Chapter 2

Option pricing

The concept of options has a very long history. The early ones were used by farmers who may agree to sell crops at a fixed price months ahead from now instead of taking a chance on the vagaries of market price. The cornerstone of pricing options is the seminar paper of Black and Scholes (1973). Options on stocks were first traded on an organized exchange in the year which coincides with the paper was published. Now options and other derivatives are widely traded on exchanges throughout the world. Their underlying assets consist of currencies, interest rates, stocks, stock indices, debt instruments, commodities and future contracts. The financial market of derivatives is larger than the market in stock securities. Namely, money invested in derivatives is much more than that invested in their underlying assets like stocks. This chapter introduces options and studies option pricing. We first consider option pricing in the discrete time set-up and then we move to the continuous time model of geometric Brownian motion and derive the Black-Scholes formula for European options. We also consider option pricing under models with stochastic volatility and jumps. Finally, we will discuss pricing American options.

2.1 Options

Definition 1 *Options are classified as call and put options. A call option gives the holder the right (not obligation) to buy a specified number of shares (usually 100 shares) of a certain asset (such as a stock) for a certain price by a future date, while a put option gives the right (not obligation) to sell the asset for a certain price by a future date. The price specified in the option is called the exercise price or strike price, and the date in the* *contract is known as expiration date, exercise date or maturity. Two common options are European options and American options. A European option can exercise only at expiration date, while an American option can exercise any time before or at expiration date.*

For every option contract, there are two sides. One is the investor who has bought or own the option and usually referred to as taking the long position, while the other is the investor who has sold or written the option and usually referred to as taking the short position. The buyer pays cash to the writer of an option for owning the option. The writer of an option receives cash up front but has liabilities later when the owner exercise the option. The profit or loss of the writer is the reverse of the buyer.

Example 1 *Suppose that an investor owns a European call option to buy 100 shares of stock A at \$25 per share expired in six months. Suppose that the option costs \$3 per share. The owner paid \$300 premium to buy the option. The risk-free interest rate* *is 8% per annum. The profit of the option depends on the stock price at expiration date. We consider three cases.*

(1). Suppose the stock is selling for \$30 per share six months later. The option is quite valuable. It allows you to purchase 100 shares of the stocks at \$25 per share. With market price \$30 per share, you immediately sell them at the market price and gain \$500. With \$300 paid up front, the net profit in dollar value at the time of purchase is

$$\exp(-0.08/2) \times \$500 - \$300 = \$180.40.$$

At the exercise date the dollar value of the net profit is

$$\$500 - \exp(0.08/2) \times \$300 = \$187.76.$$

(2). If stock price falls below the exercise price at the maturity, a call option is worthless. Suppose the stock is selling for \$24 per share six months later. The owner will not exercise the call option, since the owner can buy the stock \$24 per share from the market and exercising the option would amount to

purchasing the stock \$1 more per share than its market price. The net loss at the time of purchase is the option premium \$300.

(3). Suppose the stock is selling for \$27 per share six months later. The owner can exercise the option to buy 100 shares of stocks at \$25 per share and immediately sell them at \$27 per share. The gain is \$200. With \$300 premium paid for the option, the owner has a net loss in dollar value at the time of purchase

$$\$300 - \exp(-0.08/2) \times \$200 = \$107.84.$$

and has a net loss in dollar value at the exercise date

$$\exp(0.08/2) \times \$300 - \$200 = \$112.24.$$

At maturity, if stock price gets higher than the strike price of an option, the option will be exercised. However, as above case demonstrated, the amount gained by exercising the option may be less than the premium paid for the option, and the owner may still lose money. When stock price is higher *than the strike price, by exercising an option the holder of the option may still have loss, but exercising the option makes less loss than not exercising.*

Example 2 *Suppose that an investor owns a European put option to sell 100 shares of stock B at \$40 per share expired in three months. Suppose that the option costs \$1 per share. The owner paid \$100 premium to buy the option. The risk-free interest rate is 6% per annum. Again the profit of the option depends on the stock price at expiration date.*

(1). Suppose the stock has a market price \$38 per share three months later. The owner can buy 100 shares of the stocks at \$38 per share from the market. The option allows him or her to immediately sell them to the writer at \$40 per share. The gain is \$200. The net profit in dollar value at the time of purchase is

$$\exp(-0.06/4) \times \$200 - \$100 = \$97.02.$$

At the exercise date the dollar value of the net profit is

$$\$200 - \exp(0.06/4) \times \$100 = \$98.49.$$

(2). If stock price is higher than the exercise price at the maturity, a put option is worthless. Suppose the stock is selling for $41 per share six months later. The owner will not exercise the put option, since the owner can sell the stocks $41 per share in the market and exercising the option would amount to sell them $40 per share. The net loss at the time of purchase is the option premium $100.

