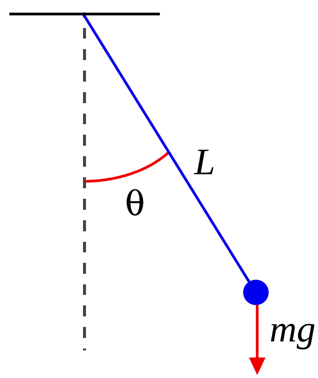
The simple plane pendulum

Definition:

• pendulum bob of mass m attached to rigid rod of length L and negligible mass;

• pendulum confined to swing in a plane.



Prerequisites:

• fundamentals of Newtonian mechanics;

• energy;

• the harmonic oscillator;

• rotational motion.

Why study it?

• it is one of the simplest dynamical systems exhibiting periodic motion;

• a small modification makes it into one of the simplest systems exhibiting chaos.

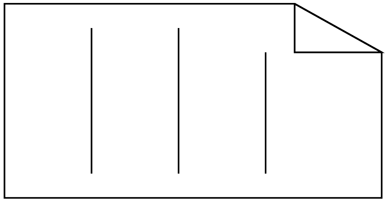
Summary:

The equation of motion is

$$\boxed{\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0}$$

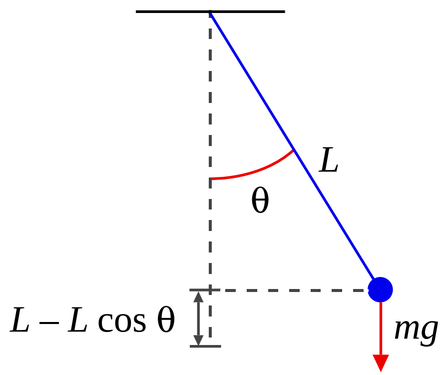
Go to derivation.

Go to Java™ applet



Introduction

The pendulum is free to swing in one plane only, so we don't need to worry about a second angle. We will neglect the mass of the rod, for simplicity.



The easiest way to approach this problem is from the point of view of energy. That way, we don't have to talk about any forces or analyze their components in various directions.

We know from our section on rotational motion that the speed of the pendulum bob is

$$v = L\dot{\theta} \ .$$

The kinetic energy of the bob is then

$$T = \frac{1}{2}mv^2 = \frac{1}{2}mL^2\dot{\theta}^2 \ ,$$

while the potential energy is

$$V = mg\left(L - L\cos\theta\right) \ .$$

We have chosen the potential to be zero when the pendulum is at the bottom of its swing, $\theta=0$. This choice is arbitrary.

Then the total energy is

$$E = \frac{1}{2}mL^2\dot{\theta}^2 + mg\left(L - L\cos\theta\right) \ .$$

The equation of motion is most easily found by using the conservation of energy. Setting

$$\frac{dE}{dt} = 0$$

leads to

$$mL^2\dot{\theta}\ddot{\theta} + mgL\dot{\theta}\sin\theta = 0 \ .$$

This has two solutions; either $\dot{\theta}=0$ always, or

$$\boxed{\ddot{\theta} + \frac{g}{L}\sin\theta = 0} \ .$$

The first solution corresponds to the pendulum hanging straight down without swinging, or just balancing straight up. The second corresponds to any other kind of motion.

This differential equation can't be solved exactly, so we will have to explore its properties in some other way.

Description of the motion

What do we expect for the motion? Well, if the energy E is less than a certain critical value, then the pendulum will just swing back and forth. This kind of periodic motion is called *libration*. In contrast, if E is greater than the critical value, the pendulum will swing around and around. This kind of periodic motion is called *rotation*.

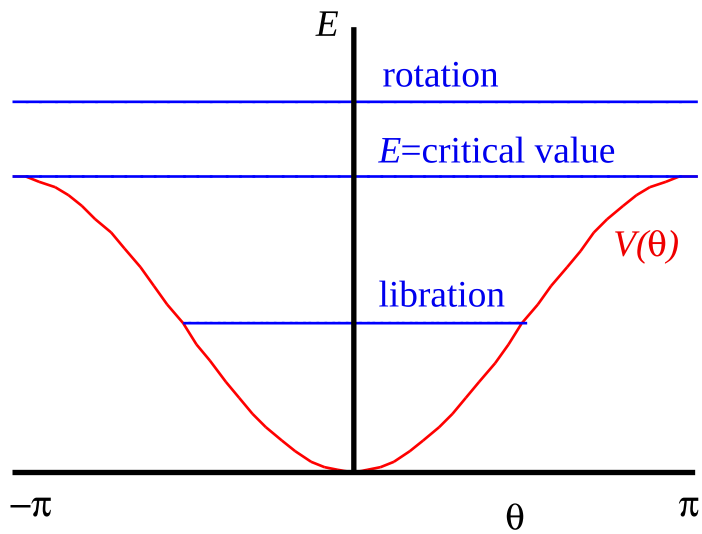
If the energy is just equal to the critical value, there will be two possibilities. If the pendulum starts out in motion, it will approach its vertical position ever more closely, without reaching it in any finite time. Or, the pendulum could start out perched exactly in the vertical position. It will remain there indefinitely.

