Numerical Analysis MATH50003 (2023–24) Problem Sheet 10

Problem 1 What are the upper 3×3 sub-block of the multiplication matrix X / Jacobi matrix J for the monic and orthonormal polynomials with respect to the following weights on $[-1, 1]$:

$$1 - x, \sqrt{1 - x^2}, 1 - x^2$$

SOLUTION

Monic We know that for monic ($b_n = 1$) orthogonal polynomials we can write the upper 3x3 block in the form

$$X_3 = \begin{bmatrix} a_0 & c_0 & 0 \\ 1 & a_1 & c_1 \\ 0 & 1 & a_2 \end{bmatrix}$$

1.

$$w(x) = 1 - x$$

Take $\pi_0(x) = 1$ (monic) and note

$$\|\pi_0\|^2 = \int_{-1}^{1} (1 - x) \mathrm{d}x = 2$$

From

$$x\pi_0(x) = a_0\pi_0(x) + \pi_1(x)$$

we deduce

$$a_0 = \langle x\pi_0, \pi_0 \rangle / \|\pi_0\|^2 = \frac{\int_{-1}^{1} (1 - x)x \mathrm{d}x}{2} = -\frac{1}{3}$$

i.e.

$$\pi_1(x) = (x - a_0)\pi_0(x) = x + 1/3.$$

and note that

$$\|\pi_1\|^2 = \int_{-1}^{1} (1 - x)(x + 1/3)^2 \mathrm{d}x = 4/9.$$

From

$$x\pi_1(x) = c_0\pi_0(x) + a_1\pi_1(x) + \pi_2(x)$$

we deduce

$$c_0 = \langle x\pi_1, \pi_0 \rangle / \|\pi_0\|^2 = \frac{\int_{-1}^{1} (1 - x)x(x + 1/3) \mathrm{d}x}{2} = \frac{2}{9}$$

and

$$a_1 = \langle x\pi_1, \pi_1 \rangle / \|\pi_1\|^2 = \frac{9}{4} \int_{-1}^{1} (1 - x)x(x + 1/3)^2 \mathrm{d}x = -\frac{1}{15}$$

Thus

$$\pi_2(x) = (x - a_1)\pi_1(x) - c_0\pi_0(x) = (x + 1/15)(x + 1/3) - 2/9 = x^2 + 2x/5 - 1/5.$$

And once again as before:

$$c_1 = \frac{\langle \pi_1, x\pi_2 \rangle}{\|\pi_1\|^2} = \frac{\int_{-1}^{1} (x + \frac{1}{3})(x^2 + \frac{2}{5}x - \frac{1}{5})x(1 - x)dx}{\int_{-1}^{1} (x + \frac{1}{3})^2(1 - x)dx} = \frac{6}{25}$$

and

$$a_2 = \frac{\langle \pi_2, x\pi_2 \rangle}{\|\pi_2\|^2} = \frac{\int_{-1}^{1}(x^2 + \frac{2}{5}x - \frac{1}{5})^2 x(1-x)dx}{\int_{-1}^{1}(x^2 + \frac{2}{5}x - \frac{1}{5})^2(1-x)dx} = -\frac{1}{35}$$

Thus we have

$$X_3 = \begin{bmatrix} -1/3 & 2/9 & \\ 1 & -1/15 & 6/25 \\ & 1 & -1/35 \end{bmatrix}$$

2.

$$w(x) = \sqrt{1-x^2}$$

Take $\pi_0(x) = k_0 = 1$ (monic) so that

$$\|\pi_0\|^2 = \int_{-1}^{1} \sqrt{1-x^2} = \frac{\pi}{2}.$$

From PS9 we know that $a_k = 0$. Thus from the recurrence we have

$$x\pi_0(x) = \pi_1(x)$$

and hence

$$\pi_1(x) = x\pi_0(x) = x.$$

Likewise for

$$x\pi_1(x) = c_0\pi_0(x) + \pi_2(x)$$

we have

$$c_0 = \frac{\langle \pi_0, x\pi_1 \rangle}{\|\pi_0\|^2} \frac{\int_{-1}^{1} x^2\sqrt{1-x^2}dx}{\pi/2} = \frac{\pi/8}{\pi/2} = \frac{1}{4}$$

i.e.

