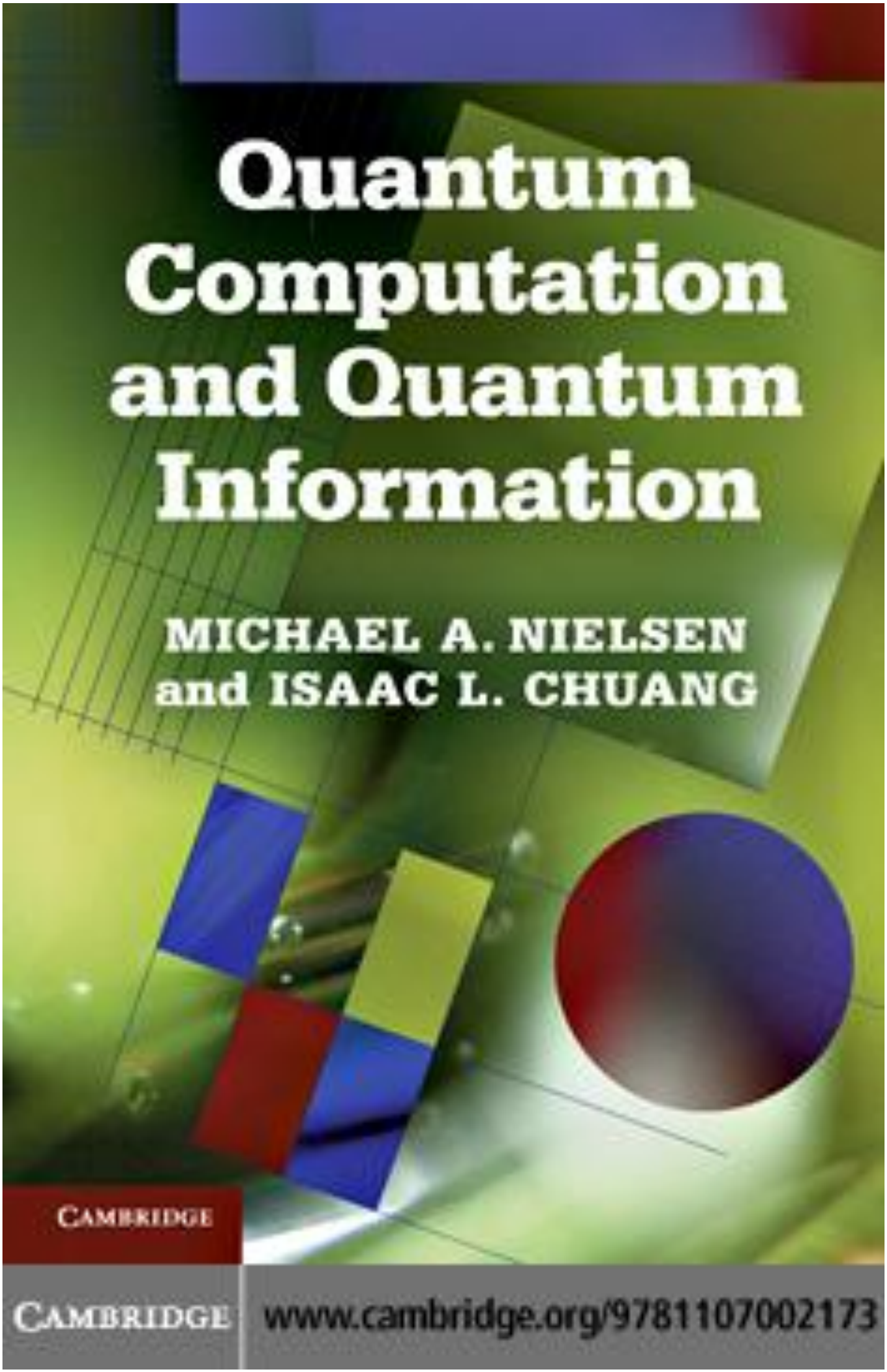
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MICHAEL A. NIELSEN and ISAAC L. CHUANG
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Michael A. Nielsen & Isaac L. Chuang

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III Quantum information

8 Quantum noise and quantum operations

Until now we have dealt almost solely with the dynamics of *closed* quantum systems, that is, with quantum systems that do not suffer any unwanted interactions with the outside world. Although fascinating conclusions can be drawn about the information processing tasks which may be accomplished *in principle* in such ideal systems, these observations are tempered by the fact that in the real world there are no perfectly closed systems, except perhaps the universe as a whole. Real systems suffer from unwanted interactions with the outside world. These unwanted interactions show up as *noise* in quantum information processing systems. We need to understand and control such noise processes in order to build useful quantum information processing systems. This is a central topic of the third part of this book, which begins in this chapter with the description of the *quantum operations formalism*, a powerful set of tools enabling us to describe quantum noise and the behavior of *open* quantum systems.

What is the distinction between an open and a closed system? A swinging pendulum like that found in some mechanical clocks can be a nearly ideal closed system. A pendulum interacts only very slightly with the rest of the world – its *environment* – mainly through friction. However, to properly describe the full dynamics of the pendulum and why it eventually ceases to move one must take into account the damping effects of air friction and imperfections in the suspension mechanism of the pendulum. Similarly, no quantum systems are ever perfectly closed, and especially not quantum computers, which must be delicately programmed by an external system to perform some desired set of operations. For example, if the state of a qubit is represented by two positions of a electron, then that electron will interact with other charged particles, which act as a source of uncontrolled *noise* affecting the state of the qubit. An open system is nothing more than one which has interactions with some other environment system, whose dynamics we wish to neglect, or average over.

The mathematical formalism of *quantum operations* is the key tool for our description of the dynamics of open quantum systems. This tool is very powerful, in that it simultaneously addresses a wide range of physical scenarios. It can be used to describe not only nearly closed systems which are weakly coupled to their environments, but also systems which are strongly coupled to their environments, and closed systems that are opened suddenly and subject to measurement. Another advantage of quantum operations in applications to quantum computation and quantum information is that they are especially well adapted to describe *discrete state changes*, that is, transformations between an initial state ρ and final state ρ', without explicit reference to the passage of time. This discrete-time analysis is rather different to the tools traditionally used by physicists for the description of open quantum systems (such as 'master equations', 'Langevin equations', and 'stochastic differential equations'), which tend to be continuous-time descriptions.

The chapter is structured as follows. We begin in Section 8.1 with a discussion of how noise is described in classical systems. The intuition gained by understanding classical noise is invaluable in learning how to think about quantum operations and quantum noise. Section 8.2 introduces the quantum operations formalism from three different points of view, enabling us to become thoroughly familiar with the basic theory of quantum operations. Section 8.3 illustrates several important examples of noise using quantum operations. These include such examples as depolarization, amplitude damping, and phase damping. A geometric approach to understanding quantum noise on single qubits is explained, using the Bloch sphere. Section 8.4 explains some miscellaneous applications of quantum operations: the connection between quantum operations and other tools conventionally used by physicists to describe quantum noise, such as *master equations*; how to *experimentally determine* the dynamics a quantum system undergoes using a procedure known as *quantum process tomography*; and an explanation of how quantum operations can be used to understand the fact that the world around us appears to obey the rules of classical physics, while it really follows quantum mechanical laws. The chapter concludes in Section 8.5 with a discussion of the limitations of the quantum operations formalism as a general approach to the description of noise in quantum systems.

8.1 Classical noise and Markov processes

To understand noise in quantum systems it is helpful to build some intuition by understanding noise in classical systems. How should we model noise in a classical system? Let's look at some simple examples to understand how this is done, and what it can teach us about noise in quantum systems.

Imagine a bit is being stored on a hard disk drive attached to an ordinary classical computer. The bit starts out in the state 0 or 1, but after a long time it becomes likely that stray magnetic fields will cause the bit to become scrambled, possibly flipping its state. We can model this by a probability p for the bit to flip, and a probability $1-p$ for the bit to remain the same. This process is illustrated in Figure 8.1.

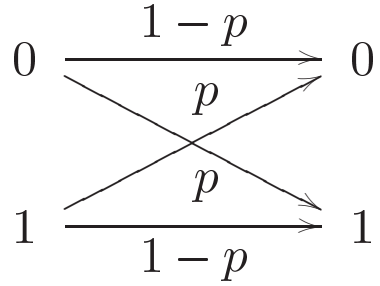


Figure 8.1. After a long time a bit on a hard disk drive may flip with probability p.

What is really going on, of course, is that the environment contains magnetic fields which can cause the bit to flip. To figure out the probability p for the bit to flip we need to understand two things. First, we need a model for the distribution of magnetic fields in the environment. Assuming that the user of the hard disk drive isn't doing anything silly like running a strong magnet near the disk drive, we can construct a realistic model by sampling the magnetic field in environments similar to the one in which the drive will be running. Second, we need a model for how magnetic fields in the environment

will interact with bits on the disk. Fortunately, such a model is already well known to physicists, and goes by the name 'Maxwell's equations'. With these two elements in hand, we can in principle calculate the probability p that a bit flip will occur on the drive over some prescribed period of time.

This basic procedure – finding a model for the environment and for the system–environment interaction – is one we follow repeatedly in our study of noise, both classical and quantum. Interactions with the environment are the fundamental source of noise in both classical and quantum systems. It is often not easy to find exact models for the environment or the system–environment interaction; however, by being conservative in our modeling and closely studying the observed properties of a system to see if it obeys our model, it is possible to attain a high degree of accuracy in the modeling of noise in real physical systems.

The behavior of the bit on the hard disk can be succinctly summarized in a single equation. Suppose p_0 and p_1 are the initial probabilities that the bit is in the states 0 and 1, respectively. Let q_0 and q_1 be the corresponding probabilities after the noise has occurred. Let X be the initial state of the bit, and Y the final state of the bit. Then the law of total probability (Appendix 1) states that

$$p(Y = y) = \sum_x p(Y = y|X = x)p(X = x)\,. \tag{8.1}$$

The conditional probabilities $p(Y = y|X = x)$ are called *transition probabilities*, since they summarize the changes that may occur in the system. Writing these equations out explicitly for the bit on a hard disk we have

$$\begin{bmatrix} q_0 \\ q_1 \end{bmatrix} = \begin{bmatrix} 1-p & p \\ p & 1-p \end{bmatrix} \begin{bmatrix} p_0 \\ p_1 \end{bmatrix}. \tag{8.2}$$

Let's look at a slightly more complicated example of noise in a classical system. Imagine that we are trying to build a classical circuit to perform some computational task. Unfortunately, we've been given faulty components to build the circuit. Our rather artificial circuit consists of a single input bit, X, to which are applied two consecutive (faulty) NOT gates, producing an intermediate bit Y, and a final bit Z. It seems reasonable to assume that whether the second NOT gate works correctly is *independent* of whether the first NOT gate worked correctly. This assumption – that the consecutive noise processes act independently – is a physically reasonable assumption to make in many situations. It results in a stochastic process $X \rightarrow Y \rightarrow Z$ of a special type known as a *Markov process*. Physically, this assumption of Markovicity corresponds to assuming that the environment causing the noise in the first NOT gate acts *independently* of the environment causing the noise in the second NOT gate, a reasonable assumption given that the gates are likely to be located a considerable distance apart in space.

Summarizing, noise in classical systems can be described using the theory of stochastic processes. Often, in the analysis of multi-stage processes it is a good assumption to use Markov processes. For a single stage process the output probabilities $\vec{q}$ are related to the input probabilities $\vec{p}$ by the equation

$$\vec{q} = E\vec{p}, \tag{8.3}$$

where E is a matrix of transition probabilities which we shall refer to as the *evolution matrix*. Thus, the final state of the system is linearly related to the starting state. This

feature of linearity is echoed in the description of quantum noise, with density matrices replacing probability distributions.

What properties must the evolution matrix E possess? We require that if $\vec{p}$ is a valid probability distribution, then $E\vec{p}$ must also be a valid probability distribution. Satisfying this condition turns out to be equivalent to two conditions on E. First, all the entries of E must be non-negative, a condition known as the *positivity* requirement. If they weren't, then it would be possible to obtain negative probabilities in $E\vec{p}$. Second, all the columns of E must sum to one, a condition known as the *completeness* requirement. Suppose this weren't true. Imagine, for example, that the first column didn't sum to one. Letting $\vec{p}$ contain a one in the first entry and zeroes everywhere else, we see that $E\vec{p}$ would not be a valid probability distribution in this case.

Summarizing, the key features of classical noise are as follows: there is a linear relationship between input and output probabilities, described by a transition matrix with non-negative entries (*positivity*) and columns summing to one (*completeness*). Classical noise processes involving multiple stages are described as Markov processes, provided the noise is caused by independent environments. Each of these key features has important analogues in the theory of quantum noise. Of course, there are also some surprising new features of quantum noise!

8.2 Quantum operations

8.2.1 Overview

The quantum operations formalism is a general tool for describing the evolution of quantum systems in a wide variety of circumstances, including stochastic changes to quantum states, much as Markov processes describe stochastic changes to classical states. Just as a classical state is described by a vector of probabilities, we shall describe quantum states in terms of the density operator (density matrix) ρ, whose properties you may wish to review by rereading Section 2.4, beginning on page 98, before continuing to read this chapter. And similar to how classical states transform as described by (8.3), quantum states transform as

$$\rho' = \mathcal{E}(\rho) . \tag{8.4}$$

The map $\mathcal{E}$ in this equation is a *quantum operation*. Two simple examples of quantum operations which we have encountered previously, in Chapter 2, are unitary transformations and measurements, for which $\mathcal{E}(\rho) = U\rho U^\dagger$, and $\mathcal{E}_m(\rho) = M_m \rho M_m^\dagger$, respectively (see Exercises 8.1 and 8.2, below). The quantum operation captures the dynamic change to a state which occurs as the result of some physical process; ρ is the initial state before the process, and $\mathcal{E}(\rho)$ is the final state after the process occurs, possibly up to some normalization factor.

Over the next few sections, we develop a general theory of quantum operations incorporating unitary evolution, measurement, and even more general processes! We shall develop *three* separate ways of understanding quantum operations, illustrated in Figure 8.2, all of which turn out to be equivalent. The first method is based on the idea of studying dynamics as the result of an interaction between a system and an environment, much as classical noise was described in Section 8.1. This method is concrete and easy to relate to the real world. Unfortunately, it suffers from the drawback of not being mathematically convenient. Our second method of understanding quantum op-

erations, completely equivalent to the first, overcomes this mathematical inconvenience by providing a powerful mathematical representation for quantum operations known as the *operator-sum representation*. This method is rather abstract, but is very useful for calculations and theoretical work. Our third approach to quantum operations, equivalent to the other two, is via a set of physically motivated axioms that we would expect a dynamical map in quantum mechanics to satisfy. The advantage of this approach is that it is exceedingly *general* – it shows that quantum dynamics will be described by quantum operations under an amazingly wide range of circumstances. However, it does not offer the calculational convenience of the second approach, nor the concrete nature of the first. Taken together, these three approaches to quantum operations offer a powerful tool with which we can understand quantum noise and its effects.

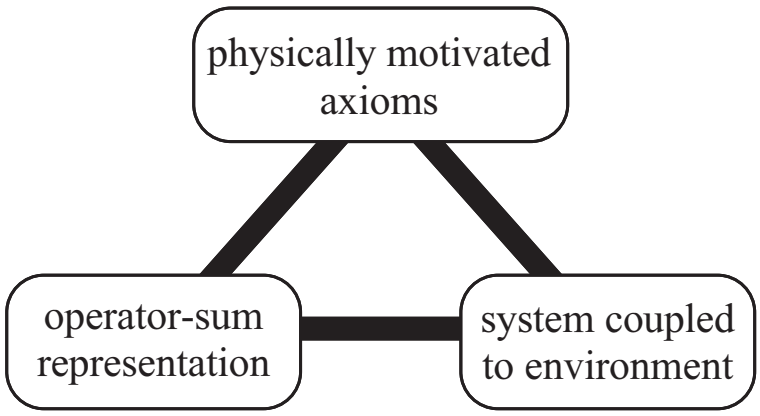


Figure 8.2. Three approaches to quantum operations which are equivalent, but offer different advantages depending upon the intended application.

Exercise 8.1: (Unitary evolution as a quantum operation) Pure states evolve under unitary transforms as $|\psi\rangle \rightarrow U|\psi\rangle$. Show that, equivalently, we may write $\rho \rightarrow \mathcal{E}(\rho) \equiv U\rho U^\dagger$, for $\rho = |\psi\rangle\langle\psi|$.

Exercise 8.2: (Measurement as a quantum operation) Recall from Section 2.2.3 (on page 84) that a quantum measurement with outcomes labeled by m is described by a set of measurement operators M_m such that $\sum_m M_m^\dagger M_m = I$. Let the state of the system immediately before the measurement be ρ. Show that for $\mathcal{E}_m(\rho) \equiv M_m \rho M_m^\dagger$, the state of the system immediately after the measurement is

$$\frac{\mathcal{E}_m(\rho)}{\mathrm{tr}(\mathcal{E}_m(\rho))}. \tag{8.5}$$

Also show that the probability of obtaining this measurement result is $p(m) = \mathrm{tr}(\mathcal{E}_m(\rho))$.

8.2.2 Environments and quantum operations

The dynamics of a closed quantum system are described by a unitary transform. Conceptually, we can think of the unitary transform as a box into which the input state enters and from which the output exits, as illustrated on the left hand side of Figure 8.3.

For our purposes, the interior workings of the box are not of concern to us; it could be implemented by a quantum circuit, or by some Hamiltonian system, or anything else.

