow A \cap B = \varnothing$

Proposition 2.20. *A and A' are disjoint.*

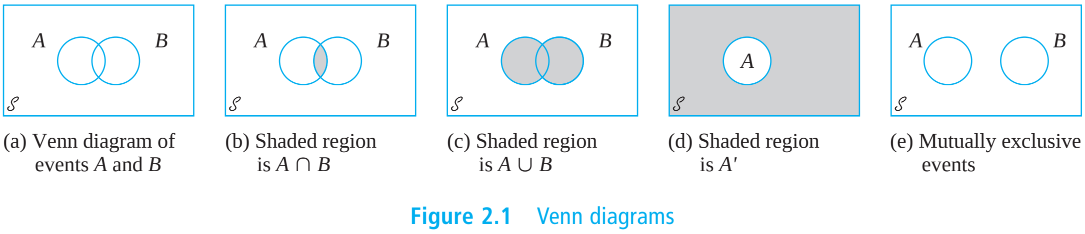


Figure 2.1: Venn diagrams

Example 2.21. $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$

2.2 Axioms, interpretations, and Properties of Probability

Probability: Given a sample space $\mathcal{S}$, for any event $A \in \mathcal{S}$, assign a number, say $P(A)$, to it.

Axiom 2.22. For every event A, $P(A) \geq 0$.

Axiom 2.23. $P(\mathcal{S}) = 1$

Axiom 2.24. If $A_1, A_2, A_3, \ldots$ is an infinite collection of disjoint events, then

$$P(A_1 \cup A_2 \cup A_3 \cup \ldots) = \sum_{i=1}^{\infty} P(A_i)$$

Proposition 2.25. $P(\varnothing) = 0$

Proof. Let $E_1 = \varnothing, E_2 = \varnothing, \ldots E_n = \varnothing$

$$P(\varnothing \cup \varnothing \cup \ldots \varnothing) = \sum_{i=1}^{n} P(\varnothing)$$

$$P(\varnothing) = nP(\varnothing)$$

$$P(\varnothing) = 0$$

■

Proposition 2.26. *If A and B are disjoint, $P(A \cup B) = P(A) + P(B)$.*

Proof. Let $E_1 = A, E_2 = B, E_3 = \varnothing \ldots E_n = \varnothing$. Then, we can prove it by Axiom 3. ■

Example 2.27. Flip a coin, $\mathcal{S} = \{H, T\}$

$$P(H) = 0.89 \qquad P(T) = 0.1$$

$$P(\mathcal{S}) = P(H \cup T) = P(H) + P(T) = \boxed{0.99} \neq 1$$

not a probability.

Example 2.28. Batteries come off an assembly line are tested one by one. The test will stop until a battery fails.

$$F : \text{faliure} \qquad S = \text{success}$$

Suppose $P(S) = 0.99 \qquad P(F) = 0.01$

$$\mathcal{S} = \{F, SF, SSF, SSSF, \dots\}$$

$$E_1 = \{F\}, E_2 = \{SF\}, E_3 = \{SSF\}, \dots$$

$$P(\mathcal{S}) = P(E_1 \cup E_2 \cup E_3 \dots) = P(E_1) + P(E_2) + P(E_3) + \dots$$

$$P(E_1) = 0.01 \qquad P(E_2) = 0.01 \times 0.99 \qquad P(E_3) = 0.01 \times 0.99^2$$

$$P(\mathcal{S}) = 0.01 + 0.99 \times 0.01 + \cdots = 0.01 \times \frac{1}{1 - 0.99} = 1$$

2.2.1 Interpreting Probability

Example 2.29. If I flip a coin 10 times, ref freq of H = # of H /10. If I flip a coin n times, ref freq of H = # of H $/n$.

The probability of flipping a coin resulted in H= relative freq of H when $n \to \infty$.

$$P(H) = \lim_{x \to \infty} \frac{\# of H}{n}$$

2.2.2 How to calculate Properties of Probability

Proposition 2.30. $P(A') = 1 - P(A)$

Proof.

$$1 = P(\mathcal{S}) = P(A \cup A') = P(A) + P(A')$$

■

Example 2.31. Components connected in a series, each component has 0.3 probability of fail, and they fail independently.

$$A = \{\text{the system fails}\}$$

$$P(A) = P(\{FSSSS, SFSSS, \dots\})$$

$$P(A) = 1 - P(\{\text{the system works}\}) = 1 - P(SSSSS) = 1 - 0.7^5$$

Proposition 2.32. *If* $A \cap B = \varnothing, P(A \cap B) = 0$

Proposition 2.33. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Proof. Let $E = B \cap A'$, A and E are disjoint

$$A \cup B = A \cup E$$

$$P(A \cup B) = P(A \cup E) = P(A) + P(E) \qquad (*)$$

Let $F = B \cap A$

$$E \cup F = B \qquad E \cap F = \varnothing$$

$$P(B) = P(E \cup F) = P(E) + P(F) = P(E) + P(A \cap B)$$

$$P(E) = P(B) - P(A \cap B)$$

Plug in (*),

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

■

Example 2.34. A card is drawn form a well-shuttled deck, what is the probability that it is a queen or a heart?

$$Q = \{\text{the card is a Queen}\}$$
$$H = \{\text{the card is a heart}\}$$

$$P(Q \cup H) = P(Q) + P(H) - P(Q \cap H) = \frac{16}{52}$$

Example 2.35. In pccw, 80% of the customers subscribed to cable TV. 30% of the customers subscribed to Internet. 25% of the customers subscribed to both. Randomly select one customer, what is the chance that the person has either TV or Internet.

$$C = \{\text{the customers subscribed to cable TV}\}$$

$$I = \{\text{the customers subscribed to Internet}\}$$

$$P(C) = 0.8 \qquad P(I) = 0.3 \qquad P(C \cap I) = 0.25$$

$$P(C \cup I) = P(C) + P(I) - P(C \cap I) = \boxed{0.85}$$

Proposition 2.36.

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$$

Example 2.37. C: Cable I: Internet T:Telephone.[2]

$$P(C) = 0.8 \qquad P(I) = 0.3 \qquad P(T) = 0.5$$
$$P(C \cap I) = 0.25 \qquad P(I \cap T) = 0.4 \qquad P(C \cap T) = 0.3$$
$$P(C \cap I \cap T) = 0.2$$

$$P(C \cup I \cup T) = P(C) + P(I) + P(T) - P(C \cap I) - P(C \cap T) - P(I \cap T) + P(C \cap I \cap T) = \boxed{0.85}$$

2.2.3 Determining Probabilities Systematically

Any event A is a union of simple events, i.e. with only one outcome. Then

$$P(A) = \sum_{E_i \in A} P(E_i),$$

and we just need to determine $P(E_i)$.

[2] Acutually a mistake $P(I \cap T) = 0.4$

Example 2.38. Toss a dice, $\mathcal{S} = \{1, 2, 3, 4, 5, 6\}$

$$P(\text{the spots} < 4)$$

$$A = \{\text{the spot} < 4\} = \{1, 2, 3\}$$

$$E_i = \{i\}; \qquad i = 1, 2, 3, 4, 5, 6$$

$$A = E_1 \cup E_2 \cup E_3 \qquad P(A) = P(E_1) + P(E_2) + P(E_3)$$

Suppose

$$P(1) = P(2) = P(6) = \frac{1}{9}$$

$$P(3) = P(4) = P(5) = \frac{2}{9}$$

$$P(A) = P(1) + P(2) + P(3) = \frac{4}{9}$$

2.2.4 Equally Likely Outcomes

Suppose $\mathcal{S}$ has N outcomes, $E_1, \ldots E_N$, they are equally likely to occur, then

$$P(E_i) = \frac{1}{N} \qquad i = 1, 2, \ldots$$

Then

$$P(A) = \frac{\text{\# of outcome in A}}{N}$$

Example 2.39. Toss a pair of fair dices. What is the chance that the sum of spots is 3?

$$N = 36 \qquad \mathcal{S} = \{(1,1), (1,2), \ldots, (6,6)\}$$

$$A = \{\text{the sum of spots is } 3\} = \{(1,2), (2,1)\}$$

$$P(A) = \frac{2}{36}$$

Example 2.40. What is the chance that the sum of spots ≤ 4?

$$B = \{\text{the sum of spots} \leq 4\} = \{(1,1), (1,2), (1,3), (2,1), (2,2), (3,1)\}$$

$$P(B) = \frac{6}{36}$$

2.3 Counting Techniques

Example 2.41. A new guy comes to HK. If there are 3 brands of cell phone, 4 telephone companies offer mobile service.

2.3.1 Product Rule

Select two elements in a row. The first element has n_1 choices, the second has n_2 choices. Then the number of pairs $= n_1 \cdot n_2$.

In general: suppose a set consists of K ordered elements (K-tuples), 1st element has n_1 choices, 2nd element has n_2 choices, 3rd element has n_3 choices,.... Then the number of different K-tuples is $n_1 n_2 \ldots n_k$.

2.3.2 Permutations and Combinations

Example 2.42. 70 students in the room choose 4 students to form a committee (secretary, treasury, officer 1, officer 2).

1. How many possible committees with position assigned?

$$70 \times 69 \times 68 \times 67 = \frac{70!}{66!}$$

2. How many possible committees without position assigned?

$$\binom{70}{4} = \frac{70!}{4! \times 66!}$$

Definition 2.43. The number of **permutation** of size k of n objects is denoted as $P_{k,n} = \frac{n!}{(n-k)!}$.
In specific, $P_{n,n} = n!$ $\quad (0! = 1)$

Definition 2.44. Given n distinct objects,any disordered subject of size k is called a **combination** of size k. The number of combination of size k of n objects, is denoted as $C_{k,n}$ or $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

Example 2.45.

$$\{A, B, C, D, E\}$$

1. Choose 3 letters, how many choices?

$$\binom{5}{3} = 10$$

2. Choose 3 letters to form a word, how many different words?

$$P_{3,5} = 60$$

Proposition 2.46.

$$k! \times C_{k,n} = \binom{n}{k} \times k! = P_{k,n}$$

Example 2.47 ("Birthday Paradox")**.** 365 different dates, n students.

$$\begin{aligned} &P\{\text{at least two students share the same birthday}\} \\ &= 1 - P\{\text{every one has a diffrent birthday}\} \\ &= 1 - \frac{P_{n,365}}{365^n} \end{aligned}$$

If $n = 50$, $P = 97\%$
If $n = 100$, $P = 99.99997\%$

2.4 Conditional Probability

Example 2.48. 52 cards. One card is dealt, and the another card is dealt.

1. P(the second card is 7 of clubs) $= \frac{1}{52}$
2. P(the second card is 7 of clubs given the first is J of spade) $= \frac{1}{51}$
3. P(the first card is J of spade, and the second card is 7 of clubs) $= \frac{1}{P_{2,52}} = \frac{1}{52 \times 51}$
4. P(the first card is J of spade) $= \frac{1}{52}$

So P(B given A)$=\frac{P(A \cap B)}{P(A)}$

Example 2.49. Fishing in the sea

	Walleye	Pike
Sam	2	3
I	1	5

Randomly pick one, found, it is s Walleye. What is the chance that it is caught by me?

$$A = \{\text{Walleye}\}$$
$$B = \{\text{Caught by me}\}$$
$$P(A) = \frac{3}{11} \qquad P(B) = \frac{6}{11}$$
$$P(B \text{ given } A) = \frac{P(A \cap B)}{P(A)} = \boxed{\frac{1}{3}}$$

2.4.1 The Definition of Conditional Probability

Definition 2.50. For any two events A and B, with $P(B) > 0$, the conditional probability of A given that B has occurred is defined by

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Example 2.51. Of all costumers purchasing computers, 60% of them include M$ Word; 50% of them include M$ Excel; 30% of them include both.

$$A = \{\text{Word is included}\}$$
$$B = \{\text{Excel is included}\}$$

$$P(A|B) = 0.6 \qquad P(B|A) = 0.5$$
$$P(A|B) \neq P(B|A)$$

Recall: Axioms of probability

1. For every event A, $P(A) \geq 0$.
2. $P(\mathcal{S}) = 1$
3. If $A_1, A_2, A_3, \ldots$ is an infinite collection of disjoint events, then

$$P(A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_n) = \sum_{i=1}^{n} P(A_i)$$

Similarly,

1. For every event A, $P(A|B) \geq 0$.
2. $P(B|B) = 1$
3. If $A_1, A_2, A_3, \ldots$ is an infinite collection of disjoint events, then

$$P(A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_n|B) = \sum_{i=1}^{n} P(A_i|B)$$

Example 2.52. A new magazine publishes 3 columns: "Art"(A), "Boobs"(B), "Cinema"(C). Research shows the reading habits:

A	B	C	$A \cap B$	$A \cap C$	$B \cap C$	$A \cap B \cap C$
0.14	0.23	0.37	0.08	0.09	0.13	0.05

.

Randomly select one reader

(1)

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \boxed{\frac{8}{23}}$$

(2)

$$\begin{aligned}
P(A|B \cup C) &= \frac{P(A \cap (B \cup C))}{P(B \cup C)} \\
&= \frac{P\big((A \cap B) \cup (A \cap C)\big)}{P(B) + P(C) - P(B \cap C)} \\
&= \frac{P(A \cap B) + P(A \cap C) - P(A \cap B \cap C)}{P(B) + P(C) - P(B \cap C)}
\end{aligned}$$

(3) $P(A|A \cup B \cup C)$: What is the probability that the reader read "Art" Column given that he/she reads at least one column?

$$\begin{aligned}
P(A|A \cup B \cup C) &= \frac{P(A \cap (A \cup B \cup C))}{P(A \cup B \cup C)} = \frac{P(A)}{P(A \cup B \cup C)} \\
&= \frac{P(A)}{P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)} \\
&= \boxed{\frac{14}{49}}
\end{aligned}$$

(4)

$$\begin{aligned}
P(A \cup B|C) =& \frac{P((A \cup B) \cap C)}{P(C)} \\
=& \frac{P(A \cap C) + P(B \cap C) - P(A \cap B \cap C)}{P(C)} = \boxed{\frac{17}{37}}
\end{aligned}$$

2.4.2 The Multiplication Rule for $P(A \cap B)$

Proposition 2.53.

$$P(A \cap B) = P(A)P(B|A) = P(B)P(A|B)$$

Example 2.54. Two cards are dealt.

$$P(\text{1st is J of spade and 2nd is 7 of heart}) = P(A)P(B|A) = \frac{1}{52} \times \frac{1}{51}$$

Example 2.55. Same scenario

$$A = \{\text{1st is a club}\}$$
$$B = \{\text{2nd is a club}\}$$
$$P(A \cap B) = P(A)P(B|A) = \frac{13}{52} \times \frac{12}{51}$$
$$C = \{\text{3rd card is a heart}\}$$
$$P(A \cap B \cap C) = P(A \cap B)P(C|A \cap B) = P(A)P(B|A)P(C|A \cap B) = \frac{13}{52} \times \frac{12}{51} \times \frac{13}{50}$$

Proposition 2.56.

$$P(A_1 \cap A_2 \cdots \cap A_k) = P(A_1)P(A_2|A_1)P(A_3|A_1 \cap A_2) \ldots P(A_k|A_1 \cap A_2 \cdots \cap A_{k-1})$$

Example 2.57. Same scenario

$$A = \{\text{1st is a club}\}$$
$$B = \{\text{2nd is A of club}\}$$
$$C = \{\text{3rd is 2 of club}\}$$

$$P(A \cap B \cap C) = P(C)P(B|C)P(A|B \cap C) = \frac{1}{52} \times \frac{1}{51} \times \frac{11}{50}$$

Introduce $D = \{\text{1st is either 3,4,...K of club}\}$

$$P(A \cap B \cap C) = P(D \cap B \cap C) = \frac{11}{52} \times \frac{1}{51} \times \frac{1}{50}$$

2.4.3 Bayes' Theorem

Theorem 2.58 (The Law of Total Probability)**.** *Let $A_1, ..., A_k$ be mutually exclusive and exhaustive events. Then for any other event B,*

$$P(B) = \sum_{i=1}^{k} P(B|A_i)P(A_i)$$

Example 2.59. A store sells 3 brands of game consoles

Brand	1	2	3
Proportion	50%	30%	20%

A one year warranty is offered, known

Brand	1	2	3
Under warranty	25%	20%	10%

$$A_i = \{\text{bought brand } i\} \qquad i = 1, 2, 3$$
$$B = \{\text{needs repaire under warranty}\}$$
$$P(A_1) = 0.5 \qquad P(A_2) = 0.3 \qquad P(A_3) = 0.2$$
$$P(B|A_1) = 0.25 \qquad P(B|A_2) = 0.2 \qquad P(B|A_3) = 0.1$$

Q1:

$$P(B'|A_1) = 1 - P(B|A_1) = 0.75$$

Q2:

$$P(B) = P(A_1)P(B|A_1) + P(A_2)P(B|A_2) + P(A_3)P(B|A_3) = 0.205$$

Q3:

$$P(A_1|B) = \frac{P(A_1 \cap B)}{P(B)} = \frac{P(A_1)P(B|A_1)}{P(B)} = 0.61$$
$$P(A_2|B) = 0.29$$
$$P(A_3|B) = 0.10$$

Theorem 2.60 (Bayes' Theorem)**.**

$$P(A_i|B) = \frac{P(A_i \cap B)}{P(B)} = \frac{P(A_i)P(B|A_i)}{\sum_{i=1}^{k} P(A_k)P(B|A_i)}$$

Example 2.61. $\frac{1}{1000}$ adults has a rare disease.

99% of people with the disease can be found positive

20% of people without the disease can be found positive

Randomly select a person and test him. Suppose the result is positive. What is the chance that he really has the disease?

Let

$$A = \{\text{the individual has the disease}\}$$

$$A' = \{\text{the individual does not have the disease}\}$$

$$B = \{\text{that positive}\}$$

$$P(A) = \frac{1}{1000} \qquad P(B|A) = 0.99 \qquad P(B|A') = 0.20$$

Question:

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)P(B|A)}{P(A)P(B|A) + P(A')P(B|A')}$$

Because A and A' are portions of $\mathcal{S}$

$$= \frac{0.001 \times 0.99}{0.001 \times 0.99 + 0.999 \times 0.2} = 0.493\%$$

2.5 Independence

Definition 2.62. Two events A and B. If A and B are **independent**, then

$$P(A \cap B) = P(A)P(B)$$

Note that $P(A \cap B) = P(A)P(B|A)$ apply for any condition.

$$P(B) = P(B|A),$$

event A has nothing to do with event B.

Example 2.63. Roll a dice once. $P(i) = \frac{1}{6}; i = 1, 2, \ldots, 6$

$$A = \{2, 4, 6\}, B = \{1, 2, 3\}, C = \{1, 2, 3, 4\}$$

$$P(A) = \frac{3}{6} = \frac{1}{2}$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{1/6}{3/6} = \frac{1}{3}$$

$$P(A|C) = \frac{P(A \cap C)}{P(C)} = \frac{2/6}{4/6} = \frac{1}{2}$$

So A and C are independent. $A \not\perp B$

Proposition 2.64. *If A and B are independent,*

- *A' and B' are independent,*
- *A' and B are independent,*
- *A and B' are independent.*

Example 2.65. Toss a fair coin repeatedly until the first H occurs.

$$A = \{\text{at least 5 tosses result in the first H}\}$$

$$P(A) = \quad ?$$

Solution. Assume the tossing are independent.

$$A = \{TTTTH\} \cup \{TTTTTH\} \cup \dots$$

$$\begin{aligned}P(A) &= P(\{TTTTH\}) + \dots = (P(T))^4 P(H) + (P(T))^5 P(H) \\ &= \left(\frac{1}{2}\right)^5 + \left(\frac{1}{2}\right)^6 + \dots = \frac{1}{16}\end{aligned}$$

■

2.5.1 Independence of More Than Two Events

Definition 2.66. $A_1, A_2, \dots A_k$ are events. If for any indices $i, \dots, i_k$

$$P(A_1 \cap A_2 \cap \dots A_k) = P(A_1)P(A_2)\dots P(A_k)$$

Then $A_1, A_2, \dots A_k$ are said to be mutually independent.

Example 2.67. Component works with probability 0.9, and they work independently.

$$P(\text{system works})$$

Solution. Let

$$\begin{aligned}&A_i = \{\text{the } i\text{th component works}\} \\ &P(A_i) = 0.9 \\ &P(\text{system works}) = P\left((A_1 \cap A_2) \cup (A_3 \cap A_4)\right) = 0.9639\end{aligned}$$

■

2.6 Challenge Question 1

Monty Hall problem. https://en.wikipedia.org/wiki/Monty_Hall_problem

2.7 Problem in Previous Mid-term Test

Example 2.68.

$$\begin{aligned}D &= \{\text{David makes a right decision}\} \\ J &= \{\text{John makes a right decision}\} \\ P &= \{\text{Peter makes a right decision}\}\end{aligned}$$

David and Peter make decision independently.

$$P(D) = 0.7 \qquad P(D|J) = 0.9 \qquad P(D'|J') = 0.8$$

$$P(J|P) = 0.3 \qquad P(J'|P') = 0.2 \qquad P(D \cap J \cap P) = 0.1$$

Question: Find (1) $P(J)$; (2) $P(P)$; (3) $P(\text{at least two make right decision})$.

