

Diffusion-Free Dynamics in Rotating Spherical Shell Convection Driven By Internal Heating and Cooling

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ABSTRACT

The bulk properties of convection in stellar and giant planet interiors are often assumed to be independent of the molecular diffusivities, which are very small. By contrast, simulations of this process in rotating, spherical shells, which are typically driven by conductive boundary heat fluxes, generally yield results that depend on the diffusivity. This makes it challenging to extrapolate these simulation results to real objects. However, laboratory models and Cartesian-box simulations suggest diffusion-free dynamics can be obtained if convection is driven using prescribed internal heating and cooling instead of boundary fluxes. Here, we apply this methodology to simulations of Boussinesq, hydrodynamic rotating spherical shell convection. We find that this set-up unambiguously yields diffusion-free behaviour for bulk ‘thermal’ properties of the convection, such as the radial temperature contrast and the convective heat transport. Moreover, the transition from prograde to retrograde equatorial zonal flow is diffusion-free and only depends on the convective Rossby number. The diffusivity dependence of other bulk ‘kinematic’ properties is regime-dependent. In simulations that are rotationally constrained, the convective velocities, and the strength and structure of the zonal flow, are diffusion-dependent, although the zonal flow appears to approach a diffusion-free state for sufficiently high supercriticality. In simulations that are uninfluenced by rotation, or are only influenced by rotation at large scales, diffusion-free convective velocities and zonal flow amplitudes are obtained. The result that many aspects of our idealised simulations are diffusion-free has promising implications for the development of realistic stellar and giant planet convection models that can access diffusion-free regimes.

Keywords: Solar physics (1476) — Planetary atmospheres (1244) — Convection zones (301)

1. INTRODUCTION

Convection in stellar and giant planet interiors is extremely turbulent, which poses a practical barrier to numerical simulation. To retain computational feasibility, simulations of this process are typically conducted in a parameter regime far from reality (A. S. Brun 2020; P. J. Käpylä et al. 2023), with the results subsequently extrapolated to real objects through scaling analysis and by appealing to theory (J. M. Aurnou et al. 2020). In order to achieve this, it is desirable that the simulations exhibit the same scaling behaviour as would a real planet or star. It is typically assumed that the bulk properties of the convection should not depend on the molecular diffusivities, which are small in real astrophysical environments (K. Julien et al. 2012; A. J. Barker

et al. 2014). This assumption underpins the derivation of mixing length theory (MLT; E. Böhm-Vitense 1958), which has been applied with noteworthy success as a parametrisation of convective heat transport in stellar evolution models (e.g., I. Baraffe et al. 2015).

However, diffusivity independent (i.e., ‘diffusion-free’) dynamics are rarely obtained in numerical simulations. For example, the vast majority of shell convection simulations yield scaling behaviour for the radial heat transport that depends upon the diffusivity (U. R. Christensen 2002; T. Gastine & J. Wicht 2012; T. Gastine et al. 2015, 2016; N. A. Featherstone & B. W. Hindman 2016; B. O’Mara et al. 2016), although a notable exception is presented by T. Gastine et al. (2016), where a subset of their simulations do obtain a diffusion-free scaling for cases with very rapid rotation (paired with moderate supercriticality). The influence of diffusivity in simulations of shell convection also affects their bulk kinematic properties. For example, simulations of

stellar convection with high diffusivity have been shown to generate prograde, ‘Solar-like’ equatorial zonal flows (or ‘differential rotation’), but they have a tendency to switch to a regime of retrograde, ‘anti-Solar’ flow as the diffusivities are reduced (T. Gastine et al. 2013, 2014; P. J. Käpylä et al. 2014; N. A. Featherstone & M. S. Miesch 2015; B. W. Hindman et al. 2020). Moreover, T. Gastine et al. (2016) obtain a diffusivity dependent scaling for the Reynolds number even for simulations that are highly supercritical.

Driving convection using a conductive heat flux through the inner and outer radial boundaries is one cause of diffusivity dependence in numerical simulations. When convection is boundary-driven, thin thermal boundary layers form, and the heat transport through the domain is throttled by the conductive flux through the boundary layer (T. Gastine et al. 2016; see also the review by G. Ahlers et al. 2009). Regarding the differential rotation, the transition from prograde to retrograde motion is understood to occur when the convective Rossby number, which for a fixed rotation rate generally increases as the diffusivities are reduced (in simulations with diffusion-dependent dynamics), exceeds unity (P. A. Gilman 1977; T. Gastine et al. 2014). M. E. Camisassa & N. A. Featherstone (2022) provide a geometric interpretation for this transition, while other authors (e.g., B. W. Hindman et al. 2020) argue it is directly related to the thinning of the thermal boundary layer. It is interesting to note that simulations of deep stellar convection driven entirely by internal heating and cooling also exhibit diffusivity dependent differential rotation (A. S. Brun & J. Toomre 2002; M. S. Miesch et al. 2008; M. K. Browning 2008); however, in these models conductive heat transport is likely still important near the upper boundary, owing to the thinness of the cooling layer.

It can be argued ‘boundary-driven’ simulations are not particularly relevant to real astrophysical environments, as the boundary layers that dominate the dynamics over large swathes of parameter space likely bear little resemblance to those of the real objects (A. J. Barker et al. 2014; B. W. Hindman et al. 2020). This has led some authors to pursue an alternative approach, where convection is driven by radiative heating and cooling (with a specified spatial structure) applied directly to the fluid. In the context of rapidly rotating f -plane convection in a Cartesian geometry, A. J. Barker et al. (2014) demonstrate that this approach yields diffusion-free bulk scaling behaviour for both thermal (e.g., the convective heat transport) and kinematic (e.g., the fluctuating kinetic energy) properties. L. K. Currie et al. (2020) generalise this result to the case where the di-

rections of the acceleration due to gravity and the rotational axis are misaligned (as is the case for the non-polar latitudes of a sphere). However, T. Joshi-Hartley et al. (2025) illustrate that it is possible to simultaneously obtain diffusion-free heat transport, but diffusion-dependent convective velocities (using simulations of internally heated and cooled, rotating convection in an idealised Cartesian geometry with 2.5 dimensions). Additionally, they show that the realisation of diffusion-free dynamics in their model is sensitive to the kinematic boundary condition (free-slip configurations more readily yield diffusion-free behaviour). S. Kazemi et al. (2022) show that diffusion-free heat transport can be obtained in Cartesian simulations of non-rotating convection driven by internal heating and cooling (i.e., a realisation of the ‘ultimate regime’). Finally, laboratory experiments driven by internal heating and cooling have obtained diffusion-free heat transport for both the rotating case and the non-rotating case (S. Lepot et al. 2018; V. Bouillaut et al. 2021; G. Hadjerici et al. 2024).

Building on these studies, our objective is to investigate the dynamics and scaling behaviour of convection driven by internal heating and cooling in a spherical geometry. We aim to determine whether internal heating and cooling gives rise to ‘diffusion-free’ dynamics, focusing on the convective heat transport, convective velocities, and the zonal flows (differential rotation). We also investigate the extent to which our simulation results can be described by standard scaling theories (whether diffusion-free or otherwise). In this work, we study the dynamics of a Boussinesq, hydrodynamic fluid, deferring inclusion of the effects of compressibility and magnetism to future work.

2. METHODS

We study the dynamics of Boussinesq, rotating spherical shell convection driven by internal heating and cooling. Our simulations are conducted using the pseudo-spectral code *Dedalus* (K. J. Burns et al. 2020). All variables are expanded in terms of spherical harmonics in the horizontal directions and Chebyshev polynomials in the radial direction using a standard 3/2-dealiasing.

2.1. Model

The governing equations are

$$\frac{D\mathbf{u}}{Dt} + 2\boldsymbol{\Omega} \times \mathbf{u} = -\nabla p + \tilde{g}(r)T\mathbf{e}_r + \nu\nabla^2\mathbf{u} \quad (1)$$

$$\nabla \cdot \mathbf{u} = 0 \quad (2)$$

$$\frac{DT}{Dt} = q(r) + \kappa\nabla^2 T, \quad (3)$$

where $\mathbf{u} = (u_r, u_\theta, u_\phi)$ is the fluid velocity in spherical polar coordinates, p is a pressure, and T is a scaled

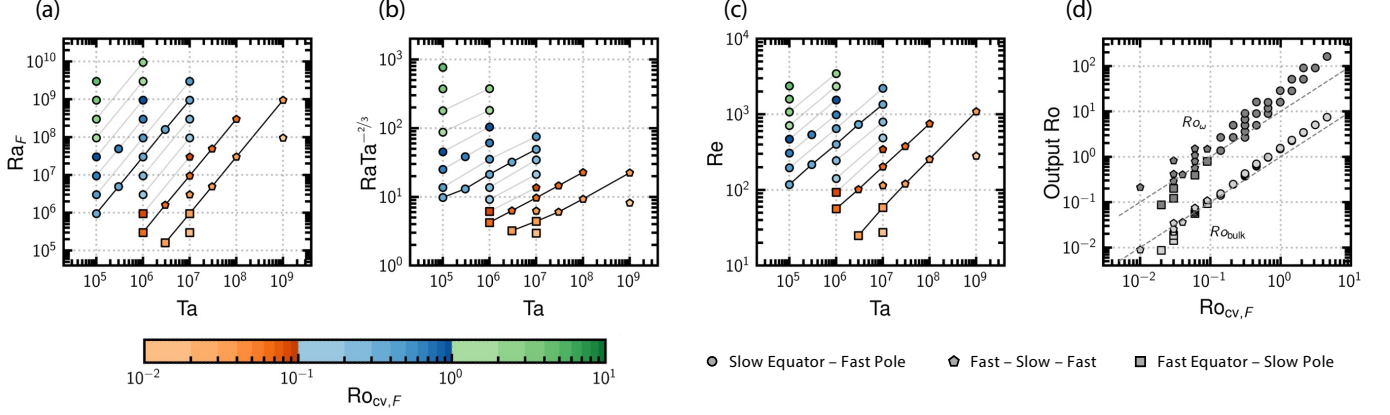


Figure 1. Summary of simulations presented in this study. Panel a) shows the input parameters Ra_F and Ta used for each experiment. Panel b) shows the combination $Ra_F Ta^{-2/3}$ as a proxy for the supercriticality (computed using the ‘output’ Rayleigh number given by Equation 7). Panel c) shows Re (defined by Equation 8), computed using the r.m.s. fluctuating velocity, as a measure of the degree of turbulence obtained in each simulation. Finally, panel d) shows the output Rossby numbers Ro_{bulk} and Ro_{ω} (defined in Equation 9) as a measure of the degree of rotational constraint at large- and small-scales, respectively, plotted against $Ro_{cv,F}$. The marker colour denotes the dynamical regime occupied by a simulation, and the marker shape denotes the configuration of the differential rotation. In panels a)–c), simulations that share the same $Ro_{cv,F}$ are connected by lines. Three lines are emphasised in bold, corresponding to $Ro_{cv,F} = 3.11 \times 10^{-2}$, 6.70×10^{-2} , and 3.11×10^{-1} (from bottom to top in each panel). In panel d), the grey dashed lines are proportional to $Ro_{cv,F}$ and included as an eye guide.

temperature variable, related to the real temperature by $T = g\alpha\delta T_{real}$ (where g is the magnitude of gravitational acceleration and α is the coefficient of thermal expansion, and δT_{real} is the Boussinesq temperature perturbation). The symbol ν denotes the kinematic viscosity and κ denotes the thermal conductivity. The dimensionless gravity profile is given by $\tilde{g}(r) = (r_o/r)^2$ where r_o is the outer radius of the convection zone (see below for definition). The unit vector in the radial direction is given by \mathbf{e}_r , and the rotation vector $\mathbf{\Omega} = \Omega_0(\cos\theta, -\sin\theta, 0)$, where θ is co-latitude and Ω_0 is the rotation rate.