$$\pi_2(x) = x\pi_1(x) - c_0 = x^2 - \frac{1}{4}.$$

Finally:

$$x\pi_2(x) = c_1\pi_1(x)\pi_3(x)$$

and thus

$$c_1 = \frac{\langle \pi_1, x\pi_2 \rangle}{\|\pi_1\|^2} = \frac{\int_{-1}^{1}(x^2 - \frac{1}{4})x^2\sqrt{1-x^2}dx}{\int_{-1}^{1} x^2\sqrt{1-x^2}dx} = \frac{\pi/32}{\pi/8} = \frac{1}{4}$$

Thus we have

$$X_3 = \begin{bmatrix} 0 & 1/4 & \\ 1 & 0 & 1/4 \\ & 1 & 0 \end{bmatrix}$$

3.

$$w(x) = 1 - x^2$$

Take $\pi_0(x) = k_0 = 1$ (monic). Again due to $w(x) = w(-x)$ from recurrence we have

$$x\pi_0(x) = \pi_1(x)$$

Then from

$$x\pi_1(x) = c_0\pi_0(x) + \pi_2(x)$$

we find

$$c_0 = \frac{\langle \pi_0, x\pi_1 \rangle}{\|\pi_0\|^2} \frac{\int_{-1}^1 x^2(1-x^2)dx}{4/15} = \frac{4/15}{4/3} = \frac{1}{5}$$

Finally,

$$x\pi_2(x) = c_1\pi_1(x) + \pi_3(x)$$

and thus

$$c_1 = \frac{\langle \pi_1, x\pi_2 \rangle}{\|\pi_1\|^2} = \frac{\int_{-1}^1 (x^2 - \frac{1}{5})x^2(1-x^2)dx}{\int_{-1}^1 x^2(1-x^2)dx} = \frac{32/525}{4/15} = \frac{8}{35}$$

Thus we have

$$X_3 = \begin{bmatrix} 0 & 1/5 & \\ 1 & 0 & 8/35 \\ & 1 & 0 \end{bmatrix}$$

Orthonormal The hard way to solve this problem is to compute $\|\pi_n\|$ for each case. Instead, we use a trick for computing the orthonormal variants: III.3 Corollary 6 tells us that if we find constants α_n and define

$$q_n(x) := \alpha_n \pi_n(x)$$

so that $\|q_0\| = 1$ and the resulting Jacobi matrix is symmetric then q_n must be orthonormal. Note that the three-term recurrence for q_n satisfies

$$xq_0 = x\alpha_0\pi_0 = \alpha_0 a_0 \pi_0 + \alpha_0 \pi_1 = a_0 q_0 + \frac{\alpha_0}{\alpha_1} q_1$$

$$xq_m = x\alpha_n\pi_n = \alpha_n c_{n-1}\pi_{n-1} + a_n\alpha_n\pi_n + \alpha_n\pi_{n+1} = \frac{\alpha_n c_{n-1}}{\alpha_{n-1}} q_{n-1} + a_n q_n + \frac{\alpha_n}{\alpha_{n+1}} q_{n+1}$$

This is easier to see using linear algebra:

$$\begin{aligned} x[q_0|q_1|\ldots] &= x[\pi_0|\pi_1|\ldots] \begin{bmatrix} \alpha_0 & & \\ & \alpha_1 & \\ & & \ddots \end{bmatrix} = [\pi_0|\pi_1|\ldots] X \begin{bmatrix} \alpha_0 & & \\ & \alpha_1 & \\ & & \ddots \end{bmatrix} \\ &= [q_0|q_1|\ldots] \begin{bmatrix} \alpha_0^{-1} & & \\ & \alpha_1^{-1} & \\ & & \ddots \end{bmatrix} X \begin{bmatrix} \alpha_0 & & \\ & \alpha_1 & \\ & & \ddots \end{bmatrix} \\ &= [q_0|q_1|\ldots] \underbrace{\begin{bmatrix} a_0 & c_0\alpha_1/\alpha_0 & & \\ \alpha_0/\alpha_1 & a_1 & c_1\alpha_2/\alpha_1 & \\ & \alpha_1/\alpha_2 & a_2 & \ddots \\ & & \ddots & \ddots \end{bmatrix}}_{\tilde{X}} \end{aligned}$$

Thus to make this symmetric we need $\tilde{c}_n := c_n\alpha_{n+1}/\alpha_n = \alpha_n/\alpha_{n+1} =: \tilde{b}_n$, i.e., $\alpha_{n+1} = \alpha_n/\sqrt{c_n}$, in other words,

$$\alpha_n = \frac{\alpha_0}{\prod_{k=0}^{n-1} \sqrt{c_k}}.$$

Moreover, we see with this choice that $\tilde{c}_n = \sqrt{b_n} = \sqrt{c_n}$.