The examples show that there is some chance that an option is worthless and some chance that it leads to a gain. The option has a value and one needs to pay a premium to buy it. The basic question in this chapter is what is the fair premium to pay for purchasing an option. Option pricing is to find the fair premium. The price of a European option depends on the six factors: current asset price S_0, strike price K, time to maturity T, asset price S_T at maturity, risk-free interest rate r, and volatility σ of asset price. For a European call option, the pay-off at the maturity is

$$\max(S_T - K, 0) = (S_T - K)_+,$$

and for a European put option, the pay-off at the maturity is

$$\max(K - S_T, 0) = (K - S_T)_+.$$

2.2 Arbitrage

Arbitrage is often referred to as “free lunch” in plain English. It means locking in a guaranteed risk-free profit by trading in the market without investing capital. Arbitrage often involves with simultaneously entering transactions in two or more markets.

Example 3 *Suppose that a stock is traded on the New York Stock Exchange at \$88 per share and on the London Stock Exchange at £50 per share when exchange rate is \$1.80 per pound. An arbitrage opportunity occurs: simultaneously purchase 100 shares of the stock in the New York Stock Exchange and sell them in the London Stock Exchange and obtain a risk-free profit*

$$100 \times (\$1.80 \times £50 - \$88) = \$200.$$

Example 4 *Suppose that a stock price is \$20 per share at current time and at the end of three months the stock price may either move up to \$22 per share or down to \$18 per share. Consider a European call option which allows to buy 100 shares of the stocks at \$21 per share at the end of three months. The risk free rate is 12% per annum.*

(1). Suppose that the option can be brought or sold at \$85 for 100 shares (or \$0.85 per share). You can make a risk-free profit. Sell one option for \$85, use the money and borrow \$415 to buy 25 shares of the stock. At the end of three months, if the stock price moves up to \$22 per share, the option holder wants to exercise the option. With the 25 shares, you sell him 100 shares of the stocks at \$21 per share by purchasing another 75 shares from the market at \$22 per share and pay back the borrowed \$415 plus interest. You obtain the profit

$$100 \times 21 - 75 \times \$22 - \exp(0.12/4) \times \$415 = \$22.36.$$

On the other hand, if the stock price is \$18 per share at the *end of three months, the option is worthless, and its owner will not exercise. You sell 25 shares of the stocks at $18 per share and pay back $415 plus interest, with the profit*

$$25 \times \$18 - \exp(0.12/4) \times \$415 = \$22.36.$$

Regardless of the movement of stock price, the net profit of $22.36 is guaranteed.

(2). Suppose the option can be brought or sold at $45 for 100 shares (or $0.40 per share). An arbitrage opportunity exists. Sell 25 shares of the stock short with $500 cash in hand. Purchase one option with $40 and leave rest $460 in the bank. At the end of three months, if the stock price is $22 per share, you exercise the option by purchasing 100 shares of the stock at $21 per share. Return 25 shares and sell the other 75 shares at market price $22 per share. Your net profit is

$$75 \times \$22 - 100 \times \$21 + \exp(0.12/4) \times 460 = \$24.01.$$

However, if the stock price goes down to $18 per share. You *will not exercise the option. Simply return 25 shares of stock by purchasing 25 shares from the market at \$18 per share. Your net profit is still*

$$\exp(0.12/4) \times 460 - 25 \times \$18 = \$24.01.$$

We will show below that the fair price of the option is \$63.3 for 100 shares (or \$0.633 per share). Prices higher (or lower) than \$0.633 per share for the option will provide arbitrage opportunities by creating trading strategies with selling (or buying) the option at expensive (or cheap) price.

Major arbitrage opportunities occur rarely and if occur, they usually last for a short period of time. In Example 4.3, the buying of the stock in the New York Stock Exchange will cause its dollar price to rise, while selling of the stock at the London Stock Exchange will drive down the sterling price of the stock. That is, the forces of supply and demand will quickly make the two prices equivalent at the current exchange rate. In Example 4.4, with the option price at \$85, one can lock a guaranteed profit of \$22.36. Many profit hungry arbitrageurs will observe this arbitrage opportunity and rush in to lock the profit. Because of competition, some one will decide to sell the same option at less expensive price say at \$75 and still make a guaranteed profit. The better deal will draw all option buyers. But then some one else will sell the same option even lower price of \$70 but still make a risk-free profit. This would continue as long as the option price is above the fair price \$63.3. Similarly, with the option price below the fair price \$63.3, the competition will drive its price up to the fair price.

It is a wide belief that without taking a risk, one can not make a profit on free competitive markets. We make general assumption that no arbitrage exists in financial markets. The arbitrage price of an asset is referred to as the price under which no arbitrage opportunities exist. The no arbitrage assumption is also equivalent to the so called the law of one price, which means that two financial instruments have the same price if their payoffs are exactly the same. One can price an option by finding a portfolio with a known price and having exactly the same payoffs as the option. By the law of one price, the price of the option must be equal to the known price of the portfolio. Pricing by the law of one price is also called arbitrage pricing.

2.3 Put-Call parity

The prices of European call and put options are functions of the six factors: current asset price S_0, strike price K, time to maturity T, asset price S_T at maturity, risk-free interest rate r, and volatility σ of asset price. Denote by C and P the prices of the European call and put options, respectively. Then C and P obey the following important rule,

$$C + K\,e^{-r\,T} = P + S_0. \tag{2.1}$$

Such a relationship is often referred to as put-call parity. The put-call parity is often used to compute put option price from call option price or vice versa. To show the identity, we consider two portfolios:

Portfolio A: one European call option plus cash of amount equal to $K\,e^{-r\,T}$;

Portfolio B: one European put option plus one share of the underlying asset.