Convection is driven by a prescribed heating and cooling function $q(r)$, configured so that heat is deposited and removed from regions of depth δ at the base and top of the simulated domain, respectively. We use the notation r_i and r_o to denote the radii of the inner and outer boundaries of the convection zone (CZ), within which there is no imposed heating or cooling. Mathematically, q is defined (following A. J. Barker et al. 2014; L. K. Currie et al. 2020):

$$q(r) = \frac{F}{\delta} \frac{r_i^2}{r^2} \begin{cases} 1 + \cos\left[\frac{2\pi(r-r_i+\frac{\delta}{2})}{\delta}\right]; & r_i - \delta \leq r \leq r_i, \\ 0; & r_i < r < r_o, \\ -1 - \cos\left[\frac{2\pi(r-r_o-\frac{\delta}{2})}{\delta}\right]; & r_o \leq r \leq r_o + \delta, \end{cases} \quad (4)$$

where F is the flux injected at the base of the CZ (corresponding to a shell-integrated flux $\mathcal{F}_{tot} = 4\pi r_i^2 F$; \mathcal{F}_{tot} is independent of r within the CZ). The geometric prefactor in the definition of q ensures that the cooling layer

removes the same amount of heat as is injected by the heating layer.

At the inner and outer radii of the simulated domain (located at $r_i - \delta$ and $r_o + \delta$, respectively) we apply a free-slip, impenetrable, and insulating boundary condition (consistent with $\iiint_V q(r) dV = 0$). Applying the heating and cooling function $q(r)$ in combination with these boundary conditions has the effect of replacing the conductive boundary layers that exist in boundary driven simulations with heating and cooling regions of fixed size.

2.2. Non-dimensionalisation

To non-dimensionalise the governing equations, we use the depth of the convection zone $d \equiv r_o - r_i$ as a reference length scale, a flux-based temperature scale, given by $\Lambda \equiv Fd/\kappa$, and a timescale given by the free-fall time $\tau = \sqrt{d/\Lambda}$.

Under this non-dimensionalisation, the dynamics of the system are determined by: a flux-based Rayleigh number, Ra_F ; the Taylor number, Ta ; the Prandtl number, Pr ; the CZ radius ratio, η ; and the non-dimensional heating/cooling depth $\tilde{\delta}$. These numbers are defined by:

$$Ra_F \equiv \frac{d^4 F}{\nu \kappa^2}, \quad Ta \equiv \frac{4\Omega_0^2 d^4}{\nu^2}, \quad Pr \equiv \frac{\nu}{\kappa}, \quad (5)$$

$$\eta \equiv \frac{r_i}{r_o}, \quad \text{and} \quad \tilde{\delta} \equiv \frac{\delta}{d}.$$

In all simulations, we set $Pr = 1$, $\eta = 0.8$, and $\tilde{\delta} = 0.2$. From the definitions of Ra_F , Ta , and Pr , it is possible to define a flux-based convective Rossby number (e.g.,

U. R. Christensen 2002; U. R. Christensen & J. Aubert 2006; B. W. Hindman et al. 2020; P. J. Käpylä 2024), given by:

$$Ro_{\text{cv},F} \equiv \left(\frac{Ra_F}{Ta^{\frac{3}{2}} Pr^2} \right)^{\frac{1}{3}} = (dF)^{\frac{1}{3}} \frac{1}{2\Omega_0 d}. \quad (6)$$

which, notably, is independent of the diffusivities ν and κ . Therefore, for a fixed flux, variation of $Ro_{\text{cv},F}$ corresponds to varying Ω_0 , while varying Ra_F and Ta so as to keep $Ro_{\text{cv},F}$ fixed is equivalent to varying ν and κ (recall $\kappa = \nu$ when $Pr = 1$). We note that this definition of $Ro_{\text{cv},F}$ is closely related to estimates of the Rossby number commonly employed in stellar astronomy (e.g., R. W. Noyes et al. 1984), $Ro \sim 1/(\Omega_0 \tau_c)$, assuming that the convective overturning time τ_c is given by MLT.

2.3. Summary of experiments and numerical details

Results from 38 simulations are presented. The parameter survey has been constructed to produce sets of simulations with constant Ta (within which Ra_F , and thus $Ro_{\text{cv},F}$, varies), as well as some sets with constant $Ro_{\text{cv},F}$ (within which Ra_F and Ta vary). The parameter survey spans a range of $Ro_{\text{cv},F}$ between 0.01 and 3.1, Ta between 10^5 and 10^9 , and Ra_F between 3×10^5 and 9.5×10^9 . A full list of our experiments is given in Appendix A (along with details of the resolution used), and a visual summary is shown in Figure 1.

Figure 1a shows the input parameters Ra_F and Ta , Figure 1b shows $RaTa^{-\frac{2}{3}}$ as a proxy for the supercriticality Ra/Ra_c ($Ra_c \sim Ta^{\frac{2}{3}}$ for large Ta ; S. Chandrasekhar 1961), and Figure 1c shows the output Reynolds number to illustrate the degree of turbulence in each experiment. Above, Ra is the usual Rayleigh number, which is an output parameter for a fixed-flux set-up. It is defined as

$$Ra \equiv \frac{d^3 \Delta T}{\nu \kappa}, \quad (7)$$

where $\Delta T = T(r=r_o) - T(r=r_i)$ is the output radial temperature contrast, evaluated between the boundaries of the CZ. The Reynolds number is defined

$$Re \equiv \frac{UL}{\nu} \quad (8)$$

where we take U to be the root-mean-square (r.m.s.) of the output fluctuating (azimuthal average removed) velocity vector within the CZ, and $L = d$. Also shown in Figure 1d are two different output Rossby numbers, defined

$$Ro_{\text{bulk}} = \frac{U}{\Omega_0 d} \quad \text{and} \quad Ro_{\omega} = \frac{\omega}{\Omega_0}, \quad (9)$$

where for Ro_{ω} , ω is the r.m.s fluctuating vorticity within the CZ. These two output Ro are intended to indicate the degree of rotational constraint at the dominant scale of convection, and at small scales, respectively. Scaling results obtained from our experiments will show that it is useful to categorise them in terms of three regimes: (1) a ‘rotationally-constrained’ regime defined by $Ro_{\text{cv},F} < 0.1$, for which *both* large and small scales are rotationally constrained (i.e., $Ro_{\omega} \lesssim 1$ and $Ro_{\text{bulk}} \ll 1$); (2) a ‘rotationally-influenced’ regime defined by $0.1 \leq Ro_{\text{cv},F} < 1$, where only the large-scales are rotationally constrained ($Ro_{\omega} \gtrsim 1$ and $Ro_{\text{bulk}} < 1$); and (3) a ‘rotationally-uninfluenced’ regime defined by $Ro_{\text{cv},F} \geq 1$, for which the rotational constraint is lost entirely ($Ro_{\omega} \gg 1$ and $Ro_{\text{bulk}} > 1$).

All simulations were integrated until the volume averaged kinetic energy and convective heat transport reached a steady state. In practise, this means that each simulation was run for multiple thermal diffusion times (corresponding to $10^3 - 10^4$ free-fall times). To ensure numerical convergence, the simulations have been inspected to verify closure of the horizontally-averaged heat budget. The kinetic energy spectra have also been inspected to confirm there is no significant accumulation of energy at the smallest scales so that they are spatially resolved.

3. RESULTS

3.1. Simulation results

Summary statistics for our experiments are presented in Figure 2. All quantities shown in this figure are pseudodimensional (specifically, they have been re-dimensionalised using $F = 1$ and $d = 1$ to set the units⁴), and are plotted against $Ro_{\text{cv},F}$ (equivalent to $1/(2\Omega_0)$ for our choice of re-dimensionalisation; cf. Equation 6). In these panels, diffusivity-dependence manifests as scatter on the y -axis for a given value $Ro_{\text{cv},F}$ (note: not all $Ro_{\text{cv},F}$ have multiple simulations; those with only one entry are marked with a \dagger symbol). Conversely, if the dynamics are diffusion-free, then all the data for a given $Ro_{\text{cv},F}$ will collapse onto one point.

Figure 2a shows the shell-averaged temperature difference ΔT between the inner and outer radii of the CZ. For $Ro_{\text{cv},F} < 1$, ΔT displays a strong dependence on $Ro_{\text{cv},F}$, increasing with Ω_0 as the rotational constraint acts to throttle convective heat transport. For each $Ro_{\text{cv},F}$, the scatter between experiments with different Ra_F is very

⁴ Taking $F = 1$ and $d = 1$, dimensional variables are obtained from their non-dimensional counterparts using $T_{\text{dim}} = \Lambda T$ and $\mathbf{u}_{\text{dim}} = \mathbf{u}/\tau$, where $\Lambda = 1/\kappa$ and $\tau = \kappa^{\frac{1}{2}}$ (Section 2.2). κ is obtained from the definition of Ra_F (using $Pr = 1$).

