How a basketball's internal air pressure affects its bounciness

1. Introduction

I once went to play a basketball game where there were already four basketballs on the court. Before the game began, the bounciest ball was chosen, for it "has the most pressure". Ever since I have been wondering: How does a basketball's internal air pressure affect its bounciness?

There is already a considerable body of work on the bouncing basketball in research literature and web resources. However, almost all of them chose to ignore the effects of air drag and air buoyance on the basketball; for the few that did mention such forces, there were neither experimental data nor modelling. My following research takes into consideration of such forces and attempts to establish a more comprehensive model, first in theory, then to validate it with experimentation.

Research Question

How does a basketball's gauged internal air pressure (P_G) affect its Coefficient of Restitution $(\boldsymbol{e})$?

(The definitions of P_G and e will be covered in the following section.)

2. Background

When a ball drops from a height, its gravity potential energy (G.P.E.) converts to kinetic energy (K.E.). When impacting the ground, its internal air compresses with heightened pressure, following the ideal gas law. With heightened pressure, the ball's kinetic energy (K.E.) converts to elastic potential energy (E.P.E). When the ball decompresses, its E.P.E converts to K.E. and reduces its pressure, like a spring.

2.1 Analysis of Forces

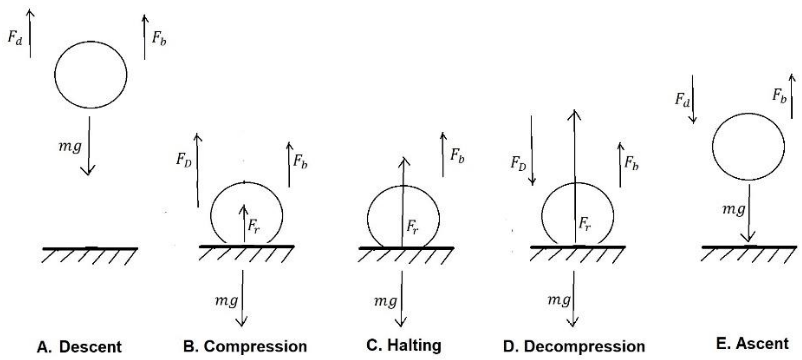

Figure 1. Analysis of forces on the dropped ball at different stages.

During the descent stage (Figure 1A), there are three forces on it: the downward gravity (mg), the upward buoyant force (F_b), and the upward air drag (F_d).

$$F_b = \rho V g \qquad \dots (1)$$

Where ρ is the density of air, V is the volume of the ball, g is the gravity constant.

$$Fd = \frac{1}{2} C_d \rho A v^2 \quad \ldots (2)$$

Where ρ is the air density, v is the speed of the ball, A is the ball's cross-sectional area, C_d is the drag coefficient which depends on shape and size of the object, and air properties. A standard sized basketball's C_d is 0.54 (Cross).

During the compression stage (Figure 1B), the ball has 4 forces on it: the downward gravity (mg), the upward dissipative force (F_D), the upward air buoyant force (F_b), the upward restoring force F_r (aka, the normal force). Previous study suggested that F_r consist of two parts: the restoring normal force F_p arising from the gauge pressure of ball, and the restoring force F_w arising from the deformation of the basketball's wall.

$$F_r = F_p + F_w = kx \quad \ldots (3)$$

where k is the Hooke constant of the ball, x is the displacement (Hubbard and Stronge).

At the halting point (Figure 1C), the ball comes to a stop with 0 net force. The ball is max compressed, with highest E.P.E. There are three forces acting on the ball: the downward gravity (mg), the upward air buoyant force (F_b), the upward restoring force F_r.

During the decompression stage (Figure 1D), the ball has four forces on it: the downward gravity (mg), the downward dissipative force (F_D), the upward air buoyant force (F_b), the upward restoring force F_r.

During the descent stage (Figure 1E), there are three forces acting on it: the downward gravity (mg), the downward air drag (F_d), the upward buoyant force (F_b).

2.2 Theories and a model

Supposing the basketball is a hollow sphere with radius R, wall thickness D_w, internal pressure P_i and external air pressure P_0, it gets dropped from an initial height of H_i, undergoes descent before impact and reaches maximum downward velocity of v_i; after impact the ball obtains max upward velocity of v_f when it restores to its full shape. Subsequently it ascends to a max height of H_f. According to published papers (Georgallas, & et al), the bounciness of a ball is measured by the **Coefficient of Restitution** e:

$$e = \frac{|v_f|}{|v_i|} \quad \ldots (4)$$

With K_i and K_f denoting the ball's K.E. before and after the impact, we have:

$$K_i = \frac{1}{2} m v_i^2, \; K_f = \frac{1}{2} m v_f^2 \;, \longrightarrow K_f = e^2 K_i \quad \ldots (5)$$

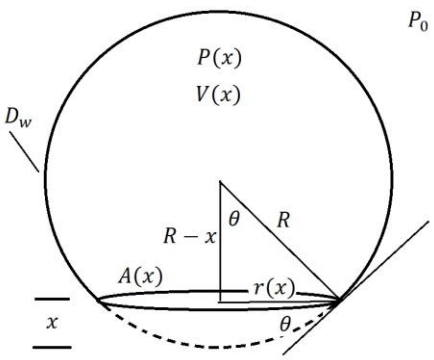

Figure 2. Illustration of the basketball upon impact, redrawn from published research (Georgallas, & et al)

As shown in Figure 2, upon impact, the ball will compress by an amount x, resulting in a circular region of radius $r(x)$ and area $A(x)$ in contact with ground; meanwhile the internal pressure and volume will change to $P(x)$ and $V(x)$, respectively. For an isothermal compression, supposing the ball's volume is V_i at pressure P_i , the following holds true, according to ideal gas law:

$$P(x)V(x) = P_i V_i$$

Appendix-D contains reasonings and formulas on how to calculate such values.

The gauge pressure P_G measures the difference between the internal pressure P_i and the external pressure P_0 of the basketball. So, we have

$$P_G = P_i - P_0 \qquad \ldots (6)$$

P_G can be measured, so can the drop height H_i and the first rebound height H_f. To calculate e, I need to figure out how to get values of v_i and v_f.