1.

$$w(x) = 1 - x$$

. We know $q_0(x) = \alpha_0 = 1/\|\pi_0\| = 1/\sqrt{2}$. Then $\alpha_1 = 1/\sqrt{2c_0} = 3/2$ (hence $q_1(x) = \alpha_1\pi_1(x) = 3x/2 + 1/2$),

which tells us

$$\tilde{c}_0 = c_0\alpha_1/\alpha_0 = \sqrt{2}/3 = \tilde{b}_0 (= \sqrt{c_0}.)$$

Then $\alpha_2 = \alpha_1/\sqrt{c_1} = 15/(2\sqrt{6})$ which tells us $\tilde{c}_1 = c_1\alpha_2/\alpha_1 = \sqrt{6}/5 = \tilde{b}_1 (= \sqrt{c_1})$. In other words we have,

$$\tilde{X}_3 = \begin{bmatrix} -1/3 & \sqrt{2}/3 & \\ \sqrt{2}/3 & -1/15 & \sqrt{6}/5 \\ & \sqrt{6}/5 & -1/35 \end{bmatrix}$$

2.

$$w(x) = \sqrt{1 - x^2}$$

We can just jump ahead since we know the answer is just with $\sqrt{c_n}$ in place of b_n and c_n:

$$\tilde{X}_3 = \begin{bmatrix} 0 & 1/2 & \\ 1/2 & 0 & 1/2 \\ & 1/2 & 0 \end{bmatrix}$$

3.

$$w(x) = 1 - x^2$$

:

$$\tilde{X}_3 = \begin{bmatrix} 0 & 1/\sqrt{5} & \\ 1/\sqrt{5} & 0 & \sqrt{8/35} \\ & \sqrt{8/35} & 0 \end{bmatrix}$$

END

Problem 2 Compute the roots of the Legendre polynomial $P_3(x)$, orthogonal with respect to $w(x) = 1$ on $[-1, 1]$, by computing the eigenvalues of a 3×3 truncation of the Jacobi matrix.

SOLUTION

We have, $P_0(x) = 1$. Though recall that in order to use Lemma (zeros), the Jacobi matrix must be symmetric and hence the polynomials orthonormal. So Take $Q_0(x) = 1/||P_0(x)|| = \frac{1}{\sqrt{2}}$. Then we have, by the three term recurrence relationship,

$$xQ_0(x) = a_0Q_0(x) + b_0Q_1(x),$$

and taking the inner product of both sides with $Q_0(x)$ we get,

$$a_0 = \langle xQ_0(x), Q_0(x)\rangle = \int_{-1}^{1} x/2dx = 0.$$

Next recall that $P_1(x) = x$ and so $Q_1(x) = x/||P_1(x)|| = \sqrt{\frac{3}{2}}x$. We then have, taking the innner product of the first equation above with $Q_1(x)$,

$$b_0 = \langle xQ_0(x), Q_1(x)\rangle = \int_{-1}^{1} \frac{\sqrt{3}}{2}x^2dx = \frac{1}{\sqrt{3}},$$

and also $b_0 = c_0$ by the Corollary 8 (Jacobi matrix). We have,

$$a_1 = \langle xQ_1(x), Q_1(x)\rangle = \int_{-1}^{1} \frac{3}{2}x^3 dx = 0.$$

Recall that $P_2(x) = \frac{1}{2}(3x^2 - 1)$, so that $Q_2(x) = P_2(x)/||P_2(x)|| = \sqrt{\frac{5}{8}}(3x^2 - 1)$, and that,

$$xQ_1(x) = c_0Q_0(x) + a_1Q_1(x) + b_1Q_2(x).$$

Taking inner the inner product of both sides with $Q_2(x)$, we see that,

$$c_1 = b_1 = \langle xQ_1(x), Q_2(x)\rangle = \int_{-1}^{1} \sqrt{\frac{5}{8}} \cdot \sqrt{\frac{3}{2}}(3x^2 - 1) \cdot x \cdot xdx = \frac{2}{\sqrt{15}}.$$