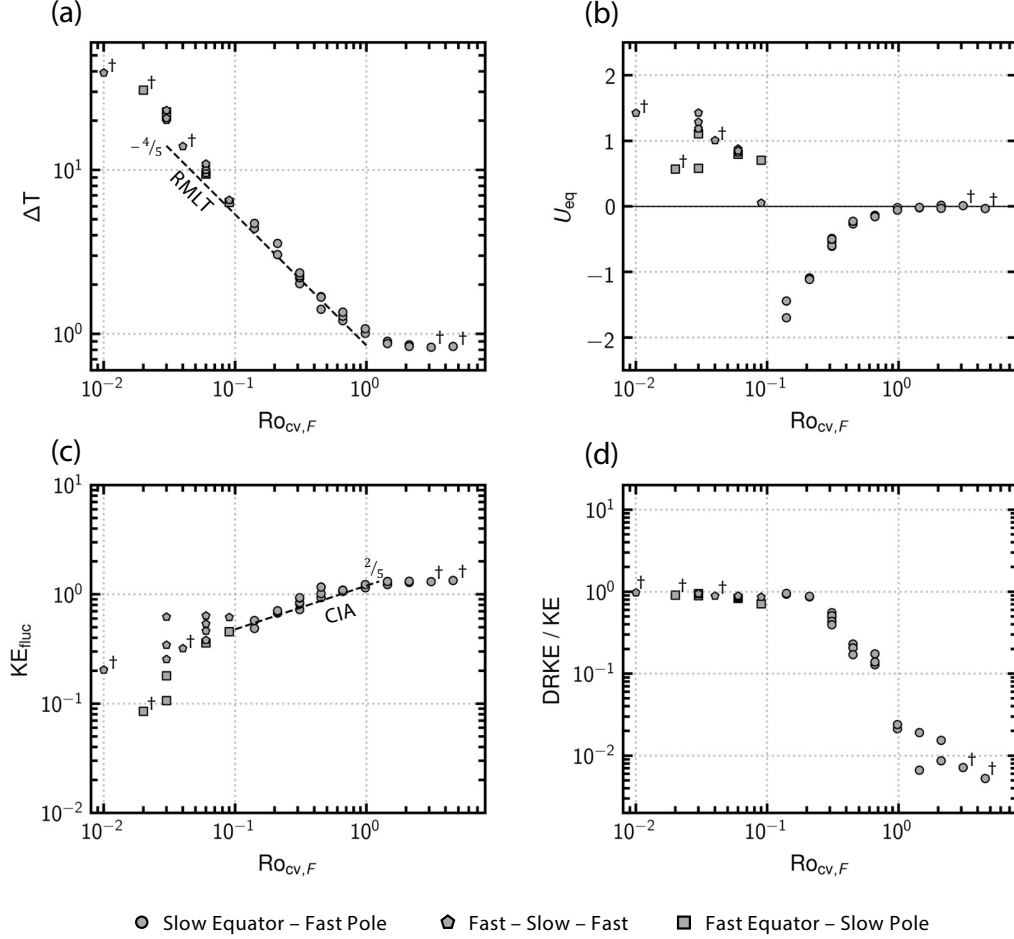


Figure 2. Pseudodimensional output data from each experiment. Panel a) shows the shell-averaged temperature difference across the CZ. Panel b) shows the equatorial zonal velocity, averaged between $\pm 5^\circ$ latitude and over the depth of the CZ. Panels c) and d) show the kinetic energy, averaged over the volume of the CZ. KE_{fluc} is computed using the fluctuating component of the velocity, and $DRKE$ is computed using the axially-symmetric component of the azimuthal velocity. In panels a) and c), lines corresponding to scaling predictions from RMLT and due to CIA balance (see text), respectively, are shown as eye guides. See footnote 4 for details of the re-dimensionalisation procedure. Instances for which only one experiment exists for a given $Ro_{\text{cv},F}$ are marked with the \dagger symbol.

small. In the rotationally-influenced regime, the dependence of ΔT on $Ro_{\text{cv},F}$ ($\Delta T \propto \Omega_0^{4/5} \sim Ro_{\text{cv},F}^{-4/5}$) is consistent with rotating mixing length theory (RMLT; see Section 3.2; D. J. Stevenson 1979; A. J. Barker et al. 2014), which is a diffusion-free theory. For the cases categorised as rotationally-constrained with $Ro_{\text{cv},F} < 0.1$, ΔT depends more strongly on $Ro_{\text{cv},F}$ than predicted by RMLT, while remaining independent of the diffusivity (for the values of $Ro_{\text{cv},F}$ for which there are multiple experiments). In the rotationally-uninfluenced regime where $Ro_{\text{cv},F} > 1$, the rotational constraint is lost, causing ΔT to become independent of $Ro_{\text{cv},F}$. Our simulations yield results that are consistent with this expectation, and thus by extension are consistent with diffusion-free behaviour.

The azimuthal velocity averaged within $\pm 5^\circ$ of the equator and throughout the depth of the CZ, U_{eq} , is shown in Figure 2b. The dependence of U_{eq} on $Ro_{\text{cv},F}$ we obtain is consistent with previous work (T. Gastine et al. 2014). In the absence of a rotational constraint ($Ro_{\text{cv},F} > 1$), differential rotation does not develop ($U_{\text{eq}} \approx 0$). In the rotationally-influenced regime ($Ro_{\text{cv},F} < 1$; $Ro_\omega \gtrsim 1$), anti-Solar (retrograde) differential rotation develops, with an amplitude that increases as the rotation rate is increased. In this regime, the amplitude of the equatorial zonal flow is independent of the diffusivity. When rotation becomes dominant ($Ro_{\text{cv},F} < 0.1$; $Ro_\omega \lesssim 1$), the differential rotation switches direction from retrograde to prograde (Solar-like). In this regime, the data in general do not collapse onto a single curve for U_{eq} vs. $Ro_{\text{cv},F}$ indicating that, at least for the supercriticalities considered in the

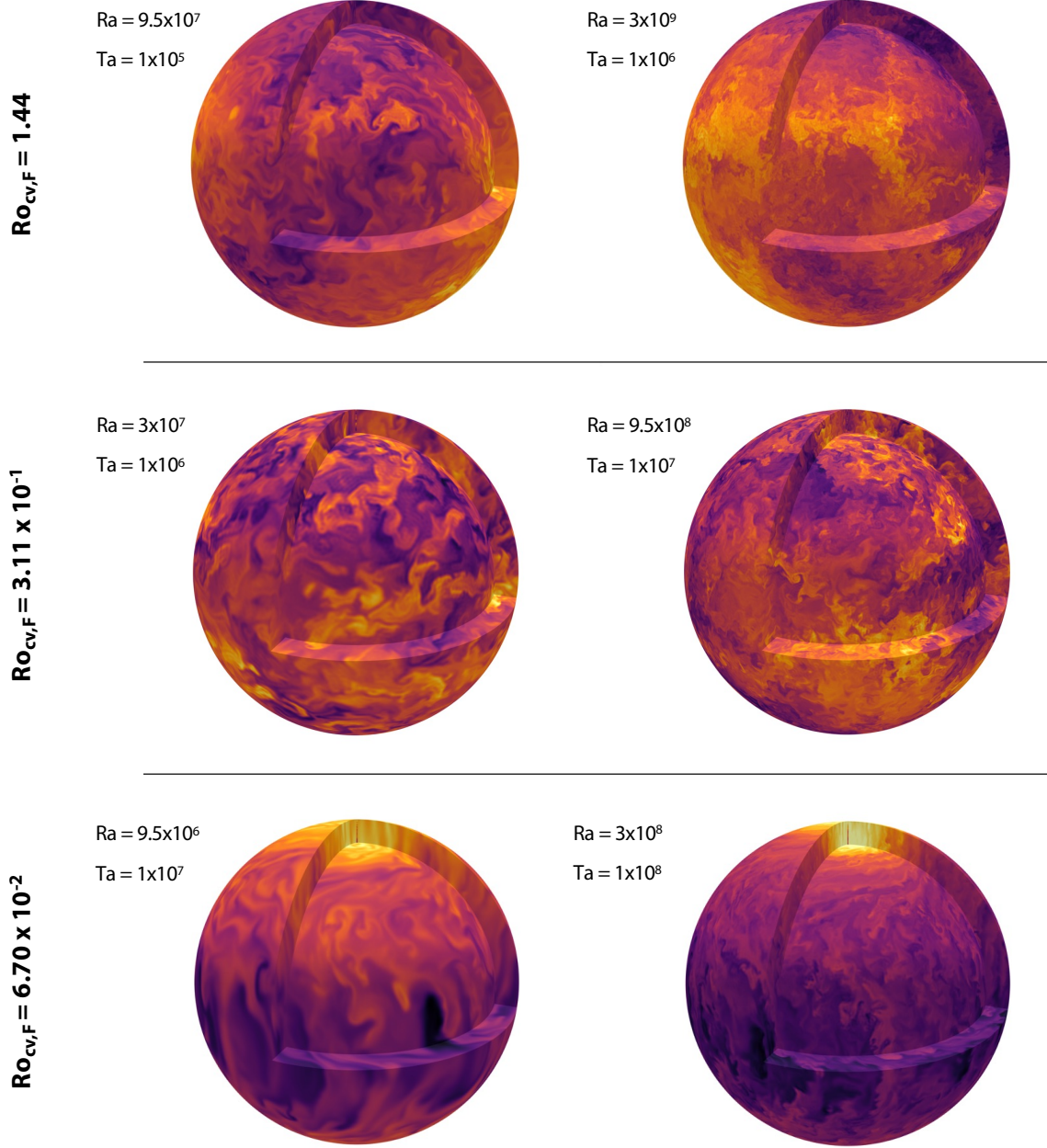


Figure 3. Snapshots of the perturbation temperature for six experiments. The experiments shown in the top row are in the rotationally-uninfluenced regime, those in the middle row are in the rotationally-influenced regime, and those in the bottom row are in the rapidly rotating regime. In each panel, the inner surface corresponds to $r = 4.2$ and the outer surface corresponds to $r = 4.8$.

present work, the strength of U_{eq} depends on the diffusivities. We note that there are only two values of $Ro_{cv,F} < 0.1$ for which we ran a significant number of experiments with varying Ra_F (5 experiments each for $Ro_{cv,F} = 3.11 \times 10^{-2}$ and 6.70×10^{-2}). For the series with $Ro_{cv,F} = 6.70 \times 10^{-2}$, U_{eq} does not vary significantly as the diffusivity is varied, but for the series with $Ro_{cv,F} = 3.11 \times 10^{-2}$ there is a diffusivity dependence.