Denote by S_T the asset price at expiration of the options. Then at expiration of the options, portfolio A has value $\max(S_T - K, 0) + e^{rT} \times K\,e^{-rT} = \max(S_T, K)$; while the value of portfolio B is $\max(K - S_T, 0) + S_T = \max(K, S_T)$. That is, the two portfolios have the same payoff at expiration. As European options can not be exercised before the expiration date, the law of one price implies that the two portfolios must have the same value at the current time. The current values of portfolios A and B are equal to the left and right hand sides of equation (2.1), respectively.

2.4 Binomial tree pricing

Binomial tree is a popular and useful technique for pricing an option in the discrete time set-up. It uses a tree like diagram to represent possible paths that the price of the underlying asset may move along over the life of the option.

2.4.1 One-step binomial model

We start with a simple example to see how a European call option can be priced.

Example 5 *Suppose that a stock has price \$50 at current time and its price at the end of six months will be either \$45 or \$55. Consider a European call option with maturity six months and strike price \$53. The risk-free interest rate is 6% per annum.*

At the end of six months, the option will have value \$2 if the stock price moves up to \$55 and zero value if the stock price turns out to be \$45. The scenario is illustrated in Figure 5

Stock price \$55

Option price=\$2

$\nearrow$

Stock price=\$50

$\searrow$

Stock price \$45

Option price=\$0

To price the option we consider a portfolio of buying Δ *shares of the stock and selling one call option, which is often referred to as a long position in* Δ *shares of the stock and a short position in one call option. We settle the value of* Δ *by making the portfolio risk-free. At the expiration date, if the stock price goes to \$55, the shares of stock has value* $\$55\,\Delta$*; the selling of the option results in \$2 loss. The \$2 is paid to the option owner who exercises the option. The portfolio has total value* $\$55\,\Delta - 2$ *at expiration. On the other hand, if the stock price moves down from \$50 to \$45, the option has zero value and hence portfolio has value* $\$45\,\Delta$*. To make the portfolio riskless, the portfolio value at expiration has to be the same for both alternative stock prices, that is*

$$55\,\Delta - 2 = 45\,\Delta.$$

Solve for Δ *and obtain* $\Delta = 0.2$*. The riskless portfolio consists of* 0.2 *shares of the stock and short one option. Regardless of the stock price moves up or down, the portfolio has a fixed value \$9 at the maturity of the option. With no arbitrage assumption, the* *riskless portfolio will earn risk-free rate of interest. Discounting \$9 back to today, we obtain the portfolio value at present time*

$$\exp(-0.6 \times 0.5) \times \$9 = \$8.734.$$

Denote by C_0 *the fair option price at current time. As the stock has value \$50 today. At current time the portfolio has dollar value*

$$50 \times 0.2 - C_0 = 10 - C_0,$$

which must be equal to \$8.734. We easily arrive at $C_0 = \$1.266$. *With the absence of arbitrage, the option has fair value \$1.266. As shown in Section 4.2, if the option had value less or more than \$1.266, arbitrage opportunities would exist. With option value more than \$1.266, the portfolio would cost less than \$9 to set up and would earn interest at rate higher than the risk-free rate, while with option value less than \$1.266, shorting the portfolio would mean borrowing money at interest rate lower than the risk-free rate.*

Above argument can be used to derive option price in general situations. Suppose that a non-dividend-paying stock has current price S_0 and at the end of time period T, its price can either move up from S_0 to a new level $S_0\,u$ or down from S_0 to a new level $S_0\,d$. Consider an European call option with maturity T and strike price K. Suppose the risk-free interest rate is r. Denote by C_0 the fair value of the option at current time, and C_d and C_u the respective values of the option at expiration when stock prices are $S_0\,d$ and $S_0\,u$. Then $C_u = \max(S_0\,u - K, 0)$ and $C_u = \max(S_0\,d - K, 0)$. To make the problem realistic we need to assume $0 < d < 1 \leq \exp(r\,T) < u$.

Set up a portfolio with Δ_0 shares of stock and short a call option. At the end of the life of the option, the portfolio has value $S_0\,u\,\Delta_0 - C_u$ when the stock price moves up to $S_0\,u$ and $S_0\,d\,\Delta_0 - C_d$ when the stock price moves down to $S_0\,d$. To make the portfolio riskless, we equate the two portfolio values

$$S_0\,u\,\Delta_0 - C_u = S_0\,d\,\Delta_0 - C_d.$$

and solve for Δ_0

$$\Delta_0 = \frac{C_u - C_d}{S_0\,(u-d)}.$$

With above given Δ_0 the portfolio has value at expiration

$$S_0\,u\,\Delta_0 - C_u = S_0\,d\,\Delta_0 - C_d = \frac{C_u\,d - C_d\,u}{u-d},$$