Finally,

$$a_2 = \langle Q_2(x), xQ_2(x)\rangle = \frac{5}{8}\int_{-1}^{1} (3x^2 - 1)^2 xdx = 0.$$

This gives us the truncated Jacobi matrix,

$$X_3 = \begin{bmatrix} a_0 & b_0 & 0 \\ b_0 & a_1 & b_1 \\ 0 & b_1 & a_2 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{15}} \\ 0 & \frac{2}{\sqrt{15}} & 0 \end{bmatrix},$$

whose eigenvalues are the zeros of $Q_3(x)$, and hence the zeros of $P_3(x)$ since they are the same up to a constant. To work out the eigenvalues, we have,

$$\begin{aligned} |X_3 - \lambda I| = \begin{vmatrix} -\lambda & \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{3}} & -\lambda & \frac{2}{\sqrt{15}} \\ 0 & \frac{2}{\sqrt{15}} & -\lambda \end{vmatrix} &= 0 \\ \Leftrightarrow -\lambda(\lambda^2 - \frac{4}{15}) - \frac{1}{\sqrt{3}} \cdot \frac{-\lambda}{\sqrt{3}} &= 0 \\ \Leftrightarrow -\lambda^3 + \frac{3}{5}\lambda &= 0, \end{aligned}$$

which has solutions $\lambda = 0, \pm\sqrt{\frac{3}{5}}$

END

Problem 3 Compute the 2-point interpolatory quadrature rule associated with roots of orthogonal polynomials for the weights $\sqrt{1-x^2}$, 1, and $1 - x$ on $[-1, 1]$ by integrating the Lagrange bases.

SOLUTION For $w(x) = \sqrt{1-x^2}$ the orthogonal polynomial of degree 2 is $U_2(x) = 4x^2 - 1$, with roots $\boldsymbol{x} = \{x = \pm\frac{1}{2}\}$. The Lagrange polynomials corresponding to these roots are,

$$\begin{aligned} \ell_1(x) &= \frac{x - 1/2}{-1/2 - 1/2} = \frac{1}{2} - x, \\ \ell_2(x) &= \frac{x + 1/2}{1/2 + 1/2} = x + \frac{1}{2} \end{aligned}$$

We again work out the weights

$$w_j = \int_{-1}^{1} \ell_j(x)w(x)dx,$$

to find,

$$w_1 = w_2 = \frac{\pi}{4},$$

and thus the interpolatory quadrature rule is,

$$\Sigma_2^{w,\boldsymbol{x}}(f) = \frac{\pi}{4}(f(-1/2) + f(1/2)).$$

For $w(x) = 1$, the orthogonal polynomial of degree 2 is, using Legendre Rodriguez formula:

$$P_2(x) = \frac{1}{(-2)^2 2!}\frac{d^2}{dx^2}\left(1 - x^2\right)^2 = -\frac{1}{2} + \frac{3}{2}x^2.$$

This has roots $\boldsymbol{x} = \left\{\pm\frac{1}{\sqrt{3}}\right\}$. We then have,

$$\begin{aligned}\ell_1(x) &= -\frac{\sqrt{3}}{2}x + \frac{1}{2} \\ \ell_2(x) &= \frac{3}{2}x + \frac{1}{2},\end{aligned}$$

from which we can compute the weights,

$$w_1 = w_2 = 1,$$

which give the quadrature rule:

$$\Sigma_2^{w,\boldsymbol{x}}(f) = \left[f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)\right]$$