Figures 2c and d show the kinetic energy (KE) averaged over the volume of the CZ. Figure 2c shows KE_{fluc} , computed using the r.m.s. fluctuating velocity, and Figure 2d shows the ratio $DRKE/KE$, where KE is the total kinetic energy, and $DRKE$ is the ‘differential rotation kinetic energy’, computed using the axially-symmetric component of the azimuthal velocity. The behaviour of KE_{fluc} in Figure 2c tells a similar story to U_{eq} , namely that the kinematic proper-

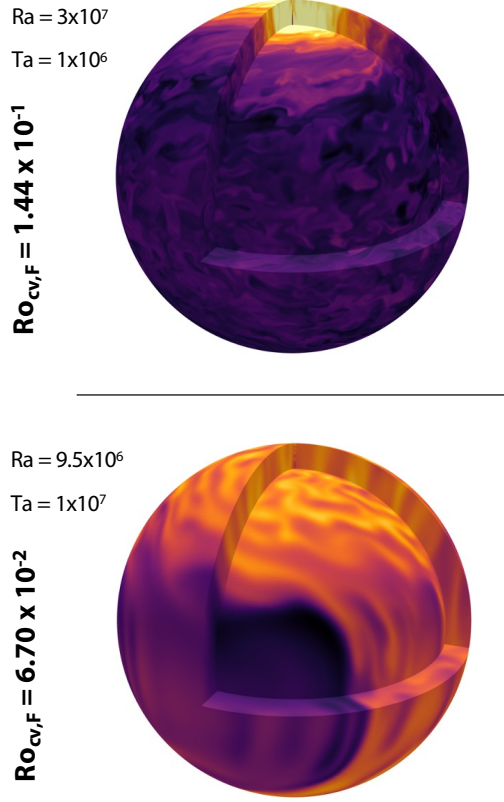


Figure 4. Snapshots of the perturbation temperature for two further experiments. The upper panel shows a simulations with $Ro_{cv,F} = 1.44 \times 10^{-1}$ that is classified as rotationally-influenced. In this experiment, the influence on the dynamics is more apparent visually than is the case for the two rotationally-influenced simulations with $Ro_{cv,F} = 3.11 \times 10^{-1}$ shown in Figure 3. The lower panel shows a rotationally constrained case with $Ro_{cv,F} = 6.70 \times 10^{-2}$ (as in Figure 3, bottom row). This simulation is more dissipative than those with $Ro_{cv,F} = 6.70 \times 10^{-2}$ shown in Figure 3. In the upper panel, the colourbar maximum has been rescaled by 0.7 to enhance the visibility of features at lower latitudes.

ties of the flow display a weak dependence on the diffusivity for $Ro_{cv,F} > 0.1$ (rotationally-uninfluenced or rotationally-influenced), but a stronger dependence once the influence of rotation becomes dominant. There are two additional features of Figure 2c,d that are worth emphasising. First, we note that in the rotationally-influenced regime, the dependence of KE_{fluc} on $Ro_{cv,F}$ is consistent with the diffusion-free ‘Coriolis-inertial-Archimedean’ (CIA) scaling (e.g., T. Gastine et al. 2016; see Section 3.2). Second, we highlight that while KE_{fluc} (Figure 2c) and the DRKE depend on the diffusivity in the rotationally-constrained regime, they do so in such a way that the ratio $DRKE/KE$ is roughly diffusion-free (Figure 2d). In this regime, the DRKE dominates the total KE so that their ratio saturates at $DRKE/KE \approx 1$.

The dependence of the summary statistics shown in Figure 2 on $Ro_{cv,F}$ (and the diffusivities, where applicable) arise from the influence of these parameters on the convective flow patterns. To illustrate this, Figure 3 shows the perturbation temperature T' (horizontal average removed) for six selected experiments. The upper row shows two experiments with $Ro_{cv,F} = 1.44$ that occupy the rotationally-uninfluenced regime, the middle row shows two experiments with $Ro_{cv,F} = 3.11 \times 10^{-1}$ that fall within the rotationally-influenced regime, and finally the bottom row shows two experiments with $Ro_{cv,F} = 6.70 \times 10^{-2}$ that occupy the rotationally-constrained regime.

In the rotationally-uninfluenced regime, convective plumes extend outwards in the radial direction at all latitudes and the effect of rotation is undetectable. In this regime, the temperature field is organised into large ‘warm’ and ‘cool’ patches that are dominated by spherical harmonic degree $\ell = 1$ or 2. The morphology in the rotationally-influenced cases with $Ro_{cv,F} = 3.11 \times 10^{-1}$ is similar, although the convection at higher latitudes now exhibits a weak alignment with the rotation axis and some low-latitude features are elongated in the azimuthal direction. However, the impact of rotation is hard to detect, consistent with $Ro_{bulk} > 1$. Figure 4 shows an additional case in the rotationally-influenced regime with $Ro_{cv,F} = 1.44 \times 10^{-1}$ (i.e., a slightly faster rotation rate), which more clearly shows that the convection is aligned cylindrically, and zonally-elongated at lower latitudes. We highlight that the pair of experiments with $Ro_{cv,F} = 3.11 \times 10^{-1}$ are morphologically very similar, as are the pair with $Ro_{cv,F} = 1.44$, which is consistent with the inference from Figure 2 that the diffusivity has a limited influence on the dynamics in the rotationally-uninfluenced and rotationally-influenced regimes.

The effect of rotation in the rotationally-constrained regime is clear. In mid-latitudes and the polar regions, convection is aligned cylindrically (i.e., with the rotation axis). Closer to the equator, large, modulated thermal Rossby wave-like features can be identified, that propagate in the prograde direction (with respect to the mean zonal flow). These waves are present in all experiments with $Ro_{cv,F} < 0.1$ but in none with $Ro_{cv,F} \geq 1$. The transition to a regime of equatorial dynamics dominated by wave-like features occurs when the small scales become rotationally constrained ($Ro_{\omega} < 1$). For the relatively turbulent cases shown in Figure 3 there is a strong polar vortex. This is not present in simulations that are more strongly dissipative (an example is given by the lower panel of Figure 4).

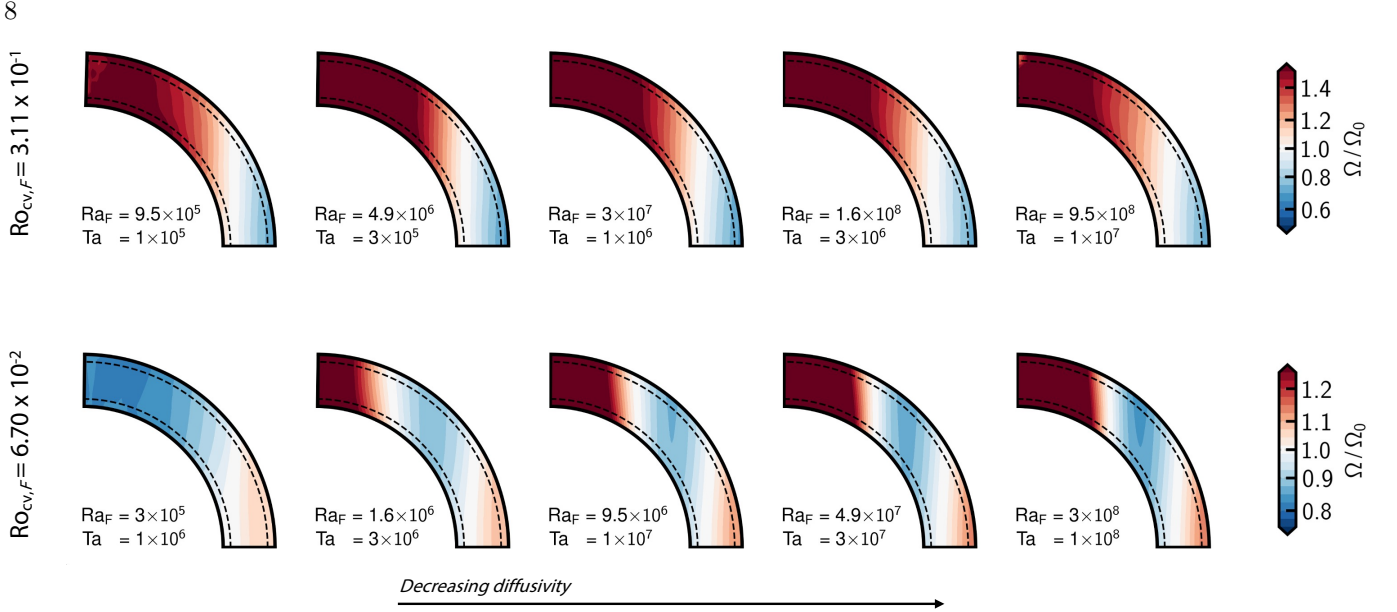


Figure 5. Azimuthally-averaged angular velocity Ω normalised by the rotation rate Ω_0 (see Equation 10) for two series of simulations with fixed $Ro_{cv,F}$. The top row shows $Ro_{cv,F} = 3.11 \times 10^{-1}$ and the bottom row shows $Ro_{cv,F} = 6.70 \times 10^{-2}$. For a given $Ro_{cv,F}$, different combinations of Ra_F and Ta correspond to different diffusivities. In the slices above, the diffusivity decreases from left to right. The dashed lines show the edges of the CZ at $r_i = 4$ and $r_o = 5$.