Solution. (1)

$$\begin{aligned}0.9 &= P(D|J) = \frac{P(D \cap J)}{P(J)} \\ 0.8 &= P(D'|J') = \frac{P(D' \cap J')}{P(J')} = \frac{P(D \cup J)'}{1 - P(J)} = \frac{1 - P(D \cup J)}{1 - P(J)}\end{aligned}$$

$$P(D \cup J) = P(D) + P(J) - P(D \cap J) = 0.7 + P(J) - P(D \cap J)$$

$$P(J) = \frac{5}{7}$$

(2) Similar to (1)
(3)

$$P(\text{at least two make right decision})$$

Method A:

$$\begin{aligned}
&= 1 - P(\text{no more than 1 make right decisions}) \\
&= 1 - P(D \cap J \cap P') - P(D' \cap J \cap P') - P(D' \cap J' \cap P) - P(D' \cap J' \cap P')
\end{aligned}$$

Method B:

$$= P(A \cap B) + P(B \cap C) + P(C \cap A) - 2P(A \cap B \cap C)$$

■

Chapter 3

Discrete Random Variables

3.1 Random Variable

Definition 3.1. For a given sample space $\mathcal{S}$ of some experiment, a **random variable (rv)** is any rule that associates a number with each outcome in $\mathcal{S}$. In mathematical language, a random variable is a **function** whose domain is the sample space and whose range is the set of real numbers.

Example 3.2. Flip a coin, $\mathcal{S} = \{\mathrm{H}, \mathrm{T}\}$

$$X(H) = 1 \qquad X(T) = 0$$

Example 3.3. Randomly pick a student, height

$$X(\text{height} \geq 6 \text{ feet}) = 1 \qquad X(\text{height} \leq 6 \text{ feet}) = 0$$

Definition 3.4. Any r.v. whose possible values are 0 and 1 is called a **Bernoulli random variable**.

Example 3.5. Randomly pick a student, phone brand

$$X(\text{Apple}) = 1 \qquad X(\text{Samsung}) = 0$$

Example 3.6. Waiting MTR at Kowloon Tong

$$X(\text{waiting time}) = \text{waiting time}$$

3.1.1 Two Types of Random Variables

Definition 3.7. a **discrete** r.v. whose possible values are either finite or countable.

a **continuous** r.v. is a r.v. whose possible values consist of an entire interval on the real lines.

3.2 Probability Distributions for Discrete Random Variables

Definition 3.8. $\mathcal{S}$ is a sample space, $X(s)$ is a r.v. $p(x) = P(s \in \mathcal{S}; X(s) = x)$ is called the probability mass function (p.m.f) or probability distribution function (p.d.f) of x.

Example 3.9.

$$\mathcal{S} = (5 \text{ feet}, 7 \text{ feet})$$

$$X(s) = \begin{cases} 1, & \text{if} \quad s \geq 6 \text{ feet} \\ 0. & \text{if} \quad s \leq 6 \text{ feet} \end{cases}$$

$$P(X = 1) = P(s \geq 6 \text{ feet})$$

Example 3.10. Six lots of components that the # of defectives are listed as follows

lot	1	2	3	4	5	6
# of defectives	0	2	0	1	2	0

One of those is randomly selected. $X = \#$ of defectives in the selected lot

$$P(X = 0) = P(\{1, 3, 6\}) = \frac{1}{2}$$

$$P(X = 1) = P(\{4\}) = \frac{1}{6}$$

$$P(X = 2) = P(\{2, 5\}) = \frac{1}{3}$$

Example 3.11. Five person 1,2,3,4,5 are blood donors. Among them, only 1 and 2 have "O" type. Collect their blood in a random segment, $X = \#$ of typing neccessary to get the first "O" type.

$$X = 1, 2, 3, 4$$

$$P(X = 1) = P(\text{typing after the first trail}) = \frac{2}{5}$$

Review

X is a discrete r.v.

1. Support $x \in \mathcal{D}$
2. p.m.f $p(x) = P(s \in \mathcal{S}; X(s) = x), \quad \forall x \in \mathcal{D}$

3.2.1 The Cumulative Distribution Function

Example 3.12. Roll a dice. Let x= # of spots. What is the probability that $x \leq 5$

$$\mathcal{D} = \{1, 2, 3, 4, 5, 6\}$$

$$P(1) = P(2) = \cdots = P(6) = \frac{1}{6}$$

$$P(X \leq 5) = P(\{1, 2, 3, 4, 5\}) = P(1) + P(2) + P(3) + P(4) + P(5) = \frac{5}{6}$$

$$F(x) = \begin{cases} P(X \leq x) = 0 & \text{if } x < 1 \\ P(X \leq x) = \frac{1}{6} & \text{if } 1 \leq x < 2 \\ \dots \\ P(X \leq x) = 1 & \text{if } x \geq 6 \end{cases}$$

It is called step function.

Definition 3.13. The Cumulative Distribution Function (c.d.f) of a r.v X is defined as

$$F(x) = P(X \leq x) = \sum_{y \leq x} p(y)$$

Example 3.14. a r.v Y

y	1	2	3	4
$p(y)$	0.4	0.3	0.2	0.1

$$F(y) = P(Y \leq y) = \begin{cases} 0 & \text{if } y < 1 \\ 0.4 & \text{if } 1 \leq y < 2 \\ 0.7 & \text{if } 2 \leq y < 3 \\ 0.9 & \text{if } 3 \leq y < 4 \\ 1 & \text{if } y \geq 4 \end{cases}$$

Example 3.15. Toss a coin until the first head. Suppose $P(\text{Head}) = p$, $P(\text{Tail}) = q = 1 - p$, $x =$ # of toses until the first head

$$\mathcal{D} = \{1, 2, 3, \dots\}$$

$$p(x) = q^{x-1}p, \qquad x = 1, 2, 3, \dots$$

$$F(x) = P(X \leq x) = \sum_{y \leq x} p(y) = \sum_{y \leq x} q^{y-1}p = p\frac{1 - q^{\lfloor x \rfloor}}{1 - q} = 1 - q^{\lfloor x \rfloor}$$

where $\lfloor x \rfloor$ is the largest integer $\leq x$ (floor function).

$$F(x) = \begin{cases} 0, & \text{if } x < 0 \\ 1 - q^{\lfloor x \rfloor}. & \text{if } x \geq 0 \end{cases}$$

How do we get p.m.f from c.d.f

In examples thus far, the cdf has been derived from the pmf. This process can be reversed to obtain the pmf from the cdf whenever the latter function is available.

$$P(X = 3) = P(x \leq 3) - P(x \leq 2) = F(3) - F(2)$$

Suppose X takes integer values, for any integers a and b,

$$P(a \leq X \leq b) = P(X \leq b) - P(X \leq a - 1) = F(b) - F(a - 1)$$

Generally, for a and b

$$P(a \leq X \leq b) = F(b) - F(a_-)$$

Here a_- is the largest integer value that is strictly less than a. If $a = 2$, $\lfloor a \rfloor = 2$, $a_- = 1$

3.3 Expected Values

Example 3.16 ("Russian roulette")**.** Bet even or odd. Bet \$1 on even, I will win \$1 if indeed it is even, and I will lose \$1 if it is odd, or 0, or 00.

Expected value

$$\frac{18}{38} \times 1 + \frac{20}{38} \times (-1) = -\frac{2}{38}$$

3.3.1 The Expected Value of X

Definition 3.17. Let X be a discrete rv with set of possible values $\mathcal{D}$ and pmf $p(x)$. The expected value or mean value of X, denoted by $E(X)$ or μ_X or just μ, is

$$E(X) = \sum_{x \in \mathcal{D}} xp(x)$$

$$x = \begin{cases} 1, & \text{w.p.} \frac{18}{38} \\ -1, & \text{w.p.} \frac{20}{38} \end{cases}$$

$$E(X) = -\frac{2}{38}$$

Example 3.18. X is a Bernoulli r.v

$$p(x) = \begin{cases} p, & \text{if} \quad x = 1 \\ 1 - p. & \text{if} \quad x = 0 \end{cases}$$

$$E(X) = 1p + 0(1 - p) = p$$

Example 3.19. A newly-wed couple want a girl. Their plan is to keep having children until they get a girl.

$$X = \#\text{ of children when the girl is born}$$

$$P(\text{boy}) = p \qquad P(\text{girl}) = 1 - p = q$$

$$p(x) = p^{x-1}q \qquad x = 1, 2, 3, \ldots$$

$$E(X) = \sum_{x=1}^{\infty} xp^{x-1}q = q\sum_{x=1}^{\infty} xp^{x-1}$$

$$S = p^0 + 2p^1 + 3p^2 + 4p^3 + \ldots$$

$$pS = p^1 + 2p^2 + 3p^3 + 4p^4 + \ldots$$

$$(1-p)S = p^0 + p^1 + p^2 + p^3 + p^4 + \cdots = \frac{1}{1-p}$$

$$E(X) = q\frac{1}{(1-p)^2} = \frac{1}{q}$$

Another method to calculate $\sum_{x=1}^{\infty} xp^{x-1}q$

$$\sum_{x=1}^{\infty} xp^{x-1} = \sum_{x=1}^{\infty}(p^x)' = \Big(\sum_{x=1}^{\infty} p^x\Big)'$$

Example 3.20.

$$p(k) = \frac{1}{k^2}\frac{6}{\pi^2} \qquad k = 1, 2, 3, \ldots$$

Verify

$$\sum_{k=1}^{\infty} p(k) = 1$$

$$E(x) = \sum_{k=1}^{\infty} k\frac{1}{k^2}\frac{6}{\pi^2} = \frac{6}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{k} = \infty$$

3.3.2 The Expected Value of a Function

Proposition 3.21.

$$E(h(X)) = \sum_{x\in\mathcal{D}} h(x)p(x)$$

Example 3.22. # of cylinders in the engine of the next car to be turned up.

Cost for x cylinders

$$h(x) = 20 + 3x + 0.5x^2$$

History shows that

x	4	6	8
$p(x)$	0.5	0.3	0.2
$h(x)$	40	56	76

$$E(h(x)) = 40 \times 0.5 + 56 \times 0.3 + 76 \times 0.2 = \boxed{52}$$

3.3.3 Rules of Expected Value

Proposition 3.23. *Let a and b be two constant, X r.v*

$$E(aX+b) = aE(X)+b$$

Particularly,

$$\text{if} \quad b=0, \quad E(aX) = aE(X)$$
$$\text{if} \quad a=0, \quad E(X+b) = E(X)+b$$

Example 3.24. A computer store has purchased three computers of a certain type at \$500 apiece. It will sell them for \$1000 apiece. The manufacturer has agreed to repurchase any computers still unsold after a specified period at \$ 200 apiece. Let X denote the number of computers sold.

x	0	1	2	3
$p(x)$	0.1	0.2	0.3	0.4

$$E(X) = 2$$

$$Y = 1000X + 200(3-X) - 1500 = 800X - 900$$
$$E(Y) = 800E(X) - 900 = 700$$

3.3.4 The Variance of X

Example 3.25.

x	1	2	3	4	5	6
$p(x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

$$E(X) = \frac{7}{2}$$

x	3	4
$p(x)$	$\frac{1}{6}$	$\frac{1}{6}$

$$E(X) = \frac{7}{2}$$

Definition 3.26. X is a discrete random variable $E(X) = \mu$,

$$\sigma_x^2 = Var(X) = E((X-\mu)^2) = \sum_{x\in\mathcal{D}} (x-\mu)^2 p(x)$$

$$\sigma_x = s.d(X) = \sqrt{Var(X)}$$

For Ex(1), $Var(X) = 2.92$; For Ex(2), $Var(X) = 0.25$.

3.3.5 Short-cut Formula

Proposition 3.27.

$$Var(X) = E(X^2) - (E(X))^2$$

Proof.

$$\begin{aligned} Var(X) =& E((X-\mu)^2) = E(x^2 - 2X\mu + \mu^2) \\ =& E(X^2) + E(-2X\mu) + E(\mu^2) = E(X^2) - 2\mu E(X) + \mu^2 \\ =& E(X^2) - 2\mu\mu + \mu^2 = E(X^2) - (E(X))^2 \end{aligned}$$

■

3.3.6 Rules

Proposition 3.28.

$$Var(aX+b) = a^2Var(X)$$

$$s.d(aX+b) = |a|s.d(X)$$

since a could be negative.

Example 3.29. Computer store

$$Y = 800X - 900$$

x	0	1	2	3
$p(x)$	0.1	0.2	0.3	0.4

$$Var(Y) = Var(800X - 900) = 800^2Var(X)$$

$$E(X) = 2 \qquad E(X^2) = 5$$

$$Var(Y) = 800^2(5 - 2^2) = 640000$$

3.4 The Binomial Probability Distribution

Recall $X \sim Bernobli(p)$

$$p(0) = 1 - p \qquad p(1) = p$$

Example 3.30. Flip a coin 3 times independently. X=# of Heads. What's the distribution of X?

$$\mathcal{D} = \{0, 1, 2, 3\}$$

	x	$p(x)$
TTT	0	$(1-p)^3$
HTT, THT, TTH	1	$3p(1-p)^2$
HHT, HTH, THH	2	$3p^2(1-p)$
HHH	3	p^3

$$\sum p(x) = 1$$

3.4.1 The Binomial Random Variable and Distribution

Generally, n Bernouli trails, independently. the success rate of each trail is constant p, then the # of success out of these n trails is a **Binomial** r.v, denoted as $X \sim Bin(n,p)$

If $X \sim Bin(n,p)$,

$$P(X = x) = \binom{n}{x} p^x (1-p)^{n-x} \qquad x = 0, 1, \dots n$$

Back to the example,

$$P(X = 0) = \binom{3}{0} p^0 (1-p)^3 = (1-p)^3$$

3.4.2 The Mean and Variance of X

Proposition 3.31. *If $X \sim Bin(n,p)$,*

$$E(X) = np \qquad Var(X) = np(1-p)$$

Proof.

$$\begin{aligned}
E(X) &= \sum_{k=0}^{n} k \binom{n}{k} p^k (1-p)^{n-k} = \sum_{k=0}^{n} k \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k} \\
&= \sum_{k=1}^{n} \frac{n \cdot (n-1)!}{(k-1)!(n-k)!} p^k (1-p)^{n-k} \\
&= np \sum_{k=1}^{n} \frac{(n-1)!}{(k-1)!(n-k)!} p^{k-1} (1-p)^{n-k} \\
&= np \sum_{k=1}^{n} \binom{n-1}{k-1} p^{k-1} (1-p)^{n-k} \\
&= np \sum_{k'=0}^{n'} \binom{n'}{k'} p^{k'} (1-p)^{n'-k'} = np
\end{aligned}$$

■

Example 3.32. Six cola drinkers. Two brand: C, P. $X = \#$ of cola C they choose.

$$P(C) = \frac{1}{2} \qquad P(P) = \frac{1}{2}$$

$$X \sim Bin\left(6, \frac{1}{2}\right)$$

$$\begin{aligned}
P(X=3) &= \binom{6}{3} \left(\frac{1}{2}\right)^3 \left(1 - \frac{1}{2}\right)^{6-3} = 0.313 \\
P(X \le 1) &= P(X=0) + P(X=1) = 0.109 \\
P(X \ge 3) &= 1 - P(X \le 2)
\end{aligned}$$

3.4.3 Using Binomial Tables

3.5 Hypergeometric and Negative Binomial Distributions

Example 3.33. 5 balls in a box, 3 red, 2 blue. Randomly choose 3 balls out of the box with replacement. What is the chance of getting 2 red and 1 blue balls?

$$X = \# \text{ of red balls out of } 3$$

$$X \sim Bin\left(3, \frac{3}{5}\right)$$

$$P(X=2) = \binom{3}{2} = \frac{54}{125}$$

Example 3.34. Same step. without replacement.

$$X = \# \text{ of red balls out of } 3$$

$$X \quad \not\sim Bin\left(3, \frac{3}{5}\right)$$

$$\begin{aligned}
P(X=2) =& \frac{\# \text{ of outcome in } E}{\# \text{ of outcomes in } \mathcal{S}} \\
=& \frac{\binom{3}{2}\binom{2}{1}}{\binom{5}{3}} = \frac{3}{5}
\end{aligned}$$

3.5.1 Hypergeometric

Proposition 3.35. *In general, M of type "1", $N-M$ of type "2 in a box, choose n items.*

$$Y \sim hypergeometric(N, M, n)$$

$$P(Y=k) = \frac{\binom{M}{k}\binom{N-M}{n-k}}{\binom{N}{n}} \qquad k = (0 \vee n-(N-M)), 1, 2, \ldots, (n \wedge M)$$

3.5.2 The Mean and Variance of X

Proposition 3.36. *If $X \sim hypergeometric(N, M, n)$,*

$$E(X) = n \cdot \frac{M}{N}$$

$$Var(X) = \frac{N-n}{N-1} \cdot n \cdot \frac{M}{N}\left(1 - \frac{M}{N}\right)$$

Example 3.37. Five wolves are caught in a forest. Tagged and released to mix with other wolves. After a while, 10 wolves are caught.

Assume there are 25 such wolves in the forest. $P(X=2)=$?

$$X \sim h.g.(25, 5, 10)$$

$$P(X=2) = \frac{\binom{5}{2}\binom{20}{8}}{\binom{25}{10}} = 0.385$$
$$E(X) = 2$$
$$Var(X) = 1$$

If we have no idea about the number of wolves in the forest. But $X=3$, how to estimate the # of wolves in the forest ?

$$N = \#\text{ of wolves in total}$$

$$10 \cdot \frac{5}{N} = E(X) \approx 3$$

$$N \approx 10 \cdot \frac{5}{3} \approx 17$$

3.5.3 The Negative Binomial Distribution

Example 3.38. A couple wants 3 girls. How many children they need to have to have fulfil his planning?

$$P(\text{girl}) = p \qquad P(\text{boy}) = 1-p$$

$$\text{X=\# of children to attain this planning}$$

$$x \geq 3 \qquad \mathcal{D} = \{3, 4, \ldots\}$$

$$P(X=k) = \binom{k-1}{2} p^3 (1-p)^{k-3}$$

This is called "Negative binomial r.v"

Proposition 3.39. *In general, $X \sim$ Negative Binomial(r, p)*

$$P(X = k) = \binom{k-1}{r-1} p^r (1-p)^{k-r} \qquad k = r, r+1, \ldots$$

$$E(X) = \frac{r}{p} \qquad Var(X) = \frac{r(1-p)}{p^2}$$

Example 3.40. Roll a dice repeatedly until the first "one" occurs. $X = \#$ of rollings.

$$X \sim n.b(1, \frac{1}{6})$$

$$E(X) = \frac{1}{\frac{1}{6}} = 6 \qquad Var(X) = 30$$

3.6 The Poisson Probability Distribution

Definition 3.41. A r.v. X takes value 0,1,2,3,...

$$P(X = k) = \frac{\lambda^k}{k!} e^{-\lambda} \qquad k = 0, 1, 2 \ldots$$

where $\lambda > 0$. Then we say $X \sim Poisson(\lambda)$

Check

$$\sum_{k=0}^{\infty} \frac{\lambda^k}{k!} e^{-\lambda} = e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = 1$$

Since

$$e^{\lambda} = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \qquad \text{(Taylor expansion)}$$

3.6.1 The Mean and Variance of X

Proposition 3.42. *If $X \sim Poisson(\lambda)$, $E(X) = \lambda$, $Var(X) = \lambda$.*

Proof.

$$\begin{aligned} E(X) &= \sum_{k=0}^{\infty} kP(X = k) = \sum_{k=0}^{\infty} k \frac{\lambda^k}{k!} e^{-\lambda} \\ &= \lambda \sum_{k=0}^{\infty} \frac{\lambda^{k-1}}{(k-1)!} e^{-\lambda} = \lambda \sum_{k'=0}^{\infty} \frac{\lambda^{k'}}{k'!} e^{-\lambda} = \lambda \end{aligned}$$

∎

3.6.2 The Poisson Distribution as a Limit

Proposition 3.43. *If $X \sim Bin(n, p)$, n is large, p is small. Then $X \sim Poisson(\lambda)$ with $\lambda = np$.*

Example 3.44. A publisher is publishing a non-technical book. P(making at least one error in a page) = 0.005. The book has 400 pages, independent from page to page.
$X =\#$ of pages with errors $\sim Bin(400, 0.005)$

$$P(X=2) = \binom{400}{2} 0.005^2 (1-0.005)^{400-2}$$

$$X \sim Poisson(2)$$

$$P(X=2) = \frac{2^2}{2!} e^{-2} = 0.27$$

Rule of Thumb

When $n \geq 50$, $np \leq 5$, we consider n is large enough, p is small enough.

3.6.3 The Poisson Process

Example 3.45. Counting the number of customers at a bank counter. Suppose

1. $\exists \alpha > 0$ such that
$$P(\text{exact one customer in } \Delta t) = \alpha \Delta t + o(\Delta t)$$

2. $$P(\text{more than one customer in } \Delta t) = o(\Delta t)$$

3. Number of customers during Δt is independent of that prior to this period

Then $P(k \text{ customers during } (0,t)) = \frac{(\alpha t)^k}{k!} e^{-\alpha t}$. Let $X_t = \#$ of customers during $(0,t)$.

$$X_t \sim Poisson(\alpha t)$$

$$E(X_t) = \alpha t \qquad Var(X_t) = \alpha t$$

Chapter 4

Continuous Random Variables and Probability Distributions

4.1 Probability Density Functions

Example 4.1. Study the ecology of a lake, measure the depth of the lake. Denote L_{max} as the largest depth of the lake.

$$X = \text{depth of the lake}$$

The support of X is $(0, L_{max}]$

This is a continues r.v., but it shares some properties of a discrete r.v.