A natural way to describe the dynamics of an *open* quantum system is to regard it as arising from an interaction between the system of interest, which we shall call the *principal system*, and an *environment*, which together form a closed quantum system, as illustrated on the right hand side of Figure 8.3. In other words, suppose we have a system in state ρ, which is sent into a box which is coupled to an environment. In general the final state of the system, $\mathcal{E}(\rho)$, may *not* be related by a unitary transformation to the initial state ρ. We *assume* (for now) that the system–environment input state is a product state, $\rho \otimes \rho_{\text{env}}$. After the box's transformation U the system no longer interacts with the environment, and thus we perform a partial trace over the environment to obtain the reduced state of the system alone:

$$\mathcal{E}(\rho) = \text{tr}_{\text{env}} \left[U \left(\rho \otimes \rho_{\text{env}} \right) U^\dagger \right] . \tag{8.6}$$

Of course, if U does not involve any interaction with the environment, then $\mathcal{E}(\rho) = \tilde{U} \rho \tilde{U}^\dagger$, where $\tilde{U}$ is the part of U which acts on the system alone. Equation (8.6) is our first of three equivalent *definitions* of a quantum operation.

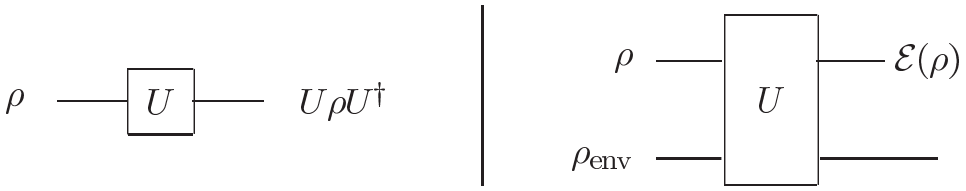


Figure 8.3. Models of closed (left) and open (right) quantum systems. An open quantum system consists of two parts, the principal system and an environment.

An important assumption is made in this definition – we assume that the system and the environment start in a product state. In general, of course, this is not true. Quantum systems interact constantly with their environments, building up correlations. One way this expresses itself is via the exchange of heat between the system and its environment. Left to itself a quantum system will relax to the same temperature as its environment, which causes correlations to exist between the two. However, in many cases of practical interest it *is* reasonable to assume that the system and its environment start out in a product state. When an experimentalist *prepares* a quantum system in a specified state they undo all the correlations between that system and the environment. Ideally, the correlations will be completely destroyed, leaving the system in a pure state. Even if this is not the case, we shall see later that the quantum operations formalism can even describe quantum dynamics when the system and environment do *not* start out in a product state.

Another issue one might raise is: how can U be specified if the environment has nearly infinite degrees of freedom? It turns out, very interestingly, that in order for this model to properly describe any possible transformation $\rho \rightarrow \mathcal{E}(\rho)$, if the principal system has a Hilbert space of d dimensions, then it suffices to model the environment as being in a Hilbert space of no more than d^2 dimensions. It also turns out not to be necessary for the environment to start out in a mixed state; a pure state will do. We shall return to these points in Section 8.2.3.

As an explicit example of the use of Equation (8.6), consider the two qubit quantum circuit shown in Figure 8.4, in which U is a controlled-NOT gate, with the principal system the control qubit, and the environment initially in the state $\rho_{\rm env} = |0\rangle\langle 0|$ as the target qubit. Inserting into Equation (8.6), it is easily seen that

$$\mathcal{E}(\rho) = P_0 \rho P_0 + P_1 \rho P_1 \,, \tag{8.7}$$

where $P_0 = |0\rangle\langle 0|$ and $P_1 = |1\rangle\langle 1|$ are projection operators. Intuitively, this dynamics occurs because the environment stays in the $|0\rangle$ state only when the system is $|0\rangle$; otherwise the environment is flipped to the state $|1\rangle$. In the next section we give a derivation of this equation as an example of the operator-sum representation.

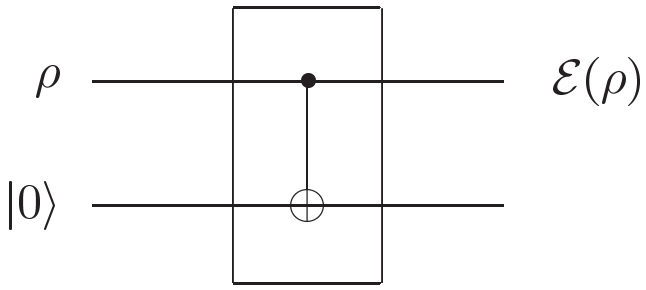


Figure 8.4. Controlled-NOT gate as an elementary example of a single qubit quantum operation.

We have described quantum operations as arising from the interaction of a principal system with an environment; however, it is convenient to generalize the definition somewhat to allow different input and output spaces. For example, imagine that a single qubit, which we label A, is prepared in an unknown state ρ. A three-level quantum system ('qutrit') labelled B is prepared in some standard state $|0\rangle$, and then interacts with system A via a unitary interaction U, causing the joint system to evolve into the state $U(\rho \otimes |0\rangle\langle 0|)U^\dagger$. We then discard system A, leaving system B in some final state ρ'. By definition, the quantum operation $\mathcal{E}$ describing this process is

$$\mathcal{E}(\rho) = \rho' = {\rm tr}_A(U(\rho \otimes |0\rangle\langle 0|)U^\dagger). \tag{8.8}$$

Notice that $\mathcal{E}$ maps density operators of the input system, A, to density operators of the output system, B. Most of our discussion of quantum operations below is concerned with quantum operations 'on' some system A, that is, they map density operators of system A to density operators of system A. However it is occasionally useful in applications to allow a more general definition. Such a definition is provided by defining quantum operations as the class of maps which arise as a result of the following processes: some initial system is prepared in an unknown quantum state ρ, and then brought into contact with other systems prepared in standard states, allowed to interact according to some unitary interaction, and then some part of the combined system is discarded, leaving just a final system in some state ρ'. The quantum operation $\mathcal{E}$ defining this process simply maps ρ to ρ'. We will see that this extension to allow different input and output spaces gels naturally with our treatment of quantum operations via the operator-sum representation, and also our axiomatic study. Nevertheless, for the most part it simplifies discussion if we assume that the input and output spaces of a quantum operation are the same, using the convenient distinction between 'principal system' and 'environment' which disappears

in the general case, and giving occasional exercises to indicate the necessary extensions when the input and output spaces are different.

8.2.3 Operator-sum representation

Quantum operations can be represented in an elegant form known as the *operator-sum representation*, which is essentially a re-statement of Equation (8.6) explicitly in terms of operators on the principal system's Hilbert space alone. The main result is motivated by the following simple calculation. Let $|e_k\rangle$ be an orthonormal basis for the (finite dimensional) state space of the environment, and let $\rho_{\rm env} = |e_0\rangle\langle e_0|$ be the initial state of the environment. There is no loss of generality in assuming that the environment starts in a pure state, since if it starts in a mixed state we are free to introduce an extra system purifying the environment (Section 2.5). Although this extra system is 'fictitious', it makes no difference to the dynamics experienced by the principal system, and thus can be used as an intermediate step in calculations. Equation (8.6) can thus be rewritten as

$$\mathcal{E}(\rho) = \sum_k \langle e_k|U \left[\rho \otimes |e_0\rangle\langle e_0|\right] U^\dagger |e_k\rangle \tag{8.9}$$

$$= \sum_k E_k \rho E_k^\dagger , \tag{8.10}$$

where $E_k \equiv \langle e_k|U|e_0\rangle$ is an operator on the state space of the principal system. Equation (8.10) is known as the operator-sum representation of $\mathcal{E}$. The operators $\{E_k\}$ are known as *operation elements* for the quantum operation $\mathcal{E}$. The operator-sum representation is important; it will be used repeatedly for the remainder of the book.

The operation elements satisfy an important constraint known as the *completeness relation*, analogous to the completeness relation for evolution matrices in the description of classical noise. In the classical case, the completeness relation arose from the requirement that probability distributions be normalized to one. In the quantum case the completeness relation arises from the analogous requirement that the trace of $\mathcal{E}(\rho)$ be equal to one,

$$1 = {\rm tr}(\mathcal{E}(\rho)) \tag{8.11}$$

$$= {\rm tr}\left(\sum_k E_k \rho E_k^\dagger\right) \tag{8.12}$$

$$= {\rm tr}\left(\sum_k E_k^\dagger E_k \rho\right) . \tag{8.13}$$

Since this relationship is true for *all* ρ it follows that we must have

$$\sum_k E_k^\dagger E_k = I. \tag{8.14}$$

This equation is satisfied by quantum operations which are *trace-preserving*. There are also non-trace-preserving quantum operations, for which $\sum_k E_k^\dagger E_k \leq I$, but they describe processes in which extra information about what occurred in the process is obtained by measurement, as we explain in more detail shortly. Maps $\mathcal{E}$ of the form of (8.10) for which $\sum_k E_k^\dagger E_k \leq I$ provide our second *definition* of a quantum operation. We show below that this definition is essentially equivalent to the first, Equation (8.6), and in fact is slightly more general, since it allows for non-trace-preserving operations. We

will often have occasion to move backwards and forwards between these two definitions; it should be clear from context which definition we are working from at any given moment.

Exercise 8.3: Our derivation of the operator-sum representation implicitly assumed that the input and output spaces for the operation were the same. Suppose a composite system AB initially in an unknown quantum state ρ is brought into contact with a composite system CD initially in some standard state $|0\rangle$, and the two systems interact according to a unitary interaction U. After the interaction we discard systems A and D, leaving a state ρ' of system BC. Show that the map $\mathcal{E}(\rho) = \rho'$ satisfies

$$\mathcal{E}(\rho) = \sum_k E_k \rho E_k^\dagger, \tag{8.15}$$

for some set of linear operators E_k from the state space of system AB to the state space of system BC, and such that $\sum_k E_k^\dagger E_k = I$.

The operator-sum representation is important because it gives us an *intrinsic* means of characterizing the dynamics of the principal system. The operator-sum representation describes the dynamics of the principal system without having to explicitly consider properties of the environment; all that we need to know is bundled up into the operators E_k, which act on the principal system alone. This simplifies calculations and often provides considerable theoretical insight. Furthermore, many different environmental interactions may give rise to the same dynamics on the principal system. If it is only the dynamics of the principal system which are of interest then it makes sense to choose a representation of the dynamics which does not include unimportant information about other systems.

In the remainder of this section, we explore the properties of the operator-sum representation, and in particular, three features. First, we give it a physical interpretation, in terms of the operation elements E_k. A natural question which arises from this is how an operator-sum representation can be determined for any open quantum system (given, for example, the system–environment interaction or other specification). This is answered in the second topic addressed below, and the converse, how to construct a model open quantum system for any operator-sum representation, concludes.

Exercise 8.4: **(Measurement)** Suppose we have a single qubit principal system, interacting with a single qubit environment through the transform

$$U = P_0 \otimes I + P_1 \otimes X\,, \tag{8.16}$$

where X is the usual Pauli matrix (acting on the environment), and $P_0 \equiv |0\rangle\langle 0|, P_1 \equiv |1\rangle\langle 1|$ are projectors (acting on the system). Give the quantum operation for this process, in the operator-sum representation, assuming the environment starts in the state $|0\rangle$.

Exercise 8.5: **(Spin flips)** Just as in the previous exercise, but now let

$$U = \frac{X}{\sqrt{2}} \otimes I + \frac{Y}{\sqrt{2}} \otimes X\,, \tag{8.17}$$

Give the quantum operation for this process, in the operator-sum representation.

Exercise 8.6: (Composition of quantum operations) Suppose $\mathcal{E}$ and $\mathcal{F}$ are quantum operations on the same quantum system. Show that the composition $\mathcal{F} \circ \mathcal{E}$ is a quantum operation, in the sense that it has an operator-sum representation. State and prove an extension of this result to the case where $\mathcal{E}$ and $\mathcal{F}$ do not necessarily have the same input and output spaces.

Physical interpretation of the operator-sum representation

There is a nice interpretation that can be given to the operator-sum representation. Imagine that a measurement of the environment is performed in the basis $|e_k\rangle$ after the unitary transformation U has been applied. Applying the principle of implicit measurement, we see that such a measurement affects only the state of the environment, and does not change the state of the principal system. Let ρ_k be the state of the principal system given that outcome k occurs, so

$$\rho_k \propto \mathrm{tr}_E(|e_k\rangle\langle e_k|U(\rho \otimes |e_0\rangle\langle e_0|)U^\dagger|e_k\rangle\langle e_k|) = \langle e_k|U(\rho \otimes |e_0\rangle\langle e_0|)U^\dagger|e_k\rangle \quad (8.18)$$

$$= E_k \rho E_k^\dagger \,. \quad (8.19)$$

Normalizing ρ_k,

$$\rho_k = \frac{E_k \rho E_k^\dagger}{\mathrm{tr}(E_k \rho E_k^\dagger)} \,, \quad (8.20)$$

we find the probability of outcome k is given by

$$p(k) = \mathrm{tr}(|e_k\rangle\langle e_k|U(\rho \otimes |e_0\rangle\langle e_0|)U^\dagger|e_k\rangle\langle e_k|) \quad (8.21)$$

$$= \mathrm{tr}(E_k \rho E_k^\dagger). \quad (8.22)$$

Thus

$$\mathcal{E}(\rho) = \sum_k p(k)\rho_k = \sum_k E_k \rho E_k^\dagger \,. \quad (8.23)$$

This gives us a beautiful physical *interpretation* of what is going on in a quantum operation with operation elements $\{E_k\}$. The action of the quantum operation is equivalent to taking the state ρ and randomly replacing it by $E_k \rho E_k^\dagger / \mathrm{tr}(E_k \rho E_k^\dagger)$, with probability $\mathrm{tr}(E_k \rho E_k^\dagger)$. In this sense, it is very similar to the concept of noisy communication channels used in classical information theory; in this vein, we shall sometimes refer to certain quantum operations which describe quantum noise processes as being noisy quantum channels.

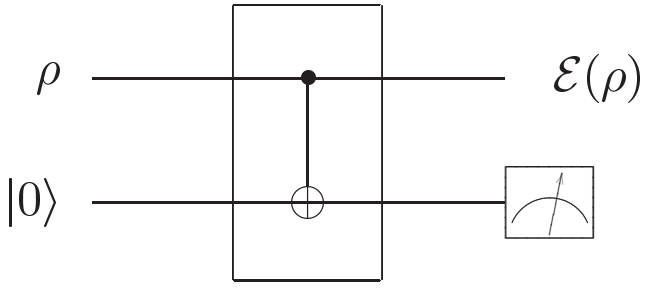


Figure 8.5. Controlled-NOT gate as an elementary model of single qubit measurement.

A simple example, based on Figure 8.4, illustrates this interpretation of the operator-sum representation. Suppose we choose the states $|e_k\rangle = |0_E\rangle$ and $|1_E\rangle$, where we include the E subscript to make it clear that the state is a state of the environment. This can be interpreted as doing a measurement in the computational basis of the environment qubit, as shown in Figure 8.5. Doing such a measurement does not, of course, change the state of the principal system. Using subscripts P to denote the principal system, the controlled-NOT may be expanded as

$$U = |0_P 0_E\rangle\langle 0_P 0_E| + |0_P 1_E\rangle\langle 0_P 1_E| + |1_P 1_E\rangle\langle 1_P 0_E| + |1_P 0_E\rangle\langle 1_P 1_E| \,. \quad (8.24)$$

Thus

$$E_0 = \langle 0_E|U|0_E\rangle = |0_P\rangle\langle 0_P| \quad (8.25)$$

$$E_1 = \langle 1_E|U|0_E\rangle = |1_P\rangle\langle 1_P| \,, \quad (8.26)$$

and therefore

$$\mathcal{E}(\rho) = E_0\rho E_0 + E_1\rho E_1 \,, \quad (8.27)$$

in agreement with Equation (8.7).

Measurements and the operator-sum representation

Given a description of an open quantum system, how do we determine an operator-sum representation for its dynamics? We have already found one answer: given the unitary system–environment transformation operation U, and a basis of states $|e_k\rangle$ for the environment, the operation elements are

$$E_k \equiv \langle e_k|U|e_0\rangle \,. \quad (8.28)$$

It is possible to extend this result even further by allowing the possibility that a *measurement* is performed on the combined system–environment after the unitary interaction, allowing the acquisition of information about the quantum state. It turns out that this physical possibility is naturally connected to non-trace-preserving quantum operations, that is, maps $\mathcal{E}(\rho) = \sum_k E_k\rho E_k^\dagger$ such that $\sum_k E_k^\dagger E_k \leq I$.

Suppose the principal system is initially in a state ρ. For convenience we denote the principal system by the letter Q. Adjoined to Q is an environment system E. We suppose that Q and E are initially independent systems, and that E starts in some standard state, σ. The joint state of the system is thus initially

$$\rho^{QE} = \rho \otimes \sigma \,. \quad (8.29)$$

We suppose that the systems interact according to some unitary interaction U. After the unitary interaction a projective measurement is performed on the joint system, described by projectors P_m. The case where no measurement is made corresponds to the special case where there is only a single measurement outcome, $m = 0$, which corresponds to the projector $P_0 \equiv I$.