Figure 5 shows cross-sections of the differential rotation, quantified by the azimuthally-averaged angular velocity normalised by the rotation rate Ω_0 ,

$$\frac{\Omega}{\Omega_0} = 1 + \frac{\bar{u}_\phi}{\Omega_0 r \sin \theta}, \quad (10)$$

for two series of experiments at $Ro_{cv,F} = 3.11 \times 10^{-1}$ (rotationally-influenced; shown in the upper row) and $Ro_{cv,F} = 6.70 \times 10^{-2}$ (rotationally-constrained; lower row). For a given $Ro_{cv,F}$, larger Ra_F corresponds explicitly to reduced diffusivity (decreasing across panels moving from left to right). The experiments in the $Ro_{cv,F} = 3.11 \times 10^{-1}$ set exhibit ‘anti-Solar’ differential rotation with a slow equator and a fast pole (denoted SF). The differential rotation in this set of experiments has little dependence on the diffusivity, with the spatial structure and amplitude of Ω/Ω_0 remaining essentially unchanged as Ra_F is increased. In contrast, the differential rotation in the $Ro_{cv,F} = 6.70 \times 10^{-2}$ set does depend on the diffusivity. All of these experiments have a fast equator, but the structure of the zonal flow at higher latitudes varies. The most diffusive experiment (left-most panel in Figure 5, bottom row) features a Solar-like fast equator and slow pole (denoted FS), while the remaining four experiments have prograde flow at the equator, flanked by retrograde flow in mid-latitudes, and then finally a polar vortex (cf. Figure 3d) at high-latitudes (i.e., fast-slow-fast; denoted FSF). Once the transition from FS to FSF differential rotation occurs, the $Ro_{cv,F} = 6.70 \times 10^{-2}$ experiments appear to converge towards an asymptotic ‘diffusion-free’ structure and am-

plitude as the diffusivity is reduced further (see also the small spread in U_{eq} when $Ro_{cv,F} = 6.70 \times 10^{-2}$ in Figure 2).

Each of the scatter charts in this article use the marker shape to illustrate the occurrence of the SF, FSF, and FS configurations of differential rotation in parameter space (circles for SF, pentagons for FSF, and squares for FS). From Figure 1a–c, we identify that solar-like differential rotation (FS) is only obtained for the most diffusive experiments ($Ra_F \lesssim 10^6$), while more turbulent simulations with a strong rotational constraint exhibit FSF differential rotation. Figure 1d shows that the transition from a fast equator (either FS or FSF) to a slow equator occurs at the boundary between the rotationally-constrained and rotationally-influenced regimes, when $Ro_{cv,F} = 0.1$, which corresponds to $Ro_\omega \approx 1$, consistent with previous work (T. Gastine et al. 2014). The fact that all of the FS or FSF cases have $Ro_{cv,F} < 0.1$, and all of the SF cases have $Ro_{cv,F} \geq 0.1$, implies that the transition from a fast equator to a slow equator is diffusion-free (even if the zonal flow structure in the fast-equator cases themselves displays a dependence on the diffusivity). We believe that this transition is associated with the existence of thermal Rossby wave-like features at low latitudes, which are present in all simulations in the rotationally-constrained regime but absent otherwise. However, a complete analysis of the zonal momentum budget is left for future work.

Finally, we note that the experiments we categorise as rotationally-constrained were run with a lower supercrit-

icality than those categorised as rotationally-influenced or rotationally-uninfluenced (Figure 1b). While we do not observe it here, we would expect the anti-Solar differential rotation in these regimes to display a diffusivity-dependence for sufficiently low supercriticality (i.e., higher diffusivity).

3.2. Comparison with scaling predictions

3.2.1. Nusselt number scaling

Radial heat transport is often characterized using the Nusselt Number, defined as the ratio of the shell-integrated total heat flux (convective plus conductive) to the conductive flux. In this work, we evaluate this as an integral over the CZ, which yields:

$$Nu \equiv \frac{\mathcal{F}_{\text{tot}}}{\mathcal{F}_{\text{cond}}} = \frac{r_i^2}{\int_{r_i}^{r_o} \partial_r \langle T \rangle r^2 dr}. \quad (11)$$

All quantities above are non-dimensional, $\langle T \rangle$ is the shell-averaged temperature, and the second relation follows from the fact that $\mathcal{F}_{\text{tot}} = 4\pi r_i^2$ within the CZ. Mixing length theory, which essentially equates the kinetic energy of fluid parcels to the buoyancy work done over a characteristic length l (the mixing length), states that the Nusselt number should scale as:

$$Nu \propto Ra^{\frac{1}{2}}, \quad (12)$$

for $Pr = 1$. A central assumption in MLT is that the total dimensional heat transport is independent of the diffusivities, and thus the scaling above is diffusion-free. We note that a simple dimensional analysis for a non-rotating system yields the scaling above under this assumption⁵. For a rotating system, the analogous theory to MLT is rotating mixing length theory (RMLT; D. J. Stevenson 1979), which assumes that convection is dominated by modes with a growth rate well described by linear theory and amplitudes determined by a balance between their growth rate and the non-linear cascade rate. RMLT predicts that Nu scales as

$$Nu \propto \frac{Ra^{\frac{3}{2}}}{TaPr^{\frac{1}{2}}}. \quad (13)$$

As with Equation 12, this is a diffusion-free scaling. It scaling can also be obtained by combining the ‘diffusion-

free heat transport assumption’ with a further assumption that the heat transport depends only on the supercriticality of the system Ra/Ra_c (recall that $Ra_c \sim Ta^{\frac{3}{2}}$ for large Ta) (K. Julien et al. 2012; A. J. Barker et al. 2014). We note that $Ra \approx Ra_F/Nu$, which yields the scaling $Nu \propto (Ra_F Ta^{-2/3})^{3/5} Pr^{-1/5}$. Using the definition of $Ro_{\text{cv},F}$, this can be re-written $Nu \propto Ra_F^{1/3} Pr^{1/3} Ro_{\text{cv},F}^{4/5}$. Therefore, with Pr constant, RMLT implies

$$Nu \propto Ra^{\frac{1}{2}} \text{ for constant } Ro_{\text{cv},F} \quad (14)$$

(i.e., as in MLT). Previous studies of boundary-driven Boussinesq convection in a spherical shell have not yielded MLT-like scaling for the Nusselt number in the regime where rotation is unimportant. Instead, shallower scaling exponents were obtained, due to the throttling of convection by the conductive heat transport in the boundary layers (e.g., T. Gastine et al. 2015). Some studies have obtained RMLT-like scaling in the rapidly rotating regime, but only for extreme parameter values (e.g., T. Gastine et al. 2016, wherein experiments with $Ta \gtrsim 10^{12}$ and $RaTa^{-2/3} \simeq 10$ exhibit a Nusselt number scaling in agreement with RMLT).

Figure 6a and 6b show Nu plotted against $RaTa^{-2/3}$ in panel a) and Ra in panel b). Dashed lines corresponding to the MLT and RMLT scaling predictions are included in each panel (note that the MLT scaling in panel a, and the RMLT scaling in panel b, apply to series of experiments with constant Ta). In panel a), the marker colour is used to show $Ro_{\text{cv},F}$.

Inspection of Figure 6a reveals that our experiments fall into three groups distinguished by different Nu vs. $RaTa^{-2/3}$ scaling behaviours. When $Ro_{\text{cv},F} \geq 1$ (indicated by a green marker colour), the Nusselt number follows the MLT scaling prediction (Equation 12). These experiments are those that we have categorised as rotationally-uninfluenced. For, $0.1 \leq Ro_{\text{cv},F} < 1$ (indicated by blue marker colours), i.e., experiments in the rotationally-influenced regime, a steeper scaling is obtained that is consistent with RMLT (Equation 13). The transition between the MLT and RMLT scalings occurs when $Ro_{\text{cv},F} = 1$, which roughly corresponds to the point where $Ro_{\text{bulk}} = 1$. The agreement between the scaling obtained by the rotationally-influenced experiments and RMLT is striking, given that the effect of rotation on the convective morphology in this regime can be relatively weak (e.g., Figure 3). These results (as well as those presented in the next subsection) inform our definition of the rotationally-uninfluenced and rotationally-influenced regimes. Finally, in the rotationally-constrained regime for which $Ro_{\text{cv},F} < 0.1$ (orange marker colours), the Nusselt num-

⁵ A diffusion-free scaling for a non-dimensional number (e.g., Nu or Re) does not mean that the scaling parameter (e.g., Ra) is diffusivity-independent. Instead, it is a scaling that is consistent with the dimensional quantities (e.g., the dimensional radial temperature gradient for Nu) being diffusivity-independent. For the specific case of the MLT Nu scaling, Nu includes a factor of κ^{-1} if T is re-dimensionalised, and so the right-hand-side must also contain a factor of κ^{-1} , which implies a scaling $Nu \propto Ra^{\frac{1}{2}} Pr^{\frac{1}{2}}$ (which is $Ra^{\frac{1}{2}}$ for $Pr = 1$).