The portfolio is risk-free and must earn the risk-free interest rate. Thus it has value at present time

$$\exp(-r\,T)\,\frac{C_u\,d - C_d\,u}{u-d}.$$

The cost of setting up the portfolio at the present time is $S_0\,\Delta_0$ for stock shares minus C_0 for shorting one option, that is,

$$S_0\,\Delta_0 - C_0.$$

Equating the two portfolio values we arrive at

$$C_0 = \exp(-r\,T)\,[q\,C_u + (1-q)\,C_d], \tag{2.2}$$

where

$$q = \frac{\exp(r\,T) - d}{u-d}. \tag{2.3}$$

Example 6 *In Example 4.5,* $S_0 = 50$*,* $u = 1.1$*,* $d = 0.9$*,* $T = 0.5$*,* $K = 53$ $r = 0.6$*,* $C_d = 0$ *and* $C_u = 2$*. From formulas in (2.2) and (2.3), we compute*

$$q = \frac{\exp(0.03) - 0.9}{1.1 - 0.9} = 0.6523, \qquad C_0 = \exp(-0.03)\times(0.6523\times 2 + 0.3477\times 0)$$

This is the same as the answer obtained in Example 4.5.

Example 7 *For Example 4.4 considered in Section 2,* $S_0 = 20$*,* $u = 1.1$*,* $d = 0.9$*,* $T = 0.25$*,* $K = 21$ $r = 0.12$*,* $C_d = 0$ *and* $C_u = 1$*. The formulas in (2.2) and (2.3) lead to*

$$q = 0.6523, \qquad C_0 = \exp(-0.03)\times(0.6523\times 1 + 0.3477\times 0) = 0.633.$$

As we mentioned there, this is the fair price for which no arbitrage opportunities exist.

Under the realistic assumption $0 < d < 1 \leq \exp(r\,T) < u$, q defined by (2.3) is between zero and one and can be interpretated as a probability that the stock price moves up from S_0 to $S_0\,u$ at the expiration date. Hence, the expression $q\,C_u + (1-q)\,C_d$ is the expected payoff of the option at maturity with respect to the probability q of moving up of the stock and probability $1 - q$ of moving down, and C_0 given by (2.2) is the discounted expected payoff of the option. The fabraticated probability q derived from no arbitrage condition is different from the probability that the stock price actually moves up or down in the real world, which is called physical probability. The option price C_0 does not depend on the physical probability or the expected stock return actually occurred in the real world. This is clearly indicated by formulas (2.2) and (2.3), or the argument showing that the option must have the same value as the replicating portfolio regardless of whether the stock moves up or down. For option pricing, it is irrelevant whether the physical probability of the stock going up is large as 0.95 or small like 0.1. The probability measure defined by q is called the risk-neutral measure (or the risk-neutral probability or the pricing measure). In such a hypothetical world, at the expiration date T, the stock price S_T is equal to $S_0\,u$ with probability q and $S_0\,d$ with probability $1-q$. Given S_0, the expected stock price with respect to the risk-neutral probability q can be easily computed as follows,

$$E_q(S_T|S_0) = q\,S_0\,u + (1-q)\,S_0\,d = q\,S_0\,(u-d) + S_0\,d = S_0\,\exp(r\,T),$$

Thus, in the risk-neutral world the stock price grows on average at the risk-free rate. Define discounted price $S_i^* = \exp(-r\,i)\,S_i$, $i = 0, T$. Then

$$E_q(S_T^*|S_0^*) = S_0^*,$$

that is, with respect to risk-neutral probability discounted price S_i^* is a martingale. Because of this, the risk-neutral measure is also referred to as the martingale measure.