Finally, with $w(x) = 1 - x$ we use the solution to PS9, which states that

$$p_2(x) = x^2 + 2x/5 - 1/5$$

which has roots, $\boldsymbol{x} = \left\{-\frac{1}{5} \pm \frac{\sqrt{6}}{5}\right\}$. The Lagrange polynomials are then,

$$\begin{aligned}\ell_1(x) &= \frac{x - (-\frac{1}{5} + \frac{\sqrt{6}}{5})}{-\frac{1}{5} - \frac{\sqrt{6}}{5} - (-\frac{1}{5} + \frac{\sqrt{6}}{5})} \\ &= \frac{x - (-\frac{1}{5} + \frac{\sqrt{6}}{5})}{-\frac{2\sqrt{6}}{5}} \\ &= -\frac{5}{2\sqrt{6}}x - \frac{1}{2\sqrt{6}} + \frac{1}{2} \\ \ell_2(x) &= \frac{x - (-\frac{1}{5} - \frac{\sqrt{6}}{5})}{\frac{2\sqrt{6}}{5}} \\ &= \frac{5}{2\sqrt{6}}x + \frac{1}{2\sqrt{6}} + \frac{1}{2}\end{aligned}$$

From which we can compute the weights,

$$\begin{aligned}w_1 &= 1 + \frac{\sqrt{6}}{9}, \\ w_2 &= 1 - \frac{\sqrt{6}}{9},\end{aligned}$$

giving the quadrature rule,

$$\Sigma_2^{w,\boldsymbol{x}}(f) = \left[\left(1+\frac{\sqrt{6}}{9}\right) f\left(-\frac{1}{5}-\frac{\sqrt{6}}{5}\right) + \left(1-\frac{\sqrt{6}}{9}\right) f\left(-\frac{1}{5}+\frac{\sqrt{6}}{5}\right)\right]$$

END

Problem 4(a) For the matrix

$$J_n = \begin{bmatrix} 0 & 1/\sqrt{2} & & & \\ 1/\sqrt{2} & 0 & 1/2 & & \\ & 1/2 & 0 & \ddots & \\ & & \ddots & \ddots & 1/2 \\ & & & 1/2 & 0 \end{bmatrix} \in \mathbb{R}^{n\times n}$$

use the relationship with the Jacobi matrix associated with $T_n(x)$ to prove that, for $x_j = \cos\theta_j$, and $\theta_j = (n-j+1/2)\pi/n$,

$$J_n = Q_n \begin{bmatrix} x_1 & & \\ & \ddots & \\ & & x_n \end{bmatrix} Q_n^\top$$

where

$$\boldsymbol{e}_1^\top Q_n \boldsymbol{e}_j = \frac{1}{\sqrt{n}}, \qquad \boldsymbol{e}_k^\top Q_n \boldsymbol{e}_j = \sqrt{\frac{2}{n}} \cos(k-1)\theta_j.$$

You may use without proof the sums-of-squares formula

$$1 + 2\sum_{k=1}^{n-1} \cos^2 k\theta_j = n.$$

SOLUTION

Recall the three term recurrence for the Chebyshev Polynomials T_n,

$$\begin{aligned} xT_0(x) &= T_1(x), \\ xT_n(x) &= \frac{T_{n-1}(x)}{2} + \frac{T_{n+1}(x)}{2}, \end{aligned}$$

and hence it has the multiplication matrix

$$x[T_0|T_1|\cdots] = [T_0|T_1|\cdots] \underbrace{\begin{bmatrix} 0 & 1/2 & & \\ 1 & 0 & 1/2 & \\ & 1/2 & 0 & \ddots \\ & & \ddots & \ddots \end{bmatrix}}_{X}$$

To find the Jacobi matrix we need to symmetrise this, that is, we write

$$[q_0(x)|q_1(x)|\cdots] = [T_0(x)|T_1(x)|\cdots] \underbrace{\begin{bmatrix} \beta_0 & & & \\ & \beta_1 & & \\ & & \beta_2 & \\ & & & \ddots \end{bmatrix}}_{K}$$

so that

$$x[q_0(x)|q_1(x)|\cdots] = [q_0(x)|q_1(x)|\cdots]\underbrace{K^{-1}XK}_{J}$$

where

$$K^{-1}XK = \begin{bmatrix} 0 & \beta_1/(2\beta_0) & & \\ \beta_0/\beta_1 & 0 & \beta_2/(2\beta_1) & \\ & \beta_1/(2\beta_2) & 0 & \ddots \\ & & \ddots & \ddots \end{bmatrix}$$

