Thomas Calculus Review

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Contents

1 Function

1.1 Function and their Graphs

DEFINITION: A function f from a set D to a set Y is a rule that assigns a unique (single) element $f(x) \in Y$ to each element $x \in D$.

Linear Functions $f(x) = mx + ba$

Power Functions $f(x) = x^a$ where a is a constant

Polynomials $p(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0$ where n is a nonnegative integer and $a_0, a_1, \cdots a_n$ are real constants (called the **coefficients** of the polynomial)

Rational Functions $f(x) = \dfrac{p(x)}{q(x)}$ where p and q are polynomials.

Algebraic Functions Any function constructed from polynomials using algebraic operations lies within the class of **algebraic functions**.

Trigonometric Function

$$\begin{cases} f(x) = \sin(x) \\ f(x) = \cos(x) \\ f(x) = \tan(x) \\ f(x) = \csc(x) \\ f(x) = \sec(x) \\ f(x) = \cot(x) \end{cases} \tag{1}$$

Exponential Functions $f(x) = a^x$ where the base a is a positive constant and $a \neq 1$

Logarithmic Functions $f(x) = \log_a x$

Transcendental Functions Functions that are not algebraic.

1.2 Combining Functions; Shifting and Scaling Graphs

Sums,Differences,Products, and Quotients

$$\begin{cases} (f+g)(x) = f(x) + g(x) \\ (f-g)(x) = f(x) - g(x) \\ (fg)(x) = f(x)g(x) \\ (\frac{f}{g})(x) = \dfrac{f(x)}{g(x)} \\ (cf)(x) = cf(x) \end{cases} \tag{2}$$

Composite Functions

DEFINITION If f and g are functions, the **composite** of function $f \circ g$ ("f composed with g") is defined by

$$(f \circ g)(x) = f(g) \tag{3}$$

Shifting a Graph of a Function Vertical Shifts and Horizontal Shifts

Scaling and Reflecting a Graph of a Function Vertical and Horizontal Scaling and Reflecting Formulas

1.3 Trigonometric Function

Angles are measured in degrees or radians.

1 radian $= \dfrac{180}{\pi}(\approx 57.3)$degrees

The Six Basic Trigonometric Functions

$$\begin{cases} \sin\theta = \dfrac{y}{r} \\ \csc\theta = \dfrac{r}{y} \\ \cos\theta = \dfrac{x}{r} \\ \sec\theta = \dfrac{r}{x} \\ \tan\theta = \dfrac{y}{x} \\ \cot\theta = \dfrac{x}{y} \end{cases} \quad \begin{cases} \tan\theta = \dfrac{\sin\theta}{\cos\theta} \\ \cot\theta = \dfrac{1}{\tan\theta} \\ \sec\theta = \dfrac{1}{\cos\theta} \\ \csc\theta = \dfrac{1}{\sin\theta} \end{cases} \tag{4}$$

Trigonometric Identities

$$x = r\cos\theta \qquad y = r\sin\theta$$
$$\sin^2\theta + \cos^2\theta = 1$$
$$1 + \tan^2\theta = \sec^2\theta$$
$$1 + \cot^2\theta = \csc^2\theta$$

Addition Formulas

$$\begin{aligned} \cos(A+B) &= \cos A\cos B - \sin A\sin B \\ \sin(A+B) &= \sin A\cos B + \cos A\sin B \end{aligned} \tag{5}$$

Double-Angle Formulas

$$\begin{aligned} \cos 2\theta &= \cos^2\theta - \sin^2\theta \\ \sin 2\theta &= 2\sin\theta\cos\theta \end{aligned} \tag{6}$$

Half-Angle Formulas

$$\begin{aligned} \cos^2\theta &= \frac{1+\cos 2\theta}{2} \\ \sin^2\theta &= \frac{1-\cos 2\theta}{2} \end{aligned} \tag{7}$$

The Law of Cosines

$$c^2 = a^2 + b^2 - 2ab\cos\theta$$

Two Special Inequalities

$$-|\theta| \leq \sin\theta \leq |\theta|$$

$$-|\theta| \leq 1-\cos\theta \leq |\theta|$$

Transformations of Trigonometric Graphs

$$y = af(b(x+c)) + d$$

a:Vertical stretch or compression; reflection about $y = d$ if negative
b:Horizontal stretch or compression; reflection about $x = -c$ if negative
c:Horizontal shift
d:Vertical shift

2 Limits and Continuity

2.1 Rates of Change and Tangents to Curves

$$\frac{\Delta y}{\Delta x}$$

DEFINITION The **average rate of change** of $y = f(x)$ with respect to x over the interval $[x_1, x_2]$ is

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(x_1 + h) - f(x_1)}{h}, \quad h \neq 0$$

2.2 Limit of a Function and Limit Laws

Limits of Function Value $\lim_{x \to c} f(x) = L$ (read "the limit of $f(x)$ as x approaches c is L")

"Informal" definition The values of $f(x)$ are close to the number L whenever is close to c (on either side of c)

The Limit Laws

Limit Laws If L, M,c and k are real numbers and $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$, then

$$\begin{aligned}
\lim_{x \to c}(f(x) + g(x)) &= L + M \\
\lim_{x \to c}(f(x) - g(x)) &= L - M \\
\lim_{x \to c}(f(x) \cdot g(x)) &= L \cdot M \\
\lim_{x \to c}(k \cdot f(x)) &= k \cdot L \\
\lim_{x \to c} \frac{f(x)}{g(x)} &= \frac{L}{M}, \quad M \neq 0 \\
\lim_{x \to c}[f(x)]^n &= L^n \\
\lim_{x \to c} \sqrt[n]{f(x)} &= \sqrt[n]{L} = L^{\frac{1}{n}}
\end{aligned} \tag{8}$$

Limits of Polynomials If $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$, then

$$\lim_{x \to c} P(x) = P(c) = a_n c^n + a_{n-1} c^{n-1} + \cdots + a_0$$

Limits of Rational Funcitions If $P(x)$ and $Q(x)$ are polynomials and $Q(c) \neq 0$, then

$$\lim_{x \to c} \frac{P(x)}{Q(x)} = \frac{P(c)}{Q(c)}$$

The Sandwich Theorem Suppose that $g(x) \leq f(x) \leq h(x)$ for all x in some open interval containing c, expect possibly at $x = c$ itself. Suppose also that

$$\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L$$

Then $\lim_{x \to c} f(x) = L$.

Theorem If $f(x) \leq g(x)$ for all x in some open interval containing c, expect possibly at $x = c$ itself, and the limits of f and g both exist as x approaches c, then

$$\lim_{x \to c} f(x) \leq \lim_{x \to c} g(x)$$

2.3 The Precise Definition of a Limit

DEFINITION Let $f(x)$ be defined on an open interval about c, except possibly at c itself. We say that the **limit of** $f(x)$ **as** x **approaches** c **is the number** L, and write

$$\lim_{x \to c} f(x) = L$$

if, for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all x,

$$0 < |x - c| < \delta \quad \Rightarrow \quad |f(x) - L| < \epsilon$$

2.4 One-Sided Limits

Two-sided limits right-hand limit and left-hand limit

THEOREM A function $f(x)$ has a limit as x approaches c if and only if it has left-hand and right-hand limits there and these one-sided limits are equal:

$$\lim_{x \to c} f(x) = L \quad \Leftrightarrow \quad \lim_{x \to c^-} f(x) = L \quad \text{and} \quad \lim_{x \to c^+} f(x) = L$$

Limits Involving $(\sin \theta)/\theta$

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$$

2.5 Continuity

DEFINITION Let c be a real number on the x-axis.
The function f is **Continuous at** c if

$$\lim_{x \to c} f(x) = f(c)$$

The function f is **right-continuous at** c **(or continuous from the right)** if

$$\lim_{x \to c^+} f(x) = f(c)$$

The function f is **left-continuous at** c **(or continuous from the left)** if

$$\lim_{x \to c^-} f(x) = f(c)$$

Properties of Continuous Functions If the functions f and g are continuous at $x = c$, then the following algebraic combinations are continuous at $x = c$.

$$\begin{gathered} f + g \\ f - g \\ k \cdot f \\ f \cdot g \\ f/g \\ f^g \\ \sqrt[n]{f} \end{gathered} \tag{9}$$

Composite of Continuous Functions If f is continuous at c and g is continuous at $f(c)$, then the composite $g \circ f$ is continuous at c.

Limits of Continuous Functions If g is continuous at the point b and $\lim_{x \to c} f(x) = b$, then

$$\lim_{x \to c} g(f(x)) = g(b) = g(\lim_{x \to c} f(x)).$$

The Intermediate Value Theorem for Continuous Functions If f is a continuous function on a closed interval $[a,b]$, and if y_0 is any value between $f(a)$ and $f(b)$, then $y_0 = f(c)$ for some c in $[a,b]$.

2.6 Limits Involving Infinity; Asymptotes of Graphs

$$\lim_{x \to \pm\infty} k = k$$

$$\lim_{x \to \pm\infty} \frac{1}{x} = 0$$

EXAMPLE

$$\lim_{x \to -\infty} \frac{11x + 2}{2x^3 - 1} = \lim_{x \to -\infty} \frac{(11/x^2) + (2/x^3)}{2 - (1/x^3)} = \frac{0 + 0}{2 - 0} = 0$$

DEFINITION A line $y = b$ is a **horizontal asymptote** of the graph of a function $y = f(x)$ if either

$$\lim_{x \to \infty} f(x) = b \quad \text{or} \quad \lim_{x \to -\infty} f(x) = b$$

EXAMPLE

$$\lim_{x \to \infty} \sin \frac{1}{x} = \lim_{t \to 0^+} \sin t = 0$$

$$\lim_{x \to -\infty} x \sin \frac{1}{x} = \lim_{t \to 0^+} \frac{\sin t}{t} = 1$$

$$\lim_{x \to \infty} x \sin \frac{1}{x} = \lim_{t \to 0^-} \frac{\sin t}{t} = 1$$

$$\begin{aligned}
\lim_{x \to -\infty} \frac{2x^5 - 6x^4 + 1}{3x^2 + x - 7} &= \lim_{x \to -\infty} \frac{2x^3 - 6x^2 + x^{-2}}{3 + x^{-1} - 7x^{-2}} \\
&= \lim_{x \to -\infty} \frac{2x^2(x - 3) + x^{-2}}{3 + x^{-1} - 7x^{-2}} \\
&= -\infty
\end{aligned} \tag{10}$$

3 Derivatives

3.1 Tangents and the Derivative at a Point

DEFINITIONS The **derivative of a function** f **at a point** x_0, denoted $f'(x_0)$, is

$$m = \lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h} \quad \text{(provided the limit exists)}$$

The **tangent line** to the curve at P is the line through P with this slope.

3.2 The Derivative as a Function

Calculating Derivatives from the Definition The process of calculating a derivative is called differentiation. To emphasize the idea that differentiation is an operation performed on a function $y = f(x)$, we use the notation

$$\frac{d}{dx} f(x)$$

as another way to denote the derivative $f'(x)$.

Notations

$$f'x = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx} f(x) = D(f)(x) = D_x f(x)$$

Differentiable on an Interval; One-Sided Derivatives

$$\begin{aligned} &\lim_{h\to 0^+} \frac{f(a+h) - f(a)}{h} \quad \textbf{Right-hand derivative at a} \\ &\lim_{h\to 0^-} \frac{f(h+h) - f(b)}{h} \quad \textbf{Left-hand derivative at b} \end{aligned} \tag{11}$$

When Does a Function *Not* Have a Derivative at a Point?
1. a corner,where the one-sided derivatives differ.
2. a cusp, where the slope of PQ approaches ∞ from one side and $-\infty$ from the other.
3. a vertical tangent, where the slope of PQ approaches ∞ from both sides or approaches $-\infty$ from both sides.
4. a *discontinuity.*

Differentiable Functions Are Continuous

Differentiability Implies Continuity If f has a derivative at $x = c$, then f is continuous at $x = c$.

3.3 Differentiation Rules

Power,Multiples, Sums, and Differences

$$\begin{aligned}
\frac{d}{dx}(c) &= 0 \\
\frac{d}{dx}(x^n) &= nx^{n-1} \\
\frac{d}{dx}(cu) &= c\frac{du}{dx} \\
\text{so } (\frac{d}{dx}(cx^n) &= cnx^{n-1}) \\
\frac{d}{dx}(u+v) &= \frac{du}{dx} + \frac{dv}{dx} \\
\frac{d}{dx}(uv) &= u\frac{dv}{dx} + v\frac{du}{dx} \\
\frac{d}{dx}(\frac{u}{v}) &= \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}
\end{aligned} \tag{12}$$

Second- and Higher-Order Derivatives

$$f^{''}(x) = \frac{d^2y}{dx^2} = \frac{d}{dx}(\frac{dy}{dx}) = \frac{dy'}{dx} = y^{''} = D^2(f)(x) = {D_x}^2 f(x)$$

3.4 The Derivative as a Rate of Change

Motion Along a Line: Displacement, Velocity, Speed, Acceleration, and Jerk

$$v(t) = \frac{ds}{dt}$$

$$\text{Speed} = |v(t)| = |\frac{ds}{dt}|$$

$$a(t) = \frac{dv}{dt} = \frac{d^2s}{dt^2}$$

$$j(t) = \frac{da}{dt} = \frac{d^3s}{dt^3}$$

3.5 Derivatives of Trigonometric Functions

$$\begin{aligned}
\frac{d}{dx}(\sin x) &= \cos x \\
\frac{d}{dx}(\cos x) &= -\sin x \\
\frac{d}{dx}(\tan x) &= \sec^2 x \\
\frac{d}{dx}(\cot x) &= -\csc^2 x \\
\frac{d}{dx}(\sec x) &= \sec x \tan x \\
\frac{d}{dx}(\csc x) &= -\csc x \cot x
\end{aligned} \tag{13}$$

3.6 The Chain Rule*

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$
$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$$

Power Chain Rule Using chain rules in loop.