The situation is summarized in Figure 8.6. Our aim is to determine the final state of Q as a function of the initial state, ρ. The final state of QE is given by

$$\frac{P_m U(\rho \otimes \sigma)U^\dagger P_m}{\mathrm{tr}(P_m U(\rho \otimes \sigma)U^\dagger P_m)} \,, \quad (8.30)$$

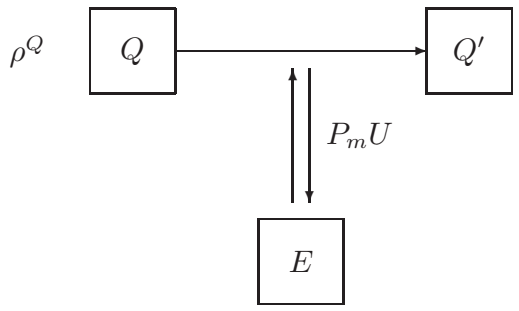


Figure 8.6. Environmental model for a quantum operation.

given that measurement outcome m occurred. Tracing out E we see that the final state of Q alone is

$$\frac{\mathrm{tr}_E(P_m U(\rho \otimes \sigma) U^\dagger P_m)}{\mathrm{tr}(P_m U(\rho \otimes \sigma) U^\dagger P_m)} . \tag{8.31}$$

This representation of the final state involves the initial state σ of the environment, the interaction U and the measurement operators P_m. Define a map

$$\mathcal{E}_m(\rho) \equiv \mathrm{tr}_E(P_m U(\rho \otimes \sigma) U^\dagger P_m) , \tag{8.32}$$

so the final state of Q alone is $\mathcal{E}_m(\rho)/\mathrm{tr}(\mathcal{E}_m(\rho))$. Note that $\mathrm{tr}[\mathcal{E}_m(\rho)]$ is the probability of outcome m of the measurement occurring. Let $\sigma = \sum_j q_j |j\rangle\langle j|$ be an ensemble decomposition for σ. Introduce an orthonormal basis $|e_k\rangle$ for the system E. Note that

$$\mathcal{E}_m(\rho) = \sum_{jk} q_j \mathrm{tr}_E(|e_k\rangle\langle e_k| P_m U(\rho \otimes |j\rangle\langle j|) U^\dagger P_m |e_k\rangle\langle e_k|) \tag{8.33}$$

$$= \sum_{jk} E_{jk} \rho E_{jk}^\dagger , \tag{8.34}$$

where

$$E_{jk} \equiv \sqrt{q_j} \langle e_k| P_m U |j\rangle . \tag{8.35}$$

This equation is a generalization of Equation (8.10), and gives an explicit means for calculating the operators appearing in an operator-sum representation for $\mathcal{E}_m$, given that the initial state σ of E is known, and the dynamics between Q and E are known. The quantum operations $\mathcal{E}_m$ can be thought of as defining a kind of measurement process generalizing the description of measurements given in Chapter 2.

Exercise 8.7: Suppose that instead of doing a projective measurement on the combined principal system and environment we had performed a general measurement described by measurement operators $\{M_m\}$. Find operator-sum representations for the corresponding quantum operations $\mathcal{E}_m$ on the principal system, and show that the respective measurement probabilities are $\mathrm{tr}[\mathcal{E}(\rho)]$.

System–environment models for any operator-sum representation

We have shown that interacting quantum systems give rise in a natural way to an operator-sum representation for quantum operations. What about the converse problem? Given a set of operators $\{E_k\}$ is there some reasonable *model environmental system and dynamics* which give rise to a quantum operation with those operation elements? By 'reasonable' we mean that the dynamics must be either a unitary evolution or a projective measurement. Here, we show how to construct such a model. We will only show how to do this for quantum operations mapping the input space to the same output space, although it is mainly a matter of notation to generalize the construction to the more general case. In particular, we show that for any trace-preserving or non-trace-preserving quantum operation, $\mathcal{E}$, with operation elements $\{E_k\}$, there exists a model environment, E, starting in a pure state $|e_0\rangle$, and model dynamics specified by a unitary operator U and projector P onto E such that

$$\mathcal{E}(\rho) = \mathrm{tr}_E(PU(\rho \otimes |e_0\rangle\langle e_0|)U^\dagger P)\,. \tag{8.36}$$

To see this, suppose first that $\mathcal{E}$ is a trace-preserving quantum operation, with operator-sum representation generated by operation elements $\{E_k\}$ satisfying the completeness relation $\sum_k E_k^\dagger E_k = I$, so we are only attempting to find an appropriate unitary operator U to model the dynamics. Let $|e_k\rangle$ be an orthonormal basis set for E, in one-to-one correspondence with the index k for the operators E_k. Note that *by definition* E has such a basis; we are trying to find a *model* environment giving rise to a dynamics described by the operation elements $\{E_k\}$. Define an operator U which has the following action on states of the form $|\psi\rangle|e_0\rangle$,

$$U|\psi\rangle|e_0\rangle \equiv \sum_k E_k|\psi\rangle|e_k\rangle\,, \tag{8.37}$$

where $|e_0\rangle$ is just some standard state of the model environment. Note that for arbitrary states $|\psi\rangle$ and $|\varphi\rangle$ of the principal system,

$$\langle\psi|\langle e_0|U^\dagger U|\varphi\rangle|e_0\rangle = \sum_k \langle\psi|E_k^\dagger E_k|\varphi\rangle = \langle\psi|\varphi\rangle\,, \tag{8.38}$$

by the completeness relation. Thus the operator U can be extended to a unitary operator acting on the entire state space of the joint system. It is easy to verify that

$$\mathrm{tr}_E(U(\rho \otimes |e_0\rangle\langle e_0|)U^\dagger) = \sum_k E_k \rho E_k^\dagger\,, \tag{8.39}$$

so this model provides a realization of the quantum operation $\mathcal{E}$ with operation elements $\{E_k\}$. This result is illustrated in Box 8.1.

Non-trace-preserving quantum operations can easily be modeled using a construction along the same lines (Exercise 8.8). A more interesting generalization of this construction is the case of a set of quantum operations $\{\mathcal{E}_m\}$ corresponding to possible outcomes from a measurement, so the quantum operation $\sum_m \mathcal{E}_m$ is trace-preserving, since the probabilities of the distinct outcomes sum to one, $1 = \sum_m p(m) = \mathrm{tr}\left[\left(\sum_m \mathcal{E}_m\right)(\rho)\right]$ for all possible inputs ρ. See Exercise 8.9, below.

Exercise 8.8: (Non-trace-preserving quantum operations) Explain how to construct a unitary operator for a system–environment model of a

non-trace-preserving quantum operation, by introducing an extra operator, E_∞, into the set of operation elements E_k, chosen so that when summing over the complete set of k, including $k = \infty$, one obtains $\sum_k E_k^\dagger E_k = I$.

Exercise 8.9: (Measurement model) If we are given a set of quantum operations $\{\mathcal{E}_m\}$ such that $\sum_m \mathcal{E}_m$ is trace-preserving, then it is possible to construct a *measurement model* giving rise to this set of quantum operations. For each m, let E_{mk} be a set of operation elements for $\mathcal{E}_m$. Introduce an environmental system, E, with an orthonormal basis $|m,k\rangle$ in one-to-one correspondence with the set of indices for the operation elements. Analogously to the earlier construction, define an operator U such that

$$U|\psi\rangle|e_0\rangle = \sum_{mk} E_{mk}|\psi\rangle|m,k\rangle\,. \tag{8.40}$$

Next, define projectors $P_m \equiv \sum_k |m,k\rangle\langle m,k|$ on the environmental system, E. Show that performing U on $\rho \otimes |e_0\rangle\langle e_0|$, then measuring P_m gives m with probability $\mathrm{tr}(\mathcal{E}_m(\rho))$, and the corresponding post-measurement state of the principal system is $\mathcal{E}_m(\rho)/\mathrm{tr}(\mathcal{E}_m(\rho))$.

Box 8.1: Mocking up a quantum operation

Given a trace-preserving quantum operation expressed in the operator-sum representation, $\mathcal{E}(\rho) = \sum_k E_k \rho E_k^\dagger$, we can construct a physical model for it in the following way. From (8.10), we want U to satisfy

$$E_k = \langle e_k|U|e_0\rangle\,, \tag{8.41}$$

where U is some unitary operator, and $|e_k\rangle$ are orthonormal basis vectors for the environment system. Such a U is conveniently represented as the block matrix

$$U = \begin{bmatrix} [E_1] & \cdot & \cdot & \cdot & \cdots \\ [E_2] & \cdot & \cdot & \cdot & \cdots \\ [E_3] & \cdot & \cdot & \cdot & \cdots \\ [E_4] & \cdot & \cdot & \cdot & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix} \tag{8.42}$$

in the basis $|e_k\rangle$. Note that the operation elements E_k only determine the first block column of this matrix (unlike elsewhere, here it is convenient to have the first label of the states be the environment, and the second, the principal system). Determination of the rest of the matrix is left up to us; we simply choose the entries such that U is unitary. Note that by the results of Chapter 4, U can be implemented by a quantum circuit.

8.2.4 Axiomatic approach to quantum operations

Until now the main motivation for our study of quantum operations has been that they provide an elegant way of studying systems which are interacting with an environment. We're going to switch to a different viewpoint now, where we try to write down physically

motivated axioms which we expect quantum operations to obey. This viewpoint is more abstract than our earlier approach, based on explicit system–environment models, but it is also extremely powerful because of that abstraction.

The way we're going to proceed is as follows. First, we're going to *forget* everything we've learned about quantum operations, and start over by *defining* quantum operations according to a set of axioms, which we'll justify on physical grounds. That done, we'll *prove* that a map $\mathcal{E}$ satisfies these axioms if and only if it has an operator-sum representation, thus providing the missing link between the abstract axiomatic formulation, and our earlier discussion.

We *define* a quantum operation $\mathcal{E}$ as a map from the set of density operators of the input space Q_1 to the set of density operators for the output space Q_2, with the following three axiomatic properties: (note that for notational simplicity in the proofs we take $Q_1 = Q_2 = Q$)

- **A1**: First, $\mathrm{tr}[\mathcal{E}(\rho)]$ is the probability that the process represented by $\mathcal{E}$ occurs, when ρ is the initial state. Thus, $0 \leq \mathrm{tr}[\mathcal{E}(\rho)] \leq 1$ for any state ρ.
- **A2**: Second, $\mathcal{E}$ is a *convex-linear map* on the set of density matrices, that is, for probabilities $\{p_i\}$,
$$\mathcal{E}\left(\sum_i p_i \rho_i\right) = \sum_i p_i \mathcal{E}(\rho_i)\,. \tag{8.43}$$
- **A3**: Third, $\mathcal{E}$ is a *completely positive* map. That is, if $\mathcal{E}$ maps density operators of system Q_1 to density operators of system Q_2, then $\mathcal{E}(A)$ must be positive for any positive operator A. Furthermore, if we introduce an extra system R of arbitrary dimensionality, it must be true that $(\mathcal{I} \otimes \mathcal{E})(A)$ is positive for any positive operator A on the combined system RQ_1, where $\mathcal{I}$ denotes the identity map on system R.

The first property is one of mathematical convenience. To cope with the case of measurements, it turns out that it is extremely convenient to make the convention that $\mathcal{E}$ does not necessarily preserve the trace property of density matrices, that $\mathrm{tr}(\rho) = 1$. Rather, we make the convention that $\mathcal{E}$ is to be defined in such a way that $\mathrm{tr}[\mathcal{E}(\rho)]$ is equal to the probability of the measurement outcome described by $\mathcal{E}$ occurring. For example, suppose that we are doing a projective measurement in the computational basis of a single qubit. Then two quantum operations are used to describe this process, defined by $\mathcal{E}_0(\rho) \equiv |0\rangle\langle 0|\rho|0\rangle\langle 0|$ and $\mathcal{E}_1(\rho) \equiv |1\rangle\langle 1|\rho|1\rangle\langle 1|$. Notice that the probabilities of the respective outcomes are correctly given by $\mathrm{tr}[\mathcal{E}_0(\rho)]$ and $\mathrm{tr}[\mathcal{E}_1(\rho)]$. With this convention the correctly normalized final quantum state is therefore

$$\frac{\mathcal{E}(\rho)}{\mathrm{tr}[\mathcal{E}(\rho)]}\,. \tag{8.44}$$

In the case where the process is deterministic, that is, no measurement is taking place, this reduces to the requirement that $\mathrm{tr}[\mathcal{E}(\rho)] = 1 = \mathrm{tr}(\rho)$, for all ρ. As previously discussed, in this case, we say that the quantum operation is a trace-preserving quantum operation, since on its own $\mathcal{E}$ provides a complete description of the quantum process. On the other hand, if there is a ρ such that $\mathrm{tr}[\mathcal{E}(\rho)] < 1$, then the quantum operation is non-trace-preserving, since on its own $\mathcal{E}$ does not provide a complete description of the processes that may occur in the system. (That is, other measurement outcomes may occur, with

some probability.) A *physical* quantum operation is one that satisfies the requirement that probabilities never exceed 1, $\mathrm{tr}[\mathcal{E}(\rho)] \leq 1$.

The second property stems from a physical requirement on quantum operations. Suppose the input ρ to the quantum operation is obtained by randomly selecting the state from an ensemble $\{p_i, \rho_i\}$ of quantum states, that is, $\rho = \sum_i p_i \rho_i$. Then we would expect that the resulting state, $\mathcal{E}(\rho)/\mathrm{tr}[\mathcal{E}(\rho)] = \mathcal{E}(\rho)/p(\mathcal{E})$ corresponds to a random selection from the ensemble $\{p(i|\mathcal{E}), \mathcal{E}(\rho_i)/\mathrm{tr}[\mathcal{E}(\rho_i)]\}$, where $p(i|\mathcal{E})$ is the probability that the state prepared was ρ_i, given that the process represented by $\mathcal{E}$ occurred. Thus, we demand that

$$\mathcal{E}(\rho) = p(\mathcal{E}) \sum_i p(i|\mathcal{E}) \frac{\mathcal{E}(\rho_i)}{\mathrm{tr}[\mathcal{E}(\rho_i)]}, \tag{8.45}$$

where $p(\mathcal{E}) = \mathrm{tr}[\mathcal{E}(\rho)]$ is the probability that the process described by $\mathcal{E}$ occurs on input of ρ. By Bayes' rule (Appendix 1),

$$p(i|\mathcal{E}) = p(\mathcal{E}|i) \frac{p_i}{p(\mathcal{E})} = \frac{\mathrm{tr}[\mathcal{E}(\rho_i)] p_i}{p(\mathcal{E})} \tag{8.46}$$

so (8.45) reduces to (8.43).

The third property also originates from an important physical requirement, that not only must $\mathcal{E}(\rho)$ be a valid density matrix (up to normalization) so long as ρ is valid, but furthermore, if $\rho = \rho_{RQ}$ is the density matrix of some joint system of R and Q, if $\mathcal{E}$ acts only on Q, then $\mathcal{E}(\rho_{RQ})$ must still result in a valid density matrix (up to normalization) of the joint system. An example is given in Box 8.2. Formally, suppose we introduce a second (finite dimensional) system R. Let $\mathcal{I}$ denote the identity map on system R. Then the map $\mathcal{I} \otimes \mathcal{E}$ must take positive operators to positive operators.

It is perhaps surprising that these three axioms are sufficient to define quantum operations. However, the following theorem shows that they are equivalent to the earlier system-environment models and the definition in terms of an operator-sum representation:

Theorem 8.1: The map $\mathcal{E}$ satisfies axioms **A1**, **A2** and **A3** if and only if

$$\mathcal{E}(\rho) = \sum_i E_i \rho E_i^\dagger, \tag{8.50}$$

for some set of operators $\{E_i\}$ which map the input Hilbert space to the output Hilbert space, and $\sum_i E_i^\dagger E_i \leq I$.

Proof

Suppose $\mathcal{E}(\rho) = \sum_i E_i \rho E_i^\dagger$. $\mathcal{E}$ is obviously linear, so to check that $\mathcal{E}$ is a quantum operation we need only prove that it is completely positive. Let A be any positive operator acting on the state space of an extended system, RQ, and let $|\psi\rangle$ be some state of RQ. Defining $|\varphi_i\rangle \equiv (I_R \otimes E_i^\dagger)|\psi\rangle$, we have

$$\langle \psi | (I_R \otimes E_i) A (I_R \otimes E_i^\dagger) | \psi \rangle = \langle \varphi_i | A | \varphi_i \rangle \tag{8.51}$$

$$\geq 0, \tag{8.52}$$

by the positivity of the operator A. It follows that

$$\langle \psi | (\mathcal{I} \otimes \mathcal{E})(A) | \psi \rangle = \sum_i \langle \varphi_i | A | \varphi_i \rangle \geq 0, \tag{8.53}$$

Box 8.2: Complete positivity versus positivity

The transpose operation on a single qubit provides an example of why complete positivity is an important requirement for quantum operations. By definition, this map transposes the density operator in the computational basis:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \xrightarrow{T} \begin{bmatrix} a & c \\ b & d \end{bmatrix}. \tag{8.47}$$

This map preserves positivity of a single qubit. However, suppose that qubit is part of a two qubit system initially in the entangled state

$$\frac{|00\rangle + |11\rangle}{\sqrt{2}}, \tag{8.48}$$

and the transpose operation is applied to the first of these two qubits, while the second qubit is subject to trivial dynamics. Then the density operator of the system after the dynamics has been applied is

$$\frac{1}{2}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{8.49}$$

A calculation shows that this operator has eigenvalues $1/2, 1/2, 1/2$ and $-1/2$, so this is not a valid density operator. Thus, the transpose operation is an example of a positive map which is not completely positive, that is, it preserves the positivity of operators on the principal system, but does not continue to preserve positivity when applied to systems which contain the principal system as a subsystem.

and thus for any positive operator A, the operator $(\mathcal{I} \otimes \mathcal{E})(A)$ is also positive, as required. The requirement $\sum_i E_i^\dagger E_i \leq I$ ensures that probabilities are less than or equal to 1. This completes the first part of the proof.