Definition 4.2. In general, X is supported on $[a, b]$. There is a $f(x)$ satisfying

1. $f(x) \geq 0, \qquad \forall x \in [a, b]$
2. $\int_a^b f(x)dx = 1$
3. $P(c < x < d) = \int_c^d f(x)dx$

Such an $f(x)$ is called the **probability distribution function(p.d.f)** of X

$$f(x) = \lim_{h \to 0} \frac{P(x \leq X \leq x + h)}{h}$$

Example 4.3. X=waiting time of a MTR at Kowloon Tong Station is

$$f(x) = \begin{cases} \frac{1}{15}, & 0 \leq x \leq 15 \\ 0. & \text{otherwise} \end{cases}$$

Check

$$\int_0^{15} f(x)dx = \int_0^{15} \frac{1}{15} dx = 1$$

$$P(5 \leq X \leq 10) = \int_5^{10} \frac{1}{15} dx = \frac{1}{3}$$

"uniform r.v"

4.2 Cumulative Distribution Functions and Expected Values

4.2.1 The Cumulative Distribution Function

Definition 4.4. Let X be a constant r.v. with c.d.f $f(x)$. Its **c.d.f.** is

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(y)dy$$

Example 4.5.

$$X \sim unif(a,b)$$

$$f(x) = \begin{cases} \frac{1}{b-a}, & a \le x \le b \\ 0. & \text{otherwise} \end{cases}$$

$$F(x) = \int_{-\infty}^{x} f(y)dy = \begin{cases} 0, & \text{if } x < a \\ \frac{x-a}{b-a}, & a \le x \le b \\ 1. & \text{if } x > b \end{cases}$$

Example 4.6. $X \sim exp(\lambda)$, "exponential r.v".

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0 \\ 0. & \text{otherwise} \end{cases}$$

$$F(x) = \int_{-\infty}^{x} f(y)dy = \begin{cases} 0, & \text{if } x < 0 \\ 1 - e^{-\lambda x}. & \text{if } x \ge 0 \end{cases}$$

If $x > 0$,

$$\begin{aligned} \int_{-\infty}^{x} f(y)dy &= \int_{0}^{x} \lambda e^{-\lambda y} dy = \int_{0}^{x} e^{-\lambda y} d(\lambda y) \\ &= -e^{-\lambda y}\Big|_{0}^{x} = 1 - e^{-\lambda x} \end{aligned}$$

Proposition 4.7. *If X is continuous. For any constant c,*

$$P(X = c) = 0$$

Furthermore, for any a, b, we have

$$P(a \le X \le b) = P(a < X \le b) = P(a \le X < b) = P(a < X < b)$$

4.2.2 Using $F(x)$ to Compute Probabilities

Let X be a constant r.v. with p.d.f $f(x)$ and c.d.f. $F(x)$,Then

$$\begin{aligned} &P(X > a) = 1 - P(X \le a) = 1 - F(a) \\ &P(X \ge a) = 1 - F(a) \\ &P(a < X < b) = 1 - F(b) \\ &P(a < X < b) = P(x < b) - P(x < b) = F(b) - F(a) \end{aligned}$$

Example 4.8. X has a p.d.f

$$f(x) = \begin{cases} \frac{1}{8} + \frac{3}{8}x, & \text{if } 0 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$$

$$F(x) \int_{-\infty}^{x} f(y)dy = \begin{cases} 0, & \text{if } x < 0 \\ \frac{x}{8} + \frac{3}{16}x^2, & \text{if } 0 \le x \le 2 \\ 1. & \text{if } x > 2 \end{cases}$$

$$\begin{aligned} &P(1 \le X \le 1.5) = F(1.5) - F(1) = 0.297 \\ &P(X \ge 1) = 1 - F(1) = \frac{11}{16} \end{aligned}$$

4.2.3 Obtaining $f(x)$ from $F(x)$

X continues with $f(x)$ and $F(x)$

$$f(x) = F'(x)$$

Example 4.9. X continues with $f(x) = 1 - e^{-\lambda x}, x > 0$

$$f(x) = F'(x) = \lambda e^{-\lambda x}, x > 0$$

4.2.4 Percentiles of a Continuous Distribution

Example 4.10. John's exam score is at the 85th percentile of the class, meaning that John's score is higher than 85% of the class.

Definition 4.11. Let $0 \leq p \leq 1$, the $(100p)$th percentile of the distribution of X, denoted by η_p is defined as

$$p = F(\eta_p)$$

Set up the equation $F(\eta_p) = p$, solve for η_p.

Example 4.12. X has $f(x) = \begin{cases} 2(1-x), & 0 \leq x \leq 1 \\ 0. & \text{o.w.} \end{cases}$

$$F(x) = \begin{cases} 0, & \text{if } x < 0 \\ 2x - x^2, & \text{if } 0 \leq x \leq 1 \\ 1. & \text{if } x > 1 \end{cases}$$

To get 90% percentile

$$F(\eta_{0.9}) = 0.9$$

Solve the equation, $\eta_{0.9} = 1 \pm \sqrt{0.1}$. Since $0 \leq \eta_{0.9} \leq 1$

$$\eta_{0.9} = 1 - \sqrt{0.1}$$

To get 50th percentile, $F(\eta_{0.5}) = 0.5$

$$\eta_{0.5} = 1 - \frac{\sqrt{2}}{2}$$

Median is the 50th percentile of the distribution of X.

4.2.5 Mean and Variance

Definition 4.13. X is continues with $f(x)$ and $F(x)$. The expected or mean value of a continuous rv X with pdf $f(x)$ is

$$E(X) = \int_{-\infty}^{\infty} x f(x) dx$$

Example 4.14.

$$f(x) = \begin{cases} \frac{2}{3}(1-x^2), & \text{if } 0 \leq x \leq 1 \\ 0. & \text{otherwise} \end{cases}$$

$$E(X) = \int_{-\infty}^{\infty} x f(x) dx = \frac{3}{8}$$

Proposition 4.15. *X is continues with $f(x)$, for any $h(x)$*

$$E(h(X)) = \int_{-\infty}^{\infty} h(x) f(x) dx$$

Particularly,

$$h(x) = ax + b \qquad E(aX + b) = aE(X) + b$$

Example 4.16. $X \sim uniform(0,1)$

$$f(x) = \begin{cases} 1, & \text{if } 0 \leq x \leq 1 \\ 0. & \text{if o.w.} \end{cases}$$

$$h(x) = max\{x, 1-x\}$$

$$E[h(x)] = \int_o^1 h(x)f(x)dx$$

$$E(2X+3) = 4 \qquad E(X) = \frac{1}{2}$$

Definition 4.17. The **variance** of a continuous random variable X with pdf $f(x)$ and mean value μ is

$$Var(X) = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)dx = E[(X-\mu)^2]$$

Proposition 4.18.

$$Var(X) = E[(X - E(X))^2] = E(X^2) - (E(X))^2$$

4.3 The Normal Distribution

If X has p.d.f

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \qquad -\infty < x < \infty.$$

Then X has a normal distribution, or $X \sim N(\mu, \sigma^2)$.

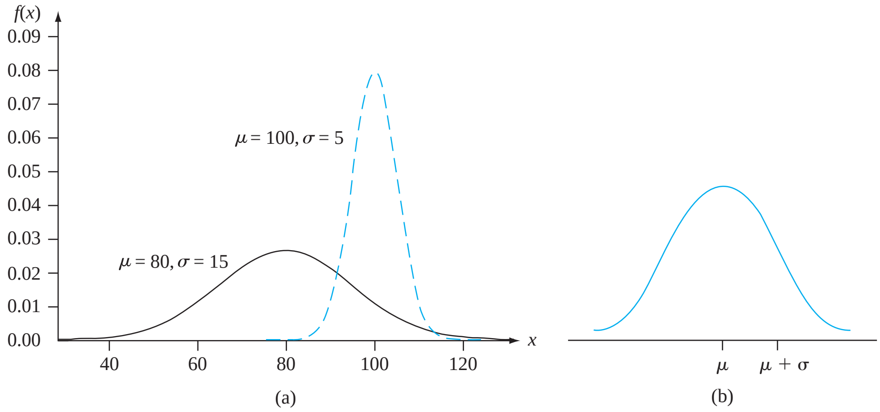


Figure 4.13 (a) Two different normal density curves (b) Visualizing μ and σ for a normal distribution

Figure 4.1: Bell-shaped curve

Symmetric about μ, μ=shift, σ=scale, large $\sigma \Rightarrow$large spread out.

Proposition 4.19 (Properties of normal distribution). *1.* $E(X) = \mu \qquad Var(X) = \sigma^2$

2. $f(x) \to 0$, *when* $x \to \pm\infty$

4.3.1 The Standard Normal Distribution

The Standard Normal Distribution, $N(0,1)$, denoted by Z,

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \qquad -\infty < x < \infty$$

c.d.f of Z

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} \, dy$$

$$\begin{aligned}
&\Phi(0) = 0.5 \\
&\Phi(1.645) = 0.95 \qquad \Phi(1.96) = 0.975 \\
&\Phi(-1.645) = 0.05 \qquad \Phi(-1.96) = 0.025
\end{aligned}$$

Example 4.20. (1)

$$\begin{aligned}
&P(-1.645 \le Z \le 1.96) \\
=&P(Z \le 1.96) - P(Z \ge -1.645) \\
=&\Phi(1.96) - \Phi(-1.645) = 0.975 - 0.05 = 0.925
\end{aligned}$$

(2)

$$\begin{aligned}
&P(-0.38 \le Z \le 1.25) \\
=&\Phi(1.25) - \Phi(-0.38) = 0.8944 - (1 - \Phi(0.38)) \\
=&0.8944 - (1 - 0.6486) = 0.5424
\end{aligned}$$

Using Standard Normal Table

4.3.2 Percentiles of the Standard Normal Distribution

$100p$ th percentile η_p of X is the solution of

$$F(\eta_p) = p$$

Example 4.21. For Z

$$\begin{aligned}
\eta_{0.975} &= 1.96 \qquad \eta_{0.95} = 1.645 \\
\eta_{0.025} &= -1.96 \qquad \eta_{0.05} = -1.645 \\
\eta_{0.9} &= 1.28
\end{aligned}$$

4.3.3 z_α Notation for z Critical Values

z_α will denote the value on the z axis for which α of the area under the z curve lies to the right of z.

$$z_{0.05} = \eta_{0.95} = 1.645$$

(lower percentile)

4.3.4 Nonstandard Normal Distributions

If $X \sim N(\mu, \sigma^2)$,then

$$Z = \frac{X - \mu}{\sigma} \sim N(0,1)$$

$$\begin{aligned}
P(a \le X \le b) &= P\left(\frac{a-\mu}{\sigma} \le \frac{X-\mu}{\sigma} \le \frac{b-\mu}{\sigma}\right) \\
&= P\left(\frac{a-\mu}{\sigma} \le Z \le \frac{b-\mu}{\sigma}\right) = \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)
\end{aligned}$$

Similarly,

$$\begin{aligned}P(X \le a) &= P\left(\frac{X-\mu}{\sigma} \le \frac{a-\mu}{\sigma}\right)\\ &= P\left(Z \le \frac{a-\mu}{\sigma}\right) = \Phi\left(\frac{a-\mu}{\sigma}\right)\\ P(X \ge b) &= 1-\Phi\left(\frac{b-\mu}{\sigma}\right)\end{aligned}$$

Example 4.22. $X \sim N(1.25, 0.46)$

$$\begin{aligned}P(1 \le X \le 1.75) &= P\left(\frac{1-1.25}{\sqrt{0.46}} \le \frac{X-1.25}{\sqrt{0.46}} \le \frac{1.75-1.25}{\sqrt{0.46}}\right)\\ &= P\left(-0.369 \le Z \le 0.737\right)\\ &= \Phi\left(0.737\right) - \Phi\left(-0.369\right) = \boxed{0.4147}\end{aligned}$$

4.3.5 Empirical Rule

If a population distribution of a r.v is roughly normal. Then

1. 68% of the values are within 1 s.d of their mean.
2. 95% of the values are within 2 s.d of their mean.
3. 99.7% of the values are within 3 s.d of their mean.

Proof.

$$\begin{aligned}LHS =& P(\mu-\sigma \le X \le \mu+\sigma) = P(-1 \le \frac{X-\mu}{\sigma} \le 1)\\ =& \Phi(1) - \Phi(-1) = 0.8413 - (1-0.8413) = 68.26\%\end{aligned}$$

∎

4.3.6 Percentiles of an Arbitrary Normal Distribution

If $X \sim N(\mu, \sigma^2)$ c.d.f $F(x)$, $(100p)$th percentile of X is the root of

$$\begin{aligned}P = F(\eta_p) =& P(X \le \eta_p)\\ =& P\left(\frac{X-\mu}{\sigma} \le \frac{\eta_p-\mu}{\sigma}\right) = \Phi\left(\frac{\eta_p-\mu}{\sigma}\right)\end{aligned}$$

So, $\frac{\eta_p-\mu}{\sigma}$ is the $(100p)$th percentile of $N(0,1)$. Therefore, $(100p)$th percentile of $N(\mu,\sigma^2)$=$\mu + \sigma \times$ [$100p$th percentile of $N(0,1)$]

Example 4.23. $X \sim N(64, 0.78^2)$. Then 99.5 percentile of X is $64 + 0.78 \times 2.58 = 66$, where 2.58 is the 99.5 percentile of Z.

4.3.7 Approximating the Binomial Distribution

If $X \sim Binom(n,p)$. When n is large, and p is not too small or too large, s.t. $np \ge 10, n(1-p) \ge 10$. Then $X \sim N(np, np(1-p))$

$$p(x) = \binom{n}{x} p^x (1-p)^{n-x} \qquad x = 0, 1, \ldots, n$$

$X \sim Binom(n,p)$[1],

$$\begin{aligned}P(a \le X \le b) &= P(a-0.5 \le X \le b+0.5)\\ P(X \le a) &= P(X \le a+0.5)\\ P(X \ge b) &= P(X \ge b-0.5)\end{aligned}$$

[1] $X \sim N(np, np(1-p))$, to avoid significant deviation

Example 4.24. 25% of all drivers in Hong Kong don't have insurance. Randomly select 50 drivers. $X = \#$ of drivers uninsured.

1. $P(X \leq 10)$
2. $P(5 \leq X \leq 15)$

First $X \sim Binom(50, 0.25) \dot{\sim} N(12.5, 12.5 \times 1.75)$.
(1)

$$\begin{aligned}P(X \leq 10) =& P(X \leq 10.5)\\ =& P\left(\frac{X - 12.5}{\sqrt{12.5 \times 1.75}} \leq \frac{10.5 - 12.5}{\sqrt{12.5 \times 1.75}}\right) = \Phi(-0.653) = 0.2578\end{aligned}$$

(2)

$$\begin{aligned}P(5 \leq X \leq 15) =& P(4.5 \leq X \leq 15.5)\\ =& P\left(\frac{4.5 - 12.5}{\sqrt{12.5 \times 1.75}} \leq \frac{X - 12.5}{\sqrt{12.5 \times 1.75}} \leq \frac{15.5 - 12.5}{\sqrt{12.5 \times 1.75}}\right)\\ =& \Phi(0.95) - \Phi(-2.61) = 0.832\end{aligned}$$

4.4 The Exponential and Gamma Distributions

4.4.1 The Gamma Function

Definition 4.25.

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx$$

This function has the following properties:

1. $\Gamma(\alpha) = (\alpha - 1)\Gamma(\alpha - 1)$
2. $\Gamma(1) = 1, \Gamma(2) = 1, \Gamma(3) = 2$
   $\Gamma(n) = (n - 1)! \qquad n = 1, 2, \ldots$
3. $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

4.4.2 The Gamma Distribution

Definition 4.26. X follows a Gamma distribution. $X \sim Gamma(\alpha, \beta)$.

$$f(x) = \begin{cases} \frac{1}{\Gamma(\alpha)} \frac{1}{\beta^\alpha} x^{\alpha-1} e^{-x/\beta}, & \text{if} \quad x \geq 0 \\ 0. & \text{if} \quad x < 0 \end{cases}$$

for $\alpha > 0, \beta > 0$

If $\beta = 1$, $X \sim Gamma(\alpha, 1)$. Standard Gamma distribution.

$$f(x) = \begin{cases} \frac{1}{\Gamma(\alpha)} x^{\alpha-1} e^{-x}, & \text{if} \quad x \geq 0 \\ 0. & \text{if} \quad x < 0 \end{cases}$$

Check

$$\int_0^\infty \frac{1}{\Gamma(\alpha)} \frac{1}{\beta^\alpha} x^{\alpha-1} e^{-x/\beta} dx = 1$$

$$\begin{aligned}L.H.S. =& \int_0^\infty \frac{1}{\Gamma(\alpha)} u^{\alpha-1} e^{-u} du\\ =& \frac{1}{\Gamma(\alpha)} \int_0^\infty u^{\alpha-1} e^{-u} du = 1\end{aligned}$$

Proposition 4.27. *If $X \sim Gamma(\alpha, \beta)$, then $E(X) = \alpha\beta$, $Var(X) = \alpha\beta^2$*

Proof.

$$\begin{aligned} E(X) &= \int_0^\infty x \frac{1}{\Gamma(\alpha)} \frac{1}{\beta^\alpha} x^{\alpha-1} e^{-x/\beta} dx = \frac{1}{\Gamma(\alpha)} \int_0^\infty \frac{1}{\beta^\alpha} x^\alpha e^{-x/\beta} dx \\ &= \frac{\beta}{\Gamma(\alpha)} \int_0^\infty \frac{1}{\beta^{\alpha+1}} x^\alpha e^{-x/\beta} dx = \frac{\beta\Gamma(\alpha+1)}{\Gamma(\alpha)} \int_0^\infty \frac{1}{\Gamma(\alpha+1)} \frac{1}{\beta^{\alpha+1}} x^\alpha e^{-x/\beta} dx \\ &= \frac{\Gamma(\alpha+1)}{\Gamma(\alpha)} \beta = \alpha\beta \end{aligned}$$

■

Example 4.28. Suppose that the reaction time X of a randomly selected individual to a certain stimulus has a standard Gamma distribution with $\alpha = 2$.

$$P(3 \le X \le 5) = F(5; 2) - F(3; 2)$$

Here $F(x; \alpha)$ is the c.d.f of $\Gamma(\alpha, 1)$

$$TableA.4 = 0.960 - 0.801 = 0.159$$

Proposition 4.29. *If $X \sim Gamma(\alpha, \beta)$, then $X/\beta \sim Gamma(\alpha, 1)$*

$$P(X \le x) = P\left(\frac{X}{\beta} \le \frac{x}{\beta}\right) = F\left(\frac{x}{\beta}; \alpha\right)$$

Example 4.30. The survival time X of a randomly selected male mouse exposed to gamma radiation has Gamma distribution with $\alpha = 8$, $\beta = 15$. Then

$$E(X) = \alpha\beta = 8 \times 15 = 120$$

$$Var(X) = \alpha\beta^2 = 8 \times 15^2 = 1800$$

$$\begin{aligned} P(60 \le X \le 120) &= P\left(\frac{60}{15} \le \frac{X}{15} \le \frac{120}{15}\right) = F(8; 8) - F(4; 8) \\ &= 0.547 - 0.051 = 0.496 \end{aligned}$$

4.4.3 Exponential distribution

If $X \sim exp(\lambda)$, $f(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0 \\ 0. & \text{otherwise} \end{cases}$. Then X has an exponential distribution with parameter λ.

Proposition 4.31. *If $X \sim exp(\lambda)$. Then $X \sim Gamma(1, 1/\lambda)$*

$$E(X) = \lambda \qquad Var(X) = \frac{1}{\lambda^2}$$

Example 4.32. $X =$ response time at some computer terminal. $X \sim exp(\lambda)$. Suppose that the expected reacting time is 5 seconds.

$$E(X) = 5 \qquad \frac{1}{\lambda} = 5 \Rightarrow \lambda = \frac{1}{5}$$

$$P(X \le 10) = \int_0^{10} \frac{1}{5} e^{-x/5} dx = e^{-x/5}\Big|_0^{10} = 1 - e^{-2}.$$

In general, if $X \sim exp(\lambda)$,

$$F(x) = \int_0^x \lambda e^{-\lambda y} dy = e^{-\lambda y}\Big|_0^x = 1 - e^{-\lambda x}$$

Two applications

A. Poisson process Suppose # of customers coming in any wait time $\sim Possion(\alpha)$, and # of customers is non-overlapping intervals are independent. Then

$$X = \text{the elapsed time between the successive customers coming in} \sim exp(\alpha).$$

Why? Let X_1 = waiting time before the 1st customer coming in. Want to show that $X_1 \sim exp()\lambda$, just need to find $f_{X_1}(x)$. Then we just need to find $F_{X_1}(x)$, as $f_{X_1}(x) = F'_{X_1}(x)$.

$$\begin{aligned} F_{X_1}(x) &= P(X_1 \le x) = P(\text{at least 1 customer in } (0,x)) \\ &= 1 - P(\text{no customer in } (0,x)) \\ &= 1 - \frac{(\alpha x)^0}{0!} e^{-\alpha x} = 1 - e^{-\alpha x}. \end{aligned}$$

$$f_{X_1}(x) = F'_{X_1}(x) = \alpha e^{-\alpha x} \qquad x > 0$$

$$X_1 \sim exp(\lambda)$$

B. Memoryless property Suppose component lifetime $\sim exp(\lambda)$. Putting this component into work, after t_0 time, check it and find it is still working. What is the probability that it will last at least another t time?