First recall that the change-of-variables $x = \cos\theta$ tells us

$$\int_{-1}^{1} \frac{\mathrm{d}x}{\sqrt{1-x^2}} = \pi$$

hence $q_0(x) = \beta_0 = 1/\sqrt{\pi}$. From this we find that

$$\frac{\beta_0}{\beta_1} = \frac{\beta_1}{2\beta_0} \Rightarrow \beta_1 = \sqrt{2/\pi}.$$

Other equations give us:

$$\frac{\beta_n}{2\beta_{n+1}} = \frac{\beta_{n+1}}{2\beta_n} \Rightarrow \beta_{n+1} = \beta_n = \sqrt{2/\pi}.$$

Hence since $\beta_1/(2\beta_0) = 1/\sqrt{2}$ and $\beta_{n+1}/(2\beta_n) = 1/2$ we have

$$J = \begin{bmatrix} 0 & 1/\sqrt{2} & & & \\ 1/\sqrt{2} & 0 & 1/2 & & \\ & 1/2 & 0 & 1/2 & \\ & & \ddots & \ddots & \ddots \end{bmatrix}$$

The roots of $q_n(x)$ are the roots of $T_n(x) = \cos n \operatorname{acos} x$, i.e., $x_j = \cos\theta_j$ for $\theta_j = (n-j+1/2)\pi/n$. Thus we know that we can diagonalise J_n as

$$J_n = Q_n \begin{bmatrix} x_1 & & \\ & \ddots & \\ & & x_n \end{bmatrix} Q_n^\top$$

where

$$Q_n = \begin{bmatrix} q_0(x_1) & \cdots & q_0(x_n) \\ \vdots & \cdots & \vdots \\ q_{n-1}(x_1) & \cdots & q_{n-1}(x_n) \end{bmatrix} \begin{bmatrix} \alpha_1^{-1} & & \\ & \ddots & \\ & & \alpha_n^{-1} \end{bmatrix}$$

where

$$\alpha_j = \sqrt{q_0(x_j)^2 + \cdots + q_{n-1}(x_j)^2} = \frac{1}{\sqrt{\pi}}\sqrt{1 + 2\sum_{k=1}^{n-1} \cos k\theta_j} = \sqrt{\frac{n}{\pi}}.$$

Thus we have

$$\begin{aligned} \boldsymbol{e}_1^\top Q_n \boldsymbol{e}_j &= \frac{q_0(x_j)}{\alpha_j} = \frac{1}{\sqrt{n}} \\ \boldsymbol{e}_k^\top Q_n \boldsymbol{e}_j &= \frac{q_{k-1}(x_j)}{\alpha_j} = \sqrt{\frac{2}{n}} \cos(k-1)\theta_j. \end{aligned}$$

END

Problem 4(b) Show for $w(x) = 1/\sqrt{1-x^2}$ that the Gaussian quadrature rule is

$$Q_n^w[f] = \frac{\pi}{n} \sum_{j=1}^n f(x_j)$$

where $x_j = \cos\theta_j$ for $\theta_j = (j - 1/2)\pi/n$.

SOLUTION This follows immediately from the previous parts as x_j are the eigenvalues of J_n and the weights in Gauss quadrature have the form

$$\frac{1}{\alpha_j^2} = \frac{\pi}{n}.$$

END

Problem 4(c) Give an explicit formula for the polynomial that interpolates $\exp x$ at the points $x_1, \ldots, x_n$ as defined above, in terms of Chebyshev polynomials with the coefficients defined in terms of a sum involving only exponentials, cosines and $\theta_j = (n - j + 1/2)\pi/n$.

SOLUTION

From Theorem 18 we know the interpolatory polynomial is

$$f_n(x) = \sum_{k=0}^{n-1} c_k^n q_k(x)$$

where $q_0(x) = 1/\sqrt{\pi}$ and $q_n(x) = \sqrt{2/\pi} T_n(x)$ and

$$c_k^n = \Sigma_n^w[\exp(x) q_k] = \frac{\pi}{n} \sum_{j=1}^n \exp(\cos\theta_j)\cos(k\theta_j)$$

for $\theta_j = (n - j + 1/2)\pi/n$.

END