Suppose next that $\mathcal{E}$ satisfies axioms **A1**, **A2** and **A3**. Our aim will be to find an operator-sum representation for $\mathcal{E}$. Suppose we introduce a system, R, with the same dimension as the original quantum system, Q. Let $|i_R\rangle$ and $|i_Q\rangle$ be orthonormal bases for R and Q. It will be convenient to use the same index, i, for these two bases, and this can certainly be done as R and Q have the same dimensionality. Define a joint state $|\alpha\rangle$ of RQ by

$$|\alpha\rangle \equiv \sum_i |i_R\rangle |i_Q\rangle. \tag{8.54}$$

The state $|\alpha\rangle$ is, up to a normalization factor, a maximally entangled state of the systems R and Q. This interpretation of $|\alpha\rangle$ as a maximally entangled state may help in understanding the following construction. Next, we define an operator σ on the state space of RQ by

$$\sigma \equiv (\mathcal{I}_R \otimes \mathcal{E})(|\alpha\rangle\langle\alpha|). \tag{8.55}$$

We may think of this as the result of applying the quantum operation $\mathcal{E}$ to one half of

a maximally entangled state of the system RQ. It is a truly remarkable fact, which we will now demonstrate, that the operator σ completely specifies the quantum operation $\mathcal{E}$. That is, to know how $\mathcal{E}$ acts on an arbitrary state of Q, it is sufficient to know how it acts on a single maximally entangled state of Q with another system!

The trick which allows us to recover $\mathcal{E}$ from σ is as follows. Let $|\psi\rangle = \sum_j \psi_j |j_Q\rangle$ be any state of system Q. Define a corresponding state $|\tilde{\psi}\rangle$ of system R by the equation

$$|\tilde{\psi}\rangle \equiv \sum_j \psi_j^* |j_R\rangle. \tag{8.56}$$

Notice that

$$\langle\tilde{\psi}|\sigma|\tilde{\psi}\rangle = \langle\tilde{\psi}| \left(\sum_{ij} |i_R\rangle\langle j_R| \otimes \mathcal{E}(|i_Q\rangle\langle j_Q|) \right) |\tilde{\psi}\rangle \tag{8.57}$$

$$= \sum_{ij} \psi_i \psi_j^* \mathcal{E}(|i_Q\rangle\langle j_Q|) \tag{8.58}$$

$$= \mathcal{E}(|\psi\rangle\langle\psi|). \tag{8.59}$$

Let $\sigma = \sum_i |s_i\rangle\langle s_i|$ be some decomposition of σ, where the vectors $|s_i\rangle$ need not be normalized. Define a map

$$E_i(|\psi\rangle) \equiv \langle\tilde{\psi}|s_i\rangle. \tag{8.60}$$

A little thought shows that this map is a linear map, so E_i is a linear operator on the state space of Q. Furthermore, we have

$$\sum_i E_i|\psi\rangle\langle\psi|E_i^\dagger = \sum_i \langle\tilde{\psi}|s_i\rangle\langle s_i|\tilde{\psi}\rangle \tag{8.61}$$

$$= \langle\tilde{\psi}|\sigma|\tilde{\psi}\rangle \tag{8.62}$$

$$= \mathcal{E}(|\psi\rangle\langle\psi|). \tag{8.63}$$

Thus

$$\mathcal{E}(|\psi\rangle\langle\psi|) = \sum_i E_i|\psi\rangle\langle\psi|E_i^\dagger, \tag{8.64}$$

for all pure states, $|\psi\rangle$, of Q. By convex-linearity it follows that

$$\mathcal{E}(\rho) = \sum_i E_i \rho E_i^\dagger \tag{8.65}$$

in general. The condition $\sum_i E_i^\dagger E_i \leq I$ follows immediately from axiom **A1** identifying the trace of $\mathcal{E}(\rho)$ with a probability. □

Freedom in the operator-sum representation

We have seen that the operator-sum representation provides a very general description of the dynamics of an open quantum system. Is it a *unique* description?

Consider quantum operations $\mathcal{E}$ and $\mathcal{F}$ acting on a single qubit with the operator-sum representations $\mathcal{E}(\rho) = \sum_k E_k \rho E_k^\dagger$ and $\mathcal{F}(\rho) = \sum_k F_k \rho F_k^\dagger$, where the operation elements for $\mathcal{E}$ and $\mathcal{F}$ are defined by

$$E_1 = \frac{I}{\sqrt{2}} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad E_2 = \frac{Z}{\sqrt{2}} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{8.66}$$

and

$$F_1 = |0\rangle\langle 0| = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad F_2 = |1\rangle\langle 1| = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} . \tag{8.67}$$

These appear to be very different quantum operations. What is interesting is that $\mathcal{E}$ and $\mathcal{F}$ are *actually the same quantum operation*. To see this, note that $F_1 = (E_1 + E_2)/\sqrt{2}$ and $F_2 = (E_1 - E_2)/\sqrt{2}$. Thus,

$$\mathcal{F}(\rho) = \frac{(E_1 + E_2)\rho(E_1^\dagger + E_2^\dagger) + (E_1 - E_2)\rho(E_1^\dagger - E_2^\dagger)}{2} \tag{8.68}$$

$$= E_1\rho E_1^\dagger + E_2\rho E_2^\dagger \tag{8.69}$$

$$= \mathcal{E}(\rho) . \tag{8.70}$$

This example shows that the operation elements appearing in an operator-sum representation for a quantum operation are *not* unique.

The freedom in the representation is very interesting. Suppose we flipped a fair coin, and, depending on the outcome of the coin toss, applied either the unitary operator I or Z to the quantum system. This process corresponds to the first operator-sum representation for $\mathcal{E}$. The second operator-sum representation for $\mathcal{E}$ (labeled $\mathcal{F}$ above) corresponds to performing a projective measurement in the $\{|0\rangle, |1\rangle\}$ basis, with the outcome of the measurement unknown. These two apparently very different physical processes give rise to exactly the same dynamics for the principal system.

When do two sets of operation elements give rise to the same quantum operation? Understanding this question is important for at least two reasons. First, from a physical point of view, understanding the freedom in the representation gives us more insight into how different physical processes can give rise to the same system dynamics. Second, understanding the freedom in operator-sum representation is crucial to a good understanding of quantum error-correction.

Intuitively, it is clear that there must be a great deal of freedom in an operator-sum representation. Consider a trace-preserving quantum operation $\mathcal{E}$ which describes the dynamics of some system such as that shown in Figure 8.3. We have shown that the operation elements $E_k = \langle e_k|U|e_0\rangle$ for $\mathcal{E}$ may be associated with an orthonormal basis $|e_k\rangle$ for the environment. Suppose that we supplement the interaction U with an additional unitary action U' on the environment alone, as shown in Figure 8.7. Clearly this does not change the state of the principal system. What are the corresponding operation elements to this new process, $(I \otimes U')U$? We obtain:

$$F_k = \langle e_k|(I \otimes U')\, U|e_0\rangle \tag{8.71}$$

$$= \sum_j \left[I \otimes \langle e_k|U'|e_j\rangle \right] \langle e_j|U|e_0\rangle \tag{8.72}$$

$$= \sum_j U'_{kj} E_j , \tag{8.73}$$

where we have used the fact that $\sum_j |e_j\rangle\langle e_j| = I$, and U'_{kj} are the matrix elements of U' with respect to the basis $|e_k\rangle$. It turns out that the freedom in the operator-sum representation yielded by this physically motivated picture captures the essence of the complete freedom available in the operator-sum representation, as proved in the following theorem.

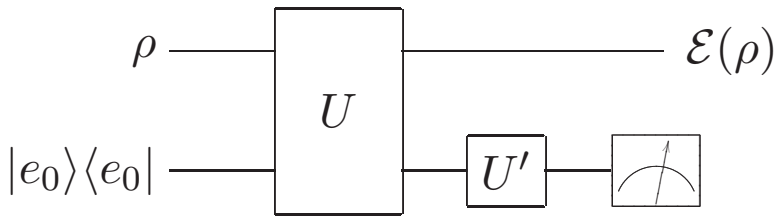


Figure 8.7. Origin of the unitary freedom in the operator-sum representation.

Theorem 8.2: (**Unitary freedom in the operator-sum representation**) Suppose $\{E_1, \ldots, E_m\}$ and $\{F_1, \ldots, F_n\}$ are operation elements giving rise to quantum operations $\mathcal{E}$ and $\mathcal{F}$, respectively. By appending zero operators to the shorter list of operation elements we may ensure that $m = n$. Then $\mathcal{E} = \mathcal{F}$ if and only if there exist complex numbers u_{ij} such that $E_i = \sum_j u_{ij} F_j$, and u_{ij} is an m by m unitary matrix.

Proof
The key to the proof is Theorem 2.6, on page 103. Recall that this result tells us that two sets of vectors $|\psi_i\rangle$ and $|\varphi_j\rangle$ generate the same operator if and only if

$$|\psi_i\rangle = \sum_j u_{ij} |\varphi_j\rangle \,, \tag{8.74}$$

where u_{ij} is a unitary matrix of complex numbers, and we 'pad' whichever set of states $|\psi_i\rangle$ or $|\varphi_j\rangle$ is smaller with additional states 0 so that the two sets have the same number of elements. This result allows us to characterize the freedom in operator-sum representations. Suppose $\{E_i\}$ and $\{F_j\}$ are two sets of operation elements for the same quantum operation, $\sum_i E_i \rho E_i^\dagger = \sum_j F_j \rho F_j^\dagger$ for all ρ. Define

$$|e_i\rangle \equiv \sum_k |k_R\rangle \left(E_i |k_Q\rangle\right) \tag{8.75}$$

$$|f_j\rangle \equiv \sum_k |k_R\rangle \left(F_j |k_Q\rangle\right) \,. \tag{8.76}$$

Recall the definition of σ in Equation (8.55), from which it follows that $\sigma = \sum_i |e_i\rangle\langle e_i| = \sum_j |f_j\rangle\langle f_j|$, and thus there exists unitary u_{ij} such that

$$|e_i\rangle = \sum_j u_{ij} |f_j\rangle \,. \tag{8.77}$$

But for arbitrary $|\psi\rangle$ we have

$$E_i|\psi\rangle = \langle\tilde{\psi}|e_i\rangle \tag{8.78}$$

$$= \sum_j u_{ij} \langle\tilde{\psi}|f_j\rangle \tag{8.79}$$

$$= \sum_k u_{ij} F_j |\psi\rangle. \tag{8.80}$$

Thus

$$E_i = \sum_j u_{ij} F_j \,. \tag{8.81}$$

Conversely, supposing E_i and F_j are related by a unitary transformation of the form $E_i = \sum_{ij} u_{ij} F_j$, simple algebra shows that the quantum operation with operation elements $\{E_i\}$ is the same as the quantum operation with operation elements $\{F_j\}$. □

Theorem 8.2 can be used to answer another interesting question: what is the maximum size of an environment that would be needed to mock up a given quantum operation?

Theorem 8.3: All quantum operations $\mathcal{E}$ on a system of Hilbert space dimension d can be generated by an operator-sum representation containing at most d^2 elements,

$$\mathcal{E}(\rho) = \sum_{k=1}^{M} E_k \rho E_k^\dagger , \tag{8.82}$$

where $1 \leq M \leq d^2$.

The proof of this theorem is simple and is left as an exercise for you.

Exercise 8.10: Give a proof of Theorem 8.3 based on the freedom in the operator-sum representation, as follows. Let $\{E_j\}$ be a set of operation elements for $\mathcal{E}$. Define a matrix $W_{jk} \equiv \mathrm{tr}(E_j^\dagger E_k)$. Show that the matrix W is Hermitian and of rank at most d^2, and thus there is unitary matrix u such that $uWu^\dagger$ is diagonal with at most d^2 non-zero entries. Use u to define a new set of at most d^2 non-zero operation elements $\{F_j\}$ for $\mathcal{E}$.

Exercise 8.11: Suppose $\mathcal{E}$ is a quantum operation mapping a d-dimensional input space to a d'-dimensional output space. Show that $\mathcal{E}$ can be described using a set of at most dd' operation elements $\{E_k\}$.

The freedom in the operator-sum representation is surprisingly useful. We use it, for example, in our study of quantum error-correction in Chapter 10. In that chapter we will see that certain sets of operators in the operator-sum representation give more useful information about the quantum error-correction process, and it behooves us to study quantum error-correction from that point of view. As usual, having multiple ways of understanding a process gives us much more insight into what is going on.

8.3 Examples of quantum noise and quantum operations

In this section we examine some concrete examples of quantum noise and quantum operations. These models illustrate the power of the quantum operations formalism we have been developing. They are also important in understanding the practical effects of noise on quantum systems, and how noise can be controlled by techniques such as error-correction.

We begin in Section 8.3.1 by considering how *measurement* can be described as a quantum operation, and in particular we consider the trace and partial trace operations. After that, we turn to noise processes, beginning in Section 8.3.2 with the presentation of a graphical method for understanding quantum operations on a single qubit. This method is used in the remainder of the section to illustrate elementary bit and phase flip error processes (in Section 8.3.3), the depolarizing channel (in Section 8.3.4), amplitude damping (in Section 8.3.5), and phase damping (in Section 8.3.6). Amplitude and phase

damping are ideal models of noise that capture many of the most important features of the noise occurring in quantum mechanical systems, and we not only consider their abstract mathematical formulation, but also how the processes arise in real-world quantum systems.

8.3.1 Trace and partial trace

One of the main uses of the quantum operations formalism is to describe the effects of measurement. Quantum operations can be used to describe both the probability of getting a particular outcome from a measurement on a quantum system, and also the state change in the system effected by the measurement.

The simplest operation related to measurement is the trace map $\rho \rightarrow \mathrm{tr}(\rho)$ – which we can show is indeed a quantum operation, in the following way. Let H_Q be any input Hilbert space, spanned by an orthonormal basis $|1\rangle \ldots |d\rangle$, and let H'_Q be a one-dimensional output space, spanned by the state $|0\rangle$. Define

$$\mathcal{E}(\rho) \equiv \sum_{i=1}^{d} |0\rangle\langle i|\rho|i\rangle\langle 0| \,, \tag{8.83}$$

so that $\mathcal{E}$ is a quantum operation, by Theorem 8.1. Note that $\mathcal{E}(\rho) = \mathrm{tr}(\rho)|0\rangle\langle 0|$, so that, up to the unimportant $|0\rangle\langle 0|$ multiplier, this quantum operation is identical to the trace function.

An even more useful result is the observation that the partial trace is a quantum operation. Suppose we have a joint system QR, and wish to trace out system R. Let $|j\rangle$ be a basis for system R. Define a linear operator $E_i : H_{QR} \rightarrow H_Q$ by

$$E_i \left(\sum_j \lambda_j |q_j\rangle|j\rangle \right) \equiv \lambda_i |q_i\rangle \,, \tag{8.84}$$

where λ_j are complex numbers, and $|q_j\rangle$ are arbitrary states of system Q. Define $\mathcal{E}$ to be the quantum operation with operation elements $\{E_i\}$, that is,

$$\mathcal{E}(\rho) \equiv \sum_i E_i \rho E_i^\dagger \,. \tag{8.85}$$

By Theorem 8.1, this is a quantum operation from system QR to system Q. Notice that

$$\mathcal{E}(\rho \otimes |j\rangle\langle j'|) = \rho\delta_{j,j'} = \mathrm{tr}_R(\rho \otimes |j\rangle\langle j'|) \,, \tag{8.86}$$

where ρ is any Hermitian operator on the state space of system Q, and $|j\rangle$ and $|j'\rangle$ are members of the orthonormal basis for system R. By linearity of $\mathcal{E}$ and tr_R, it follows that $\mathcal{E} = \mathrm{tr}_R$.

8.3.2 Geometric picture of single qubit quantum operations

There is an elegant geometric method for picturing quantum operations on a single qubit. This method allows one to get an intuitive feel for the behavior of quantum operations in terms of their action on the Bloch sphere. Recall from Exercise 2.72 on page 105 that the state of a single qubit can always be written in the Bloch representation,

$$\rho = \frac{I + \vec{r} \cdot \vec{\sigma}}{2} \,, \tag{8.87}$$

where $\vec{r}$ is a three component real vector. Explicitly, this gives us

$$\rho = \frac{1}{2} \begin{bmatrix} 1 + r_z & r_x - ir_y \\ r_x + ir_y & 1 - r_z \end{bmatrix} . \tag{8.88}$$

In this representation, it turns out that an arbitrary trace-preserving quantum operation is equivalent to a map of the form

$$\vec{r} \stackrel{\mathcal{E}}{\rightarrow} \vec{r}' = M\vec{r} + \vec{c}, \tag{8.89}$$

where M is a 3×3 real matrix, and $\vec{c}$ is a constant vector. This is an *affine map*, mapping the Bloch sphere into itself. To see this, suppose the operators E_i generating the operator-sum representation for $\mathcal{E}$ are written in the form

$$E_i = \alpha_i I + \sum_{k=1}^{3} a_{ik}\sigma_k. \tag{8.90}$$

Then it is not difficult to check that

$$M_{jk} = \sum_l \left[a_{lj}a_{lk}^* + a_{lj}^* a_{lk} + \left(|\alpha_l|^2 - \sum_p a_{lp}a_{lp}^* \right) \delta_{jk} + i \sum_p \epsilon_{jkp}(\alpha_l a_{lp}^* - \alpha_l^* a_{lp}) \right] \tag{8.91}$$

$$c_k = 2i \sum_l \sum_{jp} \epsilon_{jpk} a_{lj} a_{lp}^*, \tag{8.92}$$

where we have made use of the completeness relation $\sum_i E_i^\dagger E_i = I$ to simplify the expression for $\vec{c}$.