Let T = lifetime of this component $\sim exp(\lambda)$,

$$\begin{aligned} P(T \ge t_0 + t | T \ge t_0) &= \frac{P(T \ge t_0 + t \cap T \ge t_0)}{P(T \ge t_0)} \\ &= \frac{P(T \ge t_0 + t)}{P(T \ge t_0)} = \frac{1 - P(T \le t_0 + t)}{1 - P(T \le t_0)} \\ &= \frac{1 - (1 + e^{-\alpha(t_0+t)})}{1 - (1 + e^{-\alpha t_0})} = \frac{e^{-\alpha(t_0+t)}}{e^{-\alpha t_0}} \\ &= e^{-\alpha t}. \end{aligned}$$

4.4.4 The Chi-Squared Distribution

Let ν be an integer, if $X \sim Gamma(\frac{\nu}{2}, 2)$, then we say X has a χ^2-distribution with parameter ν, $X \sim \chi^2(\nu)$. It's p.d.f is

$$f(x;\nu) = \begin{cases} \frac{1}{2^{\nu/2}\Gamma(\nu/2)} x^{\nu/2-1} e^{-x/2} & x > 0 \\ 0 & o.w. \end{cases}.$$

Proposition 4.33. *Properties*

1. *If $X \sim N(0,1)$, then $X^2 \sim \chi^2(1)$*
2. *If $X_1 \sim \chi^2(n)$, $X_2 \sim \chi^2(m)$, independently. Then $X_1 + X_2 \sim \chi^2(m+n)$*
3. *If $X_1 \sim N(0,1)$, $X_2 \sim N(0,1)$, independently. Then $X_1 + X_2 \sim \chi^2(2)$*

4.5 Other Continuous Distributions

4.5.1 The Weibull Distribution

If $X \sim Weibull(\alpha, \beta)$

$$f(x) = \frac{1}{\beta^\alpha} x^{\alpha-1} e^{-(x/\beta)^\alpha} \qquad x > 0$$

If $\alpha = 1$, $X \sim Weibull(1, \beta)$. Then $X \sim exp(\frac{1}{\beta})$

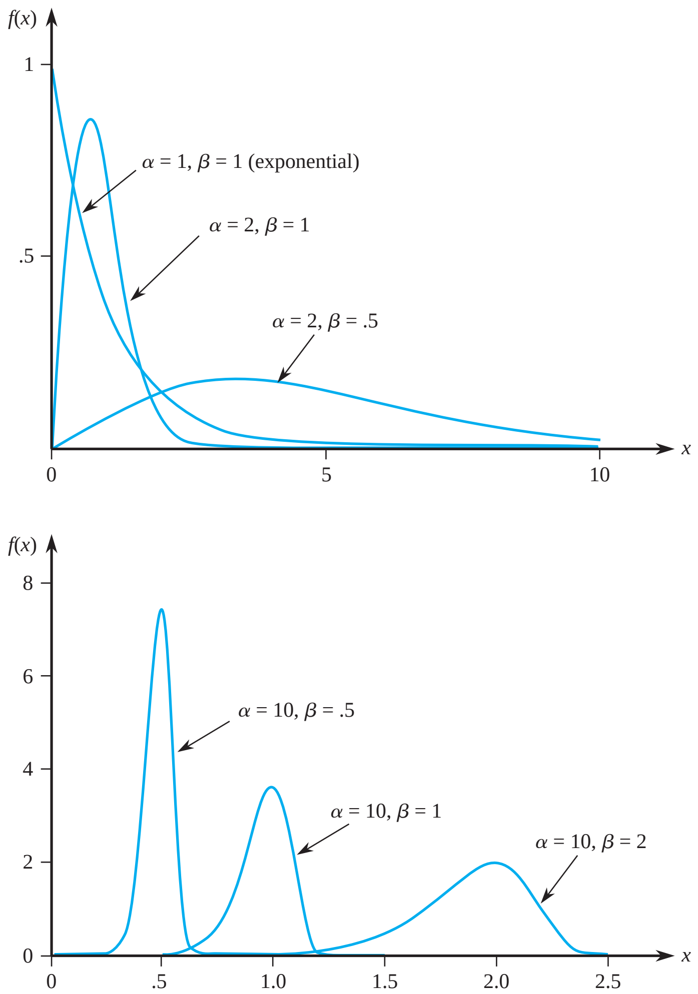


Figure 4.28 Weibull density curves

Figure 4.2: The Weibull Distribution

Proposition 4.34. $X \sim Weibull(\alpha, \beta)$

1. $E(X) = \beta\Gamma(1 + 1/\alpha)$
2. $Var(X) = \beta^2 \left(\Gamma(1 + 2/\alpha) - (\Gamma(1 + 1/\alpha))^2\right)$
3. c.d.f

$$F(x) = \begin{cases} 1 - e^{-(\alpha/\beta)^\alpha} & x \geq 0 \\ 0 & o.w. \end{cases}$$

Example 4.35. X = the strength at -20 F of a type of steel exhibiting "cold brittleness" at low temperature . $X \sim Weibull(20, 100)$

(1)

$$P(X \le 105) = F(105) = 1 - e^{-(105/100)^{20}} = 1 - 0.07 = \boxed{0.93}$$

(2)

$$P(90 \le X \le 100) = F(110) - F(90) = \left(1 - e^{-(110/100)^{20}}\right) - \left(1 - e^{-(90/100)^{20}}\right) = \boxed{\cdots}$$

4.5.2 The Lognormal Distribution

X is positive. If $\log X \sim N(\mu, \sigma^2)$ [2] , then $X \sim Lognormal(\mu, \sigma^2)$.

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(\log x - \mu)^2}{2\sigma^2}} \qquad x > 0$$

$$E(X) = e^{\mu + \frac{\sigma^2}{2}}$$
$$Var(X) = e^{2\mu+\sigma^2}(e^{\sigma^2} - 1)$$

$$P(X \le x) = P(\log X \le \log x) = P\left(\frac{\log X - \mu}{\sigma} \le \frac{\log x - \mu}{\sigma}\right) = \Phi\left(\frac{\log x - \mu}{\sigma}\right)$$

Example 4.36. X=the modulus of elasticity of some floor system.

$$X \sim Lognormal(0.375, 0.25^2)$$

$$\begin{aligned}
E(X) &= e^{0.375+0.25^2/2} = 1.5 \\
Var(X) &= e^{2\times 0.375+0.25^2}\left(e^{0.25^2} - 1\right) = 0.145 \\
P(1 \le X \le 2) &= P(\log 1 \le \log X \le \log 2) \\
&= P\left(\frac{0 - 0.375}{0.25} \le \frac{\log X - 0.375}{0.25} \le \frac{\log 2 - 0.375}{0.25}\right) \\
&= \Phi\left(\frac{\log 2 - 0.375}{0.25}\right) - \Phi\left(\frac{-0.375}{0.25}\right) = \boxed{0.8312}
\end{aligned}$$

4.5.3 The Beta Distribution

If X has a p.d.f

$$f(x) = \begin{cases} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)+\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} & 0 \le x \le 1 \\ 0. & o.w. \end{cases}$$

Then, $X \sim Beta(\alpha, \beta)$

Proposition 4.37. *Particularly, if $\alpha = \beta = 1$, $X \sim unif(0, 1)$*

Proposition 4.38. *Let $A < B$, and $Y = A + (B - A)X$, then Y be density.*

$$f(y) = \begin{cases} \frac{1}{B-A} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)+\Gamma(\beta)} \left(\frac{y-A}{B-A}\right)^{\alpha-1} \left(\frac{B-y}{B-A}\right)^{\beta-1} & A \le x \le B \\ 0. & o.w. \end{cases}$$

$$Y \sim GBeta$$

[2] $\log = \ln$

Proposition 4.39. *If* $Y \sim Beta(\alpha, \beta)$

$$E(X) = \frac{\alpha}{\alpha+\beta} \qquad Var(X) = \frac{\alpha\beta}{(\alpha+\beta)(\alpha+\beta+1)}$$

If $Y \sim GBeta(\alpha, \beta, A, B)$

$$E(Y) = A + (B-A)\frac{\alpha}{\alpha+\beta} \qquad Var(Y) = (B-A)^2\frac{\alpha\beta}{(\alpha+\beta)(\alpha+\beta+1)}$$

Because $Y = A + (B-A)X$,

$$E(Y) = E(A+(B-A)X) = A + (B-A)E(X) \qquad Var(Y) = (B-A)^2 Var(X)$$

Example 4.40.

$$X = \text{time to complete certain project}$$

$$X \sim GBeta(\alpha = 2, \beta = 3, A = 2, B = 5)$$

$$E(X) = 2 + (5-2)\frac{2}{2+3} = 3.2 \qquad Var(X) = 0.36$$

$$P(X \leq 3) = \int_2^3 \frac{1}{5-2}\frac{\Gamma(5)}{\Gamma(2)+\Gamma(3)}\left(\frac{x-2}{3}\right)^{2-1}\left(\frac{5-x}{3}\right)^{3-1} dx = \boxed{0.407}$$

4.5.4 Challenge Question 2

Cauchy distribution

$$f(x) = \frac{1}{\pi(1+x^2)}$$

https://en.wikipedia.org/wiki/Cauchy_distribution

Chapter 5

Joint Probability Distributions and Random Samples

5.1 Jointly Distributed Random Variables

5.1.1 Two Discrete Random Variables

X,Y are r.v's defined on $\mathcal{S}$. The joint p.m.f is defined as

$$p(x, y) = P(X = x, Y = y)$$

Let A be an event consisting of pairs of (x, y). Then

$$P((X, Y) \in A) = \sum_{(X,Y)\in A} p(x, y)$$

Example 5.1. Insurance company. For a newcomer, he has two insurance. Home & Cars. Deductible amount: Auto \$100, \$250; Home \$0, \$100, \$200.

		Y		
	$p(x, y)$	0	100	200
X	100	0.2	0.1	0.2
	250	0.05	0.15	0.3

An individual home-owner is randomly selected.

$$\begin{aligned} P(Y \geq 100) =& P(X = 100, Y = 100) + P(X = 250, Y = 100) + P(X = 100, Y = 200) + \\ & P(X = 250, Y = 200) \\ =& 0.1 + 0.15 + 0.2 + 0.3 = 0.75 \end{aligned}$$

Definition 5.2. "Marginal" p.m.f of X and Y, denoted by $p_X(x)$ and $p_Y(y)$ respectively, are given by

$$\begin{aligned} p_X(x) &= \sum_y p(x, y) \\ p_Y(y) &= \sum_x p(x, y) \\ p_X(x) &= P(X = x) = P(X + x, Y = \dots) + \dots \end{aligned}$$

Example 5.3. In the Insurance example,

$$P(X = x) = \begin{cases} P_X(100) = \cdots = 0.5 \\ P_X(250) = \cdots = 0.5 \end{cases}$$

$$P(Y = y) = \begin{cases} P_Y(0) = \cdots = 0.25 \\ P_Y(100) = \cdots = 0.25 \\ P_Y(200) = \cdots = 0.5 \end{cases}$$

		Y			
	$p(x,y)$	0	100	200	$p_X(x)$
X	100	0.2	0.1	0.2	0.5
	250	0.05	0.15	0.3	0.5
	$p_Y(y)$	0.25	0.25	0.5	1

5.1.2 Two Continuous Random Variables

(X,Y) continuous r.v. $f(x,y)$ is the joint p.d.f of X and Y if for any 2-dimensional set.

$$P((X,Y)\in A)=\iint_A f(x,y)\,dx\,dy$$

Particularly for $A=\{(x,y), a\le x\le b, c\le y\le d\}$,

$$\begin{aligned}P(A)&=P(a\le X\le b, c\le Y\le d)\\&=\int_a^b\int_c^d f(x,y)\,dy\,dx=\int_c^d\int_a^b f(x,y)\,dx\,dy\end{aligned}$$

Example 5.4.

$$\begin{aligned}X&=\text{right front tyre pressure}\\Y&=\text{left front tyre pressure}\end{aligned}$$

$$f(x,y)=\begin{cases}k(x^2+y^2) & 20\le x,y\le 30\\0, & o.w.\end{cases}$$

(1) What is k?

$$\begin{aligned}1=&\int_{20}^{30}\int_{20}^{30}k(x^2+y^2)dxdy=k\int_{20}^{30}\left(\left(\frac{x^3}{3}+xy^2\right)\Big|_{20}^{30}\right)dy\\=&k\int_{20}^{30}\left(\frac{19000}{3}+10y^2\right)dy=k\left(\frac{19000}{3}+\frac{19000}{3}\right)\\\Rightarrow k=&\frac{3}{38000}\end{aligned}$$

(2)

$$\begin{aligned}P(X\le 26, Y\le 26)=&\int_{20}^{26}\int_{20}^{26}k(x^2+y^2)dxdy\\=&k\int_{20}^{26}\left(\frac{26^3-20^3}{3}+6y^2\right)dy\\=&2k\cdot 6\cdot\frac{1}{3}(26^3-20^3)=0.3024\end{aligned}$$

Definition 5.5. Marginal rv

$$\begin{aligned}f_X(x)&=\int_{-\infty}^{\infty}f(x,y)dy\\f_Y(y)&=\int_{-\infty}^{\infty}f(x,y)dx\end{aligned}$$

Example 5.6. Example 5.4 (continued)
(3)

$$\begin{aligned}f_X(x)=&\int_{20}^{30}k(x^2+y^2)dy=k\left(\frac{y^3}{3}+x^2y\right)\Big|_{20}^{30}\\=&k\left(10x^2+\frac{19000}{3}\right)=\frac{3}{38000}x^2+\frac{1}{20}\qquad 20\le x\le 30\end{aligned}$$

$$f_Y(y) = \frac{3}{38000}y^2 + \frac{1}{20} \qquad 20 \le y \le 30$$

(4)

$$P(20 \le X \le 25) = \int_{20}^{25} f_X(x)dx = \cdots = 0.45$$

Definition 5.7. Expected Values

$$E[h(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x,y)f(x,y)dxdy$$

Example 5.8. Example 5.4 (continued)
(5)

$$E(X+Y) = \int_{20}^{26}\int_{20}^{26} (x+y)k(x^2+y^2)dxdy = \dots$$

5.1.3 Independent Random Variables

X and Y are said to be independent, if

$$\begin{aligned} discrete: p(x,y) &= p_X(x)p_Y(y) \text{ for all } (x,y) \\ continuous: f(x,y) &= f_X(x)f_Y(y) \text{ for all } (x,y) \end{aligned}$$

Example 5.9. Example 5.4 (continued)
(6)

$$\begin{aligned} f(x,y) =& \frac{3}{38000}(x^2+y^2) \qquad 20 \le x,y \le 30 \\ f_X(x) =& \frac{3}{38000}x^2 + \frac{1}{20} \qquad 20 \le x \le 30 \\ f_Y(y) =& \frac{3}{38000}y^2 + \frac{1}{20} \qquad 20 \le y \le 30 \end{aligned}$$

$$f(x,y) \ne f_X(x)f_Y(y) \qquad \text{for } x = 20, y = 20$$

X and Y are not independent.

Example 5.10.

$$f(x,y) = \lambda_1\lambda_2 e^{-\lambda_1 x - \lambda_2 y}, x \ge 0, y \ge 0$$

Are X and Y independent?

$$\begin{aligned} f_X(x) &= \int_0^\infty \lambda_1\lambda_2 e^{-\lambda_1 x - \lambda_2 y}dy \\ &= \lambda_1 e^{-\lambda_1 x}\int_0^\infty \lambda_2 e^{-\lambda_2 y}dy = \lambda_1 e^{-\lambda_1} \qquad x \ge 0 \end{aligned}$$

$$f_Y(y) = \lambda_2 e^{-\lambda_2} \qquad y \ge 0$$

$$f(x,y) = f_X(x)f_Y(y) \qquad \text{for any } (x,y)$$

So, $X \perp\!\!\!\perp Y$.

5.1.4 More Than Two Random Variables

Definition 5.11. $X_1, X_2, \dots, X_n$ are discrete rvs', the joint p.m.f is defined as

$$p(x_1, x_2, \dots, x_n) = P(X_1 = x_1, \dots, X_n = x_n)$$

In continuous case, the joint p.d.f $f(x_1, x_2, \dots, x_n)$ is such that

$$P(A) = \int_A \cdots \int f(x_1, x_2, \dots, x_n)dx_n \dots dx_1$$

Particularly, $A = \{a_1 \le x_1 \le b_1, \dots, a_n \le x_n \le b_n\}$

$$\begin{aligned} P(A) =& P(a_1 \le x_1 \le b_1, \dots, a_n \le x_n \le b_n) \\ =& \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} f(x_1, x_2, \dots, x_n) dx_n \dots dx_1 \end{aligned}$$

Example 5.12. A dice is rolled 100 times. $X_i =\#$ of i dots out of 100 times; $i = 1, 2, \dots, 6$

$$p_i = P(i \text{ dots}) \qquad p_1 + p_2 + \cdots + p_6 = 1$$

$$P(X_1 = x_1, \dots, X_6 = x_6) = \frac{100!}{x_1! x_2! \dots x_6!} p_1^{x_1} p_2^{x_2} \dots p_6^{x_6} \quad (0 \le x_1, \dots, x_6 \le 100, x_1 + \cdots + x_6 = 100)$$

Example 5.13. (X_1, X_2, X_3) has the joint p.d.f

$$f(x_1, x_2, x_3) = \begin{cases} kx_1x_2(1 - x_3) & 0 \le x_1, x_2, x_3 \le 1, x_1 + x_2 + x_3 \le 1 \\ 0, & o.w. \end{cases}$$

(1) What is k?

$$\begin{aligned} 1 &= \iiint kx_1x_2(1 - x_3)\, dx_3\, dx_2\, dx_1 \\ &= \int_0^1 \int_0^{1-x_1} \int_0^{1-x_2-x_1} kx_1x_2(1 - x_3)\, dx_3\, dx_2\, dx_1 \\ &= \frac{k}{144} \Rightarrow k = 144 \end{aligned}$$

(2) [1]

$$P(X_1 + X_2 \le 0.5) = \iiint kx_1x_2(1 - x_3)\, dx_3\, dx_2\, dx_1 = 0.606$$

$0 \le x_1, x_2, x_3 \le 1, x_1 + x_2 \le 0.5.$

Independence

Definition 5.14. $X_1, X_2, \dots, X_n$ are independent if

$$p(x_1, x_2, \dots, x_n) = p_{X_1}(x_1) \dots p_{X_n}(x_n)$$

or

$$f(x_1, x_2, \dots, x_n) = f_{X_1}(x_1) \dots f_{X_n}(x_n)$$

for all possible $(x_1, \dots, x_n)$

5.1.5 Conditional Distributions

Definition 5.15. (X, Y) with $f(x, y), f_X(x), f_Y(y)$, then the conditional p.d.f of Y given $X = x$,

$$f_{Y|X}(y|x) = \frac{f(x, y)}{f_X(x)}$$

for discrete case

$$p_{Y|X}(y|x) = \frac{p(x, y)}{p_X(x)}$$

Example 5.16.

$$f(x, y) = \begin{cases} \frac{6}{5}(x + y^2) & 0 \le x \le 1, 0 \le y \le 1 \\ 0 & o.w. \end{cases}$$

[1] The value might be wrong, run this in Wolfram Mathematica or Wolfram Alpha - `Integrate[144 * x y (1 - z), {x, 0, 0.5}, {y, 0, 0.5 - x}, {z, 0, 1 - x - y}]`

$$\begin{aligned} f_X(x) = &\int_0^1 \frac{6}{5}(x+y^2)\,dy = \left(\frac{6}{5}xy + \frac{6}{5}\frac{1}{3}y^3\right)\Big|_0^1 \\ =&\frac{6}{5}x + \frac{2}{5}. \qquad 0 \le x \le 1 \end{aligned}$$

$$f_{Y|X}(y|0.8) = \frac{f(0.8, y)}{f_X(0.8)} = \frac{15}{17}y^2 + \frac{12}{17} \qquad 0 \le y \le 1$$

$$E(Y|X=0.8) = \int_0^1 y f_{Y|X}(y|0.8)\,dy = \int_0^1 y\left(\frac{15}{17}y^2 + \frac{12}{17}\right)dy = \frac{39}{68}$$

5.2 Expected Values, Covariance, and Correlation

Proposition 5.17.

$$E[h(x,y)] = \begin{cases} \sum_x \sum_y h(x,y)p(x,y) & \textit{discrete} \\ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x,y)f(x,y)dxdy & \textit{continuous} \end{cases}$$

Example 5.18.

$$\begin{aligned} X &= \text{amount of almonds} \\ Y &= \text{amount of pecans} \end{aligned}$$

$$f(x) = \begin{cases} 24xy & \text{if } 0 \le x, y \le 1, x+y \le 1 \\ 0 & o.w. \end{cases}$$

Unit test: almonds: \$1.00; pecans: \$1.00; peanuts: \$ 0.50

$$h(X,Y) = X + 1.5Y + 0.5(1 - X - Y) = 0.5 + 0.5X + Y$$

$$E[h(x,y)] = \int_0^1 \int_0^{1-y} (0.5 + 0.5x + y) 24xy\,dx\,dy = 1.10$$

5.2.1 Covariance

Definition 5.19.

$$Cov(X,Y) = E\Big((X - \mu_X)(Y - \mu_Y)\Big)$$

where $\mu_X = E(X)$,$\mu_Y = E(Y)$

$$\Rightarrow Cov(X,Y) = E(XY) - E(X)E(Y)$$

Example 5.20. In Example 5.1

		Y			
X	$p(x,y)$	0	100	200	$p_X(x)$
	100	0.2	0.1	0.2	0.5
	250	0.05	0.15	0.3	0.5
	$p_Y(y)$	0.25	0.25	0.5	1

$$\begin{aligned} E(X) &= 100 \times 0.5 + 250 \times 0.5 = 175 \\ E(Y) &= \cdots = 125 \\ Cov(X,Y) &= \cdots = 1875 \end{aligned}$$

Example 5.21.

$$f(x) = \begin{cases} 24xy & \text{if } 0 \le x, y \le 1, x + y \le 1 \\ 0 & o.w. \end{cases}$$

$$Cov(X, Y) = E(XY) - E(X)E(Y)$$

$$f_X(x) = \int_0^{1-x}$$

Similarly,

$$f_Y(y) = 12y(1-y)^2, 0 \le y \le 1$$

$$E(X) = \int_0^1 x \cdot 12x(1-x)^2 dx = \frac{2}{5}$$

$$E(Y) = \frac{2}{5}$$

$$E(XY) = \int_0^1 \left(\int_0^{1-y} xy \cdot 24xy \, dx \right) dy = \frac{2}{15}$$

$$Cov(X, Y) = E(XY) - E(X) - E(Y) = -\frac{2}{75}$$

X, Y are negatively related. But Covariance cannot indicate the relation is strong or weak.