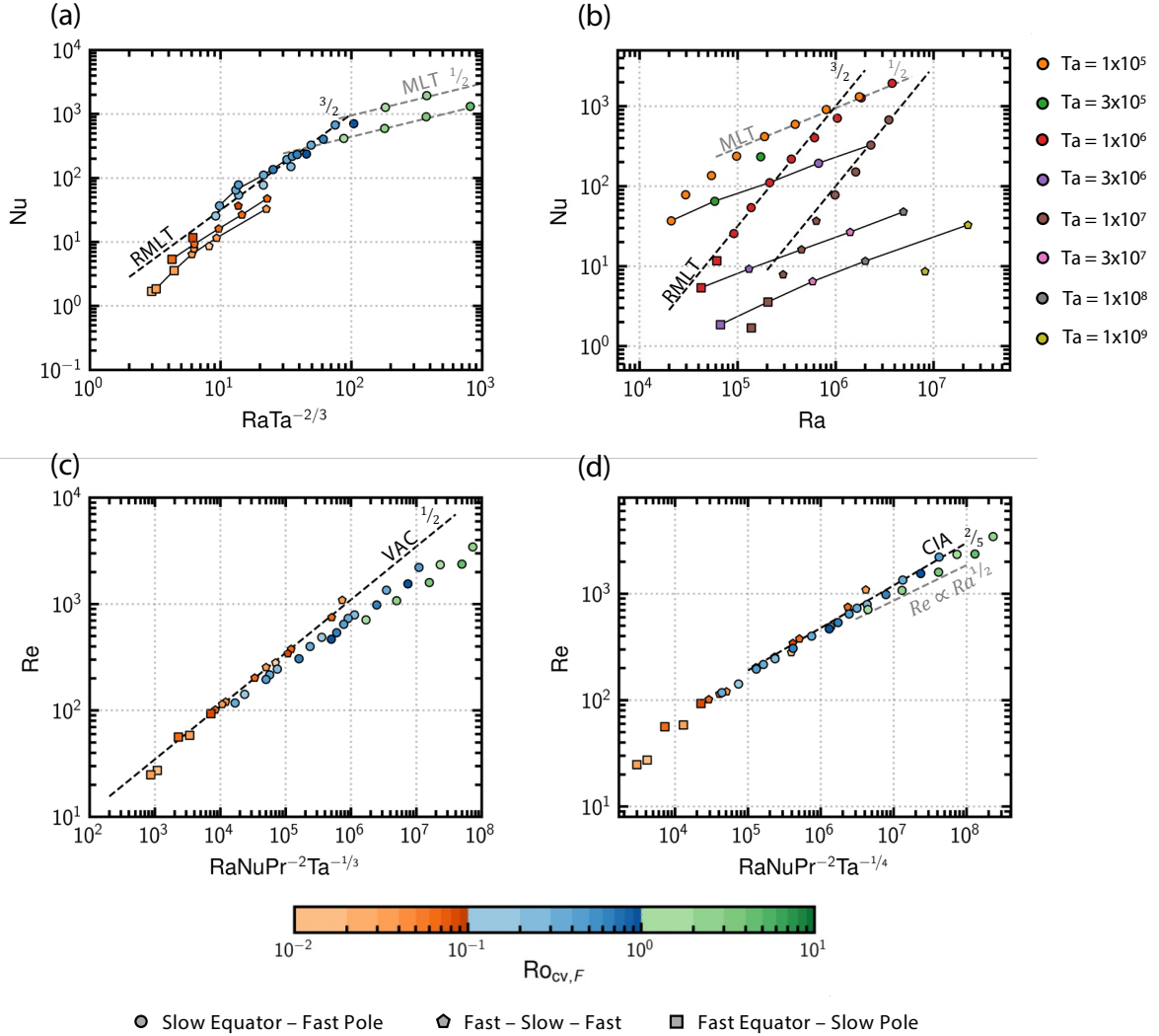


Figure 6. Scaling results for Nu (panels a and b) and Re (panels c and d) obtained with our experiments. In panels a) and b), black and grey dashed lines correspond to the mixing length theory (MLT) and rotating mixing length theory (RMLT) scaling laws given by Equations 12 and 13, respectively, and the black solid lines connect selected series of experiments with constant $Ro_{cv,F}$ (see caption of Figure 1 a–c for further information). In panel c), the scaling for Re predicted by VAC balance (Equation 17) is shown as a black dashed line. Finally, in panel d), the black dashed line shows the inertial scaling of rotating convection (associated with CIA balance; given by Equation 16), and the grey dashed line shows the diffusion-free, non-rotating ‘ultimate’ scaling $Re \propto Ra^{1/2}$ (Equation 15).

ber scaling departs from the RMLT branch occupied by the rotationally-influenced cases. This departure from RMLT is manifest as an offset from the RMLT branch (reduced Nu for a given $RaTa^{-2/3}$) that increases as $Ro_{cv,F}$ is reduced. For constant $Ro_{cv,F}$, experiments in this regime roughly follow the $Nu \propto Ra^{1/2}$ scaling noted above (Equation 14; for simulations with sufficient supercriticality, $RaTa^{-2/3} \gtrsim 5$). This indicates that the convective heat transport is still independent of the diffusivity (consistent with the pseudodimensional ΔT shown in Figure 2a), and that the offset from the rotationally-influenced RMLT branch captures a dependence of the convective heat transport on $Ro_{cv,F}$

(a diffusion-free parameter) that is not accounted for by RMLT. Hereafter, we refer to this scaling as ‘offset RMLT’. As noted previously, the transition from the rotationally-influenced regime to this new regime occurs when $Ro_{\omega} \lesssim 1$, which indicates that all spatial scales are now affected by rotation, motivating our choice to refer to this regime as rotationally-constrained.

When the same data are plotted against Ra (Figure 6b), the MLT scaling obtained by the rotationally-uninfluenced regime can be identified for experiments with $Ta = 10^5$, which unambiguously follow the $Ra^{1/2}$ prediction. The $Ta = 10^6$ series follows the RMLT scaling for moderate supercriticality, before transition-

ing towards the MLT scaling at higher supercriticality, and the experiments at $Ta = 10^7$ are approaching the RMLT scaling. Finally, the offset RMLT scaling identified for the rotationally-constrained regime is manifest as a steepening of the $Nu(Ra)$ relation for fixed Ta (but, again, note that series of experiments with constant $Ro_{cv,F}$, joined by black solid lines, follow the RMLT prediction given by Equation 14).

A preliminary analysis (not shown) of the latitudinal dependence of the heat budget and Nusselt number (cf. T. Gastine & J. M. Aurnou 2023) suggests that the transition to the offset RMLT scaling that characterises the rotationally-constrained regime may be related to the change in the dynamics that from a flow dominated by turbulent convection to a wave-dominated flow (we find wave-like features are always present when $Ro_{cv,F} < 0.1$, but are absent when $Ro_{cv,F} \geq 0.1$). However, the offset scaling could alternatively arise due to interaction between the convection and the zonal flow, although this would have to be a ‘signed’ effect, since strong (anti-Solar) zonal flows are also present in the rotationally-influenced regime for which the standard RMLT scaling is recovered. We will explore this further in future work. We note that the offset scaling persists even for the rotationally-constrained experiments with the highest supercriticality. As a result, we do not believe that it arises because the rotationally-constrained simulations generally have lower supercriticality than those categorised as rotationally-influenced.

3.2.2. Reynolds number scaling

A common metric of the degree of turbulence in fluid flow is the Reynolds number Re (given by Equation 8), which quantifies the ratio of the inertial and viscous terms in the momentum equation. In the non-rotating limit, dimensional analysis predicts the ‘ultimate’ scaling

$$Re \propto Ra^{\frac{1}{2}} \quad (15)$$

under the assumption that the velocity U (used to compute Re) and the temperature difference ΔT (used to compute Ra) are independent of the diffusivities. When instead the system is rotating, then a diffusion-free balance between the Coriolis, inertial, and Archimedean (buoyancy driving) terms in the momentum equation (CIA balance) suggests the scaling (J. Aubert et al. 2001; A. J. Barker et al. 2014; T. Gastine et al. 2016):

$$Re \propto \left(RaNuPr^{-2}Ta^{-\frac{1}{4}} \right)^{\frac{2}{5}}, \quad (16)$$

sometimes referred to as the inertial scaling of rotating convection. If alternatively, a balance is sought between the Coriolis, viscous, and Archimedean terms (i.e., VAC

balance; E. M. King et al. 2013), then the following scaling:

$$Re \propto \left(RaNuPr^{-2}Ta^{-\frac{1}{3}} \right)^{\frac{1}{2}} \quad (17)$$

is obtained. Equation 17 implies that U depends on the diffusivity. We are not aware of any study that recovers either of the diffusion-free scalings (Equations 15 and 16) from simulations of spherical shell convection that include rotation. For example, even the simulations presented in T. Gastine et al. (2016), that achieve an RMLT-like scaling for Nu in the rapidly rotating regime, exhibit a Reynolds number scaling that is not diffusion-free. There the rapidly rotating cases follow an alternate scaling in which CIA balance is obtained in the bulk, but kinetic energy dissipation is dominated by the boundary layers, while their weakly-nonlinear simulations follow the VAC scaling described above. However, we note that N. A. Featherstone & B. W. Hindman (2016) do find diffusion-free behaviour for the kinetic energy in simulations that are non-rotating.

Figure 6c and 6d present a comparison between the Reynolds number scaling obtained in our simulations, and the theoretical relations given by Equations 15–17. As with the Nusselt number, the three regimes we have identified correspond to three distinct scaling relations. Experiments with $Ro_{cv,F} > 1$ in the rotationally-uninfluenced regime, for which small scales are rotationally-unconstrained, follow the ultimate scaling given by Equation 15 (green points in Figure 6d). Meanwhile, the experiments that are rotationally-influenced ($Ro_{bulk} < 1$ but $Ro_{\omega} \gtrsim 1$) follow the inertial scaling (blue points in Figure 6d). By contrast, the rotationally-constrained $Ro_{cv,F} < 0.1$ cases display a Reynolds number scaling that closely follows the diffusivity-dependent VAC balance scaling (orange points in Figure 6c), which is consistent with the dependence of the KE on the diffusivities in Figure 2.

4. SUMMARY AND DISCUSSION

We have presented 3D simulations of rotating convection in a spherical shell, driven by an internal heating and cooling function. This function was designed so that the same flux is extracted by cooling (over a fixed region near the top of the computational domain) as is input by heating (over an identically-sized region near the base). A key finding of our work is that in some regimes, the heat transport and other aspects of the flow dynamics are well described by diffusion-free theory, with no evident dependence on the viscosity or thermal diffusivity. Below, we briefly summarise the main results of this study, relate them to prior work, and comment on their astrophysical implications.

Table 1. Summary of dynamical regimes.

Name	Ro			$Nu / \Delta T$		Re / KE_{fluc}		Differential Rotation
	$Ro_{\text{cv,F}}$	Ro_{ω}	Ro_{bulk}	Diff.-free?	Scaling	Diff.-free?	Scaling	
Rot.-constrained	$Ro < 0.1$	$Ro \lesssim 1$	$Ro \ll 1$	✓	Offset RMLT	✗	VAC	FS or FSF
Rot.-influenced	$0.1 \leq Ro < 1$	$Ro \gtrsim 1$	$Ro < 1$	✓	RMLT	✓	CIA	SF
Rot.-uninfluenced	$Ro \geq 1$	$Ro \gg 1$	$Ro \geq 1$	✓	MLT	✓	Ultimate	SF

NOTE—Experiments are classified according to their input $Ro_{\text{cv,F}}$. Two output Rossby numbers, Ro_{ω} and Ro_{bulk} , are also included in the table. These are defined in Equation 9, and are intended to indicate the degree of rotational constraint at small and large scales, respectively. Nu is the Nusselt Number (Equation 11), and ΔT is the radial temperature contrast. Both are evaluated for the convection zone (CZ; between $r_i = 4$ and $r_o = 5$). ‘MLT’ and ‘RMLT’ refer to ‘mixing length theory’ and ‘rotating mixing length theory’, respectively. The scalings for Nu associated with these theories are given by Equations 12 and 13. Re is the Reynolds number (Equation 8), and KE_{fluc} is the fluctuating kinetic energy. Both are computed using the r.m.s. fluctuating (azimuthal average removed) velocity within the CZ. ‘VAC’ and ‘CIA’ refer to ‘Viscous-Archimedean-Coriolis balance’ and ‘Coriolis-inertial-Archimedean balance’, respectively. The VAC, CIA, and ‘Ultimate’ scalings are given by Equations 15–17. Finally, the differential rotation classifications ‘FS’, ‘FSF’, and ‘SF’ correspond to configurations with: (i) a fast equator and slow pole (Solar-like), (ii) a fast equator, slow mid-latitudes, and ‘fast’ polar vortex, and (iii) a slow equator and fast pole (anti-Solar).