2.2.1 Calculation of v_i with H_i during descent phase (details in Appendix-A)

During the descent phase, the balls' acceleration is decided by the net force on it.

$$ma = mg - F_b - F_d \qquad \ldots (7a)$$

Along with equations (1) and (2), we have

$$m\frac{dv}{dt} = mg - \rho v g - \frac{1}{2}C_d \rho A v^2 \text{ , } \frac{dv}{dt} = g - \frac{4\rho\pi R^3 g}{3m} - \frac{C_d \rho \pi R^2}{2m} v^2$$

Let's set $A = g - \frac{4\rho\pi R^3 g}{3m}, B = \frac{C_d \rho \pi R^2}{2m}$, and we get

$$v = \sqrt{A/B}\,\frac{e^{2\sqrt{AB}*t} - 1}{e^{2\sqrt{AB}*t} + 1} \qquad \ldots (7c)$$

Given v_i, we can always calculate the distance of descent $y(v_i)$,

$$y(v_i) = \frac{1}{B}\ln\frac{\frac{\sqrt{A/B}+v_i}{\sqrt{A/B}-v_i}+1}{2} - \frac{1}{2B}\ln\left(\frac{\sqrt{A/B}+v_i}{\sqrt{A/B}-v_i}\right) \qquad \ldots (7e)$$

We also know that $y(v_i) = H_i$; Hence, given $y = H_i,\ v_i$ can be reversely looked up via Desmos graphing of (7e).

2.2.2 Calculation of v_f with H_f during ascent phase (details in Appendix-B)

During the ascent phase, the ball's net force decides its acceleration,

$$ma = -mg + F_b - F_d \qquad \dots (8a)$$

Similarly, we can arrive at how to calculate the distance travelled $y(v_f)$:

$$y(v_f) = \frac{1}{B}\ln\left[\sec\left(\tan^{-1}\frac{v_f}{\sqrt{A/B}}\right)\right] \qquad \dots (8e)$$

Hence, given $y = H_f$, v_f can be reversely looked up via Desmos graphing of (8e).

2.2.3 Calculation of F_p and F_w (see Appendix-D for math details)

With $R \gg x$, when $P_G > 0$,

$$F_p \approx 2\,\pi R P_G x \qquad \dots (9)$$

When $x \ll R$, with a shear modulus G, D_w as thickness of the ball's wall, we get

$$F_w \approx 4\pi G D_w x \qquad \dots (10)$$

2.2.4 A Theory on F_D

During the impact phase, the ball is subject to dissipative forces such as the air drag force and ball-ground friction forces that will dissipate its K.E. into sound energy, heat energy and other forms of energy. F_D denotes such forces collectively. The magnitude of F_D increases with the ball's velocity, which is a function of displacement x. When x is max, the ball's velocity is at its max (v_i), F_D is max; when x approaches 0, F_D quickly reduces with the ball's velocity; when $x = 0$, the ball's velocity is 0, F_D becomes 0. F_D behaves like a "leaky spring", resisting like a spring while "leaking" the K.E. to heat, sound and other energy forms. Here, I propose a "leaky spring constant" K_D to characterize it,

$$F_D = F_d + frictions = -K_D x \qquad \dots (11a)$$

Accordingly, we can have the energy loss

$$\Delta\varepsilon = \frac{1}{2}K_D x^2 \qquad \dots (11b)$$

2.2.5 Energy Model at the Impact

During the impact, according to equation (3), the restoring force acting on the ball is

$$F_r = F_p + F_w = (2\,\pi R P_G + 4\pi G D_w)x = kx \qquad \dots (12a)$$

with an associated E.P.E of $\frac{1}{2}kx^2$, as it is a Hooke's law force.

During the compression stage and decompression stages in Figure 1, we have

$$K_i = \frac{1}{2}mv_i^2 = \frac{1}{2}kx^2 - mgx + \frac{1}{2}K_D x^2 + F_b x \qquad \dots (12b)$$

$$K_f = \frac{1}{2}mv_f^2 = \frac{1}{2}kx^2 - mgx - \frac{1}{2}K_D x^2 + F_b x \qquad \dots (12c)$$

If we divide their sum $(K_i + K_f)$ by their difference $(K_i - K_f)$, with $K_f = e^2 K_i$, we get

$$\frac{1+e^2}{1-e^2} = \frac{kx^2-2mgx+2F_bx}{K_Dx^2}$$

For convenience, I will use $f(e)$ to denote $\frac{1+e^2}{1-e^2}$ in subsequent discussions.

Because the compressed ball's stored E.P.E is much greater than the change in gravity potential as well as the buoyant work, $kx^2 \gg -2mgx + 2F_bx$, we can simplify as:

$$f(e) = \frac{1+e^2}{1-e^2} \approx \frac{kx^2}{K_Dx^2} = \frac{k}{k_D}$$

Substitute with equation (12a),

$$f(e) = \frac{k}{k_D} = \frac{2\ \pi RP_G + 4\pi GD_W}{k_D}$$

Hence, we can get the following linear model:

$$\boldsymbol{f(e) = WP_G + Z} \qquad \text{… (12d)}$$

where

$$W = \frac{2\ \pi R}{k_D}, Z = \frac{4\pi GD_W}{k_D}$$

We can also solve the equation to get:

$$\boldsymbol{e} = \frac{\sqrt{\mathbf{WP_G+Z-1}}}{\sqrt{\mathbf{WP_G+Z+1}}} \qquad \text{… (12e)}$$

In the following experiment, e is determined for different values of P_G. The values of W and Z can be obtained by fitting experimental data to equation (12d).

3. Experiment

3.1 Apparatus:

- Standard Wilson Indoor Basketball (Spalding, 24 cm radius)
- Contractable Meter Stick: 3m range; with minimum scale of 1mm
- Pump and Needle
- Electronic Scale: 3kg range; with smallest digit to 0.1 gram, hence error: $\pm$0.1gram
- Digital Pressure Gauge Barometer: 20 psi (pound per square inches) range; with smallest digit to 0.1 psi, hence error $\pm$0.1 psi
- Digital Infrared Thermometer: 50°C range, w/ smallest digit to 0.1°C, hence error $\pm$0.1 °C
- IPhone 11 Pro Max: for recording the dropping and bouncing of the ball
- Tape and Scissors
- Ladder (2m)
- MacBook with LoggerPro Software to analyze the recorded video clips

3.2 Risk Assessment

- When pumping gas into the basketball, make sure its pressure is less than 20 psi as that is the range of my barometer. The ball could explode if over-pumped.
- Use caution when climbing the ladder

3.3 Experimental Variables:

- Independent variable: air pressure of the basketball (psi)
- Dependent variable: Coefficient of Restitution (e)
- Controlled variables:
  - Height of the ball release: the same height of 1.5 m was used for each drop; the ball was released by a wall marked with a sticker of 1.5m pre-determined by the meter stick
  - Temperature (^{o}C): the digital thermometer was used to monitor the ball's temperature and room temperature to make sure they stayed constant
  - Ground floor of impact: the same floor spot (about 1 m^2) was used
  - Volume of the ball: the ball's diameter was measured for each trial to make sure it stayed unchanged
  - Same basketball: the same ball was used for all experiments
  - Ball's lateral movement: the ball was held still in all directions before the drop; if there were movements, another trial was added
  - Camera angle and distance: a phone holder was placed at the same spot and same angle