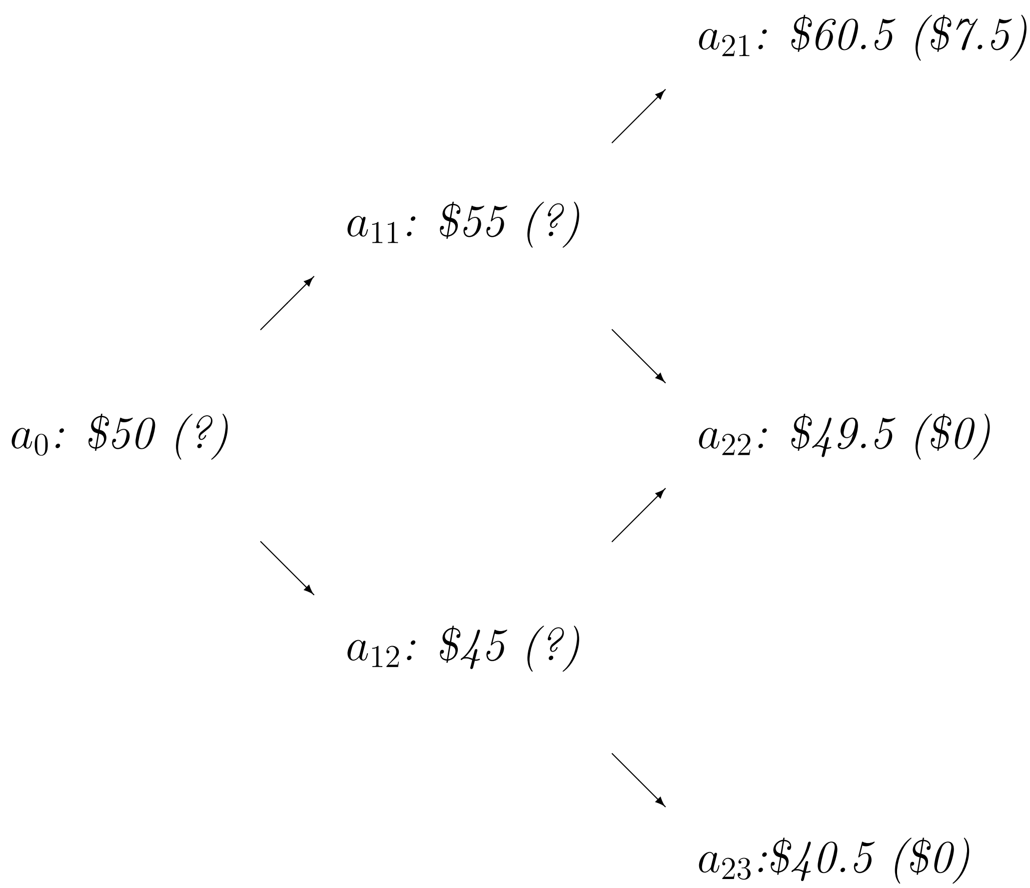
In the risk-neutral world, every investor is risk-neutral, meaning being indifferent to risk. No compensation for risk is required and stock prices are expected to rise at risk-free interest rate. When pricing an option, we can assume with complete impunity a risk-neutral world by setting q for the probability of stock price moving up, and the option price is the discounted expected payoff of the option at expiration in the risk-neutral world. Such pricing is often referred to as risk-neutral valuation. It is an important general principle in option pricing.

2.4.2 Multi-step binomial model

The analysis for one-step binomial model can be extended to multi-step binomial model. First consider an example by extending one-step binomial model in Example 4.5 to a two-step binomial tree.

Example 8 *Consider the same stock as in Example 4.5 and two time steps with six months in length for each time step. The stock price starts with \$50 and at each of two time steps may move up by 10% or down by 10%. The risk-free interest rate is 6% per annum. Consider a European call option with maturity one year and strike price \$53. The tree is shown in Figure 8, where each node lists the stock price with option payoff in parenthesis. The objective is to calculate the option price at current time corresponding to the initial node of the tree.*

At the end of one year, the stock price has three possibilities: \$60.5, \$49.5, and \$40.5, with respective payoff \$7.5, \$0, and \$0 for the option.



We break the two-step binomial tree into three one-step trees and repeatedly apply the risk-neutral valuation established for one-step binomial tree to each of the three one-step trees backward. For nodes a_{21}, a_{22} and a_{23} corresponding to the end of one year, the option prices listed in Figure are obtained easily from the payoff of the option at expiration. For node a_{11}, we compute *risk-neutral probability*

$$q = \frac{\exp(0.06/2) - 0.9}{1.1 - 0.9} = 0.6523.$$

With the option price at nodes a_{21} and a_{22} are \$7.5 and zero respectively, the option price at node a_{11} is the discounted expected payoff in the risk-neutral world: $\exp(-0.06/2) \times (0.6523 \times \$7.5 + 0.3477 \times \$0) = \$4.7477$. *Since the option price at nodes a_{22} and a_{23} are zero, the option price at node a_{12} is zero. For the initial node a_0, the risk-neutral probability is also* 0.6523, *and the option price is* $\exp(-0.06/2) \times (0.6523 \times \$4.7477 + 0.3477 \times \$0) = \$3.0054$.

For a general multi-step binomial tree model. Use index $0, 1, 2, \cdots$ to stand for present time, one-step ahead, two-step ahead, etc. Denote by τ time length between consecutive steps. Suppose the binomial tree has n steps, and the stock has initial price S_0 and then at each step, its price can only move up by a factor u or down by a factor d. Let S_i, $i = 0, 1, \cdots, n$, be the price of the stock at the end of the i-th step. Then S_i is of form $S_0\, u^j\, d^{i-j}$, where j is equal to the number of times during the past i steps where the price moves up along the tree.

Let

$$q = \frac{\exp(r\,\tau) - d}{u - d}$$

be the probability of moving up and $1 - q$ the probability of going down at each step, and denote by Q the risk-neutral probability on the whole tree. Define the discounted stock price $S_i^* = \exp(-r\,\tau)\,S_i$. Then with respect to the risk-neutral probability Q,

$$E_Q[S_{i+1}^* | S_i^*, \cdots, S_0^*] = \exp(-r\,(i+1)\tau)\,[q\,S_i\,u + (1-q)\,S_i\,d] = S_i^*,$$

that is, S_i^* is a martingale. The option price at time 0 is the discounted expected payoff with respect to risk-neutral probability. For an European call option with strike price K and expiration $T = n\,\tau$, the option price at present time

$$C_0 = \exp(-r\,T)\,E_Q[(S_n - K)_+].$$

Since S_n takes value of the form $S_0\,u^j\,d^{n-j}$ and follows a binomial distribution under the risk-neutral probability Q

$$Q[S_n = S_0\,u^j\,d^{n-j}] = \begin{pmatrix} n \\ j \end{pmatrix} q^j\,(1-q)^{n-j},$$

the option price has expression

$$C_0 = \exp(-r\,T) \sum_{j=0}^{n} \left(S_0\, u^j\, d^{n-j} - K\right)_+ \begin{pmatrix} n \\ j \end{pmatrix} q^j\,(1-q)^{n-j}.$$

The formula looks like pretty complicated. As illustrated in Example 4.6, the calculations actually can be easily processed on the tree via risk-neutral valuation one step a time backward.