The meaning of the affine map, Equation (8.89), is made clearer by considering the polar decomposition of the matrix M, $M = U|M|$, where U is unitary. Because M is real, it follows that $|M|$ is real and Hermitian, that is, $|M|$ is a symmetric matrix. Furthermore, because M is real we may assume that U is real, and is thus an orthogonal matrix, that is, $U^T U = I$, where T is the transpose operation. Thus we may write

$$M = OS\,, \tag{8.93}$$

where O is a real orthogonal matrix with determinant 1, representing a proper rotation, and S is a real symmetric matrix. Viewed this way, Equation (8.89) is just a deformation of the Bloch sphere along principal axes determined by S, followed by a proper rotation due to O, followed by a displacement due to $\vec{c}$.

Exercise 8.12: Why can we assume that O has determinant 1 in the decomposition (8.93)?

Exercise 8.13: Show that unitary transformations correspond to rotations of the Bloch sphere.

Exercise 8.14: Show that $\det(S)$ need not be positive.

8.3.3 Bit flip and phase flip channels

The geometric picture described above can be used to visualize some important quantum operations on single qubits, which will later be used in the theory of error-correction. The *bit flip* channel flips the state of a qubit from $|0\rangle$ to $|1\rangle$ (and vice versa) with probability $1 - p$. It has operation elements

$$E_0 = \sqrt{p}\, I = \sqrt{p} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad E_1 = \sqrt{1-p}\, X = \sqrt{1-p} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{8.94}$$

The effect of the bit flip channel is illustrated in Figure 8.8.

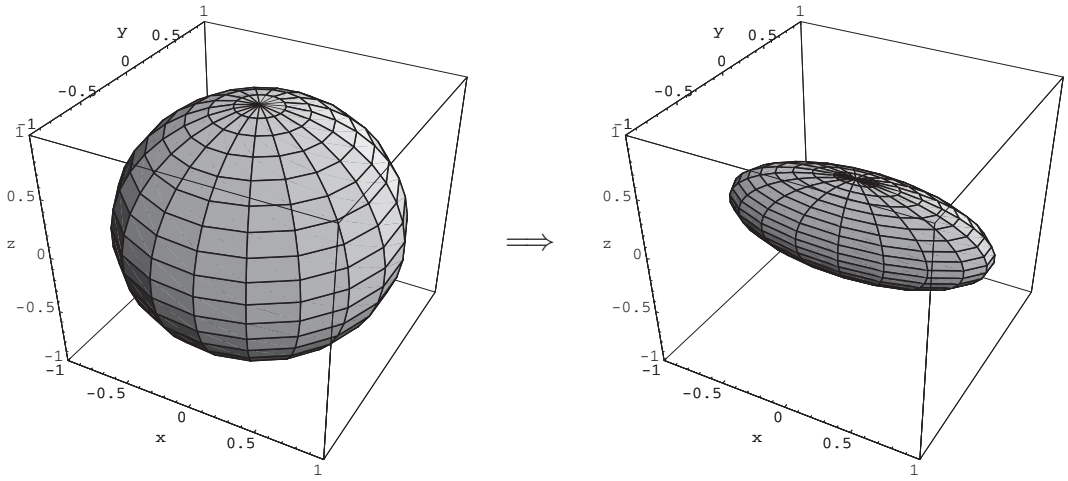


Figure 8.8. The effect of the bit flip channel on the Bloch sphere, for $p = 0.3$. The sphere on the left represents the set of all pure states, and the deformed sphere on the right represents the states after going through the channel. Note that the states on the $\hat{x}$ axis are left alone, while the $\hat{y}$-$\hat{z}$ plane is uniformly contracted by a factor of $1 - 2p$.

This geometric picture makes it very easy to verify certain facts about this quantum operation. For example, it is easy to verify that the quantity $\mathrm{tr}(\rho^2)$ for a single qubit is equal to $(1 + |r|^2)/2$, where $|r|$ is the norm of the Bloch vector. The contraction of the Bloch sphere illustrated in Figure 8.8 cannot increase the norm of the Bloch vector, and therefore we can immediately conclude that $\mathrm{tr}(\rho^2)$ can only ever decrease for the bit flip channel. This is but one example of the use of the geometric picture; once it becomes sufficiently familiar it becomes a great source of insight about the properties of quantum operations on a single qubit.

The *phase flip* channel has operation elements

$$E_0 = \sqrt{p}\, I = \sqrt{p} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad E_1 = \sqrt{1-p}\, Z = \sqrt{1-p} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{8.95}$$

The effect of the phase flip channel is illustrated in Figure 8.9. As a special case of the phase flip channel, consider the quantum operation which arises when we choose $p = 1/2$. Using the freedom in the operator-sum representation this operation may be written

$$\rho \rightarrow \mathcal{E}(\rho) = P_0 \rho P_0 + P_1 \rho P_1, \tag{8.96}$$

where $P_0 = |0\rangle\langle 0|$, $P_1 = |1\rangle\langle 1|$, which corresponds to a measurement of the qubit in the $|0\rangle, |1\rangle$ basis, with the result of the measurement unknown. Using the above prescription

it is easy to see that the corresponding map on the Bloch sphere is given by

$$(r_x, r_y, r_z) \rightarrow (0, 0, r_z)\,. \tag{8.97}$$

Geometrically, the Bloch vector is projected along the z axis, and the x and y components of the Bloch vector are lost.

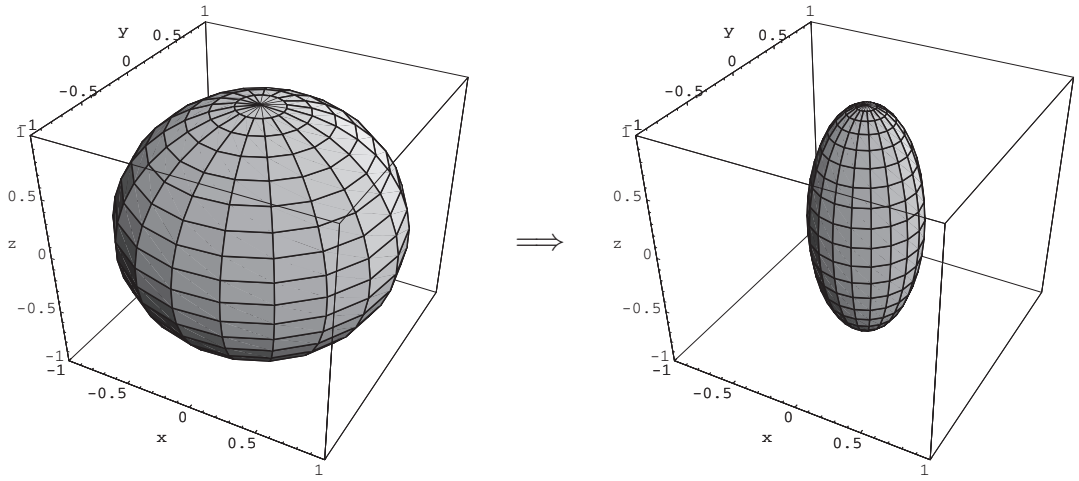


Figure 8.9. The effect of the phase flip channel on the Bloch sphere, for $p = 0.3$. Note that the states on the $\hat{z}$ axis are left alone, while the $\hat{x} - \hat{y}$ plane is uniformly contracted by a factor of $1 - 2p$.

The *bit–phase flip* channel has operation elements

$$E_0 = \sqrt{p}\, I = \sqrt{p} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad E_1 = \sqrt{1-p}\, Y = \sqrt{1-p} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}. \tag{8.98}$$

As the name indicates, this is a combination of a phase flip and a bit flip, since $Y = iXZ$. The action of the bit–phase flip channel is illustrated in Figure 8.10.

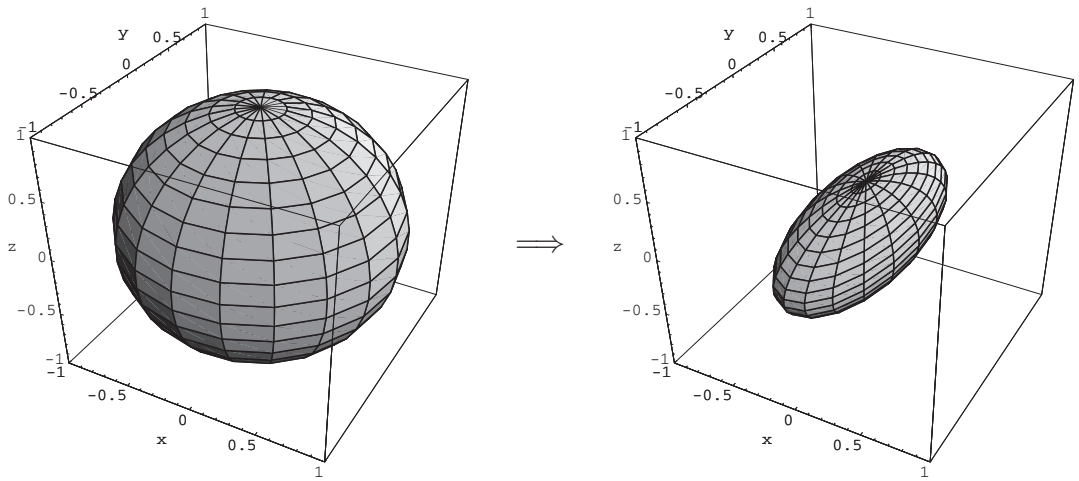


Figure 8.10. The effect of the bit–phase flip channel on the Bloch sphere, for $p = 0.3$. Note that the states on the $\hat{y}$ axis are left alone, while the $\hat{x}$-$\hat{z}$ plane is uniformly contracted by a factor of $1 - 2p$.

Exercise 8.15: Suppose a projective measurement is performed on a single qubit in the basis $|+\rangle, |-\rangle$, where $|\pm\rangle \equiv (|0\rangle \pm |1\rangle)/\sqrt{2}$. In the event that we are ignorant of the result of the measurement, the density matrix evolves according to the equation

$$\rho \rightarrow \mathcal{E}(\rho) = |+\rangle\langle+|\rho|+\rangle\langle+| \quad + \quad |-\rangle\langle-|\rho|-\rangle\langle-|. \tag{8.99}$$

Illustrate this transformation on the Bloch sphere.

Exercise 8.16: The graphical method for understanding single qubit quantum operations was derived for trace-preserving quantum operations. Find an explicit example of a non-trace-preserving quantum operation which cannot be described as a deformation of the Bloch sphere, followed by a rotation and a displacement.

8.3.4 Depolarizing channel

The *depolarizing channel* is an important type of quantum noise. Imagine we take a single qubit, and with probability p that qubit is *depolarized*. That is, it is replaced by the completely mixed state, $I/2$. With probability $1-p$ the qubit is left untouched. The state of the quantum system after this noise is

$$\mathcal{E}(\rho) = \frac{pI}{2} + (1-p)\rho. \tag{8.100}$$

The effect of the depolarizing channel on the Bloch sphere is illustrated in Figure 8.11.

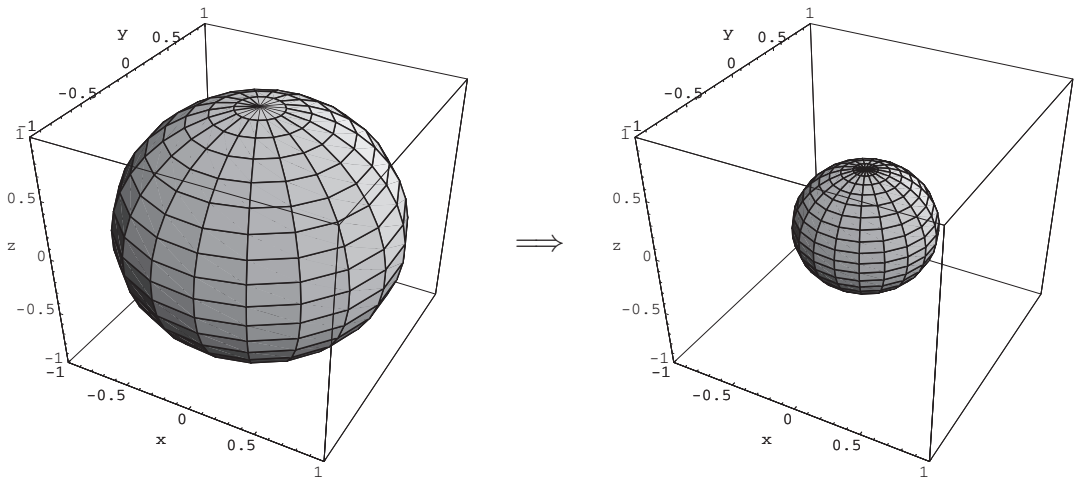


Figure 8.11. The effect of the depolarizing channel on the Bloch sphere, for $p = 0.5$. Note how the entire sphere contracts uniformly as a function of p.

A quantum circuit simulating the depolarizing channel is illustrated in Figure 8.12. The top line of the circuit is the input to the depolarizing channel, while the bottom two lines are an 'environment' to simulate the channel. We have used an environment with two mixed state inputs. The idea is that the third qubit, initially a mixture of the state $|0\rangle$ with probability $1-p$ and state $|1\rangle$ with probability p acts as a control for whether or not the completely mixed state $I/2$ stored in the second qubit is swapped into the first qubit.

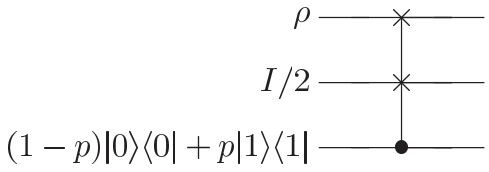


Figure 8.12. Circuit implementation of the depolarizing channel.

The form (8.100) is not in the operator-sum representation. However, if we observe that for arbitrary ρ,

$$\frac{I}{2} = \frac{\rho + X\rho X + Y\rho Y + Z\rho Z}{4} \tag{8.101}$$

and then substitute for $I/2$ into (8.100) we arrive at the equation

$$\mathcal{E}(\rho) = \left(1 - \frac{3p}{4}\right)\rho + \frac{p}{4}\left(X\rho X + Y\rho Y + Z\rho Z\right), \tag{8.102}$$

showing that the depolarizing channel has operation elements $\{\sqrt{1-3p/4}\, I, \sqrt{p}\, X/2, \sqrt{p}\, Y/2, \sqrt{p}\, Z/2\}$. Note, incidentally, that it is frequently convenient to parametrize the depolarizing channel in different ways, such as

$$\mathcal{E}(\rho) = (1-p)\rho + \frac{p}{3}\left(X\rho X + Y\rho Y + Z\rho Z\right), \tag{8.103}$$

which has the interpretation that the state ρ is left alone with probability $1-p$, and the operators X, Y and Z applied each with probability $p/3$.

Exercise 8.17: Verify (8.101) as follows. Define

$$\mathcal{E}(A) \equiv \frac{A + XAX + YAY + ZAZ}{4}, \tag{8.104}$$

and show that

$$\mathcal{E}(I) = I; \quad \mathcal{E}(X) = \mathcal{E}(Y) = \mathcal{E}(Z) = 0. \tag{8.105}$$

Now use the Bloch sphere representation for single qubit density matrices to verify (8.101).

The depolarizing channel can, of course, be generalized to quantum systems of dimension more than two. For a d-dimensional quantum system the depolarizing channel again replaces the quantum system with the completely mixed state I/d with probability p, and leaves the state untouched otherwise. The corresponding quantum operation is

$$\mathcal{E}(\rho) = \frac{pI}{d} + (1-p)\rho\,. \tag{8.106}$$

Exercise 8.18: For $k \geq 1$ show that $\mathrm{tr}(\rho^k)$ is never increased by the action of the depolarizing channel.

Exercise 8.19: Find an operator-sum representation for a generalized depolarizing channel acting on a d-dimensional Hilbert space.

8.3.5 Amplitude damping

An important application of quantum operations is the description of *energy dissipation* – effects due to loss of energy from a quantum system. What are the dynamics of an atom which is spontaneously emitting a photon? How does a spin system at high temperature approach equilibrium with its environment? What is the state of a photon in an interferometer or cavity when it is subject to scattering and attenuation?

Each of these processes has its own unique features, but the general behavior of all of them is well characterized by a quantum operation known as *amplitude damping*, which we can derive by considering the following scenario. Suppose we have a single optical mode containing the quantum state $a|0\rangle+b|1\rangle$, a superposition of zero or one photons. The scattering of a photon from this mode can be modeled by thinking of inserting a partially silvered mirror, a beamsplitter, in the path of the photon. As we saw in Section 7.4.2, this beamsplitter allows the photon to couple to another single optical mode (representing the environment), according to the unitary transformation $B = \exp\left[\theta\left(a^\dagger b - a b^\dagger\right)\right]$, where a, $a^\dagger$ and b, $b^\dagger$ are annihilation and creation operators for photons in the two modes. The output after the beamsplitter, assuming the environment starts out with no photons, is simply $B|0\rangle(a|0\rangle + b|1\rangle) = a|00\rangle + b(\cos\theta|01\rangle + \sin\theta|10\rangle)$, using Equation (7.34). Tracing over the environment gives us the quantum operation

$$\mathcal{E}_{\rm AD}(\rho) = E_0 \rho E_0^\dagger + E_1 \rho E_1^\dagger \,, \tag{8.107}$$

where $E_k = \langle k|B|0\rangle$ are

$$\begin{aligned} E_0 &= \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{bmatrix} \\ E_1 &= \begin{bmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{bmatrix} , \end{aligned} \tag{8.108}$$

the operation elements for amplitude damping. $\gamma = \sin^2\theta$ can be thought of as the probability of losing a photon.

Observe that no linear combination can be made of E_0 and E_1 to give an operation element proportional to the identity (though compare with Exercise 8.23). The E_1 operation changes a $|1\rangle$ state into a $|0\rangle$ state, corresponding to the physical process of losing a quantum of energy to the environment. E_0 leaves $|0\rangle$ unchanged, but reduces the amplitude of a $|1\rangle$ state; physically, this happens because a quantum of energy was not lost to the environment, and thus the environment now perceives it to be more likely that the system is in the $|0\rangle$ state, rather than the $|1\rangle$ state.