5.2.2 Correlation

Definition 5.22. The correlation coefficient of X and Y, denoted by $Corr(X, Y)$, $\rho_{X,Y}$, or just ρ, is defined by

$$Corr(X, Y) = \frac{Cov(X, Y)}{\sqrt{Var(X)Var(Y)}}$$

Back to Example 5.21,

$$Var(X) = E(X^2) - E(X)^2$$

$$E(X^2) = \int_0^1 x^2 12x(1-x)^2 \, gx = 12 \int_0^1 x^3(1-x)^2 \, dx = \frac{1}{5}$$

$$Var(X) = \frac{1}{5} - \left(\frac{2}{5}\right) = \frac{1}{25}$$

Similarly,

$$Var(Y) = \frac{1}{25}$$

$$Corr(X, Y) = \frac{-\frac{2}{75}}{\sqrt{\frac{1}{25}\frac{1}{25}}} = -\frac{2}{3}$$

Proposition 5.23. ***Fact:*** *for any X and Y,*

$$-1 \le Corr(X, Y) \le 1$$

Proposition 5.24. *If $ac > 0$[2], then*

$$Corr(aX + b, cX + d) = Corr(X, Y)$$

"unit free"

Proposition 5.25. *1. If X and Y are independent,then*

$$Corr(X,Y) = 0$$

But $Corr(X,Y) = 0 \not\Rightarrow X$ and Y are independent.

2. If $Corr(X,Y) = 1$ or -1 if and only if $Y = aX + b$ for some a, b with $a \neq 0$.

Example 5.26. X and Y are discrete r.v.'s

$$p(x,y) = \begin{cases} \frac{1}{4} & (x,y) = (-4,1),(4,-1),(2,2),(-2,-2) \\ 0 & o.w. \end{cases}$$

$$p_X(x) = \begin{cases} \frac{1}{4} & x = -4,-2,2,4 \\ 0 & o.w. \end{cases}$$

$$p_Y(y) = \begin{cases} \frac{1}{4} & y = -2,-1,1,2 \\ 0 & o.w. \end{cases}$$

$$E(X) = \frac{1}{4}(-4 + -2 + 2 + 4) = 0 \qquad E(Y) = 0$$

$$E(XY) = \frac{1}{4}(-4 + -4 + 4 + 4) = 0$$

$$Cov(X,Y) = 0 - 0 \times 0 = 0 \qquad Corr(X,Y) = 0$$

But $X \not\perp\!\!\!\perp Y$.

Example 5.27. $X \sim N(0,1)$, $Y = X^2 \sim \chi^2(1)$

$$E(X) = 0 \qquad E(Y) = E(X^2) = (Var(X)) = 1$$

$$E(XY) = E(X^3) = \int_{-\infty}^{\infty} x^3 \phi(x)\,dx = 0$$

$$Cov(X,Y) = E(XY) - E(X)E(Y) = 0 - 0 \times 1 = 0$$

$$Cov(X,Y) = 0 \qquad X \not\perp\!\!\!\perp Y$$

5.2.3 Properties (The Distribution of a Linear Combination)

Proposition 5.28. *(1). $X_1, X_2 \dots X_n$ are rv's. For any constant $a_1, a_2 \dots a_n$,*

$$E(a_1X_1 + \cdots + a_nX_n) = a_1E(X_1) + \cdots + a_nE(X_n)$$

$$\begin{aligned} Var(a_1X_1 + \cdots + a_nX_n) &= \sum_{i=1}^{n} a_i^2 Var(X_i) + \sum_{i \neq j} a_i a_j Cor(X_i, Y_j) \\ &= \sum_{i=1}^{n} a_i^2 Var(X_i) + 2\sum_{i=1}^{n}\sum_{j=i+1}^{n} a_i a_j Cor(X_i, Y_j) \end{aligned}$$

If $X_1, X_2 \dots X_n$ are independent

$$Var(a_1X_1 + \cdots + a_nX_n) = \sum_{i=1}^{n} a_i^2 Var(X_i)$$

$$Var(X) = Cov(X,X) = E(XX) - E(X)E(X)$$

(2). If $X_1, X_2 \dots X_n$ are independent, and $X_i \sim N(\mu_i, \sigma_i^2)$. Then

$$a_1 X_1 + \cdots + a_n X_n \sim N(\sum_{i=1}^{n} a_i \mu_i, \sum_{i=1}^{n} a_i \sigma^2)$$

Particularly, if $\mu_i = \mu, \sigma_i = \sigma, a_i = \frac{1}{n} (i = 1, \dots, n)$, then

$$\bar{X} \sim N(\mu, \frac{\sigma^2}{n})$$

5.3 Statistics and Their Distributions

Example 5.29. Number of certificate obtained by students: 2,1,4,2,0.

Sample mean: $\bar{x} = \frac{2+1+4+2+0}{5} = 1.8$

Sample variance: $s^2 = \frac{(2-1.8)^2+(1-1.8)^2+(4-1.8)^2+\dots}{4}$

Generally, $x_1, x_2, \dots, x_n$,

Sample mean:

$$\bar{x} = \frac{x_1 + \cdots + x_n}{n}$$

Sample variance:

$$s^2 = \frac{\sum_{i=1}^{n} (x_i - \bar{x})^2}{n-1}$$

Definition 5.30. A statistic is a function of data before sampling (or before data are observed). There is an uncertainty on what value the statistic will result.

Usually, we use upper-case letter to denote statistic, and lower-case letter to denote observe values of a statistic.

$$\bar{X} = \frac{X_1 + \cdots + X_n}{n} \qquad S^2 = \frac{\sum (X_i - \bar{X})^2}{n}$$

$\bar{x}, s^2$. $T = \frac{\bar{X}}{S}$ is also a statistic.

5.3.1 Random Samples

Definition 5.31. $X_1, X_2 \dots X_n$ are said to be a random sample of size n if they are independent and identically distributed (i.i.d).

5.3.2 Deriving the Sampling Distribution of a Statistic

Example 5.32. A car dealer, tune-up charge (\$40,\$45,\$50) for (4,6,8) cylinder cars. At a particular day, of all tune-up cars, (20%, 30%, 50%) are (4,6,8) cylinder cars.

The pmf of revenue is

x	40	45	50
$p(x)$	0.2	0.3	0.5

$$E(X) = 46.5 \qquad Var(X) = 15.25$$

At another day, two tune-ups are done.

$$X_1 = \text{revenue for the 1st car}$$
$$X_2 = \text{revenue for the 2nd car}$$

Then X_1, X_2 iid X with pmf $p(x)$, $\bar{X} = \frac{X_1+X_2}{2}$.

x_1	x_2	$p(x_1, x_2)$	$\bar{x}$	s^2
40	40	0.04	40	0
40	45	0.06	42.5	12.5
40	50	0.10	45	50
45	40	0.06	42.5	12.5
45	45	0.09	45	0
45	50	0.15	47.5	12.5
50	40	0.10	45	50
50	45	0.09	47.5	12.5
50	50	0.25	50	0

Distribution of $\bar{X}$

$\bar{x}$	40	42.5	45	47.5	50
$p_{\bar{X}}(\bar{x})$	0.04	0.12	0.29	0.3	0.25

Distribution of S^2

s^2	0	12.5	50
$p_{S^2}(s^2)$	0.38	0.42	0.20

$$\begin{aligned} E(\bar{X}) &= 46.5 \\ E(S^2) &= 15.25 = Var(X) \\ Var(S^2) &= \end{aligned}$$

Example 5.33. (Example 5.21 in the textbook) $X_1, X_2 \overset{iid}{\sim} exp(\lambda)$

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0 \\ 0. & \text{otherwise} \end{cases}$$

$Y = X_1 + X_2$ is the statistic of interest. $f_Y(y) = ?$.

$$\begin{aligned} F_Y(y) =& P(Y \le y) = P(X_1 + X_2 \le y) \\ =& \iint_{X_1+X_2 \le y} f(x_1, x_2)\, dx_1\, dx_2 = \int_0^y \int_0^{y-x_2} \lambda e^{-\lambda x_1} \lambda e^{-\lambda x_2} dx_1\, dx_2 \\ =& \int_0^y \int_0^{y-x_2} \lambda^2 e^{-\lambda(x_1+x_2)} dx_1\, dx_2 = \cdots = 1 - e^{-\lambda y} - \lambda y e^{-\lambda y} \qquad 0 \le y \le \infty \end{aligned}$$

$$f_Y(y) = F_Y'(y) = \lambda e^{-\lambda y} - \lambda e^{-\lambda y} + \lambda^2 y e^{-\lambda y} = \lambda^2 y e^{-\lambda y} \qquad y \ge 0$$

$$Y \sim Gamma(2, \frac{1}{\lambda})$$

$$E(Y) = \frac{2}{\lambda} \qquad Var(Y) = \frac{2}{\lambda^2}$$

5.4 The Distribution of the Sample Mean

Proposition 5.34. *Let $X_1, X_2, \ldots, X_n$ be a random sample from a distribution with mean μ and variance σ^2. Then*

$$E(\bar{X}) = \mu \qquad Var(\bar{X}) = \frac{\sigma^2}{n} \qquad s.d(\bar{X}) = \frac{\sigma}{\sqrt{n}}$$

If $T = X_1 + X_2 + \cdots + X_n$,

$$E(T) = n\mu \qquad Var(T) = n\sigma^2 \qquad s.d(T) = \sqrt{n}\sigma$$

5.4.1 The Case of a Normal Population Distribution

Example 5.35. In a previous class of MA2506, students' final exam score $\sim N(70, 20^2)$. This year, the same class, 36 students.

$$\bar{X} = \text{ average score}$$

$$P(65 \leq \bar{X} \leq 75)$$

Since $X_1, X_2, \ldots, X_n \overset{iid}{\sim} (70, 20^2)$, $\bar{X} \sim N(70, \frac{20^2}{36})$

$$P(65 \leq \bar{X} \leq 75) = P\left(\frac{65-70}{20/6} \leq \frac{\bar{X}-70}{20/6} \leq \frac{75-70}{20/6}\right) = \Phi(-1.5 \leq Z \leq 1.5) = 0.8664$$

5.4.2 The Central Limit Theorem

What if $X_1, X_2, \ldots, X_n \overset{iid}{\sim} (\mu, \sigma^2)$? No normality.

Theorem 5.36 (The Central Limit Theorem(CLT))**.** *Let $X_1, X_2, \ldots, X_n$ be a random sample from a distribution with mean μ and variance σ^2. Then if n is sufficiently large, $\bar{X}$ has approximately a normal distribution with $\mu_{\bar{X}} = \mu$ and $\sigma_{\bar{X}} = \frac{\sigma^2}{n}$, and T also has approximately a normal distribution with $\mu_T = n\mu$, $\sigma_T^2 = n\sigma^2$. The larger the value of n, the better the approximation. Usually, $n \geq 30$.*

Chapter 6

Point Estimation

6.1 Some General Concepts of Point Estimation

Example 6.1. Population $N(\mu, 1)$.

$$10.2, 9.8, 9.5, 11, 13, 9$$

A "guess" of μ can be $\frac{10.2+9.8+9.5+11+13+9}{6} = 10.4$

Definition 6.2. Generally, we need to estimate a parameter θ based on a sample data set $x_1, x_2, \ldots, x_n$. A point estimate of θ is a suitable statistic on $X_1, X_2, \ldots, X_n$.

Example 6.3. (Example 6.1 in textbook)

Example 6.4. (Example 6.2 in textbook)

6.1.1 Unbiased Estimators

Definition 6.5. An estimate $\hat{\theta}$ is said to be unbiased if

$$E(\hat{\theta}) = \theta$$

Otherwise $E(\hat{\theta}) - \theta$ is called the bias of $\hat{\theta}$.

Example 6.6. $X \sim Bin(n, p)$, $\hat{p} = \frac{X}{n}$

$$E(\hat{p}) = E\left(\frac{X}{n}\right) = \frac{1}{n}E(X) = p$$

So $\hat{p}$ is an unbiased estimate of p.

Example 6.7. $X_1, \ldots, X_n \overset{iid}{\sim} (\mu, \sigma^2)$

$$\hat{\mu} = \bar{X} \qquad \hat{\sigma^2} = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$$

$$E(\hat{\mu}) = E(\frac{X_1 + \cdots + X_n}{n}) = \frac{1}{n}E(X_1 + \cdots + X_n)$$

$$E(a_1X_1 + \cdots + a_nX_n) = a_1E(X_1) + \cdots + a_nE(X_n)$$

$$Var(a_1X_1 + \cdots + a_nX_n) = \sum_{i=1}^{n} a_i^2 Var(X_i) \qquad \text{if } X_1, \ldots, X_n \text{ are independent}$$

$$\Rightarrow E(\hat{\mu}) = \frac{1}{n}(E(X_1) + \cdots + E(X_n)) = \frac{1}{n}(\mu + \cdots + \mu) = \mu$$

$$E(S^2) = E\left(\frac{\sum_{i=1}^{n}(X_i - \bar{X})^2}{n-1}\right) = \frac{1}{n-1}E\left(\sum_{i=1}^{n}(X_i^2 - 2X_i\bar{X} + \bar{X}^2)\right)$$

$$= \frac{1}{n-1}E\left(\left(\sum_{i=1}^{n} X_i^2\right) - n\bar{X}^2\right) = \frac{1}{n-1}\left(\sum_{i=1}^{n} E(X_i^2) - nE(\bar{X}^2)\right)$$

Recall: $Var(X_i) = E(X_i^2) - (E(X_i))^2, E(X_i) = \mu, Var(X_i) = \sigma^2$ in normal distribution. $\Rightarrow E(X_i^2) = \mu^2 + \sigma^2$. $Var(\bar{X}) = E(\bar{X}^2) - (E(\bar{X}))^2 \Rightarrow E(\bar{X}^2) = \mu^2 + \frac{\sigma^2}{n}$.

$$E(S^2) = \frac{1}{n-1}\left(\sum_{i=1}^{n}(\mu^2+\sigma^2) - n\left(\mu^2 + \frac{\sigma^2}{n}\right)\right) = \sigma^2$$

So, $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$ is an unbiased estimate of σ^2. If we use $\frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2$, would generate a bias.

Example 6.8. (Example 6.4 in textbook)

Proposition 6.9. *If $X_1, X_2, \ldots, X_n$ is a random sample from a distribution with mean μ, then $\bar{X}$ is an unbiased estimator of μ. If in addition the distribution is continuous and symmetric, then $\tilde{X}$ (Sample median) and any trimmed mean are also unbiased estimators of μ.*

6.1.2 Estimators with Minimum Variance

Principle of Minimum Variance Unbiased Estimation

Among all estimators of θ that are unbiased, choose the one that has minimum variance. The resulting is called the minimum variance unbiased estimator (MVUE) of θ.

Example 6.10. (Example 6.6 in textbook)

Theorem 6.11. *$X_1, \ldots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$, then $\hat{\mu} = \bar{X}$ is the MVUE of μ.*

6.2 Methods of Point Estimation

6.2.1 The Method of Moments

Definition 6.12. $X_1, \ldots, X_n$ random sample.
k-th sample moment:

$$\frac{1}{n}\sum_{i=1}^{n} X_i^k$$

k-th population moment:

$$E(X^k)$$

Definition 6.13. Point estimation: use $\frac{1}{n}\sum_{i=1}^{n} X_i^k \to E(X^k)$

Example 6.14. $X_1, \ldots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$

$$E(X) = \mu \qquad E(X^2) = (E(X))^2 + Var(X) = \mu^2 + \sigma^2$$

$$\begin{cases} \hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} X_i \\ \hat{\mu^2 + \sigma^2} = \frac{1}{n}\sum_{i=1}^{n} X_i^2 \end{cases} \Rightarrow \begin{cases} \hat{\mu} = \bar{X} \\ \hat{\sigma^2} = \frac{1}{n}\sum_{i=1}^{n} X_i^2 - \bar{X}^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2 \end{cases}$$

Example 6.15. $X_1, \ldots, X_n \overset{iid}{\sim} Unif(0, \theta)$

$$E(X) = \frac{\theta}{2}$$

$$\frac{\hat{\theta}}{2} = \frac{1}{n}\sum_{i=1}^{n} X_i \Rightarrow \hat{\theta} = 2\bar{X}$$

Example 6.16. (Example 6.13 in textbook)
$X_1, \ldots, X_n \overset{iid}{\sim} Gamma(\alpha, \beta)$

$$\begin{cases} \hat{\alpha}\hat{\beta} = \bar{X} \\ \hat{\alpha}\hat{\beta}^2 + (\hat{\alpha}\hat{\beta})^2 = \frac{1}{n}\sum_{i=1}^n X_i^2 \end{cases} \Rightarrow \begin{cases} \hat{\alpha} = \frac{\bar{X}^2}{\frac{1}{n}\sum_{i=1}^n X_i^2 - \bar{X}^2} \\ \hat{\beta} = \frac{1}{\bar{X}}\left(\frac{1}{n}\sum_{i=1}^n X_i^2 - \bar{X}^2\right) \end{cases}$$

6.2.2 Maximum Likelihood Estimation

Example 6.17. (Similar to Example 6.15 in the textbook) A coin, $P(H) = p$, unknown.

$$X_i = \begin{cases} 1, & H \\ 0. & T \end{cases}$$

10100000001, $\hat{p}$.

The probability of the sequence happening is $p^3(1-p)^7$. try to make $p^3(1-p)^7$ large, Let $L = p^3(1-p)^7$.

$$\ln L = 3\ln p + 7\ln(1-p)$$

$$\operatorname{argmax} L = \operatorname{argmax} \ln L$$

$$\operatorname{argmax}(\log p + 7\log 1 - p) = \frac{3}{10}$$

$$(\log L)' = \frac{3}{p} - \frac{7}{1-p} \qquad \hat{p} = \frac{3}{10}$$

Definition 6.18. Let $X_1, \ldots, X_n$ have joint pmf or pdf $f(x_1, \ldots, x_n; \theta)$. The MLE of θ is the one that maximizes the joint pdf (pmf) or $f(x_1, \ldots, x_n; \widehat{\theta_{MLE}}) \geq f(x_1, \ldots, x_n; \theta)$ for any θ.