2.4.3 From binomial tree to geometric Brownian motion

The simplicity of binomial trees makes them very useful in illustrating important pricing concepts. However, stock prices in reality are continuous, at least approximately, so binomial trees of limited steps can not keep track stock price movement well. As a result, simple binomial models with limited steps can obtain very rough approximation to option prices. To make them useful in practice, we increase the number of steps in a binomial model in order to divide the life of the option into many steps with small time length between adjacent steps. This section will show that as the number of steps increases without limit, the binomial model approaches to the Black-Scholes model where the stock price is governed by geometric Brownian motion.

Consider a call option with unit expiration. It lives on unit interval $[0, 1]$. For $i = 0, 1, \cdots, n$, let $t_i = i/n$. Divide $[0, 1]$ into n steps with subintervals $[t_{i-1}, t_i]$, of length $1/n$, $i = 1, \cdots, n$. As usual, the stock has initial price S_0 and moves up by a factor u or down by a factor d, where

$$u = \exp\left(\mu/n + \sigma/\sqrt{n}\right), \qquad d = \exp\left(\mu/n - \sigma/\sqrt{n}\right),$$

μ is a real number and σ is a positive number. Later we will see that μ and σ correspond to the drift and volatility of the stock return. Denote by $S^n_{t_i}$ the stock price at the end of i-th step. Then

$$S^n_{t_i} = S_0 \exp\left(\mu\, t_i + \frac{\sigma}{\sqrt{n}} \sum_{j=1}^{i} Y_j\right),$$

where Y_j are independent random variables taking value either 1 or -1, corresponding to moving up or down of the stock price, respectively. As $\sum_{j=1}^{i} Y_j$, $i = 1, \cdots, n$, is a random walk, $S^n_{t_i}$ is a geometric random walk.

Assume under the physical probability P, the probability of moving up

$$p = P(Y_j = 1) = \frac{1}{2}\left(1 + \frac{0 - \sigma^2/2}{\sigma\sqrt{n}}\right).$$

Under P, $E_P[Y_j] = 2\,p - 1$ and $Var_P[Y_j] = 4\,p\,(1-p)$, and

$$E_P\left[\frac{\sigma}{\sqrt{n}}\sum_{j=1}^{i} Y_j\right] = \frac{\sigma\, i\,(2\,p-1)}{\sqrt{n}} = t_i\,(\mu - \sigma^2/2)$$

$$Var_P\left[\frac{\sigma}{\sqrt{n}}\sum_{j=1}^{i} Y_j\right] = \frac{4\,\sigma^2\, i\, p(1-p)}{n} \approx t_i\sigma^2.$$

As $n \to \infty$, $\{\sum_{j=1}^{i} Y_j/\sqrt{n} - t_i\,(\mu - \sigma^2/2)/\sigma, i = 1, \cdots, n\}$ as a stochastic process on $[0,1]$ converges in distribution to W_t, $t \in [0,1]$, and $\{\sigma\,\sum_{j=1}^{i} Y_j/\sqrt{n}, i = 1, \cdots, n\}$ converges in distribution to $(\mu - \sigma^2/2)\,t + \sigma\,W_t$, $t \in [0,1]$, where W_t is a standard Brownian motion. Therefore under P, $S_{t_i}^n$ converges in distribution to

$$S_t = S_0\,\exp\{(\mu - \sigma^2/2)\,t + \sigma\,W_t\}, \qquad t \in [0,1]. \tag{2.4}$$

which is called geometric Brownian motion with drift μ and volatility σ, as it geometrically relates to Brownian motion.

Denote by Q the risk-neutral probability. Under Q, at each step the stock price moves up with probability q and down with probability $1-q$, and Y_j are i.i.d. with $Q(Y_j = 1) = q = 1 - Q(Y_j = -1)$, where

$$\begin{aligned} q &= \frac{\exp(r/n) - d}{u - d} \\ &= \frac{\exp(r/n) - \exp\left(\mu/n - \sigma/\sqrt{n}\right)}{\exp\left(\mu/n + \sigma/\sqrt{n}\right) - \exp\left(\mu/n - \sigma/\sqrt{n}\right)} \end{aligned}$$

Using approximations $\exp(x) \approx 1 + x + x^2/2$ and $1/(1+x) \approx 1 - x$ for x close to zero, we obtain

$$q \approx \frac{1}{2}\left(1 - \frac{\mu - r + \sigma^2/2}{\sigma\sqrt{n}}\right).$$