3.4 Procedure:

1. Used meter to mark 1.5m, 1.0 m on an adjacent wall, for reference purpose
2. Used pump and barometer to inflat, measure and adjust the ball's pressure to a desired value; measured the ball's mass for 3 times and take the average value
3. Measured the room temperature; measured the ball's temperature by aiming the digital infrared thermometer at it within 5 cm distance for about 5 seconds
4. Placed the iPhone at about 3 meters away, same spot, same angle, started recording
5. Dropped the ball at height of 1.5 m
6. Waited for the ball to bounce to maximum height, then stopped the recording.
7. Repeated 3-6 for a total of 6 trials
8. Did steps 2-7 for different gauge pressure (10.0, 8.0, 6.0, 4.0, 2.0, 0.0; 19.9, 18.2, 16.0, 14.0, 12.1)

4. Raw Data

Throughout the experiment, the room temperature and the ball's temperature stayed constantly at 30.0 ^{o}C. The ball's mass varies slightly across different pressure. The ball's radius was also measured and recorded: it stayed constant at 24.0 cm throughout. In each case, three measurements were taken, and the average was calculated. The collected videos were uploaded to the MacPro book and subjected to LoggerPro analysis. The release height and first rebound's height for each trial were tabulated. (LoggerPro by default counts in the margin from the bottom edge of the frame to track point in each video frame, so this margin had to be deducted first). Table 1 below includes the raw data from LoggerPro.

Pressure (psi)	19.9						**mass (g):** 614.7	
Trial #	1	2	3	4	5	6	**average**	**stdev**
H_i	1.533	1.537	1.548	1.566	1.557	1.548	1.548	0.012
H_f	1.090	1.099	1.110	1.121	1.101	1.105	1.104	0.011
Pressure (psi)	18.2						**mass (g):** 614.2	
Trial #	1	2	3	4	5	6	**average**	**stdev**
H_i	1.523	1.527	1.551	1.543	1.541	1.537	1.537	0.010
H_f	1.083	1.091	1.117	1.093	1.090	1.084	1.093	0.012
Pressure (psi)	16.0						**mass (g):** 613.5	
Trial #	1	2	3	4	5	6	**average**	**stdev**
H_i	1.509	1.533	1.530	1.516	1.527	1.545	1.527	0.013
H_f	1.069	1.075	1.071	1.061	1.078	1.086	1.073	0.009
Pressure (psi)	14.0						**mass (g):** 612.8	
Trial #	1	2	3	4	5	6	**average**	**stdev**
H_i	1.523	1.511	1.503	1.543	1.510	1.547	1.523	0.018
H_f	1.056	1.050	1.051	1.062	1.054	1.063	1.056	0.005
Pressure (psi)	12.1						**mass (g):** 612.0	
Trial #	1	2	3	4	5	6	**average**	**stdev**
H_i	1.529	1.508	1.524	1.540	1.541	1.498	1.523	0.017
H_f	1.040	1.039	1.044	1.051	1.052	1.019	1.041	0.012
Pressure (psi)	10.0						**mass (g):** 611.0	
Trial #	1	2	3	4	5	6	**average**	**stdev**
H_i	1.500	1.538	1.544	1.546	1.522	1.548	1.533	0.019
H_f	1.004	1.005	1.023	1.033	1.032	1.038	1.023	0.015
Pressure (psi)	8.0						**mass (g):** 610.6	
Trial #	1	2	3	4	5	6	**average**	**stdev**
H_i	1.511	1.517	1.542	1.496	1.514	1.479	1.510	0.021
H_f	0.967	0.958	0.990	0.960	0.958	0.971	0.967	0.012
Pressure (psi)	6.0						**mass (g):** 610.0	
Trial #	1	2	3	4	5	6	**average**	**stdev**
H_i	1.510	1.490	1.521	1.490	1.524	1.496	1.505	0.015
H_f	0.887	0.872	0.929	0.893	0.929	0.910	0.903	0.023
Pressure (psi)	4.0						**mass (g):** 608.6	
Trial #	1	2	3	4	5	6	**average**	**stdev**
H_i	1.523	1.520	1.524	1.495	1.523	1.506	1.515	0.012
H_f	0.837	0.832	0.837	0.777	0.773	0.787	0.807	0.031
Pressure (psi)	2.0						**mass (g):** 607.3	
Trial #	1	2	3	4	5	6	**average**	**stdev**
H_i	1.523	1.503	1.512	1.501	1.551	1.517	1.518	0.018
H_f	0.604	0.599	0.543	0.527	0.601	0.521	0.566	0.040
Pressure (psi)	0.0						**mass (g):** 606.3	
Trial #	1	2	3	4	5	6	**average**	**stdev**
H_i	1.570	1.534	1.509	1.486	1.496	1.469	1.511	0.036
H_f	0.401	0.397	0.320	0.281	0.279	0.272	0.325	0.060

Table 1. raw data of the heights for different pressures, with average and standard deviations

5. Data Processing

Table 2 below includes the following data columns:

- ✧ mass(g): each entry is an average of 3 measurements, with a standard deviation of 0, as in each case all three measurements gave the same reading
- ✧ Columns A and B: these were calculated using the discussed formulas and values: $A = g - \frac{4\rho\pi R^3 g}{3m}, B = \frac{C_d \rho \pi R^2}{2m}, with\, \rho = 1.163\, \frac{kg}{m^3}, C_d = 0.54, \pi = 3.14, g = 9.8\, m/s^2$
- ✧ Columns H_i and H_f contain the corresponding average values with standard deviations for different pressure P_G from Table 1
- ✧ Column P_G contains the measured pressure values from Table 1, with uncertainty of $\Delta P_G = \pm 0.1$ psi
- ✧ Column v_i is looked up in the Desmos online tool after plotting the function in equation (7e) with the set of A and B values for each mass value, its uncertainty percentage ($\%\Delta v_i$) values were calculated using calculus formula: $\frac{\Delta y}{\Delta v_i} = y'$. For example, in the case of the first row in Table 2, we have $A = 10.815, B = 0.0936, y = H_i = 1.548 \pm 0.012$, $\Delta y = \pm 0.012$, through Desmos we look up $v_i = 4.856$, we now take the derivative of equation (7e) on both sides,

$$\because \frac{\Delta y}{\Delta v_i} = y' = \left(\frac{1}{B}\ln\frac{\frac{\sqrt{A/B}+v_i}{\sqrt{A/B}-v_i}+1}{2} - \frac{1}{2B}\ln\left(\frac{\sqrt{A/B}+v_i}{\sqrt{A/B}-v_i}\right)\right)'$$

$$= \frac{57.421\sqrt{10.75-v_i}-57.421\sqrt{10.75+v_i}}{\left(\sqrt{10.750-v_i}+\sqrt{10.750+v_i}\right)\left(v_i^2-115.563\right)} = 0.750$$

$$\therefore\ \Delta v_i = \frac{\Delta y}{0.75} = \frac{0.012}{0.75} = 0.016,\ \%\Delta v_\mathrm{i} = \frac{\Delta v_i}{v_i} = \frac{0.016}{4.856} \times 100\% = 0.33\%$$