Example 6.19. (Example 6.16 in the textbook)

Example 6.20. (Example 6.17 in the textbook)

6.2.3 Estimating Functions of Parameters

Proposition 6.21 (The Invariance Principle)**.** *Let $\hat{\theta}_1, \ldots, \hat{\theta}_n$ be the mle's of the parameters $\theta_1, \ldots, \theta_m$. Then the mle of any function $h(\theta_1, \ldots, \theta_m)$ of these parameters is the function $h(\widehat{\theta_1}, \ldots, \widehat{\theta_m})$ of the mle's.*

Example 6.22. (Example 6.20 in the textbook) the mle for σ is $\sqrt{\frac{1}{n}\sum_{i=1}^n (X_i - \bar{X})^2}$

$$h(\mu, \sigma^2) = \sqrt{\sigma^2}$$

Example 6.23. $X_1, \ldots, X_n \overset{iid}{\sim} f(x;\theta) = \begin{cases} (\theta+1)x^\theta & 0 \leq x \leq 1 \\ 0 & o.w. \end{cases}$, with $\theta > -1$

1. Use MM

$$E(X) = \int_0^1 x(\theta+1)x^\theta \, dx = (\theta+1)\left.\frac{x^{\theta+2}}{\theta+2}\right|_0^1 = \frac{\theta+1}{\theta+2}$$

$$\widehat{E(X)} = \frac{1}{n}\sum_{i=1}^n X_i$$

$$\frac{\theta+1}{\theta+2} = \bar{X} \Rightarrow \hat{\theta}_{MM} = \frac{2\bar{X}-1}{1-\bar{X}}$$

2. Use MLE

$$L(\theta) = f(x_1, \ldots, x_n; \theta) = \prod_{i=1}^{n} (\theta + 1) x_i^n = (\theta + 1)^n \left(\prod_{i=1}^{n} x_i \right)^{\theta}$$

$$l(\theta) = n \log \theta + 1 + \theta \sum_{i=1}^{n} \log x_i$$

$$l'(\theta) = \frac{n}{\theta + 1} + \sum_{i=1}^{n} \log x_i \Rightarrow \hat{\theta}_{MLE} = -\frac{n}{\sum_{i=1}^{n} \log X_i} - 1$$

3. Compare $\hat{\theta}_{MM}$ and $\hat{\theta}_{MLE}$ by their variance

6.2.4 Some Complications

Example 6.24. (Example 6.22 in the textbook)

Chapter 7

Statistical Intervals Based on a Single Sample

7.1 Basic Properties of Confidence Intervals

$X_1, \ldots, X_n \overset{iid}{\sim} f(x; \theta)$, use an interval to "estimate" θ.

Example 7.1. $X_1, \ldots, X_n \overset{iid}{\sim} N(\mu, \sigma_0^2)$, σ_0^2 known.

$$\bar{X} \sim N\left(\mu, \frac{\sigma_0^2}{n}\right)$$

$$Z = \frac{\bar{X} - \mu}{\sigma_0/\sqrt{n}} \sim N(0, 1)$$

$$P\left(-1.96 \leq \frac{\bar{X} - \mu}{\sigma_0/\sqrt{n}} \leq 1.96\right) = 0.95$$

$$P\left(\bar{X} - 1.96\frac{\sigma_0}{\sqrt{n}} \leq \mu \leq \bar{X} + 1.96\frac{\sigma_0}{\sqrt{n}}\right) = 0.95$$

So the chance that μ is within $\bar{x} \pm \frac{\sigma_0}{\sqrt{n}}$ is 95%. Then we call $\left(\bar{X} - 1.96\frac{\sigma_0}{\sqrt{n}}, \bar{X} + 1.96\frac{\sigma_0}{\sqrt{n}}\right)$ is the 95% CI for μ.

7.1.1 Interpreting a Confidence Level

Get 10000 such random samples independently, then 10000 different $\bar{X}$'s. Almost 9500 of such intervals will cover μ.

7.1.2 Other Levels of Confidence

Example 7.2. A swimmer adopts a new swimming style. Historical data suggests that the time he needed to swim 200 metres is μ minutes within 0.5 minutes s.d. He swims 9 times and the average he spent is 2.5 minutes. Suppose the swimming time is normally distributed. What is the 95% CI for μ?

$$X_1, \ldots, X_9 \overset{iid}{\sim} N(\mu, 0.5^2)$$

$$\bar{X} \pm 1.96\frac{\sigma_0}{\sqrt{n}} = 25 \pm 1.645\frac{0.5}{\sqrt{9}} = 2.5 \pm 0.274 = (2.226, 2.774)$$

the 97.5th percentile of Z is 1.96

$$P(Z \leq 1.96) = 0.975$$

Denote $z_{0.025} = 1.96$, as the 2.5th upper percentile of Z.

Example 7.3. The response time to do a command is normally distributed with σ_0= 25 ms. Want to estimate the μ for the system. How many times are necessary to assure that the 95% CI for μ has a width at most 10 ms?

95% CI for μ is $\bar{X} \pm z_{\alpha/2}\frac{\sigma_0}{\sqrt{n}}$. Its width is $2z_{0.025}\frac{25}{\sqrt{n}} \leq 10$.

$$\sqrt{n} \geq 9.8 \Rightarrow n \geq 96.04, \quad n = 97$$

Proposition 7.4. *In general,*

$$P\left(-z_{\alpha/2} \leq \frac{\bar{X}-\mu}{\sigma_0^2/\sqrt{n}} \leq z_{\alpha/2}\right) = 1-\alpha$$

So $\bar{x} \pm z_{\alpha/2}\frac{\sigma_0}{\sqrt{n}}$ *is the* $100(1-\alpha)\%$ *CI for* μ.

7.2 Intervals Based on a Normal Population Distribution

Assumption

The population of interest is normal, so that $X_1, \ldots, X_n$ constitutes a random sample from a normal distribution with both μ and σ unknown.

Theorem 7.5. $X_1, \ldots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$, σ *unknown.*

$$\frac{\bar{X}-\mu}{S/\sqrt{n}} \sim t(n-1)$$

where $S = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(X_i-\bar{X})^2}$ *is the sample s.d.*

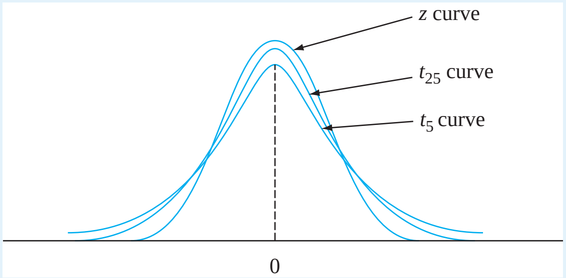


Figure 7.7 t_ν and z curves

Figure 7.1: t and Z distribution

$$P\left(-t_{\alpha/2,n-1} \leq \frac{\bar{X}-\mu}{S/\sqrt{n}} \leq t_{\alpha/2,n-1}\right) = 1-\alpha$$

where $t_{\alpha/2,n-1}$ *denotes the* $\frac{\alpha}{2}$*-th upper percentile of* $t(n-1)$.

Then $\bar{x} \pm t_{\alpha/2,n-1}\frac{S}{\sqrt{n}}$ *is the* $100(1-\alpha)\%$ *CI for* μ.

Example 7.6. The following data are believed to be sampled from normal distribution. 10490, 16620, ..., 14760 ($n = 16$)

Then the 95% CI for μ is

$$\begin{aligned}\bar{x} \pm t_{\alpha/2,n-1}\frac{S}{\sqrt{n}} =&14532.5 \pm t_{0.025,15}\frac{2055.67}{\sqrt{16}}\\ =&(13437.3, 15627.7)\end{aligned}$$

7.2.1 A Prediction Interval for a Single Future Value

See the corresponding text in the textbook.

7.2.2 Tolerance Intervals

See the corresponding text in the textbook.

7.3 Large-Sample Confidence Intervals for a Population Mean and Proportion

7.3.1 A Large-Sample Interval for μ

Proposition 7.7. *$X_1, \ldots, X_n \overset{iid}{\sim} (\mu, \sigma^2)$. μ and σ are both unknown. By CLT if n is large*

$$\frac{\bar{X} - \mu}{S/\sqrt{n}} \dot{\sim} N(0,1)$$

$$P\left(-z_{\alpha/2} \le \frac{\bar{X} - \mu}{S/\sqrt{n}} \le z_{\alpha/2}\right) \doteq 1 - \alpha$$

Then $100(1-\alpha)\%$ CI for μ is $\bar{x} \pm z_{\alpha/2}\frac{s}{\sqrt{n}}$.

Example 7.8. A random sample with $n = 48$ is as follows 62, 50, 53,..., 50, 56, 58 with $n = 48$, $\bar{x} = 54.7$, $s = 5.23$.

Then the 95% CI for μ is

$$54.7 \pm z_{0.025}\frac{5.23}{\sqrt{48}} = (53.2, 56.2)$$

7.3.2 How to Construct a Confidence Interval In General

$X_1, \ldots, X_n \overset{iid}{\sim} f(x; \theta)$. Want to construct a confidence interval for θ

1. Find a statistic (pivot) which depends on $X_1, \ldots, X_n$ and θ only;
2. Its distribution does not depend on θ or any other unknown parameters.

7.3.3 A General Large-Sample Confidence Interval

$X_1, \ldots, X_n \overset{iid}{\sim} f(x; \theta)$ and $\hat{\theta}$ is an estimate. For θ, satisfying

1. approximately normal
2. is approxiamtely unbiasd
3. $\sigma^2_{\hat{\theta}} = Var(\hat{\theta})$ is available

Then

$$P\left(-z_{\alpha/2} \le \frac{\hat{\theta} - \theta}{\sigma_{\hat{\theta}}} \le z_{\alpha/2}\right) = 1 - \alpha$$

7.3.4 A Confidence Interval for a Population Proportion

Example 7.9. A random sample of n individual is selected from $Bern(p)$. $p =$ success rate.

$$X_1, \dots, X_n \overset{iid}{\sim} Bern(p)$$

$$\boxed{Var(X_i) = p(1-p) \qquad \hat{p} = \bar{X}}$$

$$Y = \sum_{i=1}^{n} X_i \sim Bin(n,p)$$

$\hat{p} = \frac{Y}{n}$ is an estimate for p. By CLT,

$$\frac{\sqrt{n}(\hat{p}-p)}{\sqrt{p(1-p)}} \dot{\sim} N(0,1)$$

$$P\left(-z_{\alpha/2} \le \frac{\sqrt{n}(\hat{p}-p)}{\sqrt{p(1-p)}} \le z_{\alpha/2}\right) \doteq 1-\alpha$$

So, $100(1-\alpha)\%$ CI for p is $\hat{p} \pm z_{\frac{\alpha}{2}}\sqrt{\frac{p(1-p)}{n}}$, but p is unknown.

Remedy

1. If n is large, replace p by $\hat{p}$ in the CI formula $\hat{p} \pm z_{\frac{\alpha}{2}}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$

2.
$$P\left(-z_{\alpha/2} \le \frac{\sqrt{n}(\hat{p}-p)}{\sqrt{p(1-p)}} \le z_{\alpha/2}\right) \doteq 1-\alpha$$

$$p - z_{\alpha/2}\sqrt{\frac{p(1-p)}{n}} \le \hat{p} \le p + z_{\alpha/2}\sqrt{\frac{p(1-p)}{n}}$$

$$(p-\hat{p})^2 \le z_{\alpha/2}^2 \frac{p(1-p)}{n}$$

Solving the quadratic equation for p.

$$\frac{\hat{p} + \frac{z_{\alpha/2}}{2n} \pm z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n} + \frac{z_{\alpha/2}^2}{4n^2}}}{1 + z_{\alpha/2}^2/n}$$

is the $100(1-\alpha)\%$ CI for p.

7.3.5 One-Sided Confidence Intervals (Confidence Bounds)

$$X_1, \dots, X_n \overset{iid}{\sim} N(\mu, \sigma_0^2)$$

$$P\left(\frac{\bar{X}-\mu}{\sigma_0/\sqrt{n}} \le z_\alpha\right) = 1-\alpha$$

Then $P\left(\mu \le \bar{X} + z_\alpha \frac{\sigma_0}{\sqrt{n}}\right) = 1-\alpha$. So $\bar{X} + z_\alpha \frac{\sigma_0}{\sqrt{n}}$ is the $100(1-\alpha)\%$ upper confidence bound for μ. Similarly, $\bar{X} - z_\alpha \frac{\sigma_0}{\sqrt{n}}$ is the $100(1-\alpha)\%$ lower confidence bound for μ. $P\left(\mu \ge \bar{X} - z_\alpha \frac{\sigma_0}{\sqrt{n}}\right) = 1-\alpha$.

If σ is unknown, $\bar{X} + t_{\alpha,n-1}\frac{\sigma_0}{\sqrt{n}}$ is the $100(1-\alpha)\%$ upper confidence bound for μ. $\bar{X} - t_{\alpha,n-1}\frac{\sigma_0}{\sqrt{n}}$ is the $100(1-\alpha)\%$ lower confidence bound for μ.

If large sample, $\bar{x} + z_\alpha \frac{s}{\sqrt{n}}$, $\bar{x} - z_\alpha \frac{s}{\sqrt{n}}$.

Example 7.10. (Example 7.10 in the textbook)

Example 7.11. 37 helmets are tested. 24 of them shown damage: let p denote the proportions of all helmets showing damage under the same impact condition.

1. Caculate 99% CI for p.

2. What sample size is required for the width of 99% CI to be at most 0.1?

Solution.
(1) $X = \#$ of helmets with damages $\sim Bin(37, p)$. Observe $x = 24$, $\hat{p} = \frac{x}{n} = \frac{24}{37}$.
MM: $E(X) = np$, then $n\hat{p} = X$, $\hat{p} = \frac{X}{n}$
MLE: $L(p) = \binom{37}{x} p^x (1-p)^{37-x}$

$$l(p) = \log \binom{37}{x} + x \log p + (37 - x) \log (1-p)$$

$$l'(p) = 0 \qquad \hat{p} = \frac{x}{n} = \frac{24}{37}$$

$$\hat{p} = \frac{X}{n} \dot{\sim} N\left(p, \frac{p(1-p)}{n}\right)$$

99% CI for p is $\hat{p} \pm z_{0.005}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} = (0.4465, 0.8507)$.
(2) Width of 99% CI is

$$2z_{0.005}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \leq 0.1$$

$$n \geq \left(\frac{2 \times 2.575}{0.1}\right)^2 \hat{p}(1-\hat{p})$$

$$n \geq \left(\frac{2 \times 2.575}{0.1}\right)^2 \cdot \frac{1}{4}$$

7.4 Confidence Intervals for the Variance and Standard Deviation of a Normal Population

Theorem 7.12. *Then $X_1, \ldots, X_n$ are a random sample from $N(\mu, \sigma^2)$. Then*

$$\frac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1)$$

Then

$$P\left(\chi^2_{1-\alpha/2,n-1} \leq \frac{(n-1)S^2}{\sigma^2} \leq \chi^2_{\alpha/2,n-1}\right) = 1 - \alpha$$

So $100(1-\alpha)\%$ CI for σ^2 is

$$\left(\frac{(n-1)s^2}{\chi^2_{\alpha/2,n-1}}, \frac{(n-1)s^2}{\chi^2_{1-\alpha/2,n-1}}\right)$$

Then $100(1-\alpha)\%$ CI for σ is

$$\left(\sqrt{\frac{(n-1)s^2}{\chi^2_{\alpha/2,n-1}}}, \sqrt{\frac{(n-1)s^2}{\chi^2_{1-\alpha/2,n-1}}}\right)$$

Example 7.13. (Example 7.15 in the textbook)

Chapter 8

Tests of Hypotheses Based on a Single Sample

8.1 Hypotheses and Test Procedures

A test hypothesis is a method using sample data to describe between two competing claims about a population characteristic.

Example 8.1. $X_1, \ldots, X_n \overset{iid}{\sim} f(x;\theta)$
Claim 1: $\theta = 0$
Claim 2: $\theta \neq 0$

Definition 8.2. Null hypothesis (H_0): a population characteristic is usually assumed to be true. Alternative hypothesis (H_a): competing claim.

H_0 be rejected in favour of H_a, if sample evidence suggests that H_0 is false.

Example 8.3.

$$H_0 : \mu = 0.75 \qquad H_a : \mu > 0.75$$

Only if the sample data strongly suggests that θ is something different from 0.75, should H_0 be rejected. Otherwise, H_a will be rejected.

Usually, $H_0 : \theta = \theta_0$

1. $H_a : \theta > \theta_0$ (One-sided alternative)
2. $H_a : \theta < \theta_0$ (One-sided alternative)
3. $H_a : \theta \neq \theta_0$

Example 8.4.

$$H_0 : \mu = 0.75 \qquad H_a : \mu > 0.75$$

$x_1 = 0.01, x_2 = 0.03, x_3 = 0.02$. Even though the dataset indicates that$\hat{\mu}$ should be very small, if we have to choose one from H_0, H_a, choose H_0. "Not reject H_0".

8.1.1 Test Procedures

A test procedure: a rule, based on sample data, for deciding whether to reject H_0.

Example 8.5. X=# of defective among 200 randomly selected products.

$$H_0 : p = 0.1 \qquad H_a : p < 0.1$$

Here p is the defective rate.

$$X \sim Bin(200, p)$$

Under $H_0 \Rightarrow E(X) = 20$. If H_0 is true, we would expect < 20 deflective products.

If $x = 19, 18, 17$, they are not strong enough for us to make a decision.

If $x = 1, 2, 3$, they are very strong.

Test Procedure

1. A test statistic: a function of sample data on which the decision is made.
2. Rejection Region (RR): the set of all the statistic values for which H_0 will be rejected.

8.1.2 Errors in Hypothesis Testing

	H_0 True	H_0 False
Reject H_0	Type I Error	✓
Not Reject H_0	✓	Type II Error

Denote $\alpha = P(\text{Type I Error})$, $\beta = P(\text{Type II Error})$.

Example 8.6. (Example 8.1 in textbook)

Example 8.7. (Example 8.2 in textbook)
As the μ become smaller and smaller, the probability of Type II error is getting down.

Proposition 8.8. *Suppose the sample size is fixed, and a test statistic is chosen. Then decreasing the size of RR to obtain a small α result in a larger β for any particular parameter consisting with H_a.*

8.1.3 Level-α Test

A type I error is usually more serious than a type II error. The approach adhered to by most statistical practitioners is then to specify the largest value of α that can be tolerated and find a rejection region having that value of α rather than anything smaller. This makes β as small as possible subject to the bound on α. The resulting value of α is often referred to as the **significance level** of the test. Traditional levels of significance are 0.10, 0.05, and 0.01, though the level in any particular problem will depend on the seriousness of a type I error—the more serious this error, the smaller should be the significance level. The corresponding test procedure is called a **level α test** (e.g., a level 0.05 test or a level 0.01 test). A test with significance level α is one for which the type I error probability is controlled at the specified level.

Example 8.9. (Example 8.5 in textbook)

$$\begin{aligned}\beta(1.55) =& P(\bar{X} \leq 1.56 \text{ if } H_0 \text{ is false })\\ =& P(\bar{X} \leq 1.56) \qquad \bar{X} \sim N\left(1.55, \frac{0.2^2}{32}\right)\\ =& P\left(\frac{\bar{X}-1.55}{\frac{0.2}{\sqrt{32}}} \leq \frac{1.56-1.55}{\frac{0.2}{\sqrt{32}}}\right) = 0.6103\end{aligned}$$

8.2 Tests About a Population Mean

8.2.1 Case I: A Normal Population with Known σ_0^2

$$H_0 : \mu = \mu_0$$

Test statistic

$$Z = \frac{\bar{X}-\mu_0}{\sigma_0/\sqrt{n}}$$

Use of the following sequence of steps is recommended when testing hypotheses about a parameter.

1. With $H_a : \mu > \mu_0$, $RR : Z \geq c$.

   Level-α test

$$P(Z \geq c) \leq 0.05 \Rightarrow x \geq z_{0.05} = 1.645 \Rightarrow c = 1.645$$

2. With $H_a : \mu < \mu_0$, $RR : Z \le c$.
   Level-α test
$$P(Z \le c) \le 0.05 \Rightarrow x \le -z_{0.05} = -1.645 \Rightarrow c = -1.645$$

3. With $H_a : \mu \ne \mu_0$, $RR : Z \ge c$ or $Z \le -c$.
   Level-α test
$$P(Z \ge c \text{ or } Z \le -c) \le 0.05 \Rightarrow x \ge z_{0.025} = 1.96 \Rightarrow c = 1.96$$

Conclusion

$H_0 : \mu = \mu_0$. Test statistic $Z = \frac{\bar{X}-\mu_0}{\sigma_0/\sqrt{n}}$

1. $H_a : \mu < \mu_0$, $RR : Z \le -z_\alpha$
2. $H_a : \mu > \mu_0$, $RR : Z \ge z_\alpha$
3. $H_a : \mu \ne \mu_0$, $RR : |Z| \ge z_{\alpha/2}$

Procedure

1. identify the parameter of interest
2. determine the null value & state H_0
3. state the "appropriate" H_a
4. construct a test statistic
5. for the given significance level α, state RR
6. compare the observed test statistic' value
7. decide whether to reject H_0, give conclusion

Example 8.10. (Example 8.6 in textbook)

β and Sample Size Determination

$H_0 : \mu = \mu_0$.

$$H_a : \mu > \mu_0$$

$$Z = \frac{\bar{X} - \mu_0}{\sigma_0/\sqrt{n}} \overset{H_0}{\sim} N(0,1) \qquad RR : Z \ge z_\alpha$$

For $\mu' > \mu_0$:

$$\begin{aligned}
\beta(\mu') =& P(Z \le z_\alpha) \qquad \bar{X} \sim \left(\mu', \frac{\sigma_0^2}{n}\right) \\
=& P\left(\bar{X} \le \mu_0 + z_\alpha \frac{\sigma_0}{\sqrt{n}}\right) \\
=& P\left(\frac{\bar{X} - \mu'}{\sigma_0/\sqrt{n}} \le \frac{\mu_0 + z_\alpha \frac{\sigma_0}{\sqrt{n}} - \mu'}{\sigma_0/\sqrt{n}}\right) \\
=& \Phi\left(\frac{\mu_0 - \mu'}{\sigma_0/\sqrt{n}} + +z_\alpha\right)
\end{aligned}$$

Recall that Φ increases.
$\beta(\mu')$ decreases if μ' increases, n increases.
If $\beta(\mu') \le \beta$, β is given

$$\Phi\left(\frac{\mu_0 - \mu'}{\sigma_0/\sqrt{n}} + +z_\alpha\right) \le \beta$$

$$n \geq \left(\frac{z_\alpha + z_\beta}{\mu_0 - \mu'} \cdot \sigma_0 \right)^2$$

For two-sided H_a:

$$n \geq \left(\frac{z_{\alpha/2} + z_\beta}{\mu_0 - \mu'} \cdot \sigma_0 \right)^2$$

Example 8.11. (Example 8.7 in textbook)

8.2.2 Case II: Large-Sample Tests

$X_1, \ldots, X_n \overset{iid}{\sim} (\mu, \sigma^2)$ with large n ($n \geq 30$)

$H_0 : \mu = \mu_0$, $Z = \frac{\bar{X} - \mu_0}{S/\sqrt{n}} \dot{\sim} N(0,1)$

1. With $H_a : \mu > \mu_0$, $RR : Z \geq z_\alpha$.
2. With $H_a : \mu < \mu_0$, $RR : Z \leq -z_\alpha$.
3. With $H_a : \mu \neq \mu_0$, $RR : |Z| \geq z_{\alpha/2}$.