The scaling results presented in the previous section suggest that our simulations can be grouped into three distinct regimes, which we term rapidly rotating ($Ro_{\text{cv,F}} < 0.1$), rotationally-influenced ($0.1 \leq Ro_{\text{cv,F}} < 1$), and rotationally-uninfluenced ($Ro_{\text{cv,F}} \geq 1$). In the rotationally-constrained regime, both the small scales and the bulk are constrained by rotation. The rotationally-influenced regime is an intermediate regime where large scales are constrained by rotation, but small scales are not (Figure 1d). Finally, there is no discernible impact of rotation on the dynamics of simulations in the rotationally-uninfluenced regime. The key features of each regime are summarised in Table 1 and described below.

The main result of this paper is that all of our experiments (aside from those that are very weakly supercritical) exhibit Nusselt number scalings and a radial temperature contrast that are independent of the diffusivities (Figure 2a, Figure 6a,b). Simulations in the non-rotating and rotationally-influenced regimes follow scalings that are consistent with MLT and RMLT, respectively; the rotationally-constrained cases also follow a scaling that is RMLT-like, but offset by a factor of the (diffusion-free) convective Rossby number ($Ro_{\text{cv,F}}$). In addition, diffusion-free scaling is also obtained for the Reynolds number in the non-rotating and rotationally-influenced regimes (Figure 6d), but in the rotationally-constrained regime the experiments follow a Reynolds number scaling consistent with VAC balance (which depends on the diffusivity; Figure 6c). Finally, in the rotationally-constrained regime, the morphology of the differential rotation depends on the supercriticality; weakly supercritical simulations ($RaTa^{-2/3} \lesssim 4$) feature Solar-like differential rotation (denoted FS),

whereas moderately supercritical simulations additionally feature a strong, prograde polar vortex (denoted FSF) (as in Figure 5, bottom row). In the rotationally-influenced and non-rotating regimes, anti-Solar differential rotation (denoted SF) is obtained (as in Figure 5, top row), with a structure and amplitude that is independent of the diffusivity.

An important result from our simulations is that it is possible to obtain zonally-averaged zonal flow statistics that are independent of the diffusivity. This result is demonstrated by rotationally-influenced cases with finite amplitude anti-Solar differential rotation (Figure 2b and Figure 5). In the rotationally-constrained regime, the differential rotation does depend on the diffusivity, although we suggest that too few of our experiments have sufficient supercriticality (only six experiments with $RaTa^{-2/3} > 10$ and $Ro_{\text{cv,F}} < 0.1$) to explore the possibility for diffusion-free prograde differential rotation. Notwithstanding the diffusivity-dependence of the differential rotation in the rotationally-constrained regime, we have found that the sign of the equatorial zonal flow (i.e., prograde – FS or FSF, or retrograde – SF), does not depend on the diffusivity.

Previous work has shown that ‘local Nusselt numbers’ defined for the equatorial and polar regions can yield different scaling results that are set by the local dynamics (T. Gastine & J. M. Aurnou 2023). Therefore, the fact that RMLT does not fully capture the convective heat transport scaling in the rapidly-rotating experiments is not surprising, given that the morphology of the convection depends strongly on latitude (e.g., wave-like dynamics at low latitudes but convective turbulence at high latitudes) but this effect is not accounted for by RMLT. A full analysis of the latitudinal heat budget, as

well as the dependence of the size of the equatorial region on $Ro_{cv,F}$, is required to form a complete understanding of the Nu scaling in the rapidly-rotating regime. A second, puzzling feature of the rapidly-rotating regime is that the convective velocity scaling (i.e., the Re scaling) is diffusivity-dependent, but the heat transport scaling is not. Our analysis in this article has focused on the CZ in isolation; however, by analogy with the influence that boundary layers can have on the bulk in boundary-driven convection (S. Grossmann & D. Lohse 2000; E. M. King et al. 2012; T. Gastine et al. 2016), it is plausible that the dynamics of the heated and cooled regions can exert an influence on the CZ. We note that forming a volume-integrated relation between Re and the viscous dissipation rate is one route to obtaining the inertial scaling for Re (Equation 15; T. Gastine et al. 2016). As we are interested in the dynamics integrated over the CZ, this integral relation will not be exact, but instead contain ‘boundary terms’ that describe the exchange of kinetic energy between the CZ and the heated and cooled regions. Understanding the details of this exchange are likely important to understanding the scaling behaviour of Re in the CZ.

Several aspects of the flows here are unlike those realised in a star like the Sun. We believe most of these discrepancies result from choices and simplifications made in our modeling (rather than from a direct dependence on diffusivity), but testing this assertion will require future work. For example, none of the zonal flows in our calculations are particularly ‘Solar-like’; those that exhibit a fast equator typically also have a ‘polar vortex’, consisting of retrograde flow at high latitudes. This may reflect the narrow aspect ratio of the convective shell considered here (see, e.g., P. A. Gilman 1979). More generally, we have restricted ourselves to the simplest case of Boussinesq, hydrodynamic convection. This setup has the advantage of being easily relatable to theory. However, both magnetism and compressibility have important effects on the dynamics of rotating, spherical shell convection (e.g., T. Gastine & J. Wicht 2012; R. K. Yadav et al. 2016; P. J. Käpylä 2023; H. Hotta et al. 2015, 2022; K. M. Soderlund et al. 2025). Therefore, incorporating these effects is an obvious and important follow-up to this work. In addition, we have not analysed the sensitivity of our results to the functional form of the prescribed internal heating and cooling profile. We intend to study these effects in future work.

In spite of these limitations, we argue that our findings have very promising implications for the numerical modeling of convection in stellar (and giant planet) interiors. All simulations of interior convection operate with diffusivities (whether explicit or numerical) that are orders of

magnitude larger than those of real objects, so it is vital to understand how these diffusivities alter the results of the simulations. Our demonstration here that aspects of the heat transfer and flow dynamics are diffusion-free (i.e., independent of the diffusivities) will, we believe, enable more confident extrapolation from our results to real environments. To be explicit, we think that if a simulation in the rotationally-uninfluenced or rotationally-influenced regimes could be conducted with vastly lower diffusivities (but at the same value of $Ro_{cv,F}$, and in a otherwise identical setup), its temperature gradient, integrated fluctuating kinetic energy, and differential rotation would not differ from those achieved in our simulations at high supercriticality. Moreover, we believe that some aspects of the rotationally-constrained cases are approaching a diffusion-free state for which a similar conclusion could be drawn, but simulations with a higher supercriticality are required to demonstrate this conclusively.

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All authors contributed to the interpretation of results and to writing the manuscript.

Software: Dedalus v3 (K. J. Burns et al. 2020).

AUTHOR CONTRIBUTIONS

NTL, MKB, LKC, and ST designed the research. NTL performed the numerical simulations and data analysis.

APPENDIX

A. SIMULATION INPUT PARAMETERS

Table 2. All simulations were run with $Pr = 1$ and $\eta = 0.8$, where $\eta = r_i/r_o$ is the aspect ratio of the convection zone, and use a non-dimensional depth $\tilde{\delta} = 0.2$ for the heated and cooled regions. Below, N_r is the number of Chebyshev modes used, and ℓ_{\max} is the maximum spherical harmonic degree. n_r and n_θ are the corresponding number of radial and latitudinal grid points (using 3/2 dealiasing).

Input Parameters						
$Ro_{cv,F}$	Ra_F	Ta	N_r	ℓ_{\max}	n_r	n_θ
1.44×10^{-2}	9.5×10^7	1×10^9	139	383	210	576
2.12×10^{-2}	3×10^5	1×10^7	99	127	150	192
3.11×10^{-2}	1.6×10^5	3×10^6	99	127	150	192
	9.5×10^5	1×10^7	99	127	150	192
	4.9×10^6	3×10^7	119	255	180	384
	3×10^7	1×10^8	119	255	180	384
	9.5×10^8	1×10^9	139	383	210	576
4.56×10^{-2}	3×10^6	1×10^7	119	255	180	384
6.70×10^{-2}	3×10^5	1×10^6	99	127	150	192
	1.6×10^6	3×10^6	99	127	150	192
	9.5×10^6	1×10^7	119	255	180	384
	4.9×10^7	3×10^7	119	255	180	384
	3×10^8	1×10^8	139	383	210	576
9.83×10^{-2}	9.5×10^5	1×10^6	99	127	150	192
	3×10^7	1×10^7	119	255	180	384
1.44×10^{-1}	3×10^6	1×10^6	99	127	150	192
	9.5×10^7	1×10^7	139	383	210	576
2.12×10^{-1}	9.5×10^6	1×10^6	99	127	150	192
	3×10^8	1×10^7	139	383	210	576
3.11×10^{-1}	9.5×10^5	1×10^5	79	63	120	96
	4.9×10^6	3×10^5	99	127	150	192

Table 2 *continued*

Table 2 (continued)

$Ro_{cv,F}$	Input Parameters					
	Ra_F	Ta	N_r	ℓ_{\max}	n_r	n_θ
	3×10^7	1×10^6	119	255	180	384
	1.6×10^8	3×10^6	139	383	210	576
	9.5×10^8	1×10^7	159	511	240	768
4.56×10^{-1}	3×10^6	1×10^5	99	127	180	384
	9.5×10^7	1×10^6	139	383	210	576
	3×10^9	1×10^7	159	511	240	768
6.70×10^{-1}	9.5×10^6	1×10^5	119	255	180	384
	4.9×10^7	3×10^5	139	383	210	576
	3×10^8	1×10^6	139	383	210	576
9.83×10^{-1}	3×10^7	1×10^5	119	255	180	384
	9.5×10^8	1×10^6	159	511	240	768
1.44	9.5×10^7	1×10^5	139	383	210	576
	3×10^9	1×10^6	159	511	240	768
2.12	3×10^8	1×10^5	159	511	240	768
	9.5×10^9	1×10^6	159	511	240	768
3.11	9.5×10^8	1×10^5	159	511	240	768
4.56	3×10^9	1×10^5	159	511	240	768