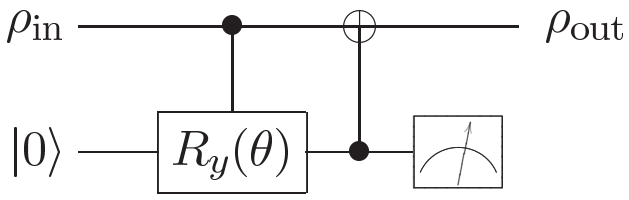


Figure 8.13. Circuit model for amplitude damping

Exercise 8.20: (Circuit model for amplitude damping) Show that the circuit in

Figure 8.13 models the amplitude damping quantum operation, with $\sin^2(\theta/2) = \gamma$.

Exercise 8.21: (**Amplitude damping of a harmonic oscillator**) Suppose that our principal system, a harmonic oscillator, interacts with an environment, modeled as another harmonic oscillator, through the Hamiltonian

$$H = \chi(a^\dagger b + b^\dagger a) \tag{8.109}$$

where a and b are the annihilation operators for the respective harmonic oscillators, as defined in Section 7.3.

(1) Using $U = \exp(-iH\Delta t)$, denoting the eigenstates of $b^\dagger b$ as $|k_b\rangle$, and selecting the vacuum state $|0_b\rangle$ as the initial state of the environment, show that the operation elements $E_k = \langle k_b|U|0_b\rangle$ are found to be

$$E_k = \sum_n \sqrt{\binom{n}{k}} \sqrt{(1-\gamma)^{n-k}\gamma^k}\, |n-k\rangle\langle n| \,, \tag{8.110}$$

where $\gamma = 1 - \cos^2(\chi\Delta t)$ is the probability of loosing a single quantum of energy, and states such as $|n\rangle$ are eigenstates of $a^\dagger a$.

(2) Show that the operation elements E_k define a trace-preserving quantum operation.

Exercise 8.22: (**Amplitude damping of single qubit density matrix**) For the general single qubit state

$$\rho = \begin{bmatrix} a & b \\ b^* & c \end{bmatrix} \tag{8.111}$$

show that amplitude damping leads to

$$\mathcal{E}_{\rm AD}(\rho) = \begin{bmatrix} 1-(1-\gamma)(1-a) & b\sqrt{1-\gamma} \\ b^*\sqrt{1-\gamma} & c(1-\gamma) \end{bmatrix} . \tag{8.112}$$

Exercise 8.23: (**Amplitude damping of dual-rail qubits**) Suppose that a single qubit state is represented by using two qubits, as

$$|\psi\rangle = a\,|01\rangle + b\,|10\rangle \,. \tag{8.113}$$

Show that $\mathcal{E}_{\rm AD} \otimes \mathcal{E}_{\rm AD}$ applied to this state gives a process which can be described by the operation elements

$$E_0^{\rm dr} = \sqrt{1-\gamma}\, I \tag{8.114}$$

$$E_1^{\rm dr} = \sqrt{\gamma}\left[|00\rangle\langle 01| + |00\rangle\langle 10|\right] , \tag{8.115}$$

that is, either nothing ($E_0^{\rm dr}$) happens to the qubit, or the qubit is transformed ($E_1^{\rm dr}$) into the state $|00\rangle$, which is orthogonal to $|\psi\rangle$. This is a simple error-detection code, and is also the basis for the robustness of the 'dual-rail' qubit discussed in Section 7.4.

Exercise 8.24: (**Spontaneous emission is amplitude damping**) A single atom coupled to a single mode of electromagnetic radiation undergoes spontaneous emission, as was described in Section 7.6.1. To see that this process is just

amplitude damping, take the unitary operation resulting from the Jaynes–Cummings interaction, Equation (7.77), with detuning $\delta = 0$, and give the quantum operation resulting from tracing over the field.

A general characteristic of a quantum operation is the set of states that are left invariant under the operation. For example, we have seen how the phase flip channel leaves the $\hat{z}$ axis of the Bloch sphere unchanged; this corresponds to states of the form $p|0\rangle\langle 0| + (1-p)|1\rangle\langle 1|$ for arbitrary probability p. In the case of amplitude damping, only the ground state $|0\rangle$ is left invariant. That is a natural consequence of our modeling the environment as starting in the $|0\rangle$ state, as if it were at zero temperature.

What quantum operation describes the effect of dissipation to an environment at finite temperature? This process, $\mathcal{E}_{\rm GAD}$, called *generalized amplitude damping*, is defined for single qubits by the operation elements

$$E_0 = \sqrt{p}\begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{bmatrix} \tag{8.116}$$

$$E_1 = \sqrt{p}\begin{bmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{bmatrix} \tag{8.117}$$

$$E_2 = \sqrt{1-p}\begin{bmatrix} \sqrt{1-\gamma} & 0 \\ 0 & 1 \end{bmatrix} \tag{8.118}$$

$$E_3 = \sqrt{1-p}\begin{bmatrix} 0 & 0 \\ \sqrt{\gamma} & 0 \end{bmatrix}, \tag{8.119}$$

where the stationary state

$$\rho_\infty = \begin{bmatrix} p & 0 \\ 0 & 1-p \end{bmatrix}, \tag{8.120}$$

satisfies $\mathcal{E}_{\rm GAD}(\rho_\infty) = \rho_\infty$. Generalized amplitude damping describes the 'T_1' relaxation processes due to coupling of spins to their surrounding lattice, a large system which is in thermal equilibrium at a temperature often much higher than the spin temperature. This is the case relevant to NMR quantum computation, where some of the properties of $\mathcal{E}_{\rm GAD}$ described in Box 8.3 become important.

Exercise 8.25: If we define the temperature T of a qubit by assuming that in equilibrium the probabilities of being in the $|0\rangle$ and $|1\rangle$ states satisfy a Boltzmann distribution, that is $p_0 = e^{-E_0/k_{\rm B}T}/\mathcal{Z}$ and $p_1 = e^{-E_1/k_{\rm B}T}/\mathcal{Z}$, where E_0 is the energy of the state $|0\rangle$, E_1 the energy of the state $|1\rangle$, and $\mathcal{Z} = e^{-E_0/k_{\rm B}T} + e^{-E_1/k_{\rm B}T}$, what temperature describes the state ρ_∞?

We can visualize the effect of amplitude damping in the Bloch representation as the Bloch vector transformation

$$(r_x, r_y, r_z) \rightarrow \left(r_x\sqrt{1-\gamma}, r_y\sqrt{1-\gamma}, \gamma + r_z(1-\gamma)\right). \tag{8.122}$$

When γ is replaced with a time-varying function like $1 - e^{-t/T_1}$ (t is time, and T_1 just some constant characterizing the speed of the process), as is often the case for real physical processes, we can visualize the effects of amplitude damping as a *flow* on the Bloch sphere, which moves every point in the unit ball towards a fixed point at the north pole, where $|0\rangle$ is located. This is shown in Figure 8.14.

Box 8.3: Generalized amplitude damping and effective pure states

The notion of 'effective pure states' introduced in Section 7.7 was found to be useful in NMR implementations of quantum computers. These states behave like pure states under unitary evolution and measurement of traceless observables. How do they behave under quantum operations? In general, non-unitary quantum operations ruin the effectiveness of these states, but surprisingly, they can behave properly under generalized amplitude damping.

Consider a single qubit effective pure state $\rho = (1-p)I + (2p-1)|0\rangle\langle 0|$. Clearly, traceless measurement observables acting on $U\rho U^\dagger$ produce results which are proportional to those on the pure state $U|0\rangle\langle 0|U^\dagger$. Suppose ρ is the stationary state of $\mathcal{E}_{\rm GAD}$. Interestingly, in this case,

$$\mathcal{E}_{\rm GAD}(U\rho U^\dagger) = (1-p)I + (2p-1)\mathcal{E}_{\rm AD}(U\rho U^\dagger)\,. \tag{8.121}$$

That is, under generalized amplitude damping, an effective pure state can remain such, and moreover, the 'pure' component of ρ behaves as if it were undergoing amplitude damping to a reservoir at zero temperature!

Similarly, generalized amplitude damping performs the transformation

$$(r_x, r_y, r_z) \to \left(r_x\sqrt{1-\gamma}, r_y\sqrt{1-\gamma}, \gamma(2p-1) + r_z(1-\gamma)\right)\,. \tag{8.123}$$

Comparing (8.122) and (8.123), it is clear that amplitude damping and generalized amplitude damping differ only in the location of the fixed point of the flow; the final state is along the $\hat{z}$ axis, at the point $(2p-1)$, which is a mixed state.

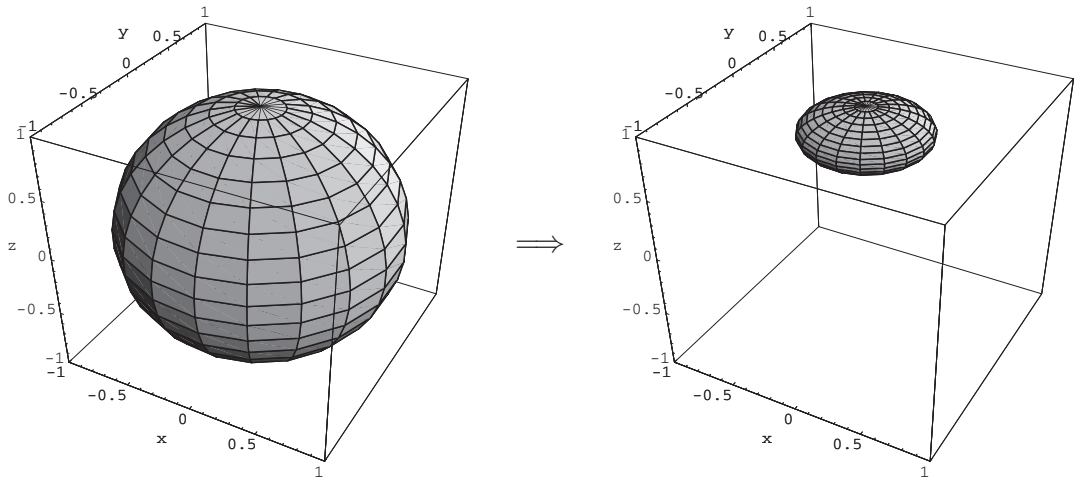


Figure 8.14. The effect of the amplitude damping channel on the Bloch sphere, for $p = 0.8$. Note how the entire sphere shrinks towards the north pole, the $|0\rangle$ state.

8.3.6 Phase damping

A noise process that is uniquely quantum mechanical, which describes the loss of quantum information without loss of energy, is *phase damping*. Physically it describes, for example,

what happens when a photon scatters randomly as it travels through a waveguide, or how electronic states in an atom are perturbed upon interacting with distant electrical charges. The energy eigenstates of a quantum system do not change as a function of time, but do accumulate a phase which is proportional to the eigenvalue. When a system evolves for an amount of time which is not precisely known, partial information about this quantum phase – the *relative* phases between the energy eigenstates – is lost.

A very simple model for this kind of quantum noise is the following. Suppose that we have a qubit $|\psi\rangle = a\,|0\rangle + b\,|1\rangle$ upon which the rotation operation $R_z(\theta)$ is applied, where the angle of rotation θ is random. The randomness could originate, for example, from a deterministic interaction with an environment, which never again interacts with the system and thus is implicitly measured (see Section 4.4). We shall call this random R_z operation a *phase kick*. Let us assume that the phase kick angle θ is well represented as a random variable which has a Gaussian distribution with mean 0 and variance 2λ.

The output state from this process is given by the density matrix obtained from averaging over θ,

$$\rho = \frac{1}{\sqrt{4\pi\lambda}} \int_{-\infty}^{\infty} R_z(\theta)|\psi\rangle\langle\psi|R_z^\dagger(\theta) e^{-\theta^2/4\lambda}\, d\theta \tag{8.124}$$

$$= \begin{bmatrix} |a|^2 & ab^*\, e^{-\lambda} \\ a^*b\, e^{-\lambda} & |b|^2 \end{bmatrix}. \tag{8.125}$$

The random phase kicking causes the expected value of the off-diagonal elements of the density matrix to decay exponentially to zero with time. That is a characteristic result of phase damping.

Another way to derive the phase damping quantum operation is to consider an interaction between two harmonic oscillators, in a manner similar to how amplitude damping was derived in the last section, but this time with the interaction Hamiltonian

$$H = \chi\, a^\dagger a (b + b^\dagger)\,, \tag{8.126}$$

Letting $U = \exp(-iH\Delta t)$, considering only the $|0\rangle$ and $|1\rangle$ states of the a oscillator as our system, and taking the environment oscillator to initially be $|0\rangle$, we find that tracing over the environment gives the operation elements $E_k = \langle k_b|U|0_b\rangle$, which are

$$E_0 = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-\lambda} \end{bmatrix} \tag{8.127}$$

$$E_1 = \begin{bmatrix} 0 & 0 \\ 0 & \sqrt{\lambda} \end{bmatrix}, \tag{8.128}$$

where $\lambda = 1 - \cos^2(\chi\Delta t)$ can be interpreted as the probability that a photon from the system has been scattered (without loss of energy). As was the case for amplitude damping, E_0 leaves $|0\rangle$ unchanged, but reduces the amplitude of a $|1\rangle$ state; unlike amplitude damping, however, the E_1 operation destroys $|0\rangle$ and reduces the amplitude of the $|1\rangle$ state, and does not change it into a $|0\rangle$.

By applying Theorem 8.2, the unitary freedom of quantum operations, we find that a unitary recombination of E_0 and E_1 gives a new set of operation elements for phase damping,

$$\tilde{E}_0 = \sqrt{\alpha} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{8.129}$$

$$\tilde{E}_1 = \sqrt{1-\alpha} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \tag{8.130}$$

where $\alpha = (1 + \sqrt{1-\lambda})/2$. Thus the phase damping quantum operation is *exactly* the same as the phase flip channel which we encountered in Section 8.3.3!

Since phase damping is the same as the phase flip channel, we have already seen how it is visualized on the Bloch sphere, in Figure 8.9. This corresponds to the Bloch vector transformation

$$(r_x, r_y, r_z) \rightarrow \left(r_x \sqrt{1-\lambda}, r_y \sqrt{1-\lambda}, r_z \right), \tag{8.131}$$

which has the effect of shrinking the sphere into ellipsoids. Phase damping is often referred to as a 'T_2' (or 'spin-spin') relaxation process, for historical reasons, where $e^{-t/2T_2} = \sqrt{1-\lambda}$. As a function of time, the amount of damping increases, corresponding to an inwards flow of all points in the unit ball towards the $\hat{z}$ axis. Note that states along the $\hat{z}$ axis remain invariant.

Historically, phase damping was a process that was almost always thought of, physically, as resulting from a random phase kick or scattering process. It was not until the connection to the phase flip channel was discovered that quantum error-correction was developed, since it was thought that phase errors were *continuous* and couldn't be described as a discrete process! In fact, single qubit phase errors can *always* be thought of as resulting from a process in which either nothing happens to a qubit, with probability α, or with probability $1-\alpha$, the qubit is flipped by the Z Pauli operation. Although this might not be the actual microscopic physical process happening, from the standpoint of the transformation occurring to a qubit over a discrete time interval large compared to the underlying random process, there is no difference at all.

Phase damping is one of the most subtle and important processes in the study of quantum computation and quantum information. It has been the subject of an immense amount of study and speculation, particularly with regard to why the world around us appears to be so classical, with superposition states not a part of our everyday experience! Perhaps it is phase damping that is responsible for this absence of superposition states from the everyday (Exercise 8.31)? The pioneering quantum physicist Schrödinger was perhaps the first to pose this problem, and he did this in a particularly stark form, as discussed in Box 8.4.

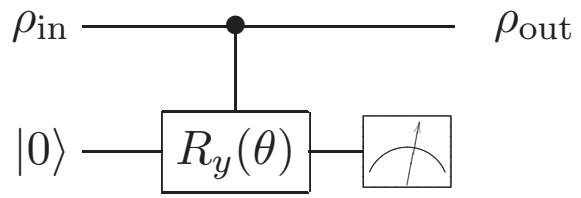


Figure 8.15. Circuit model for phase damping. The upper wire carries the input qubit with an unknown state, and the lower wire is an ancilla qubit used to model the environment.

Exercise 8.26: (Circuit model for phase damping) Show that the circuit in Figure 8.15 can be used to model the phase damping quantum operation, provided θ is chosen appropriately.

Exercise 8.27: (Phase damping = phase flip channel) Give the unitary

transformation which relates the operation elements of (8.127)–(8.128) to those of (8.129)–(8.130); that is, find u such that $\tilde{E}_k = \sum_j u_{kj} E_j$.

Exercise 8.28: (**One** CNOT **phase damping model circuit**) Show that a single controlled-NOT gate can be used as a model for phase damping, if we let the initial state of the environment be a mixed state, where the amount of damping is determined by the probability of the states in the mixture.

Exercise 8.29: (**Unitality**) A quantum process $\mathcal{E}$ is *unital* if $\mathcal{E}(I) = I$. Show that the depolarizing and phase damping channels are unital, while amplitude damping is not.

Exercise 8.30: ($T_2 \leq T_1/2$) The T_2 phase coherence relaxation rate is just the exponential decay rate of the off-diagonal elements in the qubit density matrix, while T_1 is the decay rate of the diagonal elements (see Equation (7.144)). Amplitude damping has *both* nonzero T_1 and T_2 rates; show that for amplitude damping $T_2 = T_1/2$. Also show that if amplitude and phase damping are *both* applied then $T_2 \leq T_1/2$.