3.7 Implicit Differentiation

Implicitly Defined Functions

EXAMPLE 1

$$\begin{aligned} y^2 &= x \\ 2y\frac{dy}{dx} &= 1 \\ \frac{dy}{dx} &= \frac{1}{2y} \end{aligned} \tag{14}$$

EXAMPLE 2

$$\begin{aligned} x^2 + y^2 &= 25 \\ \frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) &= \frac{d}{dx}(25) \\ 2x + 2y\frac{dy}{dx} &= 0 \\ \frac{dy}{dx} &= -\frac{x}{y} \end{aligned} \tag{15}$$

EXAMPLE 3

$$\begin{aligned} y^2 &= x^2 + \sin xy \\ \frac{d}{dx}(y^2) &= \frac{d}{dx}(x^2) + \frac{d}{dx}(\sin xy) \\ 2y\frac{dy}{dx} &= 2x + (\cos xy)\frac{d}{dx}(xy) \\ 2y\frac{dy}{dx} &= 2x + (\cos xy)(y + x\frac{dy}{dx}) \\ (2y - x\cos xy)\frac{dy}{dx} &= 2x + y\cos xy \\ \frac{dy}{dx} &= \frac{2x + y\cos xy}{2y - x\cos xy} \end{aligned} \tag{16}$$

3.8 Related Rates

Related Rate Equations

$$V = \frac{4}{3}\pi r^3$$
$$\frac{dV}{dt} = \frac{dV}{dr}\frac{dr}{dt} = 4\pi r^2\frac{dr}{dt}$$

Strategy
1. Draw a Picture and name the variables and constants.
2. Write down the numerical information.

3. Write down what you are asked to find.
4. Write an equation that relates the variables.
5. Differentiate with respect to t.
6. Evaluate.

3.9 Linearization and Differentials

Linearization If f is a differentiable at $x = a$, then the approximating function

$$L(x) = f(a) + f'(a)(x - a)$$

is the **linearization** of f at a. The approximation

$$f(x) \approx L(x)$$

of f by L is the **standard linear approximation** of f at a. The point $x = a$ is the **center** of the approximation.

Differentials Let $y = f(x)$ be a differentiable function. The **differential dx** is an independent variable. The **differential dy** is

$$dy = f'(x)dx$$

Estimating with Differentials Since

$$f(a + dx) = f(a) + \Delta y$$

the differential approximation gives

$$f(a + dx) \approx f(a) + dy$$

when $dx = \Delta x$. Thus the approximation $\Delta y \approx dy$ can be used to estimate $f(a + dx)$ when $f(a)$ is known, dx is small,and $dy = f'(a)dx$.

Error in Differential Approximation If $y = f(x)$ is differentiable at $x = a$ and x changes from a to $a + \Delta x$, the change Δy in f is given by

$$\Delta y = f'(a)\Delta x + \epsilon\Delta x$$

in which $\epsilon \to 0$ as $\Delta x \to 0$

Sensitivity to Change

	True	**Estimated**
Absolute change	$\Delta f = f(a + dx) - f(a)$	$df = f'(a)dx$
Relative change	$\dfrac{\Delta f}{f(a)}$	$\dfrac{df}{f(a)}$
Percentage change	$\dfrac{\Delta f}{f(a)} \times 100$	$\dfrac{df}{f(a)} \times 100$

4 Applications of Derivatives

4.1 Extreme Values of Functions

DEFINITIONS Let f be a function with domain D. Then f has an **absolute maximum** value on D at a point c if

$$f(x) \geq f(c) \quad \text{for all } x \text{ in } D$$

and an **absolute minimum** value on D at c if

$$f(x) \geq f(c) \quad \text{for all } x \text{ in } D$$

which are called **extreme values** of the function f.

Local(Relative) Extreme Values A function f has a $localmaximum$ value at a point c within its domain D if $f(x) \leq f(c)$ for all $x \in D$ lying in some open interval containing c.

A function f has a **local minimum** value at a point c within its domain D if $f(x) \geq f(c)$ for all $x \in D$ lying in some open interval containing c.

Finding Extrema

Theorem—The First Derivative Theorem for Local Extreme Values If f has a local maximum or minimum value at an interior point c of its domain, and if f' is defined at c, then

$$f'(c) = 0$$

Definition An interior point of the domain of a function f where f' is zero or undefined is **critical point** of f

4.2 The Mean Value Theorem*

Rolle's Theorem Suppose that $y = f(x)$ is continuous over the closed interval $[a, b]$ and differentiable at every point of its interior (a, b). If $f(a) = f(b)$, then there is at least one number c in (a, b) at which $f'(c) = 0$.

The Mean Value Theorem* Suppose $y = f(x)$ is continuous over a closed interval $[a, b]$ and differentiable on the interval's interior (a, b). Then there is at least one point c in (a, b) at which

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

Corollary 1 If $f'(x) = 0$ at each point x of an open interval (a, b), then $f(x) = C$ for all $x \in (a, b)$, where C is a constant.

Corollary 2 If $f'(x) = g'(x)$ at each point x in an open interval (a, b), then there exists a constant C such that $f(x) = g(x) + C$ for all $x \in (a, b)$. That is, $f - g$ is a constant function on (a, b).

4.3 Monotonic Functions and the First Derivative Test

Corollary 3 Suppose that f is continuous on $[a, b]$ and differentiable on (a, b).
If $f'(x) > 0$ at each point $x \in (a, b)$, then f is increasing on $[a, b]$.
If $f'(x) < 0$ at each point $x \in (a, b)$, then f is decreasing on $[a, b]$.

4.4 Concavity and Curve Sketching

Concavity The graph of a differentiable function $y = f(x)$ is
(a) **concave up** on an open interval I if f' is increasing on I.
(b) **concave down** on an open interval I if f' is decreasing on I.

The Second Derivative Test for Concavity Let $y = f(x)$ be twice-differentiable on an interval I.
1. If $f'' > 0$ on I, the graph of f over I is concave up.
2. If $f'' < 0$ on I,the graph of f over I is concave down.

Points of Inflection

Definition A point $(c, f(c))$ where the graph of a function has a tangent line and where the Concavity changes is a **point of inflection**.

*At a point of inflection $(c, f(c))$, either $f''(c) = 0$ or $f''y$

Second Derivative Test for local Extrema Suppose f'' is continuous on an open interval that contains $x = c$.
1. If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at $x = c$.
2. If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at $x = c$.
3. If $f'(c) = 0$ and $f''(c) = 0$, then the test fails. The function f may have a local maximum, a local minimum, or neither.

4.5 Applied Optimization

Solving Applied Optimization Problems
1. *Read the problem.*
2. *Draw a picture.*
3. *Introduce variables.*
4. *Write an equation for the unknown quantity.*
5. *Test the critical points and endpoints in the domain of the unknown.*

4.6 Newton's Method

Newton's Method
1. Guess a first approximation to a solution of the equation $f(x) = 0$. A graph of $y = f(x)$ may help. 2. Use the first approximation to get a second, the second to get a third, and so on, using the formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad \text{if } f'(x_n) \neq 0.$$

4.7 Antiderivative

DEFINITION A function F is an **antiderivative** of f on an interval l if $F'(x) = f(x)$ for all x in I.

THEOREM If F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is

$$F(x) + C$$

where C is an arbitrary constant.

Antiderivative formulas, k a nonzero constant

Function	General antiderivative
1. x^n	$\frac{1}{n+1}x^{n+1} + C, \quad n \neq -1$
2. $\sin kx$	$-\frac{1}{k}\cos kx + C$
3. $\cos kx$	$\frac{1}{k}\sin kx + C$
4. $\sec^2 kx$	$\frac{1}{k}\tan kx + C$
5. $\csc^2 kx$	$-\frac{1}{k}\cot kx + C$
6. $\sec kx \tan kx$	$\frac{1}{k}\sec kx + C$
7. $\csc kx \cot kx$	$-\frac{1}{k}\csc kx + C$

Antiderivative linearity rules

	Function	General antiderivative
956 1. *Constant Multiple Rule:*	$kf(x)$	$kF(x) + C$, k a constant
2. *Negative Rule:*	$-f(x)$	$-F(x) + C$
3. *Sum or Difference Rule:*	$f(x) \pm g(x)$	$F(x) \pm G(x) + C$

Indefinite Integrals

DEFINITION The collection of all antiderivatives of f is called the **indefinite integral** of f with respect to x, and is denoted by

$$\int f(x)\, dx$$

The symbol $\int$ is an **integral sign**. The function f is the integrand of the integral, and x is the **variable of integration**.

examples

$$\int 2x\, dx = x^2 + C$$

$$\int \cos x\, dx = \sin x + C$$

$$\int (\sec^2 x + \frac{1}{x\sqrt{x}})\, dx = \tan x + \sqrt{x} + C$$

5 Integrals

5.1 Area and Estimating with Finite Sums

$$\text{SUM} = \sum_{i=1}^{n} f(c_i)\Delta x$$

5.2 Sigma Notation and Limits of Finite Sums

Finite Sums and Sigma Notation

Sigma notation enables us to write a sum with many terms in the compact form

$$\textbf{Sigma Notation}: \quad \sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + \cdots + a_{n-1} + a_n$$

Algebra Rules for Finite Sums

1. *Sum Rule:*	$\sum_{k=1}^{n}(a_k + b_k) + \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$
2. *Difference Rule:*	$\sum_{k=1}^{n}(a_k - b_K) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k$
3. *Constant Multiple Rule:*	$\sum_{k=1}^{n} ca_k = c \cdot \sum_{k=1}^{n} a_k$
4. *Constant Value Rule:*	$\sum_{k=1}^{n} c = n \cdot c$

The first n squares:	$\sum_{k=1}^{n} k^2 = \dfrac{n(n+1)(2n+1)}{6}$
The first n cubes:	$\sum_{k=1}^{n} k^3 = (\dfrac{n(n+1)}{2})^2$

Riemann Sums

Riemann sum for f on the interval $[\mathbf{a}, \mathbf{b}]$.

$$S_n = \sum_{k=1}^{n} f(c_k)\Delta x_k = \sum_{k=1}^{n} f(a + k\frac{(b-a)}{n}) \cdot (\frac{b-a}{n})$$

5.3 The Definite Integral

Definition of the Definite Integral Let $f(x)$ be a function defined on a closed interval $[a, b]$. We say that a number J is the **definite integral of f over** $[a, b]$ and that j is the limit of the Riemann sums $\sum_{k=1}^{n} f(c_k)\Delta x_k$ if the following condition is satisfied.
Given any number $\epsilon > 0$ there is a corresponding number $\delta > 0$ such that for every partition $P = x_0, x_1, \cdots, x_n$ of $[a, b]$ with $\|P\| < \delta$ and any choice of c_k in $[x_{k-1}, x_k]$, we have

$$|\sum_{k=1}^{n} f(c_k)\delta x_k - J| < \epsilon$$

$$J = \lim_{\|P\| \to 0} \sum_{k=1}^{n} f(c_k)\Delta x_k$$

$$\textbf{Symbol:} \quad \int_a^b f(x)dx$$

named **"Integral of f from a to b"**.

$$\int_a^b f(x)dx = \lim_{n \to \infty} \sum_{k=1}^{n} f(a + k\frac{(b-a)}{n}) \cdot (\frac{b-a}{n})$$

Integrable and Nonintegrable Functions

Integrability of Continuous Functions If a function f is continuous over the interval $[a, b]$, or if f has at most finitely many jump discontinuities there, then the definite integral $\int_a^b f(x)dx$ exists and f is integrable over $[a, b]$.