- ✧ Column v_f is looked up in the Desmos online tool after plotting the function in equation (8e) with the set of A and B values for each mass value, its uncertainty percentage ($\%\Delta v_f$) values were calculated similarly
- ✧ Column e is calculated with $e = \frac{|v_f|}{|v_i|}$, with its uncertainty $\%\Delta e$ being the sum of the uncertainty % values of v_f and v_i: $\%\Delta e = \%\Delta v_\mathrm{i} + \%\Delta v_f$
- ✧ Column $f(e)$ is calculated using equation $f(e) = \frac{1+e^2}{1-e^2}$, with its uncertainty values decided similarly: $\frac{\Delta f}{\Delta e} = f' = \frac{4e}{(1-e^2)^2}$. For example, in the first row below,

$$e = 0.955,\ \Delta e = \pm 0.955 * 0.88\% = \pm 0.008,\quad \therefore \Delta f = \frac{4e}{(1-e^2)^2}\Delta e = \pm 3.94$$

mass (g)	A	B	H_i (m)	H_f (m)	P_G (psi)	v_i (m/s)	v_f (m/s)	e	f (e)
614.7	10.815	0.0936	1.548±0.012	1.104±0.011	19.9±0.1	4.856±0.33%	4.638±0.55%	0.955±0.88%	21.787±3.94
614.2	10.806	0.0935	1.537±0.010	1.093±0.012	18.2±0.1	4.843±0.28%	4.611±0.61%	0.952±0.89%	20.387±3.47
613.5	10.794	0.0934	1.527±0.013	1.073±0.009	16.0±0.1	4.823±0.37%	4.561±0.46%	0.946±0.83%	17.922±2.74
612.8	10.782	0.0933	1.523±0.018	1.056±0.005	14.0±0.1	4.816±0.51%	4.518±0.36%	0.938±0.87%	15.677±2.13
612.0	10.768	0.0932	1.523±0.017	1.041±0.012	12.1±0.1	4.814±0.48%	4.480±0.63%	0.931+1.11%	13.931±2.10
611.0	10.750	0.0930	1.533±0.019	1.023±0.015	10.0±0.1	4.823±0.54%	4.434±0.61%	0.919±1.15%	11.919±1.67
610.6	10.743	0.0929	1.510±0.021	0.967±0.012	8.0±0.1	4.786±0.60%	4.293±1.00%	0.897±1.68%	9.235±1.48
610.0	10.732	0.0928	1.505±0.015	0.903±0.023	6.0±0.1	4.782±0.43%	4.139±2.18%	0.866±2.61%	6.973±1.27
608.6	10.708	0.0926	1.515±0.012	0.807±0.031	4.0±0.1	4.791±0.34%	3.889±4.66%	0.812±5.10%	4.863±1.15
607.3	10.685	0.0924	1.518±0.018	0.566±0.040	2.0±0.1	4.790±0.51%	3.216±17.99%	0.671±18.5%	2.642±1.10
606.3	10.667	0.0923	1.511±0.036	0.325±0.060	0.0±0.1	4.777±1.03%	2.408±47.17%	0.504±48.2%	1.681±0.88

Table 2. Processed data.

6. Analysis and Discussion

To test our model in equation (12d), the following graph of $f(e)\ vs\ P_G$ was plotted. Note that P_G unit is in (psi), with 1 psi = 6894.76 Pa.

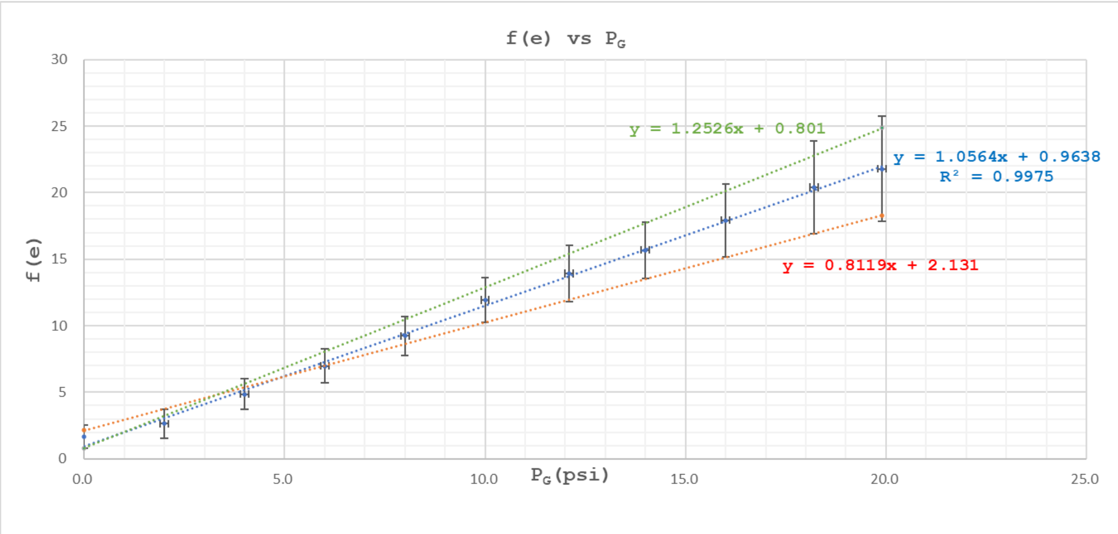

Figure 3. Plot of $f(e)\ vs\ P_G$, with 1 psi = 6894.76 Pa

Between the two worst fitting lines (red and green), the best fitting line (blue) yielded an equation $y = 1.0564x + 0.964$, with $R^2 = 0.9975$, indicating a strong linear correlation between $f(e)$ and P_G. This verifies the linear model proposed in equation (12d). From the fitting, $W = 1.0564\ (psi)^{-1}, Z = 0.964$. Accordingly, we can calculate the values of k_D, G (with $D_W = 3.1mm$ per the manufacturer) using the following:

$$W = \frac{2\,\pi R}{k_D}, Z = \frac{4\pi G D_W}{k_D}$$

We get:

$$k_D = \frac{2\,\pi R}{W} = 9.837\ e + 4 \quad \text{(N/m)}$$

$$G = 4.871e + 5\ \text{(Pa)}$$

With a "leaky spring" constant of 9.837×10^4, a tiny compression, say, 3 mm of the ball, would lead to dissipation of $1/2 \times 9.837 \times 0.003^2 \times 10^4$ =0.443 (J) of energy, about 6% of the impacted ball's K.E. (i.e., $v_i = 4.823\frac{m}{s}, K_i = \frac{1}{2}mv_i^2 = 7.106\,J$). The calculated G value is consistent with material science researcher's reported range of between 0.2 and 0.5 MP_a (Makowski, A), further verifying the model's robustness.