Example 8.12. (Example 8.8 in textbook)

β and Sample Size Determination

Determination of β and the necessary sample size for these large-sample tests can be based either on specifying a plausible value of σ and using the case I formulas (even though s is used in the test) or on using the methodology to be introduced shortly in connection with case III.

8.2.3 Case III: A Normal Population Distribution

$$X_1, \ldots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$$

$H_0 : \mu = \mu_0$. Test statistic: $T = \frac{\bar{X} - \mu_0}{S/\sqrt{n}} \sim t(n-1)$ under H_0

$H_a : \mu > \mu_0$, $RR : \{T \geq ?\}$

$$\alpha = P(\text{Type I Error}) = P(T \geq ?) \text{ if } H_0 \text{ is true} = P(T \geq t_{\alpha, n-1})$$

1. With $H_a : \mu > \mu_0$, $RR : T \geq t_{\alpha,n-1}$.
2. With $H_a : \mu < \mu_0$, $RR : T \leq -t_{\alpha,n-1}$.
3. With $H_a : \mu \neq \mu_0$, $RR : |T| \geq t_{\alpha/2,n-1}$.

Example 8.13. $N(\mu, \sigma^2)$, σ unknown. Sample: 25.8, 36.6, 26.3, 21.8, 27.2.

$$H_0 : \mu = 25, \qquad H_a : \mu > 25$$

$$T = \frac{\bar{X}}{S/\sqrt{n}} \sim t(4) \text{ under } H_0$$

$$RR : T \geq t_{0.05,4} = 2.132$$

Obviously that statistic $T^* = \frac{27.54 - 25}{5.47/\sqrt{5}} = 1.04$. $T^* \notin RR$. Fail to reject H_0.

β and Sample Size Determination

See the text in textbook.

Claim: 99.9% of MTR train will be on-time.

$$X_1, \ldots, X_n \overset{iid}{\sim} Bern(p)$$

$$H_0 : p = 0.999$$

1. $H_a : p \neq 0.999$
2. $H_a : p < 0.999$ work against MTR

3. $H_a : p > 0.999$ work for MTR

Example 8.14. (Exercise 8.32 in textbook)

8.2.4 Connection to Confidence Interval

$$X_1, \ldots, X_n \overset{iid}{\sim} N(\mu, \sigma_0^2)$$

σ_0 known. $100(1-\alpha)\%$ CI for μ is $\bar{x} \pm z_{\alpha/2}\frac{\sigma_0}{\sqrt{n}}$.

$$H_0 : \mu = \mu_0 \qquad H_a : \mu \neq \mu_0$$

$$\begin{aligned} RR : \left|\frac{\bar{X}-\mu_0}{\sigma_0/\sqrt{n}} \geq z_{\alpha/2}\right| &\Leftrightarrow \mu_0 \geq \bar{X} + z_{\alpha/2}\frac{\sigma_0}{\sqrt{n}} \text{ or } \mu_0 \leq \bar{X} - z_{\alpha/2}\frac{\sigma_0}{\sqrt{n}} \\ &\Leftrightarrow \mu_0 \notin \left(\bar{X} - z_{\alpha/2}\frac{\sigma_0}{\sqrt{n}}, \bar{X} + z_{\alpha/2}\frac{\sigma_0}{\sqrt{n}}\right) \\ &\Leftrightarrow \mu_0 \notin 100(1-\alpha)\% \text{ CI for} \mu \end{aligned}$$

However, when H_a is not two-sided.

$$H_a : \mu > \mu_0$$

$$\begin{aligned} RR : \frac{\bar{X}-\mu_0}{\sigma_0/\sqrt{n}} \geq z_\alpha &\Leftrightarrow \mu_0 \leq \bar{X} - z_\alpha\frac{\sigma_0}{\sqrt{n}} \\ &\Leftrightarrow \mu_0 \notin \left(\bar{X} - z_\alpha\frac{\sigma_0}{\sqrt{n}}, +\infty\right) \\ &\Rightarrow \text{is not a CI for } \mu_0 \end{aligned}$$

8.3 Tests Concerning a Population Proportion

8.3.1 Large-Sample Tests

Generally, for a parameter θ, if

1. sample size is large
2. $\hat{\theta}$ is approximately normal
3. $\sigma^2_{\hat{\theta}}$ is available

Test statistic: $Z = \frac{\hat{\theta}-\theta}{\sigma_{\hat{\theta}}}$.
Suppose $X \sim Bin(n,p)$, $\hat{p} = \frac{X}{n}$, $Var(\hat{p}) = Var\left(\frac{X}{n}\right) = \frac{p(1-p)}{n}$

$$Z = \frac{\hat{p}-p}{\sqrt{\frac{p(1-p)}{n}}} \dot{\sim} N(0,1)$$

$$H_0 : p = p_0 \qquad H_a : p > p_0$$

$$Z = \frac{\hat{p}-p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}} \dot{\sim} N(0,1) \text{ under } H_0$$

Reject H_0 if $Z \geq z_\alpha$.

Example 8.15. (Exercise 8.39 in textbook) A random sample of 150 recent donations at a certain blood bank reveals that 82 were type A blood. Does this suggest that the actual percentage of type A donations differs from 40%, the percentage of the population having type A blood? Carry out a test of the appropriate hypotheses using a significance level of 0.01. Would your conclusion have been different if a significance level of 0.05 had been used?

β and Sample Size Determination

$$H_0 : p = p_0 \qquad H_a : p' > p_0$$

$$RR : Z = \frac{\frac{X}{n} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}} \geq z_\alpha$$

$$\begin{aligned}
\beta(p') =& P(\text{fail to reject } H_0 \text{ if } H_0 \text{ is false}) \\
=& P(Z \leq z_\alpha) \qquad X \sim Bin(n, p') \\
=& P\left(\frac{\frac{X}{n} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}} \leq z_\alpha \right) = P\left(\frac{X}{n} \leq p_0 + z_\alpha \sqrt{\frac{p_0(1-p_0)}{n}} \right) \\
=& P\left(\frac{\frac{X}{n} - p'}{\sqrt{\frac{p'(1-p')}{n}}} \leq \frac{p_0 + z_\alpha \sqrt{\frac{p_0(1-p_0)}{n}} - p'}{\sqrt{\frac{p'(1-p')}{n}}} \right) \\
=& \Phi\left(\frac{p_0 + z_\alpha \sqrt{\frac{p_0(1-p_0)}{n}} - p'}{\sqrt{\frac{p'(1-p')}{n}}} \right) \leq \beta
\end{aligned}$$

$$\frac{p_0 + z_\alpha \sqrt{\frac{p_0(1-p_0)}{n}} - p'}{\sqrt{\frac{p'(1-p')}{n}}} \leq -z_\beta \Rightarrow n \geq \left(\frac{z_\alpha \sqrt{p_0(1-p_0)} + z_\beta \sqrt{p'(1-p')}}{p' - p_0} \right)^2$$

"One-sided" for $p' < p_0$

$$\beta(p') = 1 - \Phi\left(\frac{p_0 - p' - z_\alpha \sqrt{\frac{p_0(1-p_0)}{n}}}{\sqrt{\frac{p'(1-p')}{n}}} \right) \leq \beta$$

"Two-sided" for $p' \neq p_0$

$$\beta(p') = \Phi\left(\frac{p_0 - p' + z_{\alpha/2} \sqrt{\frac{p_0(1-p_0)}{n}}}{\sqrt{\frac{p'(1-p')}{n}}} \right) - \Phi\left(\frac{p_0 - p' - z_{\alpha/2} \sqrt{\frac{p_0(1-p_0)}{n}}}{\sqrt{\frac{p'(1-p')}{n}}} \right) \leq \beta$$

Example 8.16. (Example 8.12 in textbook)

8.3.2 Small-Sample Tests

$$H_0 : p = p_0 \qquad H_a : p > p_0$$

Observe $X \sim Bin(n, p)$, reject H_0 if $X \geq c$.

$$\begin{aligned}
P(\text{Type I error}) =& P(X \geq x) \qquad \text{if } H_0 \text{ is true} \\
=& 1 - B(c-1; n; p_0) \leq \alpha
\end{aligned}$$

$$\begin{aligned}
\beta(p') =& P(X \leq c-1) \qquad X \sim Bin(n, p') \\
=& B(c-1; n; p')
\end{aligned}$$

Example 8.17. (Example 8.13 in textbook)

8.4 P-Values

Example 8.18. In a community, the mean household water usage for Jan. '93 is 0.6. In '94, water conservation was conducted. In Jan. '95, $n = 50$ households are randomly selected. $n = 50, \bar{x} = 0.054, s = 0.016$. Does the data suggest that the water usage become less?

$$H_0 : \mu = 0.6 \qquad H_a : \mu < 0.6$$

$$Z = \frac{\bar{X} - 0.6}{S/\sqrt{n}} \overset{H_0}{\sim} N(0,1)$$

$$RR : Z \le z_{-\alpha} = \begin{cases} -1.645 & \text{if } \alpha = 0.05, \\ -2.33 & \text{if } \alpha = 0.01, \end{cases}$$

$$z^* = \frac{0.054 - 0.6}{0.016/\sqrt{50}} = -2.61$$

If $\alpha = 0.05$, reject H_0; If $\alpha = 0.01$, reject H_0.
P-value: $P(Z \le -2.61) = 0.0045$. Consider $\alpha = 0.0045$, $RR : Z \le -2.61$.

Definition 8.19. P-value is the smallest level of significance at which H_0 will be rejected when the test is used on a given database.

Conclusion

If P-value$\le \alpha$, then reject H_0. If P-value$\ge \alpha$, then fail to reject H_0.

Definition 8.20. The P-value is the probability, calculated assuming that the null hypothesis is true, of obtaining a value of the test statistic at least as contradictory to H_0 as the value calculated from the available sample. The smaller the P-value, the more contradiction is the data to H_0.

8.4.1 P-Values for z Tests

Case I: A Normal Population with Known σ_0^2

$$X_1, \ldots, X_n \overset{iid}{\sim} N(\mu, \sigma_0^2)$$

$H_0 : \mu = \mu_0$. Test statistic $Z = \frac{\bar{X} - \mu_0}{\sigma_0/\sqrt{n}}$
$H_a : \mu > \mu_0$. P-value $= P(Z \ge Z^*)$
$H_a : \mu < \mu_0$. P-value $= P(Z \le Z^*)$
$H_a : \mu \ne \mu_0$. P-value $= P(|Z| \ge |Z^*|) = 2(1 - \Phi(|Z^*|))$

Case II: Large-Sample Tests

Similar as Case I.

Example 8.21. (Example 8.17 in textbook)

8.4.2 P-Values for t Tests

$$X_1, \ldots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$$

$H_0 : \mu = \mu_0$. Test statistic: $T = \frac{\bar{X} - \mu_0}{S/\sqrt{n}} \sim t(n-1)$ under H_0
$H_a : \mu > \mu_0$. P-value $= P(T \ge T^*) = 1 - CDF_{n-1}(T^*)$
$H_a : \mu < \mu_0$. P-value $= P(T \le T^*) = CDF_{n-1}(T^*)$
$H_a : \mu \ne \mu_0$. P-value $= P(|T| \ge |T^*|) = 2(1 - CDF_{n-1}(|T^*|))$

Example 8.22. Six readings from a device: 85, 77, 82, 68, 72, 69. It is believed that the CO concentration is set at 70 ppm. Is recalibration of this device necessary? ($\alpha = 0.05$)

$H_0 : \mu = 70$. Test statistic: $T = \frac{\bar{X} - 70}{S/\sqrt{n}} \overset{H_0}{\sim} t(n-1)$
$H_a : \mu \ne 70$. $T^* = \frac{75.5 - 70}{7/\sqrt{6}} = 1.92$

$$P - Value = P(|T| \ge 1.92) = 2(1 - CDF_5(1.92)) = 0.116 > 0.05$$

Fail to reject H_0.

8.5 Hypotheses Testing For σ^2

Then $X_1, \dots, X_n$ are a random sample from $N(\mu, \sigma^2)$. μ, σ^2 unknown.

(a)

$$H_0 : \sigma^2 = \sigma_0^2 \qquad H_a : \sigma^2 \neq \sigma_0^2$$

$$\frac{(n-1)S^2}{\sigma^2} \overset{H_0}{\sim} \chi^2(n-1)$$

$$RR : \{\chi^2 \leq \chi^2_{1-\alpha/2,n-1} \text{ or } \chi^2 \geq \chi^2_{\alpha/2,n-1}\}$$

(b) $H_a : \sigma^2 > \sigma_0^2$, $RR : \{\chi^2 \geq \chi^2_{\alpha,n-1}\}$
(c) $H_a : \sigma^2 < \sigma_0^2$, $RR : \{\chi^2 \leq \chi^2_{1-\alpha,n-1}\}$

Example 8.23. A battery manufacture claims that he produce batteries have a s.d. equal to 0.9 year. A random sample is collected $n = 10$, $s = 1.2$ year. Does the data suggest that $\sigma > 0.9$? Assume normality.

$$H_0 : \sigma = 0.9 \qquad H_a : \sigma \geq 0.9$$

$$H_0 : \sigma^2 = 0.81 \qquad H_a : \sigma^2 \geq 0.81$$

$$\frac{(n-1)S^2}{0.81} \overset{H_0}{\sim} \chi^2(n-1)$$

$RR : \{\chi^2 \geq \chi^2_{0.05,9}\} = \{\chi^2 \geq 16.919\}$

$$(\chi^2)^* = \frac{(10-1)1.2^2}{0.9^2} = 16.0 \notin RR$$

Fail to reject H_0.

$$P-Value = P(\chi^2 \geq (\chi^2)^*) = P(\chi^2 \geq 16) = 0.07$$

P-Value > 0.05, fail to reject H_0.

Chapter 9

Inferences Based on Two Samples

9.1 z Tests and Confidence Intervals for a Difference Between Two Population Means

$$X_1, \ldots, X_n \overset{iid}{\sim} N(\mu_1, \sigma_1^2), \sigma_1 \text{ known.}$$

$$Y_1, \ldots, Y_n \overset{iid}{\sim} N(\mu_2, \sigma_2^2), \sigma_2 \text{ known.}$$

Proposition 9.1.

$$E(\bar{X} - \bar{Y}) = \mu_1 - \mu_2$$
$$Var(\bar{X} - \bar{Y}) = \frac{\sigma_1^2}{n} + \frac{\sigma_2^2}{n}$$

9.1.1 Test Procedures for Normal Populations with Known Variances

Case I

$$Z = \frac{\bar{X} - \bar{Y} - (\mu_1 - \mu_2)}{\sqrt{\frac{\sigma_1^2}{n} + \frac{\sigma_2^2}{n}}} \sim N(0, 1)$$

$H_0 : \mu_1 - \mu_2 = \Delta_0$

1. With $H_a : \mu_1 - \mu_2 > \Delta_0$, $RR : Z \geq z_\alpha$.
2. With $H_a : \mu_1 - \mu_2 < \Delta_0$, $RR : Z \leq -z_\alpha$.
3. With $H_a : \mu_1 - \mu_2 \neq \Delta_0$, $RR : |Z| \geq z_{\alpha/2}$.

Example 9.2. (Example 9.1 in textbook)

9.1.2 Large-Sample Tests

Case II large sample, σ^2 unknown

$$Z = \frac{\bar{X} - \bar{Y} - (\mu_1 - \mu_2)}{\sqrt{\frac{S_1^2}{n} + \frac{S_2^2}{n}}} \sim N(0, 1)$$

$100(1 - \alpha)$ CI for $\mu_1 - \mu_2$

$$\bar{X} - \bar{Y} \pm z_{\alpha/2} \sqrt{\frac{S_1^2}{n} + \frac{S_2^2}{n}}$$

9.2 The Two-Sample t Test and Confidence Interval

Case III

(a)

$$X_1, \ldots, X_n \overset{iid}{\sim} N(\mu_1, \sigma_1^2)$$
$$Y_1, \ldots, Y_n \overset{iid}{\sim} N(\mu_2, \sigma_2^2)$$

σ_1, σ_2 independent

$$T = \frac{\bar{X} - \bar{Y} - (\mu_1 - \mu_2)}{\sqrt{\frac{S_1^2}{n} + \frac{S_2^2}{n}}} \sim t(\nu) \qquad \nu = \frac{\left(\frac{S_1^2}{n} + \frac{S_2^2}{n}\right)^2}{\frac{\left(\frac{S_1^2}{m}\right)^2}{m-1} + \frac{\left(\frac{S_1^2}{m}\right)^2}{n-1}}$$

$100(1-\alpha)$ CI for $\mu_1 - \mu_2$

$$\bar{X} - \bar{Y} \pm t_{\alpha/2,\nu}\sqrt{\frac{S_1^2}{n} + \frac{S_2^2}{n}}$$

$$H_0 : \mu_1 - \mu_2 = \Delta_0 \qquad T = \frac{\bar{X} - \bar{Y} - \Delta_0}{\sqrt{\frac{S_1^2}{n} + \frac{S_2^2}{n}}} \sim t(\nu)$$

1. With $H_a : \mu_1 - \mu_2 > \Delta_0$, $RR : T \geq t_{\alpha,\nu}$.
2. With $H_a : \mu_1 - \mu_2 < \Delta_0$, $RR : T \leq -t_{\alpha,\nu}$.
3. With $H_a : \mu_1 - \mu_2 \neq \Delta_0$, $RR : |T| \geq t_{\alpha/2,\nu}$.

Example 9.3. (Example 9.6 in textbook)

9.2.1 Pooled t Procedures

(b) Small sample size, $\sigma_1^2 = \sigma_2^2$

$$T = \frac{\bar{X} - \bar{Y} - (\mu_1 - \mu_2)}{\sqrt{\frac{S_p^2}{n} + \frac{S_p^2}{n}}}$$

$$S_p = \frac{n-1}{m+n-2}S_1^2 + \frac{m-1}{m+n-2}S_2^2$$

"pooled sample variance"

$$S_1^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2 \qquad S_2^2 = \frac{1}{m-1}\sum_{i=1}^{m}(Y_i - \bar{Y})^2$$

Example 9.4. Body weight gained on animal treatment: given 1 mg/piller dose of soft steroid control: placebo.

treatment	$m = 8$	$\bar{x} = 32.8$	$s_1 = 2.6$
placebo	$n = 8$	$\bar{y} = 40.5$	$s_2 = 2.5$

Does the data suggest the average weight gain in the control group exceeds that in the treatment group by more than 5 g? $\alpha = 0.01$

1. $H_0 : \mu_1 - \mu_2 = -5 \qquad H_a : \mu_1 - \mu_2 < -5$

$$T^* = \frac{\bar{X} - \bar{Y} - (-5)}{\sqrt{\frac{S_1^2}{n} + \frac{S_2^2}{n}}} = -2.23$$

$$\nu = \frac{\left(\frac{S_1^2}{n} + \frac{S_2^2}{n}\right)^2}{\frac{\left(\frac{S_1^2}{m}\right)^2}{m-1} + \frac{\left(\frac{S_1^2}{m}\right)^2}{n-1}} = \frac{\left(\frac{2.6^2}{8} + \frac{2.5^2}{10}\right)^2}{\frac{1}{7}\left(\frac{2.6^2}{8}\right)^2 + \frac{1}{9}\left(\frac{2.5^2}{10}\right)^2} = 14.886 \approx 14$$

$$P - Value = P(T_{14} \leq T^*) = 0.022 > 0.1$$

2. Assume $\sigma_1^2 = \sigma_2^2$

   $H_0 : \mu_1 - \mu_2 = -5 \qquad H_a : \mu_1 - \mu_2 < -5$

$$T^* = \frac{\bar{X} - \bar{Y} - (-5)}{\sqrt{\frac{S_p^2}{n} + \frac{S_p^2}{n}}} = -2.24$$

   Here $S_p = \sqrt{\frac{2.6^2(8-1)}{8+10-2} + \frac{2.5^2(10-1)}{8+10-2}} = 2.54$

$$P - Value = P(T_{16} < -2.24) = 0.021 > 0.01$$

9.3 Analysis of Paired Data

n independent selected pairs $(X_1, Y_1), (X_2, Y_2), \ldots, (X_n, Y_n)$

$$E(X_i) = \mu_1 \qquad E(Y_i) = \mu_2$$

$$H_0 : \mu_1 - \mu_2 = \Delta \qquad H_a : \mu_1 - \mu_2 \neq \Delta$$

X	X_1	X_2	...	X_n
Y	Y_1	Y_2	...	Y_n
$D = X - Y$	$D_1 = X_1 - Y_1$	$D_2 = X_2 - Y_2$	...	$D_n = X_n - Y_n$

$$H_0 : \mu_D = \mu_1 - \mu_2 = \Delta \qquad H_a : \mu_D = \mu_1 - \mu_2 \neq \Delta$$

$$D_1, D_2, \ldots, D_n$$

$$T = \frac{\bar{D} - \Delta}{S_D/\sqrt{n}}$$

$$\bar{D} = \frac{1}{n}\sum_{i=1}^{n} D_i \qquad S_D^2 = \frac{1}{n-1}\sum_{i=1}^{n}(D_i - \bar{D})^2$$

9.3.1 The Paired t Test

Example 9.5. (Exercise 8.39 in textbook) reports the accompanying data on amount of milk ingested by each of 14 randomly selected infants.