Properties of Definite Integrals

$$\int_b^a f(x)dx = -\int_a^b f(x)dx$$

$$\int_a^a f(x)dx = 0$$

Theorem When f and g are integrable over the interval $[a, b]$, the definite integral satisfies the rules below:

Rules satisfied by definite integrals

1. *Order of Integration:*	$\int_b^a f(x)dx = -\int_a^b f(x)dx$
2. *Zero Width Interval:*	$\int_a^a f(x)dx = 0$
3. *Constant Multiple:*	$\int_a^b kf(x)dx = k\int_a^b f(x)dx$
4. *Sum and Difference:*	$\int_a^b (f(x) \pm g(x))dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$
5. *Additivity:*	$\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$
6. *Max-Min Inequality:*	If f has maximum value max f and minimum value min f on $[a, b]$, then $\text{min}f \cdot (b-a) \leq \int_a^b f(x)dx \leq \text{max}f \cdot (b-a)$
7. *Domination:*	$f(x) \geq g(x)$ on $[a, b] \Rightarrow \int_a^b f(x)dx \leq \int_a^b g(x)dx$
	$f(x) \geq 0$ on $\Rightarrow \int_a^b f(x)dx \geq 0$

Area Under the Graph of a Nonnegative Funcion

Definition If $y = f(x)$ is nonnegative and integrable over a closed interval $[a, b]$, then the **area under the curve $\mathbf{y = f(x)}$ over $[\mathbf{a}, \mathbf{b}]$** is the integral of f from a to b.

$$A = \int_a^b f(x)dx$$

We have the following rules:

$$\begin{aligned} \int_a^b xdx &= \frac{b^2}{2} - \frac{a^2}{2}, \quad a < b \\ \int_a^b cdx &= c(b-a), \quad c \text{ any constant} \\ \int_a^b x^2dx &= \frac{b^3}{3} - \frac{a^3}{3}, \quad a < b \end{aligned} \tag{17}$$

Average Value of a Continuous Function Revisited

$$\text{av}(f) = \frac{1}{b-a}\int_a^b f(x)dx$$

is f's **average value on $[\mathbf{a}, \mathbf{b}]$**, also called it's **mean**.

5.4 The Fundamental Theorem of Calculus

Mean Value Theorem for Definite Integrals If f is continuous on $[a, b]$, then at some point c in $[a, b]$,

$$f(c) = \frac{1}{b-a}\int_a^b f(x)dx$$

Fundamental Theorem, Part 1

If $f(t)$ is an integrable function over a finite interval I, then the integral from any fixed number $a \in I$ to another number $x \in I$ defines a new function F whose value at x is

$$F(x) = \int_a^x f(t)dt$$

For example, if f is nonnegative and x lies to the right of a, then $F(x)$ is the area under the graph from a to x. The variable x is the upper limit of integration of an integral, but F is just like any other real-valued function of a real variable. For each value of the input x, there is a well-defined numerical output, in this case the definite integral of f from a to x.

This equation gives a way to define new functions, but its importance now is the connection it makes between integrals and derivatives. If f is any continuous function, then the Fundamental Theorem asserts that F is a differentiable function of x whose derivative is f itself. At every value of x, it asserts that

$$\frac{d}{dx}F(x) = f(x).$$

To gain some insight into why this result holds, we look at the geometry behind it.

If $f \geq 0$ on $[a, b]$, then the computation of $F'(x)$ from the definition of teh derivative means taking the limit as $h \to 0$ of the difference quotient

$$\frac{F(x+h) - F(x)}{h}$$

For $h > 0$, the numerator is obtained by subtracting two areas, so it is the area under the graph of f from x to $x + h$. If h is small, this area is approximately equal to the area of the rectangle of height $f(x)$ and width h. That is

$$F(x+h) - F(x) \approx hf(x)$$

Dividing both sides of this approximation by h and letting $h \to 0$, it is reasonable to expect that

$$F'(x) = \lim_{h\to 0}\frac{F(x+h) - F(x)}{h} = f(x)$$

This result is true even if the function f is not positive, and it forms the first part of the Fundamental Theorem of Calculus. **EXAMPLE:Find dy/dx if**

$$y = \int_{1+3x^2}^{4} \frac{1}{2+t}dt$$

$$\text{let } u = 1 + 3x^2$$

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\
&= \frac{d}{du}\int_u^4 \frac{1}{2+t}dt \cdot (6x) \\
&= -\frac{d}{du}\int_4^u \frac{1}{2+t}dt \cdot (6x) \\
&= -\frac{1}{2+u} \cdot (6x) \\
&= -\frac{2x}{x^2+1}
\end{aligned} \tag{18}
$$

Proof of Theorem

$$
\begin{aligned}
F'(x) &= \lim_{h\to 0} \frac{F(x+h) - F(x)}{h} \\
&= \lim_{h\to 0} \frac{1}{h}[\int_0^{x+h} f(t)dt - \int_a^x f(t)dt] \\
&= \lim_{h\to 0} \frac{1}{h}\int_x^{x+h} f(t)dt
\end{aligned} \tag{19}
$$

According to the Mean Value Theorem for Definite Integrals, the value before taking the limit in the last expression is one of the values taken on by f in the interval between x and $x+h$. That is, for some numer c in this interval,

$$
\frac{1}{h}\int_x^{x+h} f(t)dt = f(c) \tag{20}
$$

As $h \to 0$, $x+h$ approaches x, forcing c to approach x also (because c is trapped between x and $x+h$). Since f is continuous at x, f(c) approaches $f(x)$

$$
\lim_{h\to 0} f(c) = f(x) \tag{21}
$$

In conclusion, we have

$$
\begin{aligned}
F'(x) &= \lim_{h\to 0} \frac{1}{h}\int_x^{x+h} f(t)dt \\
&= \lim_{h\to 0} f(c) \\
&= f(x)
\end{aligned} \tag{22}
$$

If $x = a$ or b, then the limit of Equation is interpreted as a one-sided limit with $h \to 0^+$ or $h \to 0^-$, respectively.

Fundamental Theorem, Part 2(The Evaluation Theorem)

The Fundamental Theorem of Calculus, Part 2 If f is continuous over $[a, b]$ and F is any antiderivative of f on $[a, b]$, then

$$
\int_a^b f(x)dx = F(b) - F(a) = F(x)\Big]_a^b = \Big[F(x)\Big]_a^b
$$

Example:

$$\begin{aligned}\int_1^4 (\frac{3}{2}\sqrt{x} - \frac{4}{x^2})dx &= \left[x^{3/2} + \frac{4}{x}\right]_1^4 \\ &= [8+1] - [5] \\ &= 4\end{aligned} \tag{23}$$

The Integral of a Rate

$$\int_a^b F'(x)dx = F(b) - F(a)$$

$$F(b) = F(a) + \int_a^b F'(x)dx$$

The Relationship Between Integration and Differrentiation

$$\frac{d}{dx}\int_a^x f(t)dt = f(x)$$

5.5 Indefinite Integrals and the Substitution Method

The indefinite integral $\int$ notation means for any antiderivative F of f,

$$\int f(x)dx = F(x) + C$$

where C is an arbitrary constant.

Substitution: Running the Chain Rule Backwards If u is a differentiable function of x and n is an number different form -1, the Chain Rule tells us that

$$\frac{d}{dx}(\frac{u^{n+1}}{n+1}) = u^n \frac{du}{dx}$$

From another point of view, this same equation says that $u^{n+1}/(n+1)$ is one of the antiderivatives of the function $u^n(du/dx)$. Therefore,

$$\int u^n \frac{du}{dx}dx = \frac{u^{n+1}}{n+1} + C$$

The integral in the equation is equal to the simpler integral

$$\int u^n du = \frac{u^{n+1}}{n+1} + C$$

$$du = \frac{du}{dx}dx$$

Example 1 Find the integral $\int (x^3+x)^5(3x^2+1)dx$

Solution We set $u = x^3 + x$. Then

$$du = \frac{du}{dx}dx = (3x^2 + 1)dx$$

so that by Substitution we have

$$\begin{aligned} \int (x^3 + x)^5 (3x^2 + 1) dx &= \int u^5 du \\ &= \frac{u^6}{6} + C \\ &= \frac{(x^3 + x)^6}{6} + C \end{aligned} \tag{24}$$

Example 2 Find $\int \sqrt{2x + 1}dx$

Solution

$$\begin{aligned} \int \sqrt{2x + 1}dx &= \frac{1}{2}\int \sqrt{2x + 1} \cdot 2dx \\ &= \frac{1}{2}\int u^{\frac{1}{2}} du \\ &= \frac{1}{2}\frac{u^{3/2}}{3/2} + C \\ &= \frac{1}{2}(2x + 1)^{\frac{3}{2}} + C \end{aligned} \tag{25}$$

The Substitution Rule If $u = g(x)$ is a differentiable function whose range is an interval I, and f is continuous on I, then

$$\int f(g(x))g'(x)dx = \int f(u)du$$

Example 3 Find $\int \sec^2(5x + 1) \cdot 5dx$

Solution We substitute $u = 5x + 1$ and $du = 5dx$. Then

$$\begin{aligned} \int \sec^2(5x + 1) \cdot 5dx &= \int \sec^2 u \ du \\ &= \tan u + C \\ &= \tan(5x + 1) + C \end{aligned} \tag{26}$$

5.6 Definite Integral Substitutions and the Area Between Curves

The Substitution Formula

Substitution in Definite Integrals If g' is continuous on the interval $[a, b]$ and f is continuous on the range of $g(x) = u$, then

$$\int_a^b f(g(x)) \cdot g'(x)dx = \int_g (a)^g (b) f(u)du$$

Proof Let F denote any antiderivative of f. Then,

$$\begin{aligned}
\int_a^b f(g(x)) \cdot g'(x)dx &= F(g(x))\Big]_{x=a}^{x=b} \\
&= F(g(b)) - F(g(a)) \\
&= F(u)\Big]_{u=g(a)}^{u=g(b)} \\
&= \int_{g(a)}^{g(b)} f(u)du
\end{aligned} \tag{27}$$

Example 1 Evaluate $\int_{-1}^{1} 3x^2\sqrt{x^3+1}dx$

Solution

$$\begin{aligned}
\int_{-1}^{1} 3x^2\sqrt{x^3+1}dx &= \int_0^2 \sqrt{u}du \\
&= \frac{2}{3}u^{3/2}\Big]_0^2 \\
&= \frac{2}{3}[2^{3/2} - 0^{3/2}] \\
&= \frac{2}{3}[2\sqrt{2}] \\
&= \frac{4\sqrt{2}}{3}
\end{aligned} \tag{28}$$

Definite Integrals of Symmetric Functions

Theorem Let f be continuous on the symmetric interval $[-a, a]$.
(a) If f is even, then $\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$
(b) If f is odd, then $\int_{-a}^{a} f(x)dx = 0$

Areas Between Curves

Definition If f and g are continuous with $f(x) \geq g(x)$ throughout $[a, b]$, then the **area of the region between the curves $\mathbf{y = f(x)}$ and $\mathbf{y = g(x)}$ from $(\mathbf{a})$ to $\mathbf{b}$** is the integral of $(f - g)$ from a to b:

$$A = \int_a^b [f(x) - g(x)]dx$$

Integration with Respect to y

$$A = \int_c^d [f(y) - g(y)]dy$$

6 Applications of Definite Integrals

6.1 Volumes Using Cross-Sections

Definition The **volume** of a solid of integrable cross-sectional area $A(x)$ from $x = a$ to $x = b$ is the integral of A from a to b.

$$V = \int_a^b A(x)dx$$

Calculating the Volume of a Solid

1. *Sketch the solid and a typical cross-section.*
2. *Find a formula for $A(x)$, the area of a typical cross-section.*
3. *Find the limits of integration.*
4. *Integrate $A(x)$ to find the volume.*

Solid of Revolution: The Disk Method The solid generated by rotating (or revolving) a plane region about an axis in its plane is called a **solid of revolution**.

$$A(x) = \pi(\text{radius})^2 = \pi[R(x)]^2$$

Volume by Disks for Rotation About the x-axis

$$V = \int_a^b A(x)\ dx = \int_a^b \pi[R(x)]^2\ dx$$

Example Find the volume of the solid generated by revolving the region bounded by $y = \sqrt{x}$ and the lines $y = 1$, $x = 4$ about the line $y = 1$.

Solution

$$\begin{aligned} V &= \int_1^4 \pi[R(x)]^2\ dx \\ &= \int_1^4 \pi[\sqrt{x} - 1]^2\ dx \\ &= \pi \int_1^4 [x - 2\sqrt{x} + 1]dx \\ &= \pi \left[\frac{x^2}{2} - 2 \cdot \frac{2}{3} x^{3/2} + x \right]_1^4 \\ &= \frac{7\pi}{6} \end{aligned} \tag{29}$$

Volume by Disks for Rotation About the y-axis

$$V = \int_c^d A(y)\ dy = \int_c^d \pi[R(y)]^2\ dy$$

Solids of Revolution: The Washer Method

Volume by Washers for Rotation About the x-axis

$$V = \int_a^b A(x)\ dx = \int_a^b \pi([R(x)]^2 - [r(x)]^2)dx$$

6.2 Volumes Using Cylindrical Shells

Slicing with Cylinders Unrolling a cylindrical shell shows that its volume is approximately that of a rectangular slab with area $A(x)$ and thickness Δx.