REFERENCES

- Ahlers, G., Grossmann, S., & Lohse, D. 2009, Reviews of Modern Physics, 81, 503, doi: [10.1103/RevModPhys.81.503](https://doi.org/10.1103/RevModPhys.81.503)
- Aubert, J., Brito, D., Nataf, H.-C., Cardin, P., & Masson, J.-P. 2001, Physics of the Earth and Planetary Interiors, 128, 51, doi: [10.1016/S0031-9201\(01\)00277-1](https://doi.org/10.1016/S0031-9201(01)00277-1)
- Aurnou, J. M., Horn, S., & Julien, K. 2020, Physical Review Research, 2, 043115, doi: [10.1103/PhysRevResearch.2.043115](https://doi.org/10.1103/PhysRevResearch.2.043115)
- Baraffe, I., Homeier, D., Allard, F., & Chabrier, G. 2015, Astronomy and Astrophysics, 577, A42, doi: [10.1051/0004-6361/201425481](https://doi.org/10.1051/0004-6361/201425481)
- Barker, A. J., Dempsey, A. M., & Lithwick, Y. 2014, The Astrophysical Journal, 791, 13, doi: [10.1088/0004-637X/791/1/13](https://doi.org/10.1088/0004-637X/791/1/13)
- Böhm-Vitense, E. 1958, Zeitschrift für Astrophysik, 46, 108
- Bouillaut, V., Miquel, B., Julien, K., Aumaitre, S., & Gallet, B. 2021, Proceedings of the National Academy of Science, 118, e2105015118, doi: [10.1073/pnas.2105015118](https://doi.org/10.1073/pnas.2105015118)
- Browning, M. K. 2008, The Astrophysical Journal, 676, 1262, doi: [10.1086/527432](https://doi.org/10.1086/527432)
- Brun, A. S. 2020, in Astrophysics and Space Science Proceedings, Vol. 57, Dynamics of the Sun and Stars; Honoring the Life and Work of Michael J. Thompson, ed. M. J. P. F. G. Monteiro, R. A. García, J. Christensen-Dalsgaard, & S. W. McIntosh, 75–89, doi: [10.1007/978-3-030-55336-4_9](https://doi.org/10.1007/978-3-030-55336-4_9)
- Brun, A. S., & Toomre, J. 2002, The Astrophysical Journal, 570, 865, doi: [10.1086/339228](https://doi.org/10.1086/339228)
- Burns, K. J., Vasil, G. M., Oishi, J. S., Lecoanet, D., & Brown, B. P. 2020, Physical Review Research, 2, 023068, doi: [10.1103/PhysRevResearch.2.023068](https://doi.org/10.1103/PhysRevResearch.2.023068)
- Camisassa, M. E., & Featherstone, N. A. 2022, The Astrophysical Journal, 938, 65, doi: [10.3847/1538-4357/ac879f](https://doi.org/10.3847/1538-4357/ac879f)
- Chandrasekhar, S. 1961, Hydrodynamic and hydromagnetic stability
- Christensen, U. R. 2002, Journal of Fluid Mechanics, 470, 115, doi: [10.1017/S0022112002002008](https://doi.org/10.1017/S0022112002002008)
- Christensen, U. R., & Aubert, J. 2006, Geophysical Journal International, 166, 97, doi: [10.1111/j.1365-246X.2006.03009.x](https://doi.org/10.1111/j.1365-246X.2006.03009.x)

- Currie, L. K., Barker, A. J., Lithwick, Y., & Browning, M. K. 2020, *Monthly Notices of the Royal Astronomical Society*, 493, 5233, doi: [10.1093/mnras/staa372](https://doi.org/10.1093/mnras/staa372)
- Featherstone, N. A., & Hindman, B. W. 2016, *The Astrophysical Journal*, 818, 32, doi: [10.3847/0004-637X/818/1/32](https://doi.org/10.3847/0004-637X/818/1/32)
- Featherstone, N. A., & Miesch, M. S. 2015, *The Astrophysical Journal*, 804, 67
- Gastine, T., & Aurnou, J. M. 2023, *Journal of Fluid Mechanics*, 954, R1, doi: [10.1017/jfm.2022.1010](https://doi.org/10.1017/jfm.2022.1010)
- Gastine, T., & Wicht, J. 2012, *Icarus*, 219, 428, doi: [10.1016/j.icarus.2012.03.018](https://doi.org/10.1016/j.icarus.2012.03.018)
- Gastine, T., Wicht, J., & Aubert, J. 2016, *Journal of Fluid Mechanics*, 808, 690, doi: [10.1017/jfm.2016.659](https://doi.org/10.1017/jfm.2016.659)
- Gastine, T., Wicht, J., & Aurnou, J. M. 2013, *Icarus*, 225, 156, doi: [10.1016/j.icarus.2013.02.031](https://doi.org/10.1016/j.icarus.2013.02.031)
- Gastine, T., Wicht, J., & Aurnou, J. M. 2015, *Journal of Fluid Mechanics*, 778, 721, doi: [10.1017/jfm.2015.401](https://doi.org/10.1017/jfm.2015.401)
- Gastine, T., Yadav, R. K., Morin, J., Reiners, A., & Wicht, J. 2014, *Monthly Notices of the Royal Astronomical Society*, 438, L76, doi: [10.1093/mnrasl/slt162](https://doi.org/10.1093/mnrasl/slt162)
- Gilman, P. A. 1977, *Geophysical and Astrophysical Fluid Dynamics*, 8, 93, doi: [10.1080/03091927708240373](https://doi.org/10.1080/03091927708240373)
- Gilman, P. A. 1979, *The Astrophysical Journal*, 231, 284, doi: [10.1086/157191](https://doi.org/10.1086/157191)
- Grossmann, S., & Lohse, D. 2000, *Journal of Fluid Mechanics*, 407, 27, doi: [10.1017/S0022112099007545](https://doi.org/10.1017/S0022112099007545)
- Hadjerci, G., Bouillaut, V., Miquel, B., & Gallet, B. 2024, *Journal of Fluid Mechanics*, 998, A9, doi: [10.1017/jfm.2024.556](https://doi.org/10.1017/jfm.2024.556)
- Hindman, B. W., Featherstone, N. A., & Julien, K. 2020, *The Astrophysical Journal*, 898, 120, doi: [10.3847/1538-4357/ab9ec2](https://doi.org/10.3847/1538-4357/ab9ec2)
- Hotta, H., Kusano, K., & Shimada, R. 2022, *The Astrophysical Journal*, 933, 199, doi: [10.3847/1538-4357/ac7395](https://doi.org/10.3847/1538-4357/ac7395)
- Hotta, H., Rempel, M., & Yokoyama, T. 2015, *The Astrophysical Journal*, 798, 51, doi: [10.1088/0004-637X/798/1/51](https://doi.org/10.1088/0004-637X/798/1/51)
- Joshi-Hartley, T., Browning, M. K., Currie, L. K., et al. 2025, *Monthly Notices of the Royal Astronomical Society*, 541, 2291, doi: [10.1093/mnras/staf1131](https://doi.org/10.1093/mnras/staf1131)
- Julien, K., Knobloch, E., Rubio, A. M., & Vasil, G. M. 2012, *Physical Review Letters*, 109, 254503, doi: [10.1103/PhysRevLett.109.254503](https://doi.org/10.1103/PhysRevLett.109.254503)
- Käpylä, P. J. 2023, *Astronomy and Astrophysics*, 669, A98, doi: [10.1051/0004-6361/202244395](https://doi.org/10.1051/0004-6361/202244395)
- Käpylä, P. J. 2024, *Astronomy and Astrophysics*, 683, A221, doi: [10.1051/0004-6361/202348325](https://doi.org/10.1051/0004-6361/202348325)
- Käpylä, P. J., Browning, M. K., Brun, A. S., Guerrero, G., & Warnecke, J. 2023, *Space Science Reviews*, 219, 58, doi: [10.1007/s11214-023-01005-6](https://doi.org/10.1007/s11214-023-01005-6)
- Käpylä, P. J., Käpylä, M. J., & Brandenburg, A. 2014, *Astronomy and Astrophysics*, 570, A43, doi: [10.1051/0004-6361/201423412](https://doi.org/10.1051/0004-6361/201423412)
- Kazemi, S., Ostilla-Mónico, R., & Goluskin, D. 2022, *Physical Review Letters*, 129, 024501, doi: [10.1103/PhysRevLett.129.024501](https://doi.org/10.1103/PhysRevLett.129.024501)
- King, E. M., Stellmach, S., & Aurnou, J. M. 2012, *Journal of Fluid Mechanics*, 691, 568, doi: [10.1017/jfm.2011.493](https://doi.org/10.1017/jfm.2011.493)
- King, E. M., Stellmach, S., & Buffett, B. 2013, *Journal of Fluid Mechanics*, 717, 449, doi: [10.1017/jfm.2012.586](https://doi.org/10.1017/jfm.2012.586)
- Lepot, S., Aumaître, S., & Gallet, B. 2018, *Proceedings of the National Academy of Science*, 115, 8937, doi: [10.1073/pnas.1806823115](https://doi.org/10.1073/pnas.1806823115)
- Miesch, M. S., Brun, A. S., DeRosa, M. L., & Toomre, J. 2008, *The Astrophysical Journal*, 673, 557, doi: [10.1086/523838](https://doi.org/10.1086/523838)
- Noyes, R. W., Hartmann, L. W., Baliunas, S. L., Duncan, D. K., & Vaughan, A. H. 1984, *The Astrophysical Journal*, 279, 763, doi: [10.1086/161945](https://doi.org/10.1086/161945)
- O'Mara, B., Miesch, M. S., Featherstone, N. A., & Augustson, K. C. 2016, *Advances in Space Research*, 58, 1475, doi: [10.1016/j.asr.2016.03.038](https://doi.org/10.1016/j.asr.2016.03.038)
- Soderlund, K. M., Wulff, P., Käpylä, P. J., & Aurnou, J. M. 2025, *Monthly Notices of the Royal Astronomical Society*, 541, 1816, doi: [10.1093/mnras/staf1081](https://doi.org/10.1093/mnras/staf1081)
- Stevenson, D. J. 1979, *Geophysical and Astrophysical Fluid Dynamics*, 12, 139, doi: [10.1080/03091927908242681](https://doi.org/10.1080/03091927908242681)
- Yadav, R. K., Gastine, T., Christensen, U. R., Duarte, L. D. V., & Reiners, A. 2016, *Geophysical Journal International*, 204, 1120, doi: [10.1093/gji/ggv506](https://doi.org/10.1093/gji/ggv506)