Exercise 8.31: (**Exponential sensitivity to phase damping**) Using (8.126), show that the element $\rho_{nm} = \langle n|\rho|m\rangle$ in the density matrix of a harmonic oscillator decays exponentially as $e^{-\lambda(n-m)^2}$ under the effect of phase damping, for some constant λ.

8.4 Applications of quantum operations

As befits a powerful tool, the quantum operations formalism has numerous applications. In this section we describe two of these applications. Section 8.4.1 describes the theory of *master equations*, a picture of quantum noise complementary to the quantum operations formalism. The master equation approach describes quantum noise in *continuous time* using differential equations, and is the approach to quantum noise most often used by physicists. In Section 8.4.2 we describe *quantum process tomography*, a procedure to experimentally determine the dynamics of a quantum system.

8.4.1 Master equations

Open quantum systems occur in a wide range of disciplines, and many tools other than quantum operations can be employed in their study. In this section, we briefly describe one such tool, the approach of *master equations*.

The dynamics of open quantum systems have been studied extensively in the field of quantum optics. The main objective in this context is to describe the time evolution of an open system with a differential equation which properly describes non-unitary behavior. This description is provided by the master equation, which can be written most generally in the *Lindblad form* as

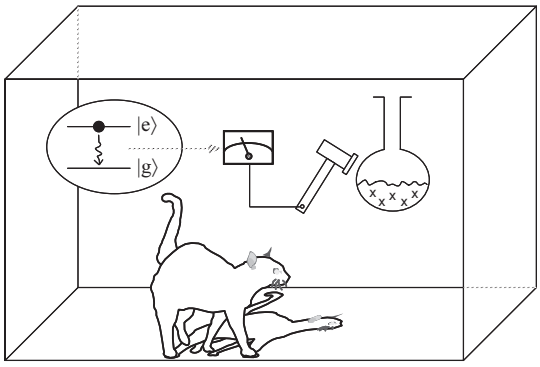
$$\frac{d\rho}{dt} = -\frac{i}{\hbar}[H, \rho] + \sum_j \left[2L_j \rho L_j^\dagger - \{L_j^\dagger L_j, \rho\} \right] , \tag{8.134}$$

where $\{x, y\} = xy + yx$ denotes an anticommutator, H is the system Hamiltonian, a Hermitian operator representing the coherent part of the dynamics, and L_j are the *Lindblad*

Box 8.4: Schrödinger's cat

When I hear about Schrödinger's cat, I reach for my gun. – Stephen Hawking

Schrödinger's infamous cat faces life or death contingent upon an automatic device which breaks a vial of poison and kills the cat if an excited atomic state is observed to decay, as illustrated here:



Schrödinger asked what happens when the atom is in a superposition state? Is the cat alive or dead? Why do superposition states such as this apparently not occur in the everyday world? This conundrum is resolved by realizing that it is very unlikely to occur in real life, because of extreme sensitivity of macroscopic superposition states to decoherence. Let the atom represent a single qubit. The joint system has the initial state $|\text{alive}\rangle|1\rangle$. Suppose that after one half-life of the atom, the state is the equal superposition $|\text{alive}\rangle(|0\rangle + |1\rangle)/\sqrt{2}$ (this represents a simplification of the actual physics, which are too involved to go into here). The apparatus kills the cat if the atom is in the $|0\rangle$ state; otherwise, the cat lives. This gives the state $|\psi\rangle = \left[|\text{dead}\rangle|0\rangle + |\text{alive}\rangle|1\rangle\right]/\sqrt{2}$, in which the cat's state has become entangled with that of the atom. This would seem to indicate the cat is simultaneously alive and dead, but suppose we consider the density matrix of this state,

$$\rho = |\psi\rangle\langle\psi| \tag{8.132}$$

$$= \frac{1}{2}\Big[|\text{alive},1\rangle\langle\text{alive},1| + |\text{dead},0\rangle\langle\text{dead},0| + |\text{alive},1\rangle\langle\text{dead},0| + |\text{dead},0\rangle\langle\text{alive},1|\Big] . \tag{8.133}$$

Now, in practice it is impossible to perfectly isolate the cat and the atom in their box, and thus information about this superposition state will leak into the external world. For example, heat from the cat's body could permeate the wall and give some indication of its state to the outside. Such effects may be modeled as phase damping, which exponentially damps out the final two (off-diagonal) terms in ρ. To a first approximation, we may model the cat–atom system as a simple harmonic oscillator. An important result about the decoherence of such a system is that coherence between states of high energy difference decays faster than between states with a lower energy difference (Exercise 8.31). Thus ρ will quickly be transformed into a nearly diagonal state, which represents an ensemble of cat–atom states which correspond to either dead or alive, and are not in a superposition of the two states.

operators, representing the coupling of the system to its environment. The differential equation takes on the above form in order that the process be completely positive in a sense similar to that described earlier for quantum operations. It is also generally assumed that the system and environment begin in a product state. Furthermore, in order to derive a master equation for a process, one usually begins with a system–environment model Hamiltonian, and then makes the Born and Markov approximations in order to determine L_j. Note that in the master equation approach, $\mathrm{tr}[\rho(t)] = 1$ at all times.

As an example of a Lindblad equation, consider a two-level atom coupled to the vacuum, undergoing spontaneous emission. The coherent part of the atom's evolution is described by the Hamiltonian $H = -\hbar\omega\sigma_z/2$. $\hbar\omega$ is the energy difference of the atomic levels. Spontaneous emission causes an atom in the excited ($|1\rangle$) state to drop down into the ground ($|0\rangle$) state, emitting a photon in the process. This emission is described by the Lindblad operator $\sqrt{\gamma}\sigma_-$, where $\sigma_- \equiv |0\rangle\langle 1|$ is the atomic lowering operator, and γ is the rate of spontaneous emission. The master equation describing this process is

$$\frac{d\rho}{dt} = -\frac{i}{\hbar}[H, \rho] + \gamma\left[2\sigma_-\rho\sigma_+ - \sigma_+\sigma_-\rho - \rho\sigma_+\sigma_-\right], \tag{8.135}$$

where $\sigma_+ \equiv \sigma_-^\dagger$ is the atomic raising operator.

To solve the equation it is helpful to move to the interaction picture, that is, make the change of variables

$$\tilde{\rho}(t) \equiv e^{iHt}\rho(t)e^{-iHt}. \tag{8.136}$$

The equation of motion for $\tilde{\rho}$ is easily found to be

$$\frac{d\tilde{\rho}}{dt} = \gamma\left[2\tilde{\sigma}_-\tilde{\rho}\tilde{\sigma}_+ - \tilde{\sigma}_+\tilde{\sigma}_-\tilde{\rho} - \tilde{\rho}\tilde{\sigma}_+\tilde{\sigma}_-\right] \tag{8.137}$$

where

$$\tilde{\sigma}_- \equiv e^{iHt}\sigma_-e^{-iHt} = e^{-i\omega t}\sigma_- \tag{8.138}$$

$$\tilde{\sigma}_+ \equiv e^{iHt}\sigma_+e^{-iHt} = e^{i\omega t}\sigma_+. \tag{8.139}$$

Our final equation of motion is thus

$$\frac{d\tilde{\rho}}{dt} = \gamma\left[2\sigma_-\tilde{\rho}\sigma_+ - \sigma_+\sigma_-\tilde{\rho} - \tilde{\rho}\sigma_+\sigma_-\right]. \tag{8.140}$$

This equation of motion is easily solved using a Bloch vector representation for $\tilde{\rho}$. The solution is

$$\lambda_x = \lambda_x(0)e^{-\gamma t} \tag{8.141}$$

$$\lambda_y = \lambda_y(0)e^{-\gamma t} \tag{8.142}$$

$$\lambda_z = \lambda_z(0)e^{-2\gamma t} + 1 - e^{-2\gamma t}. \tag{8.143}$$

Defining $\gamma' = 1 - \exp(-2t\gamma)$ we can easily check that this evolution is equivalent to

$$\tilde{\rho}(t) = \mathcal{E}(\tilde{\rho}(0)) \equiv E_0\tilde{\rho}(0)E_0^\dagger + E_1\tilde{\rho}(0)E_1^\dagger, \tag{8.144}$$

where

$$E_0 \equiv \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma'} \end{bmatrix} \tag{8.145}$$

$$E_1 \equiv \begin{bmatrix} 0 & \sqrt{\gamma'} \\ 0 & 0 \end{bmatrix} \tag{8.146}$$

are the operation elements defining the quantum operation $\mathcal{E}$. Note that the effect of $\mathcal{E}$ is amplitude damping; compare with Equation (8.108). The example we have considered is an instance of the spin-boson model, in which a small, finite dimensional quantum system interacts with a bath of simple harmonic oscillators. Physically, it is important in describing the interaction of atoms with electromagnetic radiation, as in cavity QED, or atom and ion traps.

The master equation approach is less general than the quantum operations formalism. Solving a master equation allows one to determine the time dependence of a density matrix. Knowing this, in turn, means that the result can be expressed as a quantum operation in the operator-sum representation,

$$\rho(t) = \sum_k E_k(t)\rho(0)E_k^\dagger(t)\,, \tag{8.147}$$

where $E_k(t)$ are time dependent operation elements, determined from the solution to the master equation. However, a quantum process described in terms of an operator-sum representation cannot necessarily be written down as a master equation. For example, quantum operations can describe non-Markovian dynamics, simply because they describe only state changes, not continuous time evolution. Nevertheless, each approach has its own place. In fact, even quantum operations do not provide the *most* general description; we consider in Section 8.5 some processes which are not described by quantum operations.

8.4.2 Quantum process tomography

Quantum operations provide a wonderful mathematical model for open quantum systems, and are conveniently visualized (at least for qubits) – but how do they relate to experimentally measurable quantities? What measurements should an experimentalist do if they wish to characterize the dynamics of a quantum system? For classical systems, this elementary task is known as *system identification*. Here, we show how its analogue, known as *quantum process tomography*, can be performed for finite dimensional quantum systems.

To understand process tomography we first need to understand another procedure called *quantum state tomography*. State tomography is the procedure of experimentally determining an unknown quantum state. Suppose we are given an unknown state, ρ, of a single qubit. How can we experimentally determine what the state of ρ is?

If we are given just a single copy of ρ then it turns out to be impossible to characterize ρ. The basic problem is that there is no quantum measurement which can distinguish non-orthogonal quantum states like $|0\rangle$ and $(|0\rangle + |1\rangle)/\sqrt{2}$ with certainty. However, it is possible to estimate ρ if we have a large number of copies of ρ. For instance, if ρ is the quantum state produced by some experiment, then we simply repeat the experiment many times to produce many copies of the state ρ.

Suppose we have many copies of a single qubit density matrix, ρ. The set $I/\sqrt{2}$, $X/\sqrt{2}$, $Y/\sqrt{2}$, $Z/\sqrt{2}$ forms an orthonormal set of matrices with respect to the Hilbert–Schmidt inner product, so ρ may be expanded as

$$\rho = \frac{\mathrm{tr}(\rho)I + \mathrm{tr}(X\rho)X + \mathrm{tr}(Y\rho)Y + \mathrm{tr}(Z\rho)Z}{2}\,. \tag{8.148}$$

Recall, however, that expressions like $\mathrm{tr}(A\rho)$ have an interpretation as the average value of *observables*. For example, to estimate $\mathrm{tr}(Z\rho)$ we measure the observable Z a large number of times, m, obtaining outcomes $z_1, z_2, \ldots, z_m$, all equal to $+1$ or -1. The empirical

average of these quantities, $\sum_i z_i/m$, is an estimate for the true value of $\mathrm{tr}(Z\rho)$. We can use the central limit theorem to determine how well this estimate behaves for large m, where it becomes approximately Gaussian with mean equal to $\mathrm{tr}(Z\rho)$ and with standard deviation $\Delta(Z)/\sqrt{m}$, where $\Delta(Z)$ is the standard deviation for a single measurement of Z, which is upper bounded by 1, so the standard deviation in our estimate $\sum_i z_i/m$ is at most $1/\sqrt{m}$.

In a similar way we can estimate the quantities $\mathrm{tr}(X\rho)$ and $\mathrm{tr}(Y\rho)$ with a high degree of confidence in the limit of a large sample size, and thus obtain a good estimate for ρ. Generalizing this procedure to the case of more than one qubit is not difficult, at least in principle! Similar to the single qubit case, an arbitrary density matrix on n qubits can be expanded as

$$\rho = \sum_{\vec{v}} \frac{\mathrm{tr}\left(\sigma_{v_1} \otimes \sigma_{v_2} \otimes \cdots \otimes \sigma_{v_n} \rho\right) \sigma_{v_1} \otimes \sigma_{v_2} \otimes \cdots \otimes \sigma_{v_n}}{2^n}, \tag{8.149}$$

where the sum is over vectors $\vec{v} = (v_1, \ldots, v_n)$ with entries v_i chosen from the set $0, 1, 2, 3$. By performing measurements of observables which are products of Pauli matrices we can estimate each term in this sum, and thus obtain an estimate for ρ.

We've described how to do state tomography for systems comprised of qubits. What if non-qubit systems are involved? Not surprisingly, it is easy to generalize the above prescription to such systems. We won't explicitly do so here, but instead refer you to the end of chapter 'History and further reading' for references.

Now that we know how to do quantum state tomography, how can we use it to do quantum process tomography? The experimental procedure may be outlined as follows. Suppose the state space of the system has d dimensions; for example, $d = 2$ for a single qubit. We choose d^2 pure quantum states $|\psi_1\rangle, \ldots, |\psi_{d^2}\rangle$, chosen so that the corresponding density matrices $|\psi_1\rangle\langle\psi_1|, \ldots, |\psi_{d^2}\rangle\langle\psi_{d^2}|$ form a *basis set* for the space of matrices. We explain in more detail how to choose such a set below. For each state $|\psi_j\rangle$ we prepare the quantum system in that state and then subject it to the process which we wish to characterize. After the process has run to completion we use quantum state tomography to determine the state $\mathcal{E}(|\psi_j\rangle\langle\psi_j|)$ output from the process. From a purist's point of view we are now done, since in principle the quantum operation $\mathcal{E}$ is now determined by a linear extension of $\mathcal{E}$ to all states.

In practice, of course, we would like to have a way of determining a useful representation of $\mathcal{E}$ from experimentally available data. We will explain a general procedure for doing so, worked out explicitly for the case of a single qubit. Our goal is to determine a set of operation elements $\{E_i\}$ for $\mathcal{E}$,

$$\mathcal{E}(\rho) = \sum_i E_i \rho E_i^\dagger. \tag{8.150}$$

However, experimental results involve numbers, not operators, which are a theoretical concept. To determine the E_i from measurable parameters, it is convenient to consider an equivalent description of $\mathcal{E}$ using a *fixed* set of operators $\tilde{E}_i$, which form a basis for the set of operators on the state space, so that

$$E_i = \sum_m e_{im} \tilde{E}_m \tag{8.151}$$

for some set of complex numbers e_{im}. Equation (8.150) may thus be rewritten as

$$\mathcal{E}(\rho) = \sum_{mn} \tilde{E}_m \rho \tilde{E}_n^\dagger \chi_{mn}, \tag{8.152}$$

where $\chi_{mn} \equiv \sum_i e_{im} e_{in}^*$ are the entries of a matrix which is positive Hermitian by definition. This expression, known as the *chi matrix representation*, shows that $\mathcal{E}$ can be completely described by a complex number matrix, χ, once the set of operators E_i has been fixed.

In general, χ will contain $d^4 - d^2$ independent real parameters, because a general linear map of d by d complex matrices to d by d matrices is described by d^4 independent parameters, but there are d^2 additional constraints due to the fact that ρ remains Hermitian with trace one; that is, the completeness relation

$$\sum_i E_i^\dagger E_i = I, \tag{8.153}$$

is satisfied, giving d^2 real constraints. We will show how to determine χ experimentally, and then show how an operator-sum representation of the form (8.150) can be recovered once the χ matrix is known.

Let ρ_j, $1 \leq j \leq d^2$ be a fixed, linearly independent basis for the space of $d \times d$ matrices; that is, any $d \times d$ matrix can be written as a unique linear combination of the ρ_j. A convenient choice is the set of operators $|n\rangle\langle m|$. Experimentally, the output state $\mathcal{E}(|n\rangle\langle m|)$ may be obtained by preparing the input states $|n\rangle$, $|m\rangle$, $|+\rangle = (|n\rangle + |m\rangle)/\sqrt{2}$, and $|-\rangle = (|n\rangle + i|m\rangle)/\sqrt{2}$ and forming linear combinations of $\mathcal{E}(|n\rangle\langle n|)$, $\mathcal{E}(|m\rangle\langle m|)$, $\mathcal{E}(|+\rangle\langle +|)$, and $\mathcal{E}(|-\rangle\langle -|)$, as follows:

$$\mathcal{E}(|n\rangle\langle m|) = \mathcal{E}(|+\rangle\langle +|) + i\mathcal{E}(|-\rangle\langle -|) - \frac{1+i}{2}\mathcal{E}(|n\rangle\langle n|) - \frac{1+i}{2}\mathcal{E}(|m\rangle\langle m|). \tag{8.154}$$

Thus, it is possible to determine $\mathcal{E}(\rho_j)$ by state tomography, for each ρ_j.