The Shell Method

$$\begin{aligned}
\Delta V_k &= 2\pi \times \text{average shell radius} \times \text{shell height} \times \text{thickness} \\
&= 2\pi \cdot (c_k - L) \cdot f(c_k) \cdot \Delta x_k \\
V &= \lim_{n\to\infty} \sum_{k=1}^{n} \Delta V_k \\
&= \int_a^b 2\pi(\text{shell radius})(\text{shell height})dx \\
&= \int_a^b 2\pi(x-L)f(x)\ dx
\end{aligned} \tag{30}$$

6.3 Arc Length

Length of a Curve $y = f(x)$ Suppose the curve whose length we want to find is the graph of the function $y = f(x)$ from $x = a$ to $x = b$. In order to derive an integral formula for the length of the curve, we assume that f has a continuous derivative at every point of every point of $[a, b]$. Such a function is called **smooth**, and its graph is a **smooth curve** because it does not have any breaks, corners, or cusps.

$$L_k = \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$$

$$\sum_{k=1}^{n} L_k = \sum_{k=1}^{n} \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$$

$$\Delta y_k = f'(c_k)\Delta x_k$$

$$\sum_{k=1}^{n} L_k = \sum_{k=1}^{n} \sqrt{(\Delta x_k)^2 + (f'(c_k)\Delta x_k)^2} = \sum_{k=1}^{n} \sqrt{1 + [f'(c_k)]^2}\Delta x_k$$

$$\lim_{n\to\infty} \sum_{k=1}^{n} L_k = \lim_{n\to\infty} \sum_{k=1}^{n} \sqrt{1 + [f'(c_k)]^2}\Delta x_k = \int_a^b \sqrt{1 + [f'(x)]^2}dx$$

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}dx = \int_a^b \sqrt{1 + (\frac{dy}{dx})^2}dx$$

Dealing with Discontinuities in dy/dx

Formula for the Length of $x = g(y),\ c \le y \le d$

$$L = \int_c^d \sqrt{1 + [g'(y)]^2}dy = \int_c^d \sqrt{1 + (\frac{dx}{dy})^2}dy$$

The Differential Formula for Arc Length

If $y = f(x)$ and if f' is continuous on $[a, b]$, then by the Fundamental Theorem of Calculus we can define a new function

$$s(x) = \int_a^x \sqrt{1 + [f'(t)]^2}\ dt$$

$$\frac{ds}{dx} = \sqrt{1 + [f'(x)]^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx$$

$$ds = \sqrt{dx^2 + dy^2}$$

6.4 Areas of Surfaces of Revolution

Defining Surface Area

$$\begin{aligned}\text{Frustum surface area} &= 2\pi \cdot \frac{f(x_{k-1}) + f(x_k)}{2} \cdot \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2} \\ &= \pi(f(x_{k-1}) + f(x_k))\sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}\end{aligned} \tag{31}$$

Definition If the function $f(x) \geq 0$ is continuously differentiable on $[a, b]$, the **area of the surface** generated by revolving the graph of $y = f(x)$ about the x-axis is

$$S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \int_a^b 2\pi f(x) \sqrt{1 + (f'(x))^2}\ dx$$

Revolution About the y-Axis

$$S = \int_c^d 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy = \int_c^d 2\pi g(y) \sqrt{1 + (g'(x))^2}\ dy$$

6.5 Work and Wfuid Forces

Work Done by a Constant Force

$$W = Fd \qquad \text{(Constant-force formula for work.)}$$

Work Done by a Variable Force Along a Line

$$W = \int_a^b F(x) dx$$

Hooke's Law for Springs:$F = kx$

Hooke's Law The force required to hold a stretched or compressed spring x units from its nature (unstressed) length is proportional to x. In symbol

$$F = kx$$

Lifting Objects and Pumping Liquidsfrom Containers

Fluid Pressure and Forces

The Pressure-Depth Equation

$$p == wh$$

$$F = pA = whA$$

The Integral for Fluid Force Against a Vertical Flat Plate

$$F = \int_a^b w \cdot (\text{strip depth}) \cdot L(y)\ dy$$

6.6 Moments and Centers of Mass

Masses Along a Line

$$\textbf{System Torque} = \sum_{k=1}^{n} m_k g x_k$$

$$M_0 = \text{Moment of system about origin} = \sum_{k=1}^{n} m_k x_k$$

$$\begin{aligned} \sum (x_k - \overline{x}) m_k g &= 0 \\ \overline{x} &= \frac{\sum m_k x_k}{\sum m_k} \\ &= \frac{\text{System moment about origin}}{\text{System mass}} \end{aligned} \tag{32}$$

Masses Distributed over a Plane Region

System mass: $M = \sum m_k$

Moment about x-axis: $M_x = \sum m_k y_k$

Moment about y-axis: $M_y = \sum m_k x_k$

$$\overline{x} = \frac{M_y}{M} = \frac{\sum m_k x_k}{\sum m_k}$$

$$\overline{y} = \frac{M_x}{M} = \frac{\sum m_k y_k}{\sum m_k}$$

Thin, Flat Plates

Moments, Mass and Center of Mass of a Thin Plate Covering a Region in the xy-Plane

Moment about the x-axis:	$M_x = \int \tilde{y} dm$
Moment about the y-axis:	$M_y = \int \tilde{x} dm$
Mass:	$M = \int dm$
Center of mass:	$\overline{x} = \frac{M_y}{M}, \overline{y} = \frac{M_x}{M}$

Plates Bounded by Two Curves

$$\overline{x} = \frac{1}{M}\int_a^b \delta x[f(x) - g(x)]dx$$

$$\overline{y} = \frac{1}{M}\int_a^b \frac{\delta}{2}[f^2(x) - g^2(x)]dx$$

Centroids

Fluid Forces and Centroids

$$F = w\overline{h}A$$

The Theorems of Pappus

Pappus's Theorem for Volumes If a plane region is revolved once about a line in the plane that does not cut through the regions interior, then the volume of the solid it generates is equal to the regions area times the distance traveled by the regions centroid during the revolution. If ρ is the distance from the axis of revolution to the centroid, then

Proof

$$V = \int_c^d 2\pi(\text{shell radius})(\text{shell height})\ dy = 2\pi\int_c^d yL(y)dy$$

$$\overline{y} = \frac{\int_c^d \widetilde{y}\ dA}{A} = \frac{\int_c^d yL(y)\ dy}{A}$$

$$\int_c^d yL(y)\ dy = A\overline{y}$$

Pappus's Theorem for Surface Areas If an arc of a smooth plane curve is revolved once about a line in the plane that does not cut through the arcs interior, then the area of the surface generated by the arc equals the length L of the arc times the distance traveled by the arcs centroid during the revolution. If ρ is the distance from the axis of revolution to the centroid, then

$$S = 2\pi\rho L$$

7 Transcendental Functions

7.1 Inverse Functions and Their Derivatives

One-to-One Functions A function $f(x)$ is **One-to-One** on a domain D if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$ in D.

Inverse Functions Suppose that f is a one-to-one function on a domain D with range R. The **Inverse function** f^{-1} is defined by

$$f^{-1}(b) = a \quad \text{if} \quad f(a) = b$$

The domain of f^{-1} is R and the range of f^{-1} is D.

$$\begin{aligned} (f^{-1} \circ f)(x) &= x, \quad \text{for all } x \text{ in the domain of } f \\ (f \circ f^{-1})(y) &= x, \quad \text{for all } y \text{ in the domain of } f^{-1} \end{aligned} \tag{33}$$

Finding Inverses

Derivatives of Inverses of Differentiable Functions

The Derivative Rule for Inverses If f has an interval I as domain and $f'(x)$ exists and is never zero on I, then f^{-1} is differentiable at every point in its domain (the range of f). The value of $(f^{-1})'$ at a point b in the domain of f^{-1} is the reciprocal of the value of f' at the point $a = f^{-1}(b)$:

$$(f^{-1})'(b) = \frac{1}{f'(f^{-1}(b))}$$

or

$$\left.\frac{df^{-1}}{dx}\right|_{x=b} = \frac{1}{\left.\frac{df}{dx}\right|_{x=f^{-1}(b)}}$$

Proof

$$\begin{aligned} f(f^{-1}(x)) &= x \\ \frac{d}{dx} f(f^{-1}(x)) &= 1 \\ \frac{d}{dx} f^{-1}(x) \cdot f'(f^{-1}(x)) &= 1 \\ \frac{d}{dx} f^{-1}(x) &= \frac{1}{f'(f^{-1}(x))} \end{aligned} \tag{34}$$

7.2 Natural Logarithms

Definition of the Natural Logarithm Function The **natural logarithm** is the function given by

$$\ln x = \int_1^x \frac{1}{t} dt, \quad x > 0$$

so $\ln 1 = \int_1^1 \frac{1}{t} dt = 0$

Definition The **number e** is that number in the domain of the natural logarithm satisfying

$$\ln(e) = \int_1^e \frac{1}{t}dt = 1$$

$$e \approx 2.71828$$

The Derivative of $y = \ln x$

$$\frac{d}{dx}\ln x = \frac{d}{dx}\int_1^x \frac{1}{t}dt = \frac{1}{x}$$

$$\frac{d}{dx}\ln x = \frac{1}{x}$$

$$\frac{d}{dx}\ln u = \frac{1}{u}\cdot\frac{du}{dx}, \quad u > 0 \tag{35}$$

Properties of Logarithms

Algebraic Properties of the Nature Logarithm For any numbers $b > 0$ and $x > 0$, the natural logarithm satisfies the following rules:

1. *Product Rule:*	$\ln bx = \ln b + \ln x$
2. *Quotient Rule:*	$\ln \frac{b}{x} = \ln b - \ln x$
3. *Reciprocal Rule:*	$\ln \frac{1}{x} = -\ln x$
4. *Power Rule:*	$\ln x^r = r\ln x$

The Graph and Range of $\ln x$

The Integral $\int(1/u)du$

$$\int \frac{1}{u}du = \ln|u| + C$$

If $u = f(x)$, then $du = f'(x)$ and

$$\int \frac{f'(x)}{f(x)}dx = \ln|f(x)| + C$$

whenever $f(x)$ is a differentiable function that is never zero.

The Integrals of $\tan x$, $\cot x$, $\sec x$, and $\csc x$

$$\begin{aligned}
\int \tan x \; dx &= \int \frac{\sin x}{\cos x} dx = \int \frac{-\,du}{u} \\
&= -\ln|u| + C = -\ln|\cos x| + C \\
&= \ln \frac{1}{|\cos x|} + C = \ln|\sec x| + C \\
\int \cot x \; dx &= \int \frac{\cos x dx}{\sin x} = \int \frac{du}{u} \\
&= \ln|u| + C = \ln|\sin x| + C = -\ln|\csc x| + C \\
\int \sec x \; dx &= \int \sec x \frac{(\sec x + \tan x)}{(\sec x + \tan x)} dx \\
&= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} dx \\
&= \int \frac{du}{u} = \ln|u| + C \\
&= \ln|\sec x + \tan x| + C \\
\int \csc x \; dx &= \int \csc x \frac{(\csc x + \cot x)}{(\csc x + \cot x)} dx \\
&= \int \frac{\csc^2 x + \csc x \cot x}{\csc x + \cot x} dx \\
&= \int \frac{-\,du}{u} = -\ln|u| + C \\
&= -\ln|\csc x + \cot x| + C
\end{aligned} \tag{36}$$

Logarithmic Differentiation Use laws of logarithms to simplify the formulas before differentiating.

7.3 Exponential Function

The Inverse of $\ln x$ and the Number e

Definition For every real number x, we define the **natural exponential function** to be $e^x = \exp x$.

Inverse Equations for e^x and $\ln x$

$$\begin{aligned}
e^{\ln x} &= x \qquad (\text{all } x > 0) \\
\ln(e^x) &= x \qquad (\text{all } x)
\end{aligned} \tag{37}$$

The Derivative and Integral of e^x

$$\begin{aligned}
\ln(e^x) &= x \\
\frac{d}{dx}\ln(e^x) &= 1 \\
\frac{1}{e^x}\cdot\frac{d}{dx}(e^x) &= 1 \\
\frac{d}{dx}e^x &= e^x \\
\frac{d}{dx}e^u &= e^u\frac{du}{dx} \\
\int e^u du &= e^u + C \\
\lim_{x\to-\infty} e^x &= 0 \\
\lim_{x\to\infty} e^x &= \infty
\end{aligned} \tag{38}$$

Laws of Exponents

$$\begin{aligned}
e^{x_1}\cdot e^{x_2} &= e^{x_1+x_2} \\
e^{-x} &= \frac{1}{e^x} \\
\frac{e^{x_1}}{e^{x_2}} &= e^{x_1-x_2} \\
(e^{x_1})^r &= e^{rx_1}, \quad \text{if } f \text{ is rational}
\end{aligned}$$

The General Exponential Function a^x

Definition For any number $a > 0$ and x, the **exponential function with base a** is

$$a^x = e^{x\ln a}$$

Proof of the Power Rule(General Version)

Definition For any $x > 0$ and for any real number n,

$$x^n = e^{n\ln x}$$

General Power Rule for Derivatives

For $x > 0$ and any real number n.

$$\frac{d}{dx}x^n = nx^{n-1}$$

If $x < 0$, then the formula holds whenever the derivative, x^n, and x^{n-1} all exist.