Therefore, the analyses have been able to verify the correctness of the model in equation (12d) with experimental data and industry data, with good confidence.

With values of W, Z determined, according to equation (12e), we get the theoretical equation between e and P_G:

$$e = \frac{\sqrt{WP_G+Z-1}}{\sqrt{WP_G+Z+1}} = \frac{\sqrt{1.0564 * P_G+0.964-1}}{\sqrt{1.0564 * P_G+0.964+1}}$$

Next, I plotted the Coefficient of Restitution e vs the Gauge Pressure P_G, with both theoretical and experimental data.

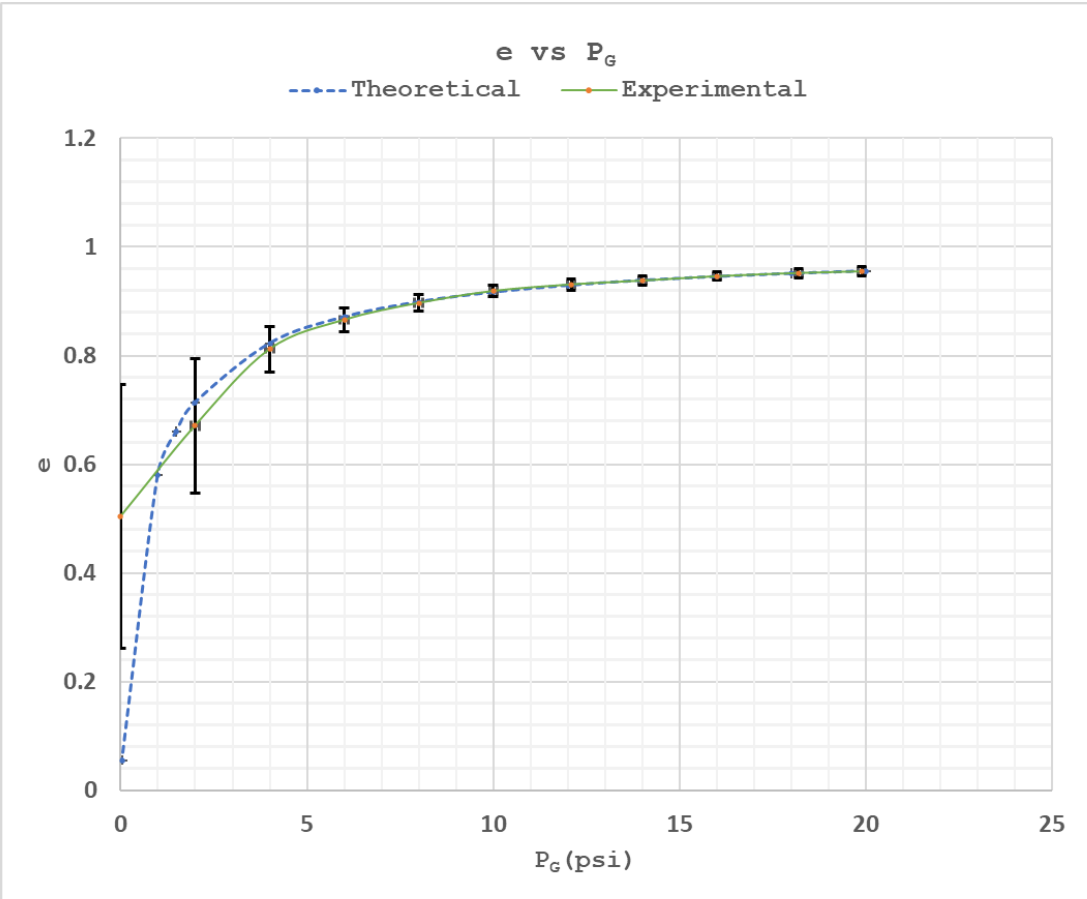

Figure 4. Plot of e vs P_G, with theoretical and experimental data. Uncertainties of the experimental data are also plotted.

As we can see, when P_G is greater than 5 psi, the experimental data points match the theoretical curve very well. Generally, the Coefficient of Restitution e goes higher when P_G increases, and it has value 0f 0.919 at 10 psi and gradually grows greater as gauge pressure increases, reaching 0.955 at 19.9 psi. Due to range limit of the barometer and also safety concern that the ball could explode if too inflated, the max gauge pressure I tried was 19.9 psi. Theoretically, the line gradually approaches to 1 as the pressure further increases.

At gauge pressure approaches 0, the theoretical curve nearly goes through (0,0), while the experimental curve's trendline substantially deviates from the former, with an e value around 0.5, with $\pm 48.2\%$ uncertainty. This make sense because 1) the ball's solid materials do have a certain degree of bounciness of their own, as evidenced by the ball's bouncing up at gauge pressure 0; 2) one of the conditions that I was able to approximate in equation (9) is that the gauge pressure is not approaching 0. When the gauge pressure is 0 or approaching 0, the other items of the polynomial in (see Appendix-D)

$$F_p = 2\,\pi R^2 P_G \left[\left(\frac{x}{R}\right) - \frac{1}{2}\left(\frac{x}{R}\right)^2 + \frac{3}{4}\left(1 + \frac{P_0}{P_G}\right)\left(\frac{x}{R}\right)^3 + \cdots\right]$$

becomes significant, which would cause the fitting line to be non-linear, and thus not go through origin (0,0).

If we look at the magnitude of air drag and buoyance, clearly their sizes are significant compared to the gravity (Appendix-C). F_b is consistently larger than 10% of the gravity, and F_d could be ~28% of the gravity. The collectively could add up to ~40% of the gravity size. During one round of descent-impact-ascent (dropped from 1.533m), the ball incurs 67.742% of its total energy loss during the descent phase, caused by air drag and buoyance. By considering such forces in my model, we can get more realistic results and thus are able to assess the relation between P_G and Coefficient of Restitution (e).

7. Conclusion

In summary, to answer the research question **"How does a basketball's gauged internal air pressure (P_G) affect its Coefficient of Restitution (e)?"**, a combined approach involving theoretical modeling and a systematic experiment was used to investigate the relationship between the gauge pressure of the basketball and its Coefficient of Restitution (e).

A hypothetical linear model was proposed: $f(e) = \frac{1+e^2}{1-e^2} = WP_G + Z$. Furthermore, their theoretical relation was established: $e = \frac{\sqrt{\mathrm{WP_G+Z-1}}}{\sqrt{\mathrm{WP_G+Z+1}}}$.

