A645: Exercise #5
Liouville and Boltzmann

Due: 08 Nov 2019

$\boxed{1}$ **Collisionless Boltzmann Equation**

Show that in a frame that rotates with constant angular velocity $\boldsymbol{\Omega}$, with $\Phi_{eff} \equiv \Phi - \frac{1}{2}|\boldsymbol{\Omega} \times \mathbf{x}|^2$, the collisionless Boltzmann equation can be written:

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f - \left[2(\boldsymbol{\Omega} \times \mathbf{v}) + \nabla \Phi_{eff}\right] \cdot \frac{\partial f}{\partial \mathbf{v}} = 0.$$

[Hint: This is B&T, Problem 4.1 on page 387.]

$\boxed{2}$ **Liouville equation and the pendulum**

Consider the pendulum of length l and mass m in a gravitational field g. The kinetic energy and potential energy are

$$T = \frac{1}{2} m l^2 \dot{\theta}^2$$

and

$$V = mgl\,(1 - \cos\theta)\,.$$

(a) Derive the Hamiltonian and write down Hamilton's equations.

(b) Transform these equations to a non-dimensional set of variables $(q, p) \to (z_1, z_2)$ such that $\tau = \omega t$ where $\omega^2 = g/l$ and the dimensionless energy is:

$$\varepsilon \equiv \frac{E}{mgl} = \frac{1}{2} z_2^2 + (1 - \cos z_1).$$

Show that $\tau_0 = 2\pi/\omega$ is the period of small oscillations for this system.

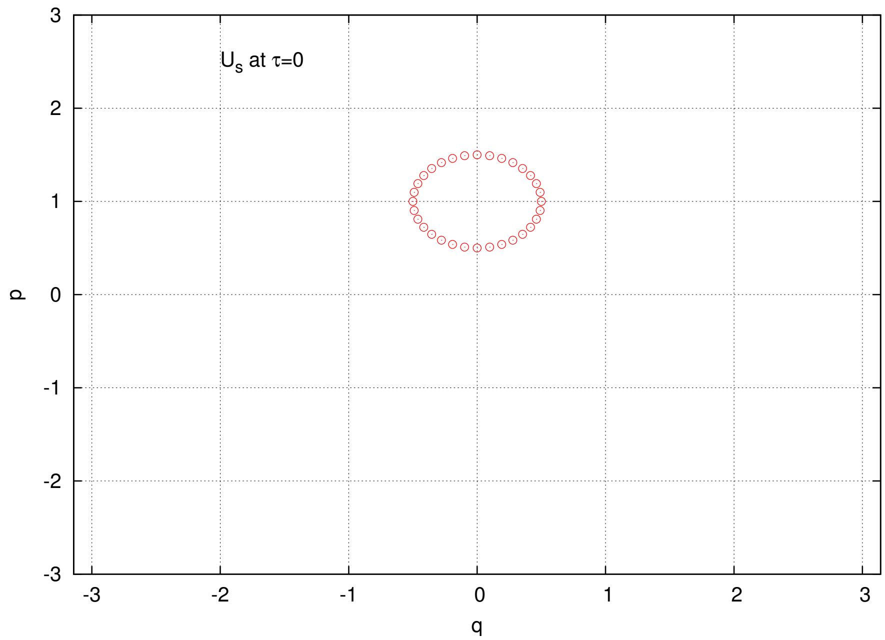

Figure 1: Initial conditions for U_s

(c) Sketch the phase portrait in these units, paying special attention to the three domains: 1) $\varepsilon < 2$, the *oscillation* or *libration* regime; 2) $\varepsilon > 2$, the *rotation* regime; and 3) $\varepsilon = 2$, the critical infinite period, the *separatrix*.

(d) Consider a disk-like domain of initial configurations U_s enclosed within the circle centered at $(q, p) = (z_1, z_2) = (0, 1)$ with radius of $1/2$ in these units. See Fig. 1 for a plot of these initial conditions. Compute the evolution of the disk from $\tau = 0$ to each of $\tau = 0.25\tau_0, 0.5\tau_0$, and τ_0

Remarks:

- Argue that it is sufficient to compute the trajectories on the circle at edge of the disk. In other words, a point on the inside of the disk can not pass through a boundary point.
- You may solve this any way you like. E.g. there is an analytic solution in terms of elliptic integrals. But I recommend an numerical solution, using your RK4 routine to solve the equations of motion for say 32 points equally spaced around the circle at the perimeter of U_s at $\tau = 0$.
- For insight consider this problem in the small angle limit. In this limit, the solution is analytic. If you have any worries about your solution, make sure it agrees with the small angle

limit.

(e) Repeat the experiment but now with the circle centered at $(q, p) = (0, 1.5)$. Note: some points will be on or near the seperatrix. Compute the evolution of the disk for $\tau = 0, 0.2\tau_0, 0.4\tau_0$ and $0.75\tau_0$. [NB: the "evolution" at $\tau = 0$ is the initial condition, of course.]

(f) Repeat the experiment but now with the circle centered at $(q, p) = (0, 2)$. Compute the evolution at times $\tau = 0, 0.1\tau_0, 0.25\tau_0$ and $0.5\tau_0$.

(g) For each of the three experiment with different disk centers, make a plot the disk at each of these times in the (q, p) plane (e.g. a phase portrait). Interpret the results. Is Liouville's theorem obeyed?