Proof Differentiating x^n with respect to x gives

$$\begin{aligned}\frac{d}{dx}x^n &= \frac{d}{dx}e^{n\ln x} \\ &= e^{n\ln x}\cdot\frac{d}{dx}(n\ln x) \\ &= x^n\cdot\frac{n}{x} \\ &= nx^{n-1}\end{aligned} \tag{39}$$

Example Differentiate $f(x) = x^x, x > 0$

$$\begin{aligned}f'(x) &= \frac{d}{dx}(e^{x\ln x}) \\ &= e^{x\ln x}\frac{d}{dx}(x\ln x) \\ &= e^{x\ln x}\cdot(\ln x + x\cdot\frac{1}{x}) \\ &= x^x(\ln x + 1)\end{aligned} \tag{40}$$

The Number e Expressed as a Limit

The Number e as a Limit

$$e = \lim_{x\to 0}(1+x)^{1/x}$$

Proof If $f(x) = \ln x$, then $f'(x) = 1/x$, so $f'(1) = 1$. But, by the definition of derivative.

$$\begin{aligned}f'(1) &= \lim_{h\to 0}\frac{f(1+h)-f(1)}{h} \\ &= \lim_{x\to 0}\frac{f(1+x)-f(1)}{x} \\ &= \lim_{x\to 0}\frac{\ln(1+x)-\ln 1}{x} \\ &= \lim_{x\to 0}\frac{1}{x}\ln(1+x) \\ &= \lim_{x\to 0}\ln(1+x)^{1/x} \\ &= \ln\left[\lim_{x\to 0}(1+x)^{1/x}\right] = 1 \\ &\lim_{x\to 0}(1+x)^{1/x} = e\end{aligned} \tag{41}$$

The Derivative of a^u

$$\frac{d}{dx}a^x = \frac{d}{dx}e^{x\ln a} = e^{x\ln a}\cdot\frac{d}{dx}(x\ln a) = a^x\ln a$$

$$\frac{d}{dx}a^u = a^u\ln a\frac{du}{dx}$$

$$\int a^u du = \frac{a^u}{\ln a} + C$$

$$\frac{d^2}{dx^2}(a^x) = \frac{d}{dx}(a^x \ln a) = (\ln a)^2 a^x$$

Logarithms with Base a

Definition For any positive number $a \neq 1$, $\log_a x$ is the inverse function of a^x.

$$a^{\log_a x} = x \quad (x > 0)$$

$$log_a(a^x) = x \quad (\text{all } x)$$

$$\log_a x = \frac{\ln x}{\ln a}$$

$$\ln xy = \ln x + \ln y$$

$$\frac{\ln xy}{\ln a} = \frac{\ln x}{\ln a} + \frac{\ln y}{\ln a}$$

$$\log_a xy = \log_a x + \log_a y$$

Derivatives and Integrals Involving $log_a x$

$$\frac{d}{dx}(\log_a u) = \frac{d}{dx}\left(\frac{\ln u}{\ln a}\right) = \frac{1}{\ln a}\frac{d}{dx}(\ln u) = \frac{1}{\ln a} \cdot \frac{1}{u}\frac{du}{dx}$$

7.4 Exponential Change and Separable Differential Equations

Exponential Change

$$\frac{dy}{dt} = ky, \qquad y(0) = y_0$$

$$y = y_0 e^{kt}$$

Separable Differential Equations

$$\frac{dy}{dx} = f(x, y)$$

$$\frac{d}{dx}y(x) = f(x, y(x))$$

$$\frac{dy}{dx} = g(x)H(y)$$

$$\frac{dy}{dx} = \frac{g(x)}{h(y)}$$

$$h(y)dy = g(x)dx$$

$$\int h(y)dy = \int g(x)dx$$

$$\begin{aligned} \int h(y)\ dy &= \int h(y(x))\frac{dy}{dx}dx \\ &= \int h(y(x))\frac{g(x)}{h(y(x))}dx \\ &= \int g(x)\ dx \end{aligned} \tag{42}$$

Unlimited Population Growth

Radioactivity

$$\text{Half-life} = \frac{\ln 2}{k}$$

Heat Transfer: Newton's Law of Cooling

$$\frac{dH}{dt} = -k(H - H_s)$$

$$\begin{aligned}\frac{dy}{dt} = \frac{d}{dt}(H - H_s) &= \frac{dH}{dt} - \frac{d}{dt}(H_s) \\ &= \frac{dH}{dt} \\ &= -k(H - H_s) \\ &= -ky\end{aligned}$$

7.5 Indeterminate Forms and L'Hôpital's Rule

Indeterminate Form $0/0$ Suppose that $f(a) = g(a) = 0$, that f and g are differentiable on an open interval I containing a, and that $g'(x) \neq 0$ on I if $x \neq a$. Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

assuming that the limit on the right side of this equation exists.

Indeterminate Forms ∞/∞, $\infty/0$, $\infty \cdot 0$, $\infty - \infty$ $x \to a$ may be replaced by the one-sided limits $x \to a^+$ or $x \to a^-$

Indeterminated Power

If $\lim\limits_{x \to a}$ in $f(x) = L$ then

$$\lim_{x \to a} f(x) = \lim_{x \to a} e^{\ln f(x)} = e^L$$

Here a may be either finite or infinite.

Example Apply l'hopital's Rule to show that $\lim\limits_{x \to 0^+} (1 + x)^{1/x} = e$

Solution

$$\ln f(x) = \ln(1 + x)^{1/x} = \frac{1}{x}\ln(1 + x)$$

$$\begin{aligned}\lim_{x \to 0^+} \ln f(x) &= \lim_{x \to 0^+} \frac{\ln(1 + x)}{x} \\ &= \lim_{x \to 0^+} \frac{\frac{1}{1 + x}}{1} \\ &= 1\end{aligned} \tag{43}$$

Proof of L'Hôpital's Rule

$$\begin{aligned}\lim_{x\to a}\frac{f(x)}{g(x)} &= \lim_{x\to a}\frac{f'(a)(x-a)+\epsilon_1(x-a)}{g'(a)(x-a)+\epsilon_2(x-a)}\\ &= \lim_{x\to a}\frac{f'(a)+\epsilon_1}{g'(a)+\epsilon_2}\\ &= \frac{f'(a)}{g'(a)}\\ &= \lim_{x\to a}\frac{f'(x)}{g'(x)}\end{aligned}\tag{44}$$

Cauchy's Mean Value Theorem Suppose functions f and g are continuous on [a,b] and differentiable throughout (a, b) and also suppose $g'(x) \neq 0$ throughout (a, b). Then there exists a number c in (a, b) at which

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

Proof

$$g'(c) = \frac{g(b)-g(a)}{b-a} = 0$$

$$F(x) = f(x) - f(a) - \frac{f(b)-f(a)}{g(b)-g(a)}[g(x)-g(a)]$$

$$F'(c) = f'(c) - \frac{f(b)-f(a)}{g(b)-g(a)}[g'(c)] = 0$$

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

Proof of L'Hôpital's Rule

$$\frac{f'(c)}{g'(c)} = \frac{f(x)-f(a)}{g(x)-g(a)}$$

$$\frac{f'(c)}{g'(c)} = \frac{f(x)}{g(x)}$$

$$\lim_{x\to a^+}\frac{f(x)}{g(x)} = \lim_{c\to a^+}\frac{f'(c)}{g'(c)} = \lim_{x\to a^+}\frac{f'(x)}{g'(x)}$$

7.6 Inverse Trigonometric Functions

Defining the Inverses

The Arcsine and Arccosine Functions

Identities Involving Arcsine and Arccosine

$$\cos^{-1}x + \cos^{-1}(-x) = \pi$$

$$\sin^{=1}x + \cos^{-1}x = \pi/2$$

Inverses of $\tan x$, $\cot x$, $\sec x$, and $\csc x$

$$\sec^{-1} x = \cos^{-1}\left(\frac{1}{x}\right) = \frac{\pi}{2} - \sin^{-1}\left(\frac{1}{x}\right)$$

The Derivative of $y = \sin^{-1} u$

$$\begin{aligned}
(f^{-1})'(x) &= \frac{1}{f'(f^{-1}(x))} \\
&= \frac{1}{\cos(\sin^{-1} x)} \\
&= \frac{1}{\sqrt{1 - \sin^2(\sin^{-1} x)}} \\
&= \frac{1}{\sqrt{1 - x^2}} \\
\frac{d}{dx}(sin^{-1}u) &= \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}, \quad |u| < 1
\end{aligned} \tag{45}$$

The Derivative of $y = \tan^{-1} u$

$$\begin{aligned}
(f^{-1})'(x) &= \frac{1}{f'(f^{-1}(x))} \\
&= \frac{1}{\sec^2(\tan^{-1} x)} \\
&= \frac{1}{1 + \tan^2(\tan^{-1} x)} \\
&= \frac{1}{\sqrt{1 + x^2}} \\
\frac{d}{dx}(sin^{-1}u) &= \frac{1}{\sqrt{1+u^2}}\frac{du}{dx}, \quad |u| < 1
\end{aligned} \tag{46}$$

The Derivative of $y = \sec^{-1} u$

$$\begin{aligned}
y &= \sec^{-1} x \\
\sec y &= x \\
\frac{d}{dx}(\sec y) &= \frac{d}{dx}x \\
\sec y \tan y \frac{dy}{dx} &= 1 \\
\frac{dy}{dx} &= \frac{1}{\sec y \tan y} \\
\frac{dy}{dx} &= \pm\frac{1}{x\sqrt{x^2-1}} \\
\frac{d}{dx}\sec^{-1} x &= \begin{cases} +\dfrac{1}{x\sqrt{x^2-1}} & \text{if} x > 1 \\ -\dfrac{1}{x\sqrt{x^2-1}} & \text{if} x < -1 \end{cases}
\end{aligned} \tag{47}$$

$$\frac{d}{dx}\sec^{-1}x = \frac{1}{|x|\sqrt{x^2-1}}$$
$$\frac{d}{dx}\sec^{-1}u = \frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx}$$

Inverse Function-Inverse Confunction Identities

$$\cos^{-1}x = \pi/2 - \sin^{-1}x$$
$$\cot^{-1}x = \pi/2 - \tan^{-1}x$$
$$\csc^{-1}x = \pi/2 - \sec^{-1}x$$

Integration Formulas

$$\begin{aligned}\int \frac{du}{\sqrt{a^2-u^2}} &= \sin^{-1}\left(\frac{u}{a}\right) + C\\ \int \frac{du}{a^2+u^2} &= \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C\\ \int \frac{du}{u\sqrt{u^2-a^2}} &= \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right| + C\end{aligned} \tag{48}$$

7.7 Hyperbolic Functions

Definitions and Identities

$$\begin{aligned}\sinh x &= \frac{e^x - e^{-x}}{2}\\ \cosh x &= \frac{e^x + e^{-x}}{2}\\ 2\sinh x\cosh x &= 2\left(\frac{e^x - e^{-x}}{2}\right)\left(\frac{e^x + e^{-x}}{2}\right)\\ &= \frac{e^{2x} - e^{-2x}}{2}\\ &= \sinh 2x\\ \tanh x &= \frac{e^x + e^{-x}}{e^x - e^{-x}}\\ \coth x &= \frac{e^x - e^{-x}}{e^x + e^{-x}}\\ \mathrm{sech} x &= \frac{2}{e^x + e^{-x}}\\ \mathrm{csch} x &= \frac{2}{e^x - e^{-x}}\\ \cosh^2 x - \sinh^2 x &= 1\\ \cosh 2x &= \cosh^2 x + \sinh^2 x\\ \sinh 2x &= 2\sinh x\cosh x\\ \cosh^2 x &= \frac{\cosh 2x + 1}{2}\\ \sinh^2 x &= \frac{\cosh 2x - 1}{2}\\ \tanh^2 x &= 1 - \mathrm{sech}^2 x\\ \coth^2 x &= 1 + \mathrm{csch}^2 x\end{aligned} \tag{49}$$