For the experiment, gauge pressure was the independent variable, and Coefficient of Restitution (e) was the dependent variable. The experimental data validated the model well by matching the line of the best fit, $f(e) = \frac{1+e^2}{1-e^2} = WP_G + Z = 1.0564P_G + 0.964$, with a very high value of $R^2(0.9975)$. This demonstrates a strong linear relationship between $f(e) = \frac{1+e^2}{1-e^2}$ and the gauge pressure. The theoretical relation $e = \frac{\sqrt{1.0564 * P_G + 0.964 - 1}}{\sqrt{1.0564 * P_G + 0.964 + 1}}$ fits well with experimental data when P_G is greater than 5 psi, but does not work well when P_G approaches 0, where some of the assumptions no longer hold true.

Further analysis suggested that my model is more comprehensive by modelling the energy changes of a bouncing basketball with consideration of air drag, air buoyance, and dissipative forces, thus is better suited for evaluation of the ball's bounciness.

8. Evaluation

Though the empirical data fits the model very well, the match is not perfect because there were random errors as well as systematic errors in the experiment.

In a frame of the ball's drop and rebound video, the bottom margin typically contains the floor's image. LoggerPro undesirably counts this margin to be part of the ball's measured height, thus manual marking and deduction of this margin from the measured height values are necessary, which could introduce errors at medium-low level of significance. Such errors could include both systematic errors and random human marking errors. To mitigate the human errors, the average of 6 trials' height readings were used in each case. To mitigate the systematic errors, usage of more direct measurement methods with better devices can help, such as a laser distance measurer to measure the ball's release and bounce heights.

The model assumes the basketball is a sphere with a single layer of leather. In reality, the ball has an internal layer. When the pressure is high, this is not an issue as the internal layer pushes tightly under the exterior ball skin. However, when the pressure is reduced to near the same level as the external air, the internal skin could deflate or deform, thus separating from the skin layer. This could have implications on our model's assumptions, for example, how to calculate the ball's internal pressure and volume under such circumstances. One possible extension of the experiment could be to use a sphere-shaped kickball to perform the same set of experiments to study more on the relation of P_G and e, especially when P_G is near 0.

Different camera locations and angles could affect positions of the ball in the video frame, which could introduce errors at low level of significance. To mitigate this, the camera was placed at a fixed location with the same angle, and the ball was dropped at the same location throughout the experiment.

Manual releases of the ball were subject to unwanted lateral or vertical movements, moving the ball closer or further from the camera, causing LoggerPro to misjudge the heights. This posed as a low-level of significance. Efforts were made to hold the ball still before each release and repeat if there were unwanted movements during the experiment. A future way to improve would be to use a machine/device to horizontally clamp the ball and hold it, then unclamp and release it with even horizontal forces from all directions.

The design can be enhanced by using a high precision laser speedometer to directly measure the ball's velocities, instead of relying on theoretical deductions to calculate the ball's velocities. My limited attempts to measure velocities with motion sensor and video based LoggerPro analysis were not very productive, though the measured v_f, v_i values seemed to fall within the right range of the calculated ones. These devices' sampling rates and precisions not fine enough for the drop and bounce scenario.

[Word count: 3,441]

References

Abel, John. “Do Basketballs That Are Fully Inflated Bounce Better than Flatter Ones?” UCSB Science Line, scienceline.ucsb.edu/getkey.php?key=3939.

Cross, Rod. “Measuring the Drag Force on a Falling Ball”. March 2014. The Physics Teacher. Vol. 52

Desmos graphing tool: https://www.desmos.com/calculator

“Effect of Pressure in Ball Bounce Height”. Dec 5th, 2020
https://www.seminarsonly.com/Engineering-Projects/Physics/rebound-height.php

Georgallas, Alex and Landry, Geatan. “The Coefficient Of Restitution of Pressurized Balls: A Mechanistic Model”. Can. J. Physics. 94: 42-26 (2-16)

Hubbard, M. and Stronge, “Bounce of hollow balls on flat surfaces”. W.J. 4, 49 (2001).doi:10.1046/j.1460-2687.2001.00073.x.

Makowski, A. Determination of Material Properties of Basketball. Finite Element Modeling of the Basketball Bounce Height. Preprints 2021, 2021010241 (doi: 10.20944/preprints202101.0241.v1).
https://www.preprints.org/manuscript/202101.0241/v1

Nagurka, Mark. “Aerodynamic Effects in a Dropped Ping-Pong Ball Experiment”. Int. J. Engng Ed. Vol. 19. No. 4, pp. 623-630, 2003

Okubo, Kiroki and Hubbard, Mont. “Identification of basketball parameters for a simulation model”. 31 January 2010, 8th Conference of the International Sports Engineering Association (ISEA). ScienctDirect, Procedia Engineering 2 (2010) 3281-3286

Tamara. “Q & A: Why Do Basketballs Bounce?” Q & A: Why Do Basketballs Bounce? | Department of Physics | University of Illinois at Urbana-Champaign, van.physics.illinois.edu/qa/listing.php?id=103.

The Drag Coefficient
https://www.grc.nasa.gov/www/k-12/airplane/dragco.html

WolframAlpha Computational Intelligence site: https://www.wolframalpha.com/

Appendix-A: Descent phase mathematic deductions of formulas

During the descent phase,

$$ma = mg - F_b - F_d \qquad \ldots (7a)$$

Along with equations (1) and (2), we have

$$m\frac{dv}{dt} = mg - F_b - F_d = mg - \rho V g - \frac{1}{2}C_d \rho A v^2$$

$$m\frac{dv}{dt} = mg - F_b - F_d = mg - \rho V g - \frac{1}{2}C_d \rho A v^2$$

$$\frac{dv}{dt} = g - \frac{4\rho\pi R^3 g}{3m} - \frac{C_d \rho \pi R^2}{2m} v^2$$

with $t0, v0$ denoting the time and speed at time instant 0,

$$t - t0 = \int_{v0}^{v} \frac{1}{g - \frac{4\rho\pi R^3 g}{3m} - \frac{C_d \rho\pi R^2}{2m} v^2} dv$$

if we set $A = g - \frac{4\rho\pi R^3 g}{3m}, B = \frac{C_d \rho \pi R^2}{2m}$, we get:

$$t - t0 = \int_{v0}^{v} \frac{dv}{A - Bv^2} = \left[\frac{1}{A}\frac{\sqrt{A/B}}{2} \ln\left(\frac{\sqrt{A/B}+v}{\sqrt{A/B}-v}\right)\right]_{v0}^{v}$$

Given $t0, v0$ both have values of 0, we have

$$t = \frac{1}{2\sqrt{AB}} \ln\left(\frac{\sqrt{A/B}+v}{\sqrt{A/B}-v}\right) \qquad \ldots (7b)$$