If the energy is zero, the pendulum just hangs straight down.

The critical value of E is just the value of the potential energy at the top, $\theta=\pm\pi$. It is

$$E_{\text{crit}} = 2mgL \quad .$$

These kinds of motion are reflected in a plot of the potential energy:



If E is less than the critical value, then the kinetic energy gets “used up” before the pendulum reaches its vertical position. It turns around and goes back again (libration). If E is more than the critical value, there is kinetic energy left over at the top, so the pendulum keeps going around (rotation).

Period of the motion

An interesting question is: what is the frequency of the libration or rotation? In general, the answer will be a complicated function of the energy E, as we have already hinted. Are there any special cases that can be treated easily?

A major simplification suggests itself in the special case where the angle θ never gets too large. Then the sine may be approximated by

$$\sin\theta \approx \theta ,$$

and we recognize an equation we have met before, the simple harmonic equation:

$$\ddot{\theta} + \frac{g}{L}\theta \approx 0 .$$

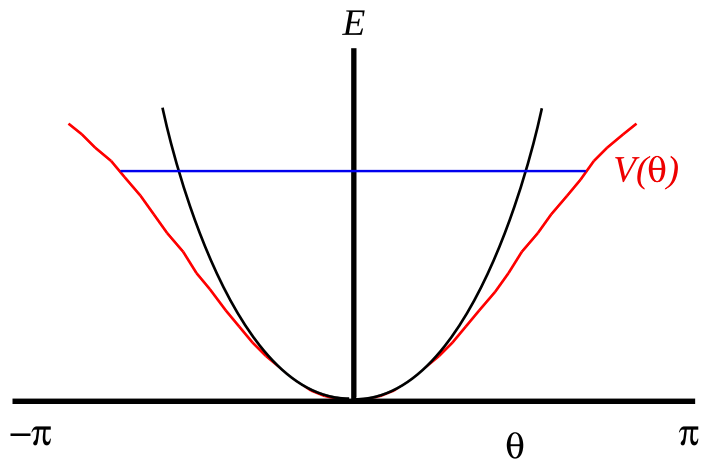
From this, we read off the *angular frequency of small oscillations*:

$$\boxed{\omega_0 = \sqrt{\frac{g}{L}}} \quad .$$

Note that this is independent of the energy of the pendulum; you may recall that this is a special property of simple harmonic motion. Here is a movie illustrating this fact.

As the amplitude of oscillation becomes larger, however, the above approximation breaks down and the frequency will depend on the energy. We know that the frequency must *decrease* as the energy is increased, until the energy reaches the critical energy, at which point the frequency is zero.

Another way to see the decrease in frequency with increasing energy is to look back at the potential for the pendulum, and compare it with the simple harmonic oscillator (shown in black in the next figure):



For a given energy, the pendulum spends more time out on the "tails" of the potential than the harmonic oscillator does. Here is a movie which shows that as the energy gets larger, the frequency decreases for the pendulum.

When the energy equals the critical energy, it turns out that we can actually solve for $\theta(t)$. The law of conservation of energy gives

$$2mgL = \frac{1}{2}mL^2\dot{\theta}^2 + mg\left(L - L\cos\theta\right) ,$$

which may be re-arranged to yield

$$\dot{\theta} = 2\omega_0 \cos\frac{\theta}{2} \quad .$$

This is easily integrated using standard tables. If we suppose that the pendulum starts out at θ=0 and moves in the positive direction, for example, then the solution is found to be