Derivatives and Integrals of Hyperbolic Functions

$$\frac{d}{dx}(\sinh u) = \cosh u \frac{du}{dx}$$

$$\frac{d}{dx}(\cosh u) = \sinh u \frac{du}{dx}$$

$$\frac{d}{dx}(\tanh u) = \text{sech}^2 u \frac{du}{dx}$$

$$\frac{d}{dx}(\coth u) = -\text{csch}^2 u \frac{du}{dx}$$

$$\frac{d}{dx}(\text{csch } u) = -\text{sech } u \tanh u \frac{du}{dx}$$

$$\frac{d}{dx}(\text{csch } u) = -\text{csch } u \coth u \frac{du}{dx}$$

$$\int \sinh u \; du = \cosh u + C$$

$$\int \cosh u \; du = \sinh u + C$$

$$\int \text{sech}^2 u \; du = \tanh u + C$$

$$\int \text{csch}^2 u \; du = -\coth u + C$$

$$\int \text{sech } u \tanh u \; du = -\text{sech } u + C$$

$$\int \text{csch } u \coth u \; du = -\text{csch } u + C$$

Inverse Hyperbolic Function

$$\text{sech}^{-1} = \cosh^{-1} \frac{1}{x}$$

$$\text{csch}^{-1} = \sinh^{-1} \frac{1}{x}$$

$$\coth^{-1} = \tanh^{-1} \frac{1}{x}$$

Derivatives of Inverse Hyperbolic Functions

$$\frac{d(\sinh^{-1} u)}{dx} = \frac{1}{\sqrt{1+u^2}} \frac{du}{dx}$$

$$\frac{d(\cosh^{-1} u)}{dx} = \frac{1}{\sqrt{1-u^2}} \frac{du}{dx} \quad u > 1$$

$$\frac{d(\tanh^{-1} u)}{dx} = \frac{1}{1-u^2} \frac{du}{dx} \quad |u| < 1$$

$$\frac{d(\coth^{-1} u)}{dx} = \frac{1}{1-u^2}\frac{du}{dx} \quad |u| > 1$$

$$\frac{d(\text{sech}^{-1} u)}{dx} = \frac{1}{u\sqrt{1-u^2}}\frac{du}{dx} \quad 0 < u < 1$$

$$\frac{d(\text{csch}^{-1} u)}{dx} = \frac{1}{|u|\sqrt{1+u^2}}\frac{du}{dx} \quad u \neq 0$$

$$\int \frac{du}{\sqrt{a^2+u^2}} = \sinh^{-1}(\frac{u}{a}) + C, \qquad a > 0$$

$$\int \frac{du}{\sqrt{u^2-a^2}} = \cosh^{-1}(\frac{u}{a}) + C, \qquad u > a > 0$$

$$\int \frac{du}{a^2-u^2} = \begin{cases} \frac{1}{a}\tanh^{-1}(\frac{u}{a}) + C, & u^2 < a^2 \\ \frac{1}{a}\coth^{-1}(\frac{u}{a}) + C, & u^2 > a^2 \end{cases}$$

$$\int \frac{du}{u\sqrt{a^2-u^2}} = -\frac{1}{a}\text{sech}^{-1}(\frac{u}{a}) + C, \quad 0 < u < a$$

$$\int \frac{du}{u\sqrt{a^2+u^2}} = -\frac{1}{a}\text{csch}^{-1}|\frac{u}{a}| + C, \qquad 0 < u < a$$

7.8 Relative Rates of Growth

Growth Rate of Functions Let $f(x)$ and $g(x)$ be positive for x sufficiently large

1. f grows faster than g as $x \to \infty$ if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty$$

2. f grows faster than g as $x \to \infty$ if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$$

3. f and g grow at the same rate as $x \to \infty$ if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L$$

where L is finite and positive.

Order and Oh-Notation

Definition A function f is **of smaller order than g** as $x \to \infty$ if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$. We indicate this by writing $f = o(g)$.

Definition Let $f(x)$ and $g(x)$ be positive for x sufficiently large. Then f is of at most the order of g as $x \to \infty$ if there is a positive integer M for which

$$\frac{f(x)}{g(x)} \leq M$$

for x sufficiently large. We indicate this by writing $f = O(g)$.

Sequential vs. Binary Search

8 Techniques of Integration

8.1 Using Basic Integration Formulas

Basic Integration formulas

$$\int k\ dx = kx + C$$

$$\int x^n\ dx = \frac{x^{n+1}}{n+1} + C$$

$$\int \frac{dx}{x} = \ln|x| + C$$

$$\int e^x\ dx = e^x + C$$

$$\int a^x\ dx = \frac{a^x}{\ln a} + C$$

$$\int \sin x\ dx = -\cos x + C$$

$$\int \cos x\ dx = \sin x + C$$

$$\int \sec^2 x\ dx = \tan x + C$$

$$\int \csc^2 x\ dx = -\cot x + C$$

$$\int \sec x \tan x\ dx = \sec x + C$$

$$\int \csc x \cot x\ dx = -\csc x + C$$

$$\int \tan x\ dx = \ln|\sec x| + C$$

$$\int \cot x\ dx = \ln|\sin x| + C$$

$$\int \sec x\ dx = \ln|\sec x + \tan x| + C$$

$$\int \csc x\ dx = -\ln|\csc x + \cot x| + C$$

$$\int \sinh x\ dx = \cosh x + C$$

$$\int \cosh x\ dx = \sinh x + C$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}(\frac{x}{a}) + C$$

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}(\frac{x}{a}) + C$$

$$\int \frac{dx}{x\sqrt{x^2 - a^2}} dx = \frac{1}{a}\sec^{-1}|\frac{x}{a}| + C$$

$$\int \frac{dx}{\sqrt{a^2+x^2}}dx = \sinh^{-1}(\frac{x}{a}) + C$$

$$\int \frac{dx}{\sqrt{x^2-a^2}}dx = \cosh^{-1}(\frac{x}{a}) + C$$

8.2 Integration by Parts

Product Rule in Integral Form

$$\int \frac{d}{dx}[f(x)g(x)]dx = \int [f'(x)g(x) + f(x)g'(x)]dx$$

$$\int \frac{d}{dx}[f(x)g(x)]dx = \int [f'(x)g(x)] + \int [f(x)g'(x)]dx$$

$$\int f(x)g'(x)dx = \int \frac{d}{dx}[f(x)g(x)]dx + \int f'(x)g(x)$$

$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx$$

Integration by Parts Formula

$$\int u\ dv = uv - \int v\ du$$

Example Find

$$\int x\cos x\ dx$$

Solution

$$u = x, \quad dv = \cos x\ dx$$

$$du = dx, \quad v = \sin x$$

Then

$$\int x\cos x\ dx = x\sin x - \int \sin x\ dx = x\sin x + \cos x + C$$

Evaluating Definite Integrals by Parts

Integration by Parts Formula for Definite Integrals

$$\int_a^b f(x)g'(x)dx = f(x)g(x)\Big]_a^b = \int_a^b f'(x)g(x)dx$$

Tabular Integration Can Simplify Repeated Integrations

8.3 Trigonometric Integrals

Products of Power of Sines and Cosinea

$$\int \sin^m x \cos^n x \; dx$$

Case 1: If **m is odd**

$$\sin^m x = \sin^{2k+1} x = (\sin^2 x)^k \sin x = (1 - \cos^2)^k \sin x$$

Then we combine the single $\sin x$ with dx in the integral and set $\sin x \; dx$ equal to $-d(cosx)$.

Case 2: If **m is even and n is odd**

$$\cos^n x = \cos^{2k+1} x = (\cos^2 x)^k \cos x = (1 - \sin^2 x)^k \cos x$$

We then combine the single $\cos x$ with dx and set $\cos x \; dx$ equal to $d(\sin x)$.

Case 3: If **both m and n are even**

$$\sin^2 x = \frac{1 - \cos 2x}{2}, \qquad \cos^2 x = \frac{1 + \cos 2x}{2}$$

to reduce the integrand to one in lower power of $\cos 2x$.

Eliminating Square Roots

Integrals of Powers of $\tan x$ **and** $\sec x$

Example Evaluate

$$\int \tan^4 x \; dx$$

Solution

$$\begin{aligned}
\int \tan^4 x \; dx &= \int \tan^2 x \cdot \tan^2 x \; dx \\
&= \int \tan^2 x \cdot (\sec^2 x - 1) dx \\
&= \int \tan^2 x \sec^2 x \; dx - \int \tan^2 x \; dx \\
&= \int \tan^2 \sec^2 \; dx - \int (\sec^2 - 1) \; dx \\
&= \int \tan^2 \sec^2 \; dx - \int \sec^2 x \; dx + \int dx
\end{aligned} \tag{50}$$

$$u = \tan x, \quad du = \sec^2 x \; dx$$

$$\int u^2 du = \frac{1}{3} u^3 + C_1$$

$$\int \tan^4 x \; dx = \frac{1}{3} \tan^3 x - \tan x + x + C$$

Products of Sines and Cosines

$$\sin mx \sin nx = \frac{1}{2}[\cos(m-n)x - \cos(m+n)x]$$

$$\sin mx \cos nx = \frac{1}{2}[\sin(m-n)x + \sin(m+n)x]$$

$$\cos mx \cos nx = \frac{1}{2}[\cos(m-n)x + \cos(m+n)x]$$

8.4 Trigonometric Substitutions

Procedure for a Trigonometric Substitution

1. Write down the substitution for x, calculate the differential dx, and specify the selected values of θ for the substitution.
2. Substitute the trigonometric expression and the calculated differential into the integrand, and then simplify the results algebraically.
3. Integrate the trigonometric integral, keeping in mind the restrictions on the angle θ for reversibility.
4. Draw an appropriate reference triangle to reverse the substitution in the integration result and convert it back to the original variable x.

8.5 Integration of Rational Functions by Partial Fractions

the method of partial fractions

$$\frac{5x-3}{x^2-2x-3} = \frac{A}{x+1} + \frac{B}{x-3}$$

$$A = 2, \qquad B = 3$$

General Description of the Method

Method of Partial Fractions when $f(x)/g(x)$ is Proper

1. Let $x-r$ be a linear factor of $g(x)$. Suppose that $(x-r)^m$ is the highest power of $x-r$ that divides $g(x)$. Then, to this factor, assign the sum of the m partial fractions:

$$\frac{A_1}{(x-r)} + \frac{A_2}{(x-r)^2} + \cdots + \frac{A_m}{(x-r)^m}$$

Do this for each distinct linear factor of $g(x)$.

2. Let x^2+px+q be an irreducible quadratic factor of $g(x)$ so that x^2+px+q has no real roots. Suppose that $(x^2+px+q)^n$ is the highest power of this factor that divides $g(x)$. Then, to this factor, assign the sum of the n partial fractions:

$$\frac{B_1x+C_1}{(x^2+px+q)} + \frac{B_2x+C_2}{(x^2+px+q)^2} + \cdots + \frac{B_nx+C_n}{(x^2+px+q)^n}$$

Do this for each distinct quadratic factor of $g(x)$.

3. Set the original fraction $f(x)/g(x)$ equal to the sum of all these partial fractions. Clear the resulting equation of fractions and arrange the terms in decreasing power of x.
4. Equate the coefficients of corresponding powers of x and solve the resulting equations for the undetermined coefficients.

The Heavisde "Cover-up" method for Linear Factors When the degree of the polynomial $f(x)$ is less than the degree of $g(x)$ and

$$g(x) = (x - r_1)(x - r_2)\cdots(x - r_n)$$

is a product of n distinct linear factors, each raised to the first power, there is a quick way to expand $f(x)/g(x)$ by partial fractions.

Heaviside Method 1. *Write the quotient with $g(x)$ factored:*

$$\frac{f(x)}{g(x)} = \frac{f(x)}{(x - r_1)(x - r_2)\cdots(x - r_n)}$$

2. *Cover the factors $(x-r_1)$ of $g(x)$ one at a time* , each time replacing all the uncovered x's by the number r_i. This gives a number A_i for each root r_i:

$$\begin{aligned} A_1 &= \frac{f(r_1)}{(r_1 - r_2)\cdots(r_1 - r_n)} \\ A_2 &= \frac{f(r_2)}{(r_2 - r_1)(r_2 - r_3)\cdots(r_2 - r_n)} \\ &\vdots \\ A_n &= \frac{f(r_n)}{(r_n - r_1)(r_n - r_2)\cdots(r_n - r_{n-1})} \end{aligned} \tag{51}$$

3. *Write the partial fraction expansion of $f(x)/g(x)$ as*

$$\frac{f(x)}{g(x)} = \frac{A_1}{(x - r_1)} + \frac{A_2}{(x - r_2)} + \cdots + \frac{A_n}{(x - r_n)}$$

Other Ways to Determine the Coefficients

8.6 Integral Tables and Computer Algebra Systems

Integral Tables

Reduction Formulas

Integration with CAS

Nonelemtary Integrals

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\,dt$$

$$\int \sin x^2 dx$$

$$\int \sqrt{1 + x^4} dx$$

8.7 Numerical integration

Trapezoidal Approximation

$$\Delta x = \frac{b-a}{n} \quad \textbf{step size} \text{ or } \textbf{mesh size}$$

$$\Delta x\left(\frac{y_{i-1}+y_i}{2}\right) = \frac{\Delta x}{2}(y_{i-1}+y_i)$$

$$\begin{aligned} T &= \frac{1}{2}(y_0+y_1)\Delta x + \frac{1}{2}(y_1+y_2)\Delta x + \cdots + \frac{1}{2}(y_{n-1}+y_n)\Delta x \\ &= \Delta x\left(\frac{1}{2}y_0 + y_1 + y_2 + \cdots + y_{n-1} + \frac{1}{2}y_n\right) \\ &= \frac{\Delta x}{2}(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n) \end{aligned} \tag{52}$$

where

$$y_0 = f(a), \quad y_1 = f(x_1), \cdots, \quad y_{n-1} = f(x_{n-1}), \quad y_n = f(b)$$

The Trapezoidal Rule says: Use T to estimate teh integral of f from a to b.

