A645/A445: Exercise #6

Distribution functions, Jeans' equations and the computation of equilibrium models

Due: 2019 Nov 29

References:
Binney & Tremaine, Chap. 4

$\boxed{1}$ **Constructing King models**

In class, I motivated the King model, which has the distribution function described by the dimensionless potential

$$f \propto e^W - 1$$

where

$$W(r) \equiv \frac{V(r_t) - V(r)}{\sigma^2}.$$

Recall that the profile shape only depends on $W_o \equiv W(0)$. Given the profile, the overall linear size and mass of the profile can be scaled as desired to any set of units.

(a) Note that Poisson's equation is a second order ordinary differential equation just like Newton's equations and therefore can be written as two coupled first order differential equations. Rewrite the Poisson equation as a system of first order ODEs with "physical boundary conditions" (that is: $V(0) =$ constant, $dV(0)/dr = 0$) in spherical coordinates.

(b) Integrate (numerically) Poisson's equation for the King model. By adjusting $W_o \equiv W(0)$, one gets models with different concentration; you can check against the table below. Plot up the density and the potential just to see what they look like. You will use these later on to generate initial conditions for a simulation. Check your procedure by computing the values in each of the four columns. The core radius of the King model is defined as $r_c = \sqrt{9\sigma^2/4\pi G\rho_o}$. In these units, the second column describes the total energy of the King model, the third column describes the concentration, c, the fourth is the the central density in units of the mean density and the fifth is the mean square particle radius in units of the tidal radius.

$\boxed{2}$ *Models with non-isotropic distribution functions.* In class, we discussed models with $f = f(E)$. One feature of isotropic models is the that velocity ellipsoid is spherical. Many times one needs models where this is not the case. Here is practice with a few simple *non-isotropic* models.

(a) Show that

$$\bar{v}_\theta^2/\bar{v}_r^2 = 1 - \beta \tag{1}$$

for the following distribution function:

$$f(E,L) = L^{-2\beta} g(E). \tag{2}$$

There are many acceptable and useful choices for isotropic part of the distribution, $g(E)$ (e.g. one could choose it to be a King model). [This is similar to B&T, Problem 4.6, pg. 387 (2nd edition).]

(b) Show that the distribution function

$$f(E,L) = \begin{cases} f_o \delta(L^2)(E_o - E)^{-1/2} & \text{for} \quad E < E_o \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

where f_o and E_o are constants, self-consistently generate a model with density

$$\rho(r) = \begin{cases} C/r^2 & \text{for} \quad r < r_o \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

where C is some constant and r_o satisfies $V(r_o) = E_o$ (similar to the tidal radius in the King models). Because of the Dirac delta function in L^2, this model consists of radial orbits (and is unstable as we will discuss later on ...). [This is B&T, Problem 4.9, pg. 388 (2nd edition); note the different conventions for the sign of E than my typical usage in class.]

Table 1: King model parameters

W_o	$E_{tot}/(GM^2/r_t)$	$\log_{10}(r_t/r_c)$	$\rho(0)/(\frac{M}{4\pi r_t^3/3})$	$\langle r^2 \rangle / r_t^2$
1.00	-0.64	0.30	3.21e+01	1.51e-01
1.25	-0.65	0.36	3.52e+01	1.46e-01
1.50	-0.67	0.41	3.89e+01	1.41e-01
1.75	-0.68	0.46	4.33e+01	1.36e-01
2.00	-0.70	0.50	4.86e+01	1.31e-01
2.25	-0.72	0.55	5.50e+01	1.26e-01
2.50	-0.74	0.59	6.27e+01	1.21e-01
2.75	-0.76	0.63	7.21e+01	1.16e-01
3.00	-0.78	0.67	8.39e+01	1.11e-01
3.25	-0.81	0.71	9.85e+01	1.05e-01
3.50	-0.84	0.75	1.17e+02	1.00e-01
3.75	-0.87	0.80	1.41e+02	9.52e-02
4.00	-0.91	0.84	1.71e+02	9.02e-02
4.25	-0.94	0.88	2.11e+02	8.53e-02
4.50	-0.99	0.93	2.64e+02	8.05e-02
4.75	-1.04	0.98	3.36e+02	7.59e-02
5.00	-1.09	1.03	4.34e+02	7.15e-02
5.25	-1.15	1.08	5.70e+02	6.74e-02
5.50	-1.21	1.14	7.63e+02	6.35e-02
5.75	-1.29	1.19	1.04e+03	6.00e-02
6.00	-1.37	1.26	1.44e+03	5.67e-02
6.25	-1.45	1.32	2.04e+03	5.38e-02
6.50	-1.55	1.39	2.95e+03	5.14e-02
6.75	-1.64	1.46	4.32e+03	4.93e-02
7.00	-1.74	1.53	6.43e+03	4.77e-02
7.25	-1.84	1.60	9.68e+03	4.66e-02
7.50	-1.94	1.68	1.46e+04	4.62e-02
7.75	-2.02	1.76	2.21e+04	4.62e-02
8.00	-2.08	1.83	3.32e+04	4.66e-02
8.25	-2.12	1.91	4.90e+04	4.75e-02
8.50	-2.13	1.98	7.08e+04	4.86e-02

$\boxed{3}$ **Practice with Maxwellian distributions**
B&T, Problem 4.18, pg. 389 (2nd edition)

$\boxed{4}$ **Practice with the virial equations**
B&T, Problem 4.35, pg. 391 (2nd edition)

$\boxed{5}$ **Virial equations with an external field**
B&T, Problem 4.38, pg. 392 (2nd edition). [This technique can be useful in practice, but mostly, it's an excuse to get you to work through the derivation for V_{jk}.]