Under the risk-neutral probability Q, $E_Q[Y_j] = 2\,q - 1$ and $Var_Q[Y_j] = 4\,q\,(1-q)$, and

$$E_Q\left[\frac{\sigma}{\sqrt{n}}\sum_{j=1}^{i} Y_j\right] = \frac{\sigma\, i\,(2\,q-1)}{\sqrt{n}} \approx t_i\,(r - \mu - \sigma^2/2)$$

$$Var_Q\left[\frac{\sigma}{\sqrt{n}}\sum_{j=1}^{i} Y_j\right] = \frac{4\,\sigma^2\, i\, q(1-q)}{n} \approx t_i \sigma^2.$$

Then under Q, as $n \to \infty$, $\{\sigma \sum_{j=1}^{i} Y_j/\sqrt{n}, i = 1, \cdots, n\}$ as a stochastic process on $[0,1]$ converges in distribution to $(r - \mu -$

$\sigma^2/2)\, t + \sigma\, W_t$, $t \in [0,1]$, where W_t is a standard Brownian. Therefore under Q, S_{t_i} converges in distribution to geometric Brownian motion

$$S_0 \exp\{(r - \sigma^2/2)\, t + \sigma\, W_t\}, \qquad t \in [0,1], \tag{2.5}$$

which corresponds to geometric Brownian motion in the physical world given by (2.4) with drift μ replaced by the risk-free interest rate r. This fact is general true for continuous time price processes when we switch from the physical measure to the risk-neutral measure.

2.5 The Black-Scholes formula

By Itô lemma we easily show that geometric Brownian motion S_t with drift μ and volatility σ satisfies stochastic differential equation

$$\frac{dS_t}{S_t} = \mu\, dt + \sigma\, dW_t. \tag{2.6}$$

As a continuous-time model the stochastic differential equation can be interpreted as a limit of the discrete model with $\Delta \to 0$,

$$\frac{S_{t+\Delta} - S_t}{S_t} = \mu\, \Delta + \sigma \sqrt{\Delta}\, \varepsilon_t,$$

where ε_t are standard normal random variables. The discrete model relates to stock return over time period Δ with mean rate μ of return and volatility σ.

Suppose that a stock price process S_t, $t \in [0, T]$, is geometric Brownian motion obeying (2.4). Consider an European call option with strike price K and maturity T. Again r is the constant risk-free interest rate.

At the expiration date T, the option has payoff $(S_T - K)_+$. According to the risk-neutral valuation, its price at present time must be

$$C_0 = \exp(-r\,T)\, E_Q[(S_T - K)_+], \tag{2.7}$$

where Q is the risk-neutral probability, and E_Q stands for the expectation taken with respect to Q. To derive the Black-Scholes formula we need to evaluate the expectation of $(S_T - K)_+$ under Q.

Under the risk-neutral probability Q, the stock price has expression (2.5), that is, replace the mean rate in the physical model by the risk-free interest rate. This can also be seen from the fact that the discounted stock price

$$S_t^* = \exp(-r\,t)\,S_0\,\exp\{(r-\sigma^2/2)\,t+\sigma\,W_t\} = S_0\,\exp(\sigma\,W_t-\sigma^2\,t/2)$$

is an exponential martingale. Moreover, it can be verified directly via Girsanov's theorem that the transformation from the physical probability P to the risk-neutral probability Q is to shift Brownian motion W_t by $t\,(\mu - r)/\sigma$.

Now we are ready to evaluate the expectation in (2.7). Let $X = log\, S_T - log\, S_0$. Then under Q, $X \sim N((r - \sigma^2/2)\,T, \sigma^2\,T)$ and $S_T = S_0\,e^X$. Expressing $(S_T - K)_+$ by X and simple direct manipulations show

$$\begin{aligned} E_Q[(S_T - K)_+] &= E_Q[(S_T - K)\,1(S_T \geq K)] \\ &= E_Q[S_T 1(S_T \geq K)] - K\,Q[S_T \geq K] \\ &= S_0\,E_Q[e^X 1(X \geq log\,(K/S_0))] - K\,Q[X \geq log\,(K/S_0)] \end{aligned} \tag{2.8}$$

Normality of X under Q leads to

$$\begin{aligned} Q[X \geq log\,(K/S_0)] &= 1 - \Phi\left(\frac{log\,(K/S_0) - (r - \sigma^2/2)\,T}{\sigma\,\sqrt{T}}\right) \\ &= \Phi\left(\frac{log\,(S_0/K) + (r - \sigma^2/2)\,T}{\sigma\,\sqrt{T}}\right), \end{aligned} \tag{2.9}$$

and

$$
\begin{aligned}
E_Q[e^X 1(X \geq log\,(K/S_0))] &= \int_{log\,(K/S_0)}^{\infty} e^x\, \phi\left(\frac{x-(r-\sigma^2/2)\,T}{\sigma\sqrt{T}}\right) dx \\
&= e^{r\,T} \int_{log\,(K/S_0)}^{\infty} \phi\left(\frac{x-(r+\sigma^2/2)\,T}{\sigma\sqrt{T}}\right) dx \\
&= e^{r\,T} \left\{1-\Phi\left(\frac{log\,(K/S_0)-(r+\sigma^2/2)\,T}{\sigma\sqrt{T}}\right)\right\} \\
&= e^{r\,T}\,\Phi\left(\frac{log\,(S_0/K)+(r+\sigma^2/2)\,T}{\sigma\sqrt{T}}\right), \quad (2.10)
\end{aligned}
$$

where the second equality in (2.10) uses the fact

$$
e^x\,\phi\left(\frac{x-(r-\sigma^2/2)\,T}{\sigma\sqrt{T}}\right) = e^{r\,T}\,\phi\left(\frac{x-(r+\sigma^2/2)\,T}{\sigma\sqrt{T}}\right).
$$