Simpson's Rule: Approximations Using Parabolas
To approximate $\int_a^b f(x)dx$, use

$$S = \frac{\Delta x}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n)$$

The y's are the values of f at the partition points

$$x_i = a + i\Delta x$$

The number n is even, and $\Delta x = (b-a)/n$.

Error Analysis

Error Estimates in the Trapezoidal and Simpson's Rules If f'' is continuous and M is any upper bound for the values of $|f''|$ on $[a,b]$

$$|E_T| \leq \frac{M(b-a)^3}{12n^2} \quad \textit{Trapezoidal Rule}$$

If $f^{(4)}$ is continuous and M is any upper bound for the values of $|f^{(4)}|$ on $[a,b]$

$$|E_S| \leq \frac{M(b-a)^5}{180n^4} \quad \textit{Simpson's Rule}$$

8.8 Improper Integrals

Infinite Limits of Integration

$$\int_a^\infty f(x)dx = \lim_{b\to\infty}\int_a^b f(x)dx$$

$$\int_{-\infty}^b f(x)dx = \lim_{b\to\infty}\int_a^b f(x)dx$$

$$\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^c f(x)dx + \int_c^\infty f(x)dx$$

The Integral $\int_1^{\infty} \dfrac{dx}{x^p}$

Solution If $p \neq 1$

$$\int_1^b \frac{dx}{x^p} = \frac{x^{-p+1}}{-p+1}\Bigg]_1^b = \frac{1}{1-p}(b^{-p+1}-1) = \frac{1}{1-p}(\frac{1}{b^{p-1}}-1)$$

Thus,

$$\begin{aligned}
\int_1^{\infty} \frac{dx}{x^p} &= \lim_{b\to\infty} \int_1^b \frac{dx}{x^p} \\
&= \lim_{b\to\infty} \left[\frac{1}{1-p}\left(\frac{1}{b^{p-1}-1}\right)\right] \\
&= \begin{cases} \dfrac{1}{p-1}, & p > 1 \\ \infty, & p < 1 \end{cases}
\end{aligned} \tag{53}$$

because

$$\lim_{b\to\infty} \frac{1}{b^{p-1}} = \begin{cases} 0, & p > 1 \\ \infty, & p < 1 \end{cases}$$

If $p = 1$, the integral also diverges:

$$\begin{aligned}
&\int_1^{\infty} \frac{dx}{x^p} \\
&= \int_1^{\infty} \frac{dx}{x} \\
&= \lim_{b\to\infty} \int_1^b \frac{dx}{x} \\
&= \lim_{b\to\infty} \ln x\Bigg]_1^b \\
&= \lim_{b\to\infty} (\ln b - \ln 1) = \infty
\end{aligned} \tag{54}$$

Integrands with Vertical Asymptotes

Imporoper Integrals of Type II

1. If $f(x)$ is continuous on $(a, b]$ and discontinuous at a, then

$$\int_a^b f(x)dx = \lim_{c\to a^+} \int_c^b f(x)dx$$

2. If $f(x)$ is continuous on $[a, b)$ and discontinuous at b, then

$$\int_a^b f(x)dx = \lim_{c\to b^-} \int_a^c f(x)dx$$

3. If $f(x)$ is discontinuous at c, where $a < c < b$, and continuous on $[a, c) \cup (c, b]$, then

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

Improper Integrals with a CAS

Tests for convergence and Divergence

Direct Comparison Test Let f and g be continuous on $[a, \infty)$ with $0 \leq f(x) \leq g(x)$ for all $x \geq a$. Then
1. $\int_a^\infty (x)dx$ converges if $\int_a^\infty (x)dx$ converges.
2. $\int_a^\infty (x)dx$ diverges if $\int_a^\infty (x)dx$ diverges.

Limit Comparison Test If the positive functions f and g are continuous on $[a, \infty)$, and if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L, \quad 0 < L < \infty$$

then

$$\int_a^\infty f(x)dx \quad \text{and} \quad \int_a^\infty g(x)dx$$

both converge or both diverge.

8.9 Probability

Random Variables A **random variable** is a function X that assigns a numerical value to each outcome in a sample place.

Random variables that have only finitely many values are called **discrete** random variables. A **continuous random variable** can take on values in an entire interval, and it is associated with a *distribution function.*

Probability Distributions A **probability density function** for a continuous random variable is a function f defined over $(-\infty, \infty)$ and having the following properties:
1. f is continuous, except possibly at a finite number of points.
2. f is nonnegative, so $f \geq 0$.
3. $\int_{-\infty}^\infty f(x)dx = 1$. If X is a continuous random variable with probability density function f, the **probability** that X assumes a value in the interval between $X = c$ and $X = d$ is the area integral

$$P(c \leq X \leq d) = \int_c^d f(X)dX$$

Exponentially Decreasing Distributions

$$f(x) = \begin{cases} 0 & x < 0 \\ ce^{-cx} & x \geq 0 \end{cases}$$

Expected Values, Means, and Medians The **expected value** or **mean** of a continuous random variable X with probability density function f is the number

$$\mu = E(X) = \int_{-\infty}^\infty Xf(X)dx$$

Exponential Density Function for a Random Variable X with Mean μ

$$f(X) = \begin{cases} a & X < 0 \\ \mu^{-1}e^{-X/\mu} & X \geq 0 \end{cases}$$

Definition The **median** of a continuous random variable X with probability density function f is the number m for which

$$\int_{-\infty}^{m} f(X)dX = \frac{1}{2} \qquad \text{and} \qquad \int_{m}^{\infty} f(X)dX = \frac{1}{2}$$

Variance and Standard Deviation The **variance** of a random variable X with probability density function f is the expected value of $(X-\mu)^2 f(X)dX$

$$\mathrm{Var}(X) = \int_{-\infty}^{\infty} (X-\mu)^2 f(X)dX$$

The **standard deviation** of X is

$$\sigma_X = \sqrt{\mathrm{Var}(X)} = \sqrt{\int_{-\infty}^{\infty} (X-\mu)^2 f(X)dX}$$

Uniform Distributions

$$f(x) = \frac{1}{b-a}, \quad a \leq x \leq b$$

Normal Distributions

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-(x-\mu)^2/2\sigma^2}$$

9 First-Order Differential Equations

9.1 Solutions, Slope Fields, and Euler's Method

A **first-order differential equation** is an equation

$$\frac{dy}{dx} = f(x, y)$$
$$\frac{d}{dx}y(x) = f(x, y(x))$$

Slope Fields: Viewling Solution Curves

Euler's Method

9.2 First-Order Linear Equations

$$\frac{dy}{dx} + P(x)y = Q(x) \quad \text{Standard form}$$

Solving Linear Equations

$$\begin{aligned}
\frac{dy}{dx} + P(x)y &= Q(x) \\
v(x)\frac{dy}{dx} + P(x)v(x)y &= v(x)Q(x) \\
\frac{d}{dx}(v(x) \cdot y) &= v(x)Q(x) \\
v(x) \cdot y &= \int v(x)Q(x)dx \\
y &= \frac{1}{v(x)} \int v(x)Q(x)dx
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dx}(vy) &= v\frac{dy}{dx} + Pvy \\
v\frac{dy}{dx} + y\frac{dv}{dx} &= v\frac{dy}{dx} + Pvy \\
y\frac{dy}{dx} &= Pvy \\
\frac{dv}{dx} &= Pv \\
\frac{dv}{v} &= P\ dx \\
\int \frac{dv}{v} &= \int P\ dx \\
\ln v &= \int P\ dx \\
e^{\ln v} &= e^{\int P\ dx} \\
v &= e^{\int P\ dx}
\end{aligned} \tag{55}$$

To solve the linear equation $y' + P(x)y = Q(x)$, multiply both sides by the integrating factor $v(x) = e^{\int P(x)dx}$ and integrate both sides.

Example Solve the equation

$$x\frac{dy}{dx} = x^2 + cy, \quad x > 0$$

Solution

$$\begin{aligned}
\frac{dy}{dx} - \frac{3}{x}y &= x \\
v(x) = e^{\int P(x)dx} &= e^{\int(-3/x)dx} \\
&= e^{-3\ln|x|} \\
&= e^{-3\ln x} \\
&= e^{\ln x^{-3}} = \frac{1}{x^3} \\
\frac{1}{x^3}\cdot\left(\frac{dy}{dx} - \frac{3}{x}y\right) &= \frac{1}{x^3}\cdot x \\
\frac{1}{x^3}\frac{dy}{dx} - \frac{3}{x^4}y &= \frac{1}{x^2} \\
\frac{d}{dx}\left(\frac{1}{x^3}y\right) &= \frac{1}{x^2} \\
\frac{1}{x^3}y &= \int \frac{1}{x^2}dx \\
\frac{1}{x^3}y &= -\frac{1}{x} + C \\
y &= -x^2 + Cx^3, \quad x > 0
\end{aligned} \tag{56}$$

RL Circuits

$$L\frac{di}{dt} + Ri = V$$

9.3 Applications

Motion with Resistance Proportional to Velocity

$$F = m\frac{dv}{dt}$$

$$m\frac{dv}{dt} = -kv$$

$$v = v_0 e^{-(k/m)t}$$

$$\frac{ds}{dt} = v_0 e^{-(k/m)t}, \quad s(0) = 0$$

$$s = -\frac{v_0 m}{k}e^{-(k/m)t} + C$$

Substituting $s = 0$ when $t = 0$ gives

$$0 = -\frac{v_0 m}{k} + C \quad \text{and} \quad C = \frac{v_0 m}{k}$$

$$s(t) = -\frac{v_0 m}{k}e^{-(k/m)t} + \frac{v_0 m}{k} = \frac{v_0 m}{k}(1 - e^{-(k/m)t})$$

$$
\begin{aligned}
\lim_{t\to\infty} s(t) &= \lim_{t\to\infty} \frac{v_0 m}{k}(1 - e^{-(k/m)t}) \\
&= \frac{v_0 m}{k}(1-0) = \frac{v_0 m}{k} \\
\text{Distance coasted} &= \frac{v_0 m}{k}
\end{aligned}
$$

Inaccuracy of the Exponential Population Growth Model

Orthogonal Trajectories An **orthogonal trajectory** of a family of curves is a curve that intersects each curve of the family at right angles, or textitorthogonally.

9.4 Graphical Solutions of Autonomous Equations

Equilibrium Values and Phase Lines If $dy/dx = g(y)$ is an autonomous differential equation, then the values of y for which $dy/dx = 0$ are called **equilibrium values** or **rest points**.

Example Draw a phase line for the equation

$$\frac{dy}{dx} = (y+1)(y-2)$$

Solution 1. *Draw a number line for y and mark the equilibrium values $y = -1$ and $y = 2$, where $dy/dx = 0$.*
2. *Identify and label the intervals where $y' > 0$ and $y' < 0$.*
3. *Calculate y'' and mark the intervals where $y'' > 0$ and $y'' < 0$.*

$$y' = (y+1)(y-2) = y^2 - y - 2$$

$$
\begin{aligned}
y'' &= \frac{d}{dx}(y') \\
&= \frac{d}{dx}(y^2 - y - 2) \\
&= 2yy' - y' \\
&= (2y-1)y' \\
&= (2y-1)(y+1)(y-2)
\end{aligned}
\tag{57}
$$

4. *Sketch an assortment of solution curves in the xy-plane.*

Stable and Unstable Equilibria

Newton's Law of Cooling

$$\frac{dH}{dt} = -k(H - H_s), \quad k > 0$$

$$\frac{d^2H}{dt^2} = -k\frac{dH}{dt}$$

A Falling Body Encountering Resistance

Logistic Population Growth

$$\frac{dP}{dt} = kP$$

$$\frac{dP}{dt} = r(M - P)P = rMP - rP^2$$

9.5 Systems of Equations and Phase Planes

Phase Planes

$$\frac{dx}{dt} = F(x, y)$$

$$\frac{dy}{dt} = G(x, y)$$

A Competitive-Hunter Model

$$\frac{dx}{dt} = (a - by)x$$

$$\frac{dy}{dt} = (m - nx)y$$

Limitations of Phase-Plane Analysis Method

Another Type of Behavior The system

$$\frac{dx}{dt} = y + x - x(x^2 + y^2)$$

$$\frac{dy}{dt} = -x + y - y(x^2 + y^2)$$

Limit cycle: $x^2 + y^2 = 1$

10 Infinite Sequences and Series

10.1 Sequence

Representing Sequences

Convergence and Divergence

Definitions The sequence $\{a_n\}$ **converges** to the number L if for every positive number ϵ there corresponds an integer N such that for all n.

$$n > N \qquad |a_n - L| < \epsilon$$

If no such number L exists, we say that $\{a_n\}$ **diverges**.

If a_n converges to L, we write $\lim_{n\to\infty} a_n = L$, or simply $a_n \to L$, and call L the **limit** of the sequence.