Conversely, at a given time t we can calculate the velocity of the falling ball:

$$v = \sqrt{A/B}\,\frac{e^{2\sqrt{AB}*t}-1}{e^{2\sqrt{AB}*t}+1} \qquad \ldots (7c)$$

To compute the distance the ball has travelled at time t,

$$y(t) = \int_0^t v\, dt = \int_0^t \sqrt{A/B}\,\frac{e^{2\sqrt{AB}*t}-1}{e^{2\sqrt{AB}*t}+1} dt$$

$$y(t) = \frac{1}{B}\ln\left(\frac{e^{2\sqrt{AB}*t}+1}{2}\right) - \sqrt{A/B} * t \qquad \ldots (7d)$$

By substituting equation (7b) into equation (7d), we get

$$y(v) = \frac{1}{B}\ln\frac{\frac{\sqrt{A/B}+v}{\sqrt{A/B}-v}+1}{2} - \frac{1}{2B}\ln\left(\frac{\sqrt{A/B}+v}{\sqrt{A/B}-v}\right)$$

When $v = v_i, y(v_i) = H_i$:

$$y(v_i) = \frac{1}{B}\ln\frac{\frac{\sqrt{A/B}+v_i}{\sqrt{A/B}-v_i}+1}{2} - \frac{1}{2B}\ln\left(\frac{\sqrt{A/B}+v_i}{\sqrt{A/B}-v_i}\right) \qquad \ldots (7e)$$

Therefore, given a descent distance y, the ball's velocity can be looked up using Desmos online calculator. For example, when the ball's pressure is 10 psi, its mass is 0.611 kg, we have: $A = g - \frac{4\rho\pi R^3 g}{3m} \approx 8.720$, $B = \frac{C_d \rho\pi R^2}{2m} \approx 0.093$

$$\therefore y(v) = 10.750 * \ln\frac{\frac{9.683+v}{9.683-v}+1}{2} - 5.375\ln\left(\frac{9.683+v}{9.683-v}\right) \qquad \ldots (7f)$$

If the ball's drop height ($y\ or\ H_i$) is 1.533 m, its velocity v_i before hitting the ground will be 4.823 m/s.

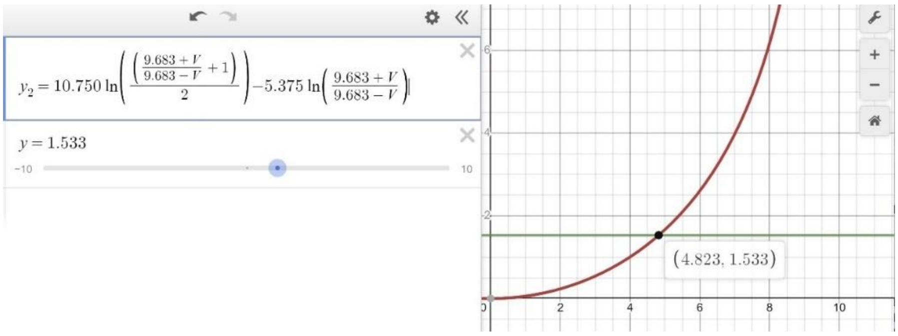

Figure A-1. Desmos plot of descent phase $y(v)$

Appendix-B: Ascent phase mathematic deductions of formulas

During the ascent phase,

$$ma = -mg + F_b - F_d \qquad \dots (8a)$$

Along with equations (1) and (2), we have

$$m\frac{dv}{dt} = -mg + F_b - F_d = -\,mg + \,\rho V g - \,\frac{1}{2}C_d \rho A v^2$$

$$\frac{dv}{dt} = -g + \frac{4\rho\pi R^3 g}{3m} - \frac{C_d \rho\pi R^2}{2m}v^2$$

with $t0, v_f$ denoting the time and speed at time instant 0 of the ascent phase,

$$t - t0 = \int_{v_f}^{v} \frac{1}{-g+\frac{4\rho\pi R^3 g}{3m}-\frac{C_d\rho\pi R^2}{2m}v^2}dv$$

Again, we set $A = g - \frac{4\rho\pi R^3 g}{3m}, B = \frac{C_d\rho\pi R^2}{2m},$ we get:

$$t - t0 = -\int_{v_f}^{v}\frac{dv}{A+Bv^2} = \left[\frac{1}{\sqrt{AB}}\tan^{-1}\frac{v}{\sqrt{A/B}}\right]_{v_f}^{v}$$

$$\therefore t = \frac{1}{\sqrt{AB}}(\tan^{-1}\frac{v_f}{\sqrt{A/B}} - \tan^{-1}\frac{v}{\sqrt{A/B}}) \qquad \dots (8b)$$

Hence with equation (8b), at given time t of the ball's ascent phase, we can calculate its velocity, and vice versa:

$$v = \sqrt{A/B}\tan(\tan^{-1}\frac{v_f}{\sqrt{A/B}} - \sqrt{AB} * t) \qquad \dots (8c)$$

To calculate the distance the ball has ascended at time t,

$$y(t) = \int_0^t v\,dt = \int_0^t \sqrt{A/B}\tan(\tan^{-1}\frac{v_f}{\sqrt{A/B}} - \sqrt{AB} * t)dt$$

$$y(t) = \frac{1}{B}\left[\ln\left(\cos(\tan^{-1}\frac{v_f}{\sqrt{A/B}} - \sqrt{AB} * t)\right)\right]_0^t$$

$$= \frac{1}{B}\ln\left(\frac{\cos(\tan^{-1}\frac{v_f}{\sqrt{A/B}}-\sqrt{AB}*t)}{\cos(\tan^{-1}\frac{v_f}{\sqrt{A/B}})}\right) \qquad \dots (8d)$$

Thus, we can calculate the relation between $y\ and\ v,$ if we substitute t with equation (8b),

$$y(v) = \frac{1}{B}\ln\left(\frac{\cos(\tan^{-1}\frac{v}{\sqrt{A/B}})}{\cos(\tan^{-1}\frac{v_f}{\sqrt{A/B}})}\right)$$

When $v = 0$, aka, the ball has peaked during the ascent, we have the relation function between $y\ and\ v_f$:

$$y(v_f) = \frac{1}{B}\ln\left(\frac{\cos(\tan^{-1}\frac{0}{\sqrt{A/B}})}{\cos(\tan^{-1}\frac{v_f}{\sqrt{A/B}})}\right) = \frac{1}{B}\ln\left[\sec(\tan^{-1}\frac{v_f}{\sqrt{A/B}})\right] \qquad \dots (8e)$$

Suppose the ball's mass is 0.611 kg, after substitution with the values of $A = 8.720, B = 0.093$ calculated previously, we get

$$y(v0) = 10.750 * \ln\left[\sec(\ln(\tan^{-1}\frac{v_f}{9.683}))\right] \qquad \dots (8f)$$

With equation (8f) plotted in Desmos, we can easily find out a ball's v_f, based on its rebound height H_f. For example, for a ball of .611 kg that has a rebound height H_f of 1.023 m, its velocity v_f right after decompression is 4.434 m/s.