Furthermore, each $\mathcal{E}(\rho_j)$ may be expressed as a linear combination of the basis states,

$$\mathcal{E}(\rho_j) = \sum_k \lambda_{jk} \rho_k, \tag{8.155}$$

and since $\mathcal{E}(\rho_j)$ is known from the state tomography, λ_{jk} can be determined by standard linear algebraic algorithms. To proceed, we may write

$$\tilde{E}_m \rho_j \tilde{E}_n^\dagger = \sum_k \beta_{jk}^{mn} \rho_k, \tag{8.156}$$

where β_{jk}^{mn} are complex numbers which can be determined by standard algorithms from linear algebra given the $\tilde{E}_m$ operators and the ρ_j operators. Combining the last two expressions and (8.152) we have

$$\sum_k \sum_{mn} \chi_{mn} \beta_{jk}^{mn} \rho_k = \sum_k \lambda_{jk} \rho_k. \tag{8.157}$$

From the linear independence of the ρ_k it follows that for each k,

$$\sum_{mn} \beta_{jk}^{mn} \chi_{mn} = \lambda_{jk}. \tag{8.158}$$

This relation is a necessary and sufficient condition for the matrix χ to give the correct quantum operation $\mathcal{E}$. One may think of χ and λ as vectors, and β as a $d^4 \times d^4$ matrix

with columns indexed by mn, and rows by jk. To show how χ may be obtained, let κ be the generalized inverse for the matrix β, satisfying the relation

$$\beta_{jk}^{mn} = \sum_{st,xy} \beta_{jk}^{st} \kappa_{st}^{xy} \beta_{xy}^{mn} . \tag{8.159}$$

Most computer packages for matrix manipulation are capable of finding such generalized inverses. We now prove that χ defined by

$$\chi_{mn} \equiv \sum_{jk} \kappa_{jk}^{mn} \lambda_{jk} \tag{8.160}$$

satisfies the relation (8.158).

The difficulty in verifying that χ defined by (8.160) satisfies (8.158) is that, in general, χ is not uniquely determined by Equation (8.158). For convenience we rewrite these equations in matrix form as

$$\beta \vec{\chi} = \vec{\lambda} \tag{8.161}$$

$$\vec{\chi} \equiv \kappa \vec{\lambda} . \tag{8.162}$$

From the construction that led to Equation (8.152) we know there exists at least one solution to Equation (8.161), which we shall call $\vec{\chi}'$. Thus $\vec{\lambda} = \beta \vec{\chi}'$. The generalized inverse satisfies $\beta \kappa \beta = \beta$. Premultiplying the definition of $\vec{\chi}$ by β gives

$$\beta \vec{\chi} = \beta \kappa \vec{\lambda} \tag{8.163}$$

$$= \beta \kappa \beta \vec{\chi}' \tag{8.164}$$

$$= \beta \vec{\chi}' \tag{8.165}$$

$$= \vec{\lambda} . \tag{8.166}$$

Thus χ defined by (8.162) satisfies the Equation (8.161), as we wanted to show.

Having determined χ one immediately obtains the operator-sum representation for $\mathcal{E}$ in the following manner. Let the unitary matrix $U^\dagger$ diagonalize χ,

$$\chi_{mn} = \sum_{xy} U_{mx} d_x \delta_{xy} U_{ny}^* . \tag{8.167}$$

From this it can easily be verified that

$$E_i = \sqrt{d_i} \sum_j U_{ji} E_j \tag{8.168}$$

are operation elements for $\mathcal{E}$. Our algorithm may thus be summarized as follows: λ is experimentally determined using state tomography, which in turn determines χ via the equation $\vec{\chi} = \kappa \lambda$, which gives us a complete description of $\mathcal{E}$, including a set of operation elements E_i.

In the case of a single qubit quantum process, only 12 parameters must be determined (Box 8.5). The dynamics of a two qubit quantum black box $\mathcal{E}_2$ pose an even greater challenge for our understanding. In this case there are 240 parameters which need to be determined in order to completely specify the quantum operation acting on the quantum system! Determining these would obviously be quite a considerable undertaking. However, as for the single qubit case, it is relatively straightforward to implement a numerical routine which will automate the calculation, provided experimental state tomography and state preparation procedures are available in the laboratory.

Box 8.5: Process tomography for a single qubit

The general method of process tomography can be simplified in the case of a one qubit operation to provide explicit formulas which may be useful in experimental contexts. This simplification is made possible by choosing the fixed operators $\tilde{E}_i$ to have commutation properties which conveniently allow the χ matrix to be determined by straightforward matrix multiplication. In the one qubit case, we use:

$$E_0 = I \tag{8.169}$$
$$\tilde{E}_1 = X \tag{8.170}$$
$$\tilde{E}_2 = -iY \tag{8.171}$$
$$\tilde{E}_3 = Z. \tag{8.172}$$

There are 12 parameters, specified by χ, which determine an arbitrary single qubit quantum operation $\mathcal{E}$.
These parameters may be measured using four sets of experiments. As a specific example, suppose the input states $|0\rangle$, $|1\rangle$, $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$ and $|-\rangle = (|0\rangle + i\,|1\rangle)/\sqrt{2}$ are prepared, and the four matrices

$$\rho'_1 = \mathcal{E}(|0\rangle\langle 0|) \tag{8.173}$$
$$\rho'_4 = \mathcal{E}(|1\rangle\langle 1|) \tag{8.174}$$
$$\rho'_2 = \mathcal{E}(|+\rangle\langle +|) - i\mathcal{E}(|-\rangle\langle -|) - (1-i)(\rho'_1 + \rho'_4)/2 \tag{8.175}$$
$$\rho'_3 = \mathcal{E}(|+\rangle\langle +|) + i\mathcal{E}(|-\rangle\langle -|) - (1+i)(\rho'_1 + \rho'_4)/2 \tag{8.176}$$

are determined using state tomography. These correspond to $\rho'_j = \mathcal{E}(\rho_j)$, where

$$\rho_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \tag{8.177}$$

$\rho_2 = \rho_1 X$, $\rho_3 = X\rho_1$, and $\rho_4 = X\rho_1 X$. From (8.156) and Equations (8.169)–(8.172) we may determine β, and similarly ρ'_j determines λ. However, due to the particular choice of basis, and the Pauli matrix representation of $\tilde{E}_i$, we may express the β matrix as the Kronecker product $\beta = \Lambda \otimes \Lambda$, where

$$\Lambda = \frac{1}{2}\begin{bmatrix} I & X \\ X & -I \end{bmatrix}, \tag{8.178}$$

so that χ may be expressed conveniently as

$$\chi = \Lambda \begin{bmatrix} \rho'_1 & \rho'_2 \\ \rho'_3 & \rho'_4 \end{bmatrix} \Lambda, \tag{8.179}$$

in terms of block matrices.

We have shown how a useful representation for the dynamics of a quantum system may be experimentally determined using a systematic procedure. This procedure of quantum process tomography is analogous to the system identification step performed in classical control theory, and plays a similar role in understanding and controlling noisy quantum systems.

Exercise 8.32: Explain how to extend quantum process tomography to the case of non-trace-preserving quantum operations, such as arise in the study of measurement.

Exercise 8.33: (Specifying a quantum process) Suppose that one wished to completely specify an arbitrary single qubit operation $\mathcal{E}$ by describing how a set of points on the Bloch sphere $\{\vec{r}_k\}$ transform under $\mathcal{E}$. Prove that the set must contain at least four points.

Exercise 8.34: (Process tomography for two qubits) Show that the χ_2 describing the black box operations on two qubits can be expressed as

$$\chi_2 = \Lambda_2 \overline{\rho}' \Lambda_2 \,, \tag{8.180}$$

where $\Lambda_2 = \Lambda \otimes \Lambda$, Λ is as defined in Box 8.5, and $\overline{\rho}'$ is a block matrix of 16 measured density matrices,

$$\overline{\rho}' = P^T \begin{bmatrix} \rho'_{11} & \rho'_{12} & \rho'_{13} & \rho'_{14} \\ \rho'_{21} & \rho'_{22} & \rho'_{23} & \rho'_{24} \\ \rho'_{31} & \rho'_{32} & \rho'_{33} & \rho'_{34} \\ \rho'_{41} & \rho'_{42} & \rho'_{43} & \rho'_{44} \end{bmatrix} P \,, \tag{8.181}$$

where $\rho'_{nm} = \mathcal{E}(\rho_{nm})$, $\rho_{nm} = T_n |00\rangle\langle 00| T_m$, $T_1 = I \otimes I$, $T_2 = I \otimes X$, $T_3 = X \otimes I$, $T_4 = X \otimes X$, and $P = I \otimes [(\rho_{00} + \rho_{12} + \rho_{21} + \rho_{33}) \otimes I]$ is a permutation matrix.

Exercise 8.35: (Process tomography example) Consider a one qubit black box of unknown dynamics $\mathcal{E}_1$. Suppose that the following four density matrices are obtained from experimental measurements, performed according to Equations (8.173)–(8.176):

$$\rho'_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \tag{8.182}$$

$$\rho'_2 = \begin{bmatrix} 0 & \sqrt{1-\gamma} \\ 0 & 0 \end{bmatrix} \tag{8.183}$$

$$\rho'_3 = \begin{bmatrix} 0 & 0 \\ \sqrt{1-\gamma} & 0 \end{bmatrix} \tag{8.184}$$

$$\rho'_4 = \begin{bmatrix} \gamma & 0 \\ 0 & 1-\gamma \end{bmatrix}, \tag{8.185}$$

where γ is a numerical parameter. From an independent study of each of these input–output relations, one could make several important observations: the ground state $|0\rangle$ is left invariant by $\mathcal{E}_1$, the excited state $|1\rangle$ partially decays to the ground state, and superposition states are damped. Determine the χ matrix for this process.

8.5 Limitations of the quantum operations formalism

Are there interesting quantum systems whose dynamics are not described by quantum operations? In this section we will construct an artificial example of a system whose evo-

lution is not described by a quantum operation, and try to understand the circumstances under which this is likely to occur.

Suppose a single qubit is prepared in some unknown quantum state, which we denote ρ. The preparation of this qubit involves certain procedures to be carried out in the laboratory in which the qubit is prepared. Suppose that among the laboratory degrees of freedom is a single qubit which, as a side effect of the state preparation procedure, is left in the state $|0\rangle$ if ρ is a state on the bottom half of the Bloch sphere, and is left in the state $|1\rangle$ if ρ is a state on the top half of the Bloch sphere. That is, the state of the system after preparation is

$$\rho \otimes |0\rangle\langle 0| \otimes \text{other degrees of freedom} \tag{8.186}$$

if ρ is a state on the bottom half of the Bloch sphere, and

$$\rho \otimes |1\rangle\langle 1| \otimes \text{other degrees of freedom} \tag{8.187}$$

if ρ is a state on the top half of the Bloch sphere.

Once the state preparation is done, the system begins to interact with the environment, in this case all the laboratory degrees of freedom. Suppose the interaction is such that a controlled-NOT is performed between the principal system and the extra qubit in the laboratory system. Thus, if the system's Bloch vector was initially in the bottom half of the Bloch sphere it is left invariant by the process, while if it was initially in the top half of the Bloch sphere it is rotated into the bottom half of the Bloch sphere.

Obviously, this process is not an affine map acting on the Bloch sphere, and therefore, by the results of Section 8.3.2, it *cannot be a quantum operation.* The lesson to be learned from this discussion is that *a quantum system which interacts with the degrees of freedom used to prepare that system after the preparation is complete will in general suffer a dynamics which is not adequately described within the quantum operations formalism*. This is an important conclusion to have reached, as it indicates that there are physically reasonable circumstances under which the quantum operations formalism may not adequately describe the processes taking place in a quantum system. This should be kept in mind, for example, in applications of the quantum process tomography procedure discussed in the previous section.

For the remainder of this book we will, however, work within the quantum operations formalism. It provides a powerful, and reasonably general tool for describing the dynamics experienced by quantum systems. Most of all, it provides a means by which concrete progress can be made on problems related to quantum information processing. It is an interesting problem for further research to study quantum information processing beyond the quantum operations formalism.

Problem 8.1: (Lindblad form to quantum operation) In the notation of Section 8.4.1, explicitly work through the steps to solve the differential equation

$$\dot{\rho} = -\frac{\lambda}{2}(\sigma_+\sigma_-\rho + \rho\sigma_+\sigma_- - 2\sigma_-\rho\sigma_+) \tag{8.188}$$

for $\rho(t)$. Express the map $\rho(0) \rightarrow \rho(t)$ as $\rho(t) = \sum_k E_k(t)\rho(0)E_k^\dagger(t)$.

Problem 8.2: (Teleportation as a quantum operation) Suppose Alice is in possession of a single qubit, denoted as system 1, which she wishes to teleport to

Bob. Unfortunately, she and Bob only share an imperfectly entangled pair of qubits. Alice's half of this pair is denoted system 2, and Bob's half is denoted system 3. Suppose Alice performs a measurement described by a set of quantum operations $\mathcal{E}_m$ with result m on systems 1 and 2. Show that this induces an operation $\tilde{\mathcal{E}}_m$ relating the initial state of system 1 to the final state of system 3, and that teleportation is accomplished if Bob can reverse this operation using a trace-preserving quantum operation $\mathcal{R}_m$, to obtain

$$\mathcal{R}_m \left(\frac{\tilde{\mathcal{E}}_m(\rho)}{\mathrm{tr}[\tilde{\mathcal{E}}_m(\rho)]} \right) = \rho \,, \tag{8.189}$$

where ρ is the initial state of system 1.

Problem 8.3: (Random unitary channels) It is tempting to believe that all unital channels, that is, those for which $\mathcal{E}(I) = I$, result from averaging over random unitary operations, that is, $\mathcal{E}(\rho) = \sum_k p_k U_k \rho U_k^\dagger$, where U_k are unitary operators and the p_k form a probability distribution. Show that while this is true for single qubits, it is untrue for larger systems.

Summary of Chapter 8: Quantum noise and quantum operations

- **The operator-sum representation**: The behavior of an open quantum system can be modeled as

$$\mathcal{E}(\rho) = \sum_k E_k \rho E_k^\dagger , \tag{8.190}$$

  where E_k are operation elements, satisfying $\sum_k E_k^\dagger E_k = I$ if the quantum operation is trace-preserving.

- **Environmental models for quantum operations**: A trace-preserving quantum operation can always be regarded as arising from the unitary interaction of a system with an initially uncorrelated environment, and vice versa. Non-trace-preserving quantum operations may be treated similarly, except an additional projective measurement is performed on the composite of system and environment, with the different outcomes corresponding to different non-trace-preserving quantum operations.

- **Quantum process tomography**: A quantum operation on a d-dimensional quantum system can be completely determined by experimentally measuring the output density matrices produced from d^2 pure state inputs.

- **Operation elements for important single qubit quantum operations**:

depolarizing channel	$\sqrt{1-\frac{3p}{4}}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \sqrt{\frac{p}{4}}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sqrt{\frac{p}{4}}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sqrt{\frac{p}{4}}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
amplitude damping	$\begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{bmatrix}, \quad \begin{bmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{bmatrix}$
phase damping	$\begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & \sqrt{\gamma} \end{bmatrix}$
phase flip	$\sqrt{p}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \sqrt{1-p}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
bit flip	$\sqrt{p}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \sqrt{1-p}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
bit–phase flip	$\sqrt{p}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \sqrt{1-p}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$

History and further reading

Quantum noise is an important topic in several fields, and there is an enormous literature on the subject. We will necessarily be restricted to citing only a small sample of

the resources available on the topic. An early treatise on quantum noise from a rather mathematical perspective is due to Davies[Dav76]. Caldeira and Leggett[CL83] did some of the first and most complete studies of an important model known as the *spin-boson* model, using an approach based upon the Feynman path integral. Gardiner[Gar91] studied quantum noise from the perspective of quantum optics. More recently, the quantum optics community has developed what is known as the *quantum trajectories* approach to quantum noise. Reviews of this subject may be found in the articles by Zoller and Gardiner[ZG97], and Plenio and Knight[PK98].

A large literature exists on the subject of quantum operations. We mention just a few key references, primarily the book by Kraus[Kra83], which contains references to much earlier work on the subject. Influential early papers on the subject include those by Hellwig and Kraus[HK69, HK70], and by Choi[Cho75]. Lindblad[Lin76] connected the quantum operations formalism to the theory of continuous time quantum evolution, introducing what is now known as the Lindblad form. Schumacher[Sch96b] and Caves[Cav99] have written excellent summaries of the quantum operations formalism from the point of view of quantum error-correction.

Quantum state tomography was suggested by Vogel and Risken[VR89]. Leonhardt[Leo97] has written a recent review containing references to other work. The need for quantum process tomography was pointed out in a paper by Turchette, Hood, Lange, Mabuchi, and Kimble[THL+95]. The theory was developed independently by Chuang and Nielsen[CN97], and by Poyatos, Cirac and Zoller[PCZ97]. Jones[Jon94] had earlier sketched out the main ideas of quantum process tomography.

An unfortunate confusion of terms has arisen with the word 'decoherence'. Historically, it has been used to refer just to a phase damping process, particularly by Zurek[Zur91]. Zurek and other researchers recognized that phase damping has a unique role in the transition from quantum to classical physics; for certain environmental couplings, it occurs on a time scale which is much faster than any amplitude damping process, and can therefore be much more important in determining the loss of quantum coherence. The major point of these studies has been this emergence of classicality due to environmental interactions. However, by and large, the usage of decoherence in quantum computation and quantum information is to refer to *any noise process* in quantum processing. In this book, we prefer the more generic term 'quantum noise' and tend towards its usage, although occasionally *decoherence* finds a proper place in the context.

A more detailed discussion of some of the limitations of the quantum operations formalism (and in particular, the assumption of a system and environment initially in a product state) is provided by Royer[Roy96].

Problem 8.2 is due to Nielsen and Caves[NC97]. Problem 8.3 is due to Landau and Streater[LS93] as part of an in-depth study of the extremal points of the convex set of doubly stochastic quantum operations.