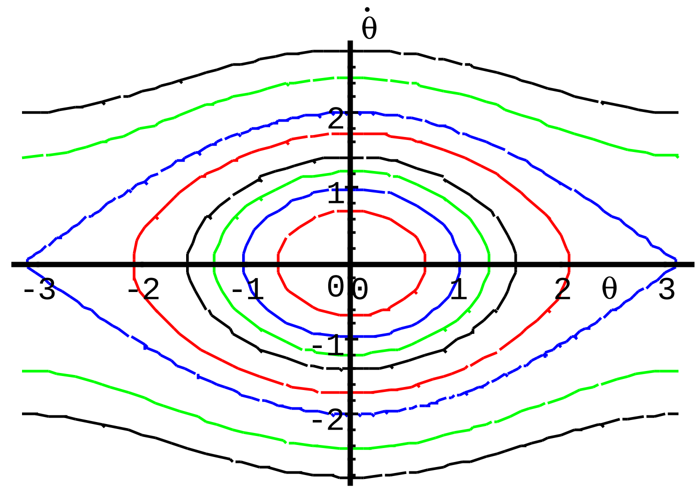
$$\theta(t) = 2\arcsin\left(\frac{1-\exp\left(-2\omega_0 t\right)}{1+\exp\left(-2\omega_0 t\right)}\right) \ .$$

As the time becomes large, θ approaches π. The pendulum has exactly enough energy to reach the top, but it never gets there in finite time. Of course, this latter feature is an artifact of our idealized treatment of the pendulum. This kind of motion can never be achieved in practice.

Nevertheless, this motion is important because it serves to separate two different kinds of motion - librations and rotations. The rotations occur when the energy is greater than the critical energy. The pendulum just spins around and around, and its frequency increases as its energy does.

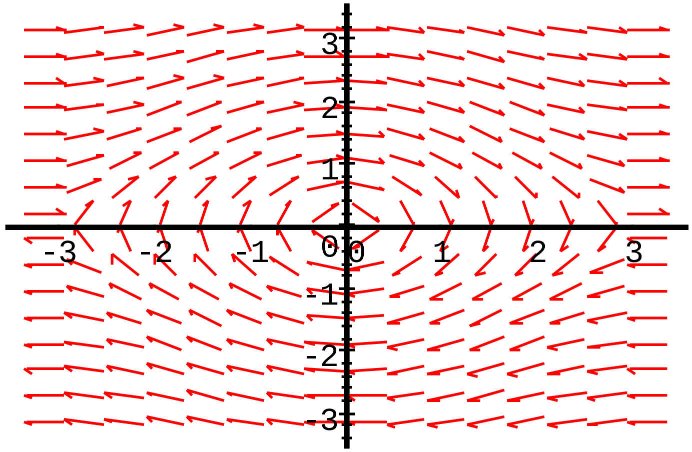
Phase portrait

An interesting way to view the motion of the pendulum is to plot the angular velocity versus the angle, as time goes on. You end up with several possibilities, depending on the energy. Some characteristic ones are shown in the following figure:

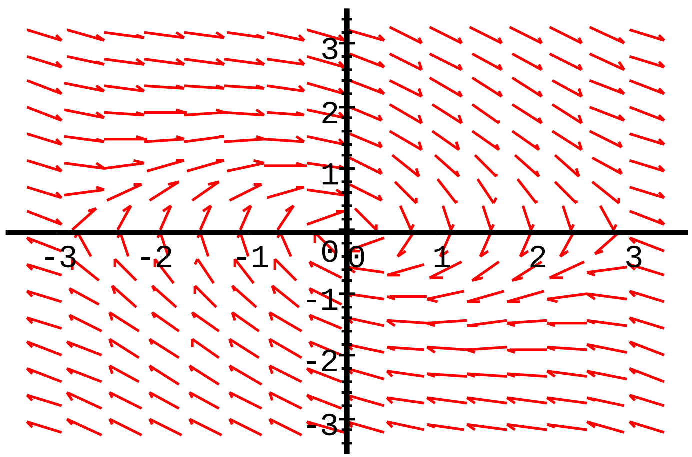


The oval-shaped trajectories in the middle correspond to the librations, while the blue one with pointed ends corresponds to motions with energy equal to the critical energy. Such a trajectory is called a *separatrix*, because it separates regions with trajectories having different character. The trajectories outside this correspond to rotations. (Note that the system is periodic in θ, so the points on the left and right edges of the above plot are the same.)

In order to show the direction of motion along the trajectories, it is useful to draw arrows tangent to the trajectories. This shows the *phase flow*. Here is a phase flow diagram for the pendulum:



It is interesting to see the effect of damping on the above phase portrait:



The trajectories now spiral in towards the origin because the pendulum comes to rest as it loses energy due to the damping.

Here is some MAPLE input code which will generate the above phase diagrams:

```
with(DEtools): damping:=0.5:
dfieldplot([diff(x(t),t)=y,
diff(y(t),t)=-sin(x)-damping*y],
[x,y],0..1,x=-Pi..Pi,y=-Pi..Pi,
grid=[15,15]);
```

What's next?

To see how chaos is introduced by a small modification to the simple pendulum, see the section on the driven pendulum.