Figure A-2. Desmos plot of ascent phase $y(v_f)$

Appendix-C: Are F_d and F_b negligible?

I decided to compute the forces first. According to equation (1), given the density of air is 1.29 kg/m^3 at 0°C, its density at 30°C (the temperature at which my experiments were performed) can be calculated using equation (2) and the density formula:

$$\rho = 1.29 * \frac{273+0}{273+30} = 1.163\ kg/m^3$$

The radius of the ball is 24 cm = 0.24m, therefore I can calculate the following:

$$F_b = \rho V g = 1.163 * \frac{4}{3}\pi R^3 * 9.8 = 0.660\ N$$

$$F_d = \frac{1}{2} C_d \rho A v^2 = \frac{1}{2} C_d \rho \pi R^2 v^2 == 0.0568 v^2$$

If I neglect the air drag and air buoyance when the ball drops from a height of 1.5m, its v_i will reach

$$v_i = \sqrt{2gH_i} = \sqrt{2 * 9.8 * 1.5} = 5.422 \text{ m/s}$$

Hence,

$$F_d = 0.0568 v^2 \approx 1.670 \text{ N}$$

The ball's gravity, given its mass is measured as 0.611 kg at 10 psi pressure, with gravity constant $g = 9.800\ m/s^2$, $mg = 0.611 * 9.8 = 5.988\ N$

If we look at the magnitude of air drag and buoyance, clearly their sizes are significant compared to the gravity. F_b is consistently larger than 10% of the gravity, and F_d could be ~28% of the gravity. The collectively could add up to ~40% of the gravity size. In other words, **they are NOT negligible**!

If my model did not account for the air drag and air buoyance, it would substantially undervalue the ball's bounciness. For example, with a 10.0 psi and a drop height of 1.533 m, the ball's first rebound height will be 1.023m. Below is an analysis on the energy loss. (G.P.E-i is initial G.P.E at drop height, G.P.E-i is the G.P.E at rebound height.)

	G.P.E-i (J)	v_i (m/s)	$\Delta\varepsilon$ (J) (descent)	v_o (m/s)	$\Delta\varepsilon$ (J) (impact)	G.P.E-o (J)	$\Delta\varepsilon$ (J) (ascent)	e
My model	9.164	4.823	**-2.058**	4.434	**-1.1**	6.126	**+0.12**	0.919
$\frac{\Delta\varepsilon}{\sum\Delta\varepsilon}$	-	-	**67.742%**	-	**32.916%**	-	**3.950%**	-

Table. energy loss analysis using my model. with $H_i = 1.533m, H_o = 1.023m, m = 0.611kg$

During one round of descent-impact-ascent (dropped from 1.533m), if we consider the influences of air drag and air buoyance, the ball incurs 67.742% of its total energy loss during the descent phase, 32.916% during the impact phase, and winds up gaining 3.950% during the ascent phase as buoyance's work beats the air drag at this speed range. If my model ignored the air drag and air buoyance and attributed 100% of energy loss to the impact phase, it would have resulted in a substantially lower Coefficient of Restitution $e = \sqrt{\frac{H_f}{H_i}} = 0.817$, masking the true bounciness ($e = 0.919$) of the ball. This further argues that my model is better suited for measuring the ball's bounciness because it is more comprehensive by considering such forces.

Appendix-D: Calculation of F_p & F_w

For Figure 2, the following relations hold true:

$$r(x) = \sqrt{R^2 - (R-x)^2} \qquad \ldots (6a)$$

$$A(x) = \pi[r(x)]^2 = \pi x(2R - x) \qquad \ldots (6b)$$

$$V(x) = \tfrac{4}{3}\pi R^3 - \tfrac{1}{3}\pi x^2(3R - x) \qquad \ldots (6c)$$

$$\cos\big(\theta(x)\big) = \tfrac{R-x}{R} \qquad \ldots (6d)$$

The gauge pressure P_G measures the difference between the internal pressure P_i and the external pressure P_0 of the basketball. So, we have

$$P_G = P_i - P_0$$

When the ball is compressed, its internal pressure is increased to $P(x)$. For an isothermal compression, according to equation (2) and equation (6c),

$$P(x)V(x) = P_i V_i$$

$$P(x) = \frac{P_i V_i}{V(x)} = \frac{4R^3(P_G + P_0)}{4R^3 - x^2(3R - x)}$$

The force $F_p$resulting from the pressure difference can be calculated as

$$F_p = (P(x) - P_0)\pi x(2R - x) = (\frac{4R^3(P_G + P_0)}{4R^3 - x^2(3R - x)} - P_0)\,\pi x(2R - x)$$

Using the Taylor expansion tool at https://www.wolframalpha.com, I get

$$F_p = 2\,\pi R^2 P_G \left[\left(\frac{x}{R}\right) - \frac{1}{2}\left(\frac{x}{R}\right)^2 + \frac{3}{4}\left(1 + \frac{P_0}{P_G}\right)\left(\frac{x}{R}\right)^3 + \cdots\right]$$

$\because R \gg x$, when $P_G > 0$ *and not approaching* 0, we can have

$$F_p \approx 2\,\pi R P_G x$$

Thus, it is clear that F_p would have the characteristics of a Hooke's Law restoring force.

According to Hubbard and Stronge's work, the impact phase will lead to deformation along the perimeters of $A(x)$ that will result in a shear strain within the ball's wall giving rise to a restoring force F_w.

The shear stress can be expressed as:

$$\tau = \frac{F_w}{2\pi r D_w} = \frac{F_w}{2\pi\sqrt{R^2 - (R-x)^2} * D_w}$$

In the field of materials science, the shear stress can also be written as the shear modulus G and the shear strain γ, which is equal to the angle of deformation θ:

$$\tau = \gamma G = \theta G = G\cos^{-1}(\frac{R-x}{R})$$

Therefore,

$$F_w = 2\pi G D_w \sqrt{R^2 - (R-x)^2} \cos^{-1}(\tfrac{R-x}{R})$$

Using the Taylor expansion tool at https://www.wolframalpha.com, we can get

$$F_w = 4\pi G D_w R \left[\left(\frac{x}{R}\right) - \frac{1}{6}\left(\frac{x}{R}\right)^2 - \frac{1}{30}\left(\frac{x}{R}\right)^3 + \cdots\right]$$

$$\because x \ll R$$

$$\therefore\ F_w \approx 4\pi G D_w x$$