Substituting (2.9) and (2.10) into (2.8) to evaluate the expectation and then combining with (2.7), we obtain the option price at current time

$$
C_0 = e^{-r\,T}\,E_Q[(S_T-K)_+] = S_0\,\Phi(d_1) - K\,e^{-r\,T}\,\Phi(d_2), \quad (2.11)
$$

where

$$
d_1 = \frac{log\,(S_0/K)+(r+\sigma^2/2)\,T}{\sigma\sqrt{T}}, \qquad d_2 = d_1 - \sigma\sqrt{T}.
$$

The option price formula (2.11) was derived in Black and Scholes (1973) and is called the Black-Scholes formula.

Example 9 *Consider the situation where the stock price six months from the expiration of an option is \$42, the exercise price of the option is \$40, the risk-free interest is 10% per annum, and the volatility is 20% per annum. This means that* $S_0 = 42, K = 40, r = 0.1, \sigma = 0.2, T = 0.5.$

$$d_1 = \frac{log(42/40) + (0.1 + 0.2^2/2)\,0.5}{0.2\sqrt{0.5}} = 0.7693,$$

$$d_2 = \frac{log(42/40) + (0.1 - 0.2^2/2)\,0.5}{0.2\sqrt{0.5}} = 0.6278,$$

and

$$C_0 = 42\,\Phi(0.7693) - 40\,e^{-0.1\times 0.5}\,\Phi(0.6278) = 4.76.$$

For the purchaser of the call option to break even, the stock price has to rise by

$$e^{0.1\times 0.5} \times \$4.76 - (\$42 - \$40) = \$3.00.$$

Example 10 *Suppose that the current price of Intel stock is \$80 per share with volatility* $\sigma = 20\%$ *per annum. The risk free interest* $r = 8\%$ *per annum. Consider a call option with strike* *price K = 90 and maturity T = 3 months.*

$$d_1 = \frac{log(80/90) + (0.08 + 0.2^2/2)\,0.25}{0.2\sqrt{0.25}} = -0.9278,$$

$$d_2 = \frac{log(80/90) + (0.08 - 0.2^2/2)\,0.25}{0.2\sqrt{0.25}} = -1.0278,$$

$$\Phi(-0.9278) = 0.1767, \qquad \Phi(-1.0278) = 0.1520,$$

$$C_0 = 80\,0.1767 - 90\,e^{-0.08\times0.25}\,0.1520 = 0.73.$$

The strike price \$90 is well beyond the current stock price. A more realistic strike price is K=\$85. In this case,

$$d_1 = \frac{log(80/85) + (0.08 + 0.2^2/2)\,0.25}{0.2\sqrt{0.25}} = -0.356246,$$

$$d_2 = \frac{log(80/85) + (0.08 - 0.2^2/2)\,0.25}{0.2\sqrt{0.25}} = -0.456246,$$

$$C_0 = 80\,\Phi(-0.356246) - 85\,e^{-0.08\times0.25}\,\Phi(-0.456246) = 1.86.$$

Impacts of variables on option price

1. Marginal effect of current price S_0: C is positively related with S_0,

$$\frac{\partial C}{\partial S_0} = e^{-r\,T}\,\Phi(d_1),$$

and $C \to 0$ as $S_0 \to 0$, $C \to \infty$ as $S_0 \to \infty$.

2. Marginal effect of strike price K: C is negatively related with K, and $C \to S_0$ as $K \to 0$, $C \to 0$ as $S_0 \to \infty$.

3. Marginal effect of time to expiration T: C is related to T in a complicated manner,

$$\frac{\partial C}{\partial T} = -r\,K\,e^{-r\,T}\,\Phi(d_2) - \frac{S_0\,\sigma\,\phi(d_1)}{2\sqrt{T}},$$

and if $S_0 < K$, then $C \to 0$ as $T \to 0$, if $S_0 > K$, then $C \to S_0 - K$ as $T \to 0$, and $C \to S_0$ as $T \to \infty$.

4. Marginal effect of volatility σ:

$$\frac{\partial C}{\partial \sigma} = \sqrt{T}\,e^{-r\,T}\,K\,\phi(d_2),$$

and if $log(S_0/K) + r\,T < 0$, then $C \to 0$ as $\sigma \to 0$, and if $log(S_0/K) + r\,T > 0$, then $C \to S_0 - K\,e^{-r\,T}$ as $\sigma \to 0$, and $C \to S_0$ as $\sigma \to \infty$.

5. Joint effects of (K, σ):

Implied volatility