Does it appear that the true average difference between intake values measured by the two methods is something other than zero? Determine the P-value of the test, and use it to reach a conclusion at significance level 0.05.

$100(1-\alpha)\%$ CI for $\mu_1 - \mu_2 = \mu_0$

$$\bar{D} \pm t_{\alpha/2, n-2}\frac{S_D}{\sqrt{n}}$$

9.4 Inferences Concerning a Difference Between Population Proportions

Proposition 9.6. *Let $X \sim Bin(n, p_1)$, $Y \sim Bin(m, p_2)$ with X and Y independently. $\hat{p_1} = \frac{X}{n}$,$\hat{p_2} = \frac{Y}{m}$*

$$E(\hat{p_1} - \hat{p_2}) = p_1 - p_2$$

$$Var(\hat{p_1} - \hat{p_2}) = \frac{p_1(1-p_1)}{n} + \frac{p_2(1-p_2)}{m}$$

As n and m get larger,

$$Z = \frac{\hat{p_1} - \hat{p_2} - (p_1 - p_2)}{\sqrt{\frac{p_1(1-p_1)}{n} + \frac{p_2(1-p_2)}{m}}} \overset{iid}{\sim} N(0,1)$$

$100(1-\alpha)\%$ CI for $p_1 - p_2$

$$\hat{p_1} - \hat{p_2} \pm z_{\alpha/2}\sqrt{\frac{\hat{p_1}(1-\hat{p_1})}{n} + \frac{\hat{p_2}(1-\hat{p_2})}{m}}$$

9.4.1 A Large-Sample Test Procedure

To test H_0: $p_1 - p_2 = 0$

Test statistic:

$$Z = \frac{\hat{p_1} - \hat{p_2} - 0}{\sqrt{\hat{p}(1-\hat{p})\left(\frac{1}{n} + \frac{1}{m}\right)}}$$

where $\hat{p} = \frac{X+Y}{n+m}$

1. With $H_a : p_1 - p_2 > 0$, $RR : Z \geq z_\alpha$.
2. With $H_a : p_1 - p_2 < 0$, $RR : Z \leq -z_\alpha$.
3. With $H_a : p_1 - p_2 \neq 0$, $RR : |Z| \geq z_{\alpha/2}$.

Example 9.7.

	plea guilty	plea not guilty
Judged guilty	$m = 191$	$n = 64$
Sentenced to prison	$x = 101$	$y = 56$

$$H_0 : p_1 - p_2 = 0 \qquad p_1 \neq p_2$$

$$\hat{p_1} = \frac{101}{191} = 0.53 \qquad \hat{p_2} = \frac{56}{64} = 0.875$$

$$Z = \frac{\hat{p_1} - \hat{p_2} - 0}{\sqrt{\hat{p}(1-\hat{p})\left(\frac{1}{n} + \frac{1}{m}\right)}} \qquad \hat{p} = \frac{101+56}{191+64} = 0.616$$

$$Z^* = -4.91 \qquad RR : \{|Z| \geq z_{\alpha/2} = 2.58\} \quad \alpha = 0.01$$

$$Z^* \in RR \Rightarrow \text{Reject } H_0$$

Conclusion:

9.5 Challenge Question 4

Example 9.8. For the sample median, $\tilde{X}_n$, from a symmetric distribution with location θ, where the distribution median is θ, we consider $x = \theta$ and $p = F_X(\theta) = 1/2$, so

$$\sqrt{n}(\tilde{X}_n - \theta) \xrightarrow{d} X \sim N\left(0, \frac{1}{4\{f_X(\theta)\}^2}\right).$$

Proof.

$$\lim_{n\to\infty} P(\sqrt{n}(\tilde{X}_n - \theta) \leq a) = P(Z \leq 2f(\theta)a)$$

Let $Y_i = I\left(X_i \leq \theta + \frac{a}{\sqrt{n}}\right) \qquad i = 1, 2, \ldots, n$

$$Y_i = \begin{cases} 1, & x_i \leq \theta + \frac{a}{\sqrt{n}} \\ 0. & x_i \geq \theta + \frac{a}{\sqrt{n}} \end{cases}$$

Clearly, $Y_1, Y_2, \ldots, Y_n \overset{iid}{\sim} Bern(p_n)$

$$p_n = P\left(X_i \leq \theta + \frac{a}{\sqrt{n}}\right) = F\left(\theta + \frac{a}{\sqrt{n}}\right)$$

$$\begin{aligned}
P(\sqrt{n}(\tilde{X}_n-\theta)\le a) =& P\left(\tilde{X}_n \le \theta+\frac{a}{\sqrt{n}}\right)\\
=& P\left(\sum_{i=1}^n Y_i \ge \frac{n+1}{2}\right)\\
=& P\left(\frac{\sum_{i=1}^n Y_i - np_n}{\sqrt{np_n(1-p_n)}} \ge \frac{\frac{n+1}{2}-np_n}{\sqrt{np_n(1-p_n)}}\right)
\end{aligned}$$

Note that $p_n \to \frac{1}{2}, n\to\infty$. By CLT,

$$\frac{\sum_{i=1}^n Y_i - np_n}{\sqrt{np_n(1-p_n)}} \xrightarrow{d} N(0,1)$$

$$\lim_{n\to\infty}\frac{F\left(\theta+\frac{a}{\sqrt{n}}-F(\theta)\right)}{a/\sqrt{n}} = F'(\theta)=f(\theta)$$

$$\frac{n\left(p_n-\frac{1}{2}\right)}{\sqrt{n}} \longrightarrow f(\theta)\cdot a$$

$$\frac{np_n-\frac{n+1}{2}}{\sqrt{n}} \longrightarrow f(\theta)\cdot a \qquad \sqrt{p_n(1-p_n)} \longrightarrow \frac{1}{2} \qquad n\to\infty$$

$$\frac{\frac{n+1}{2}-np_n}{\sqrt{np_n(1-p_n)}} \longrightarrow -2af(\theta)$$

■

Another proof: Bootstrap.

Proof. $X_1,\dots,X_n \overset{iid}{\sim} f(x)$. Distribution of $\tilde{X_n}$. Bootstrap: if $f(x)$ is "known".

$$\begin{aligned}
\tilde{X_n^{(1)}} \leftarrow & X_1^{(1)}\dots X_n^{(1)} \overset{iid}{\sim} f(x)\\
\tilde{X_n^{(2)}} \leftarrow & X_1^{(2)}\dots X_n^{(2)} \overset{iid}{\sim} f(x)\\
\tilde{X_n^{(3)}} \leftarrow & X_1^{(3)}\dots X_n^{(3)} \overset{iid}{\sim} f(x)\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
\tilde{X_n^{(B)}} \leftarrow & X_1^{(B)}\dots X_n^{(B)} \overset{iid}{\sim} f(x)
\end{aligned}$$

$$\tilde{f(x)} = \frac{1}{n} \qquad \text{if } x = x_i$$

Sample n values from $\{x_1,\dots,x_n\}$ with replacement. ■

Appendix A

Moment generating function

A.1 Definition

Definition A.1. The moment generating function (MGF) of a r.v. X is defined as

$$M_X(\theta) = E(e^{\theta x}) = \begin{cases} \sum_{x\in\mathcal{D}} e^{\theta x} p(x) & \text{if } X \text{ is discrete} \\ \int_{-\infty}^{\infty} e^{\theta x} f(x)\, dx & \text{if } X \text{ is continues} \end{cases}$$

Example A.2. If $Z \sim N(0,1)$. Find the mgf $M_Z(\theta)$

$$\begin{aligned} M_Z(\theta) = E(e^{\theta x}) = & \int_{-\infty}^{\infty} e^{\theta z} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}\, dz \\ = & \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{\theta z - \frac{1}{2}z^2}\, dz \\ = & \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\theta^2 + \theta z - \frac{1}{2}z^2} e^{\frac{1}{2}\theta^2}\, dz \\ = & e^{\frac{1}{2}\theta^2} \end{aligned}$$

A.2 Properties of $M_X\theta$

Proposition A.3. *1. There is unique distribution with mgf $M_X\theta$*

2.

$$\begin{aligned} M_X\theta = & E(e^{\theta x}) \\ = & E\left(1 + \frac{\theta X}{1!} + \frac{\theta^2 X^2}{2!} + \dots\right) \\ = & 1 + \frac{\theta E(X)}{1!} + \frac{\theta^2 E(X)^2}{2!} + \dots \end{aligned}$$

3.

$$\frac{dM_X(\theta)}{d\theta} = \frac{dE(e^{\theta x})}{d\theta} = E\left(\frac{de^{\theta x}}{d\theta}\right) = E(e^{\theta X} X)$$

$$\left.\frac{dM_X(\theta)}{d\theta}\right|_{\theta=0} = E(X)$$

Similarly,

$$\left.\frac{d^r M_X(\theta)}{d\theta^r}\right|_{\theta=0} = E(X^r) \qquad r = 1, 2, \dots,$$

4. Let $Y = a + bX$, then $M_Y(\theta) = e^{a\theta} M_X(b\theta)$

$$M_Y(\theta) = E(E^{\theta Y}) = E(e^{\theta(a+bX)}) = E(e^{a\theta + b\theta x}) = e^{a\theta} E(e^{b\theta x}) = e^{a\theta} M_X(b\theta)$$

Example A.4. If $X \sim N(\mu, \sigma^2)$, find $M_Y(\theta)$

$$Z = \frac{X-\mu}{\sigma} \sim N(0,1) \text{ then } X = \mu + \sigma Z$$

by (4)

$$M_Y(\theta) = e^{\mu\theta} M_Z(\sigma\theta) = e^{\mu\theta + \frac{1}{2}\sigma^2\theta^2}$$

$$E(Y) = \left.\frac{dM_Y(\theta)}{d\theta}\right|_{\theta=0} = \mu$$

$$E(Y^2) = \left.\frac{d^2M_Y(\theta)}{d\theta^2}\right|_{\theta=0} = \mu^2 + \sigma^2$$

$$E(Y^3) = \left.\frac{d^3M_Y(\theta)}{d\theta^3}\right|_{\theta=0} = \dots$$

Theorem A.5. *X and Y are two independent r.v. with mgf $M_X(\theta)$ and $M_Y(\theta)$ respectively. Then*

$$M_{X+Y}(\theta) = M_X(\theta)M_Y(\theta)$$

Proof.

$$M_{X+Y}(\theta) = E(e^{\theta(X+Y)}) = E(e^{\theta X}e^{\theta Y}) = M_X(\theta)M_Y(\theta)$$

■

Corollary A.6. *If $X_1, \dots, X_n$ are independent r.v.'s*

$$M_{X_1+\dots+X_n}(\theta) = M_{X_1}(\theta)\dots M_{X_n}(\theta)$$

Example A.7. 1. $Z^2 \sim \chi^2(1)$

2. $Z_1, \dots, Z_n \overset{iid}{\sim} N(0,1)$, then

$$Z_1^2 + Z_2^2 + \dots + Z_n^2 \sim \chi^2(n)$$

Proof. 1.

$$\begin{aligned} M_{Z^2}(\theta) = E(e^{\theta z^2}) &= \int_{-\infty}^{\infty} e^{\theta z^2} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}\,dz \\ &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{\left(\theta-\frac{1}{2}\right)z^2}\,dz \\ &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(-2\theta+1)z^2}\,dz \\ &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} \frac{1}{\sqrt{1-2\theta}}\,dy \\ &= \frac{1}{\sqrt{1-2\theta}} \qquad \theta < \frac{1}{2} \end{aligned}$$

Assume $\theta < \frac{1}{2}$, Let $y = \sqrt{1-2\theta}z$.

Let $A \sim \chi^2(1)$, then

$$f_A(x) = \frac{1}{2^{1/2}\Gamma(1/2)} a^{-\frac{1}{2}} e^{-\frac{a}{2}}$$

Then

$$
\begin{aligned}
M_A(\theta) = E(e^{\theta A}) = & \int_0^\infty e^{\theta a} \frac{1}{2^{1/2}\Gamma(1/2)} a^{-\frac{1}{2}} e^{-\frac{a}{2}}\, da \\
= & \int_0^\infty \frac{1}{2^{1/2}\Gamma(1/2)} a^{-\frac{1}{2}} e^{\left(\theta - \frac{1}{2}\right)a}\, da \\
= & \int_0^\infty \frac{1}{\Gamma(1/2)} (1-2\theta)^{\frac{1}{2}} t^{-\frac{1}{2}} e^{-t}\, dt \\
= & (1-2\theta)^{-\frac{1}{2}} \qquad \theta < \frac{1}{2}
\end{aligned}
$$

Let $t = \frac{1}{2}(1-2\theta)a$, $\theta < \frac{1}{2}$
Since $M_{Z^2}(\theta) = M_A(\theta) \Rightarrow Z^2 \sim A \sim \chi^2(1)$

2. Let $S = Z_1^2 + Z_2^2 + \cdots + Z_n^2$

$$M_S(\theta) = (1-2\theta)^{-\frac{n}{2}}$$

Let $B \sim \chi^2(n)$

$$f_B(b) = \frac{1}{2^{n/2}\Gamma(n/2)} b^{\frac{n}{2}-1} e^{-\frac{b}{2}}$$

$$M_B(\theta) = \int_0^\infty e^{\theta b} \frac{1}{2^{n/2}\Gamma(n/2)} b^{\frac{n}{2}-1} e^{-\frac{b}{2}}\, b = (1-2\theta)^{-\frac{n}{2}}$$

$$M_S(\theta) = M_B(\theta) \Rightarrow S \sim B \sim \chi^2(n)$$

■

A.3 Application

Theorem A.8. *Let $Y_1, Y_2, \ldots, Y_n$ be a sequence of rv's with cdf $F_{Y_1}(y)$, $F_{Y_2}(y)$, ...and mgf $M_{Y_1}(\theta)$, $M_{Y_2}(\theta), \ldots$. Suppose as $n \to \infty$*

$$M_{Y_n}(\theta) \to M_Y(\theta) \text{ for any } \theta$$

where $M_Y(\theta)$ is the mgf of Y with cdf $F(y)$ that

$$F_{Y_n} \to F_Y(y) \text{ for any } y \text{ as } n \to \infty$$

or $Y_n \xrightarrow{d} Y$.[1]

Example A.9. If $X_n \sim Bin(n,p)$. $np = \lambda > 0$. fixed

$$
\begin{aligned}
M_{X_n}(\theta) = & E(e^{\theta X_n}) = \sum_{i=1}^{n} e^{\theta k} \binom{n}{k} p^k (1-p)^{n-k} \\
= & \sum_{k=0}^{n} \binom{n}{k} (pe^\theta)^k (1-p)^{n-k} \\
= & (pe^\theta + 1 - p)^n \\
= & \left(1 + \frac{\lambda}{n}(e^\theta - 1)\right)^n
\end{aligned}
$$

Let $n \to \infty (p \to 0)$

$$M_{X_n}(\theta) = \left(1 + \frac{\lambda}{n}(e^\theta - 1)\right)^n \longrightarrow e^{\lambda(e^\theta - 1)},$$

since $\lim_{n\to\infty} \left(1 + \frac{a}{n}\right)^n = e^a$.

Let $Y \sim Poisson(\lambda)$, $P(Y=k) = \frac{e^{-\lambda}\lambda^k}{k!}$

$$\begin{aligned} M_Y\theta =& E(e^{\theta Y}) = \sum_{k=0}^{\infty} e^{\theta k}\frac{e^{-\lambda}\lambda^k}{k!} \\ =& \sum_{k=0}^{\infty} \frac{e^{-\lambda}(e^\theta\lambda)^k}{k!} \\ =& \sum_{k=0}^{\infty} \frac{e^{-\lambda e^\theta}(e^\theta\lambda)^k}{k!} e^{\lambda e^\theta} e^{-\lambda} \\ =& e^{\lambda(e^\theta - 1)} \end{aligned}$$

$$X_1, X_2, \dots, X_n$$

$$X_n \sim Bin(n,p) \xrightarrow{d} Y \sim Poisson(\lambda)$$

Theorem A.10 (Central Limit Theorem)**.** *Let* $X_1, \dots, X_n \overset{iid}{\sim} (\mu, \sigma^2)$. $S_n = X_1 + \dots + X_n$.

$$\bar{X} = \frac{S_n}{n} \qquad Z_n = \frac{\sqrt{n}(\bar{X} - \mu)}{\sigma} = \frac{S_n - n\mu}{\sqrt{n}\sigma}$$

Proof. Let $Y_i = X_i - \mu$, then $Y_1, Y_2, \dots, Y_n \overset{iid}{\sim} (0, \sigma^2)$.

$$S_n - n\mu = Y_1 + Y_2 + \dots + Y_n$$

$$M_{S_n - n\mu}(\theta) = M_{Y_1}(\theta)\dots M_{Y_n}(\theta)$$

$$\begin{aligned} M_{Z_n}(\theta) =& E(e^{\theta Z_n}) = E\left(e^{\theta\frac{S_n - n\mu}{\sqrt{n}\sigma}}\right) \\ =& E\left(e^{\frac{\theta}{\sqrt{n}\sigma}(S_n - n\mu)}\right) = M_{S_n - n\mu}\left(\frac{\theta}{\sqrt{n}\sigma}\right) \\ =& M_{Y_1}\left(\frac{\theta}{\sqrt{n}\sigma}\right)\dots M_{Y_n}\left(\frac{\theta}{\sqrt{n}\sigma}\right) \\ =& \left(M_{Y_1}\left(\frac{\theta}{\sqrt{n}\sigma}\right)\right)^n \end{aligned}$$

Note that $E(Y_1) = 0$, $E(Y_1^2) = Var(Y_1) + (E(Y_1))^2 = \sigma^2$

$$\begin{aligned} M_{Y_1}(\theta) =& 1 + E(Y_1)\frac{\theta}{1!} + E(Y_2)\frac{\theta^2}{2!} + \dots \\ =& 1 + \sigma^2\frac{\theta^2}{2} + \mathcal{O}(\theta^2), \end{aligned}$$

where $\mathcal{O}(\theta^2)$ denotes a function $g(\theta)$ such that $\frac{g(\theta)}{\theta^2} \to 0$, as $\theta \to 0$.

$$\begin{aligned} M_{Z_n}(\theta) =& \left(1 + \frac{1}{2}\left(\frac{\theta}{\sqrt{n}\sigma}\right)^2 + \mathcal{O}\left(\frac{\theta^2}{n\sigma^2}\right)\right)^n \\ =& \left(1 + \frac{\frac{1}{2}\theta^2}{n} + \mathcal{O}\left(\frac{1}{n}\right)\right)^n \longrightarrow e^{\frac{1}{2}\theta^2} \text{ as } n \to \infty \end{aligned}$$

So, by theorem, $Z_n \xrightarrow{d} N(0,1)$ ∎

1. $X_1, X_2, \dots, X_n \sim Bern(p)$. $E(X_1) = p$, $Var(X_1) = p(1-p)$.

By CLT, $\frac{X-np}{\sqrt{np(1-p)}} \xrightarrow{d} N(0,1)$.

$$X \xrightarrow{d} N(np, np(1-p))$$

2. What if $X_1, X_2, \ldots, X_n \sim Bern(p_n)$?

   Modified CLT

$$X_1, X_2, \ldots, X_n \sim (\mu_n, \sigma_n^2)$$

$$\frac{\sqrt{n}(\bar{X} - \mu_n)}{\sigma_n} \xrightarrow{d} N(0,1)$$

Happy $\TeX$($\LaTeX$, $\LaTeX\,2_\varepsilon$)ing!