Definitions The sequence $\{a_n\}$ **diverges to infinity** if for every number M there is an integer N such that for all n larger than N, $a_n > M$. If this condition holds we write

$$\lim_{n\to\infty} a_n = \infty \quad \text{or} \quad a_n \to \infty$$

Similarly, if for every number m there is an integer N such that for all $n > N$ we have $a_n < m$, then we say a_n **diverges to negative infinity** and write

$$\lim_{n\to\infty} a_n = -\infty \quad \text{or} \quad a_n \to -\infty$$

Calculating Limits of Sequences

$$\begin{aligned}
&\lim_{n\to\infty} (a_n + b_n) = A + B \\
&\lim_{n\to\infty} (a_n - b_n) = A - b \\
&\lim_{n\to\infty} (k \cdot b_n) = k \cdot B \\
&\lim_{n\to\infty} (a_n \cdot b_n) = A \cdot B \\
&\lim_{n\to\infty} (\frac{a_n}{b_n}) = \frac{A}{B}
\end{aligned} \tag{58}$$

The Sandwich Theorem for Sequences Let $\{a_n\}, \{b_n\}, and \{c_n\}$ be sequences of real numbers. If $a_n \le b_n \le c_n$ holds for all n beyond some index N, and if $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$ also.

The Continuous Function Theorem for Sequences Let $\{a_n\}$ be a sequence of real numbers. If $a_n \to L$ and if f is a function that is continuous at L and defined at all a_n, then $f(a_n) \to f(L)$.

Using L'Hopital's Rule Suppose that $f(x)$ is a function defined for all $x \ge n_0$ and that $\{a_n\}$ is a sequence of real numbers such that $a_n = f(n)$ for $n \ge n_0$. Then

$$\lim_{x\to\infty} f(x) = L \quad \Rightarrow \quad \lim_{n\to\infty} a_n = L$$

Commonly Occuring Limits

$$\lim_{n\to\infty} \frac{\ln n}{n} = 0$$
$$\lim_{n\to\infty} \sqrt[n]{n} = 1$$
$$\lim_{n\to\infty} x^{1/n} = 1$$
$$\lim_{n\to\infty} x^n = 0$$
$$\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x$$
$$\lim_{n\to\infty} \frac{x^n}{n!} = 9$$

Bounded Monotonic Sequences

The Monotonic Sequence Theorem If a sequence $\{a_n\}$ is both bounded and monotonic, then the sequence converges.

10.2 Infinite Series

$$a_1 + a_2 + \cdots + a_n + \cdots = \sum_{n=1}^{\infty} a_n = L$$

Geometric Series

$$a + ar + ar^2 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1}$$

If $|r| < 1$

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}, \qquad |r| < 1$$

If $|r| \geq 1$, the series diverges.

The nth-Term Test for a Divergent Series

Theorem If $\sum_{n=1}^{\infty} a_n$ converges, then $a_n \to 0$.

The nth-Term Test for Divergence $\sum_{n=1}^{\infty} a_n$ diverges if $\lim\limits_{n\to\infty} a_n$ fails to exist or is different from zero.

Combining Series

Theorem

$$\sum (a_n + b_n) = \sum a_n + \sum b_n = A + B$$
$$\sum (a_n - b_n) = \sum a - n - \sum b_n = A - B$$
$$\sum ka_n = k \sum a_n = kA \quad \text{(any number } k\text{)}$$

Adding or Deleting Terms

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots + a_{k-1} + \sum_{n=k}^{\infty} a_n$$

Reindexing

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1+h}^{\infty} a_{n-h} = a_1 + a_2 + \cdots$$

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1-h}^{\infty} a_{n+h} = a_1 + a_2 + \cdots$$

10.3 The Integral Test

Nondecreasing Partial Sum A series $\sum_{n=1}^{\infty} a_n$ of nonnegative terms converges if and only if its partial sums are bounded from above.

The Integral Test Let $\{a_n\}$ be a sequence of positive terms. Suppose that $a_n = f(n)$, where f is a continuous, positive,decreasing function of x for all $x \geq N$(N a positive integer). Then the series $\sum_{n=N}^{\infty} = a_n$ and the integral $\int_N^{\infty} f(x)dx$ both converge or both diverge.

***p*-series**

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots$$

converges if $p > 1$, and diverges if $p \leq 1$

Solution If $p > 1$, then $f(x) = 1/x^p$ is a positive decreasing function of x. Since

$$\begin{aligned}
\int_1^{\infty} \frac{1}{x^p} dx &= \int_1^{\infty} x^{-p} dx \\
&= \lim_{b\to\infty} \left[\frac{x^{-p+1}}{-p+1}\right]_1^b \\
&= \frac{1}{1-p} \lim_{b\to\infty} \left(\frac{1}{b^{p-1}} - 1\right) \\
&= \frac{1}{1-p}(0-1) \\
&= \frac{1}{p-1}
\end{aligned} \tag{59}$$

If $p \leq 0$, the series diverges by the nth-term test.
If $0 < p < 1$,

$$\int_1^{\infty} \frac{1}{x^p} dx = \frac{1}{1-p} \lim_{b\to\infty} (b^{1-p} - 1) = \infty$$

If $p = 1$,

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots$$

is the **Harmonic Series**.

Error Estimation

Bounds for the Remainder in the Integral Test

Suppose $\{a_k\}$ is a sequence of positive terms with $a_k = f(k)$, where f is a continuous positive decreasing function of x for all $x \geq n$, and that $\sum a_n$ converges to S. Then the remainder $R_n = S - s_n$ satisfies the inequalities.

$$\int_{n+1}^{\infty} f(x)dx \leq R_n \leq \int_{n}^{\infty} f(x)dx$$

10.4 Comparison Tests

The Comparison Test Let $\sum a_n$, $\sum c_n$, and $\sum d_n$ be series with nonnegative terms. Suppose that for some integer N

$$d_n \leq a_n \leq c_n \quad \text{for all} \quad n > N$$

(a) If $\sum c_n$ converges, then $\sum a_n$ also converges.
(b) If $\sum d_n$ diverges, then $\sum a_n$ also diverges.

The Limit Comparison Test Suppose that $a_n > 0$ and $b_n > 0$ for all $n \geq N (N$ an integer).

1. If $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = c > 0$, then $\sum a_n$ and $\sum b_n$ both converge or both diverge.
2. If $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges.
3. If $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.

10.5 Absolute Convergence; The Ratio and Root Tests

Definition A series $\sum a_n$ **converges absolutely** (is **absolutely convergent**) if the corresponding series of absolute values, $\sum |a_n|$, converges.

The Absolute Convergence Test If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

The Ratio Test Let $\sum a_n$ be any series and suppose that

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \rho$$

Then
(a) the series converges absolutely if $\rho < 1$,
(b) the series diverges if $\rho > 1$ or ρ is infinite,
(c) the test is inconclusive if $\rho = 1$.

The Root Test Let $\sum a_n$ be any series and suppose that

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \rho$$

Then
(a) the series converges absolutely if $\rho < 1$,
(b) the series diverges if $\rho > 1$ or ρ is infinite,
(c) the test is inconclusive if $\rho = 1$

10.6 Alternating Series and Conditional Convergence

The Alternating Series Test The series

$$\sum_{n=1}^{\infty}(-1)^{n+1}u_n = u_1 - u_2 + u_3 - u_4 + \cdots$$

converges if all three of the following conditions are satisfied:

1. The u_n's are all positive.
2. The positive u_n's are (eventually) nonincreasing: $u_n \geq u_{n+1}$ for all $n \geq N$, for some integer N.
3. $u_n \to 0$

The Alternating Series Estimation Theorem If the alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}u_n$ satisfies the three conditions of The Alternating Series Test, then for $n \geq N$,

$$s_n = u_1 - u_2 + \cdots + (-1)^{n+1}u_n$$

approximates the sum L of the series with an error whose absolute value is less than u_{n+1}, the absolute value of the first unused term. Furthermore, the sum L lies between any two successive partial sums s_n and s_{n+1}, and the remainder, $L - s_n$, has the same sign as the first unused term.

Conditional Convergence

Definition A convergent series that is not absolutely convergent is **conditionally convergent**.

Rearranging Series

The Rearrangement Theorem for Absolutely Convergent Series If $\sum_{n=1}^{\infty} a_n$ converges absolutely, and b_1, b_2,$\cdots$,b_n,$\cdots$ is any arrangement of the sequence $\{a_n\}$, then $\sum b_n$ converges absolutely and

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} a_n$$

10.7 Power Series

Power Series and Convergence

Definitions **A power series about $\mathbf{x = 0}$** is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots$$

A power series about $\mathbf{x = a}$ is a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n + \cdots$$

in which the **center** a and the **coefficients** $c_0, c_1, c_2, \cdots, c_n, \cdots$ are constants.

The convergence Theorem for Power Series If the power series $\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$ converges at $x = c \neq 0$, then it converges absolutely for all x with $|x| < |c|$. If the series diverges at $x = d$, then it diverges for all x with $|x| > |d|$.

The Radius of Convergence of a Power Series The convergence of the series $\sum c_n(x-a)^n$ is described by one of the following three cases:

1. There is a positive number R such that the series diverges for x with $|x-a|>R$ but converges absolutely for x with $|x-a|<R$. The series may or may not converge at either of the endpoints $x=a-R$ and $x=a+R$.
2. The series converges absolutely for every x ($R=\infty$).
3. The series converges at $x=a$ and diverges elsewhere ($R=0$)

R is called the **radius of convergence** of the power series, and the interval of radius R centered at $x=a$ is called the **interval of convergence**.

How to Test a Power Series for Convergence

1. *Use the Ratio Test(or Root Test) to find the interval where the series converges absolutely.* Ordinarily, this is an open interval

$$|x-a|<R \quad \text{or} \quad a-R<x<a+R$$

2. *If the interval of absolute convergence is finite, test for convergence or divergence at each endpoint.* Use a Comparison Test, the Integral Test, or the Alternating Series Test.
3. *If the interval of absolute convergence is $a-R<x<a+R$, the series diverges for* $|x-a|>R$(it does not even converge conditionally) because the nth term does not approach zero for those values of x.

Operations on Power Series

The Series Multiplication Theorem for Power Series If $A(x)=\sum_{n=0}^{\infty} a_n x^n$ and $B(x)=\sum_{n=0}^{\infty} b_n x^n$ converge absolutely for $|x|<R$, and

$$c_n = a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \cdots + a_{n-1} b_1 + a_n b_0 = \sum_{k=0}^{n} a_k b_{n-k}$$

then $\sum_{n=0}^{\infty} c_n x^n$ converges absolutely to $A(x)B(x)$ for $|x|<R$:

$$\left(\sum_{n=0}^{\infty} a_n x^n\right)\cdot\left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty} c_n x^n$$

Theorem If $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely for $|x|<R$, then $\sum_{n=0}^{\infty} a_n (f(x))^n$ converges absolutely for any continuous function f on $|f(x)|<R$.

The Term-by-Term Differentiation Theorem If $\sum c_n(x-a)^n$ has radius of convergence $R>0$, it defines a function

$$f(x)=\sum_{n=0}^{\infty} c_n(x-a)^n \qquad \text{on the interval} \qquad a-R<x<a+R$$

This function f has derivatives of all orders inside the interval, and we obtain the derivatives by differentiating the original series term by term:

$$f'(x)=\sum_{n=1}^{\infty} n c_n (x-a)^{n-1}$$

$$f''(x)=\sum_{n=2}^{\infty} n(n-1) c_n (x-a)^{n-2}$$

and so on. Each of these derived series converges at every point of the interval $a-R<x<a+R$.

The Term-by-Term Integration Theorem Suppose that

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

converges for $a - R < x < a + R (R > 0)$. Then

$$\sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$$

converges for $a - R < x < a + R$ and

$$\int f(x)dx = \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1} + C$$

for $a - R < x < a + R$.

10.8 Taylor and Maclaurin Series

Series Representations

$$f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n = a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots + a_n(x-a)^n + \cdots$$

$$f'(x) = a_1 + 2a_x(x-a) + 3a_3(x-a)^2 + \cdots + na_n(x-a)^{n-1} + \cdots$$

$$f''(x) = 1 \cdot 2a_x + 2 \cdot 3a_3(x-a) + 3 \cdot 4a_4(x-a)^2 + \cdots$$

$$f'''(x) = 1 \cdot 2 \cdot 3a_3 + 2 \cdot 3 \cdot 4a_4(x-a) + 3 \cdot 4 \cdot 5a_5(x-a)^2 + \cdots$$

$$f^{(n)}(x) = n!a_n + \text{a sum of terms with (x-a) as a factor.}$$

Since these equations all hold at $x = a$, we have

$$f'(a) = a_1, \quad f''(a) = 1 \cdot 2a_2, \quad f'''(a) = 1 \cdot 2 \cdot 3a_3$$

$$f^{(n)}(a) = n!a_n$$

$$a_n = \frac{f^{(n)}(a)}{n!}$$

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots$$

Taylor and Maclaurin Series

Definitions Let f be a function with derivatives of all orders throughout some interval containing a as an interior point. Then the **Taylor series generated by f at x = a** is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots$$

The **Maclaurin series of f** is the Taylor series generated by f at $x = 0$, or

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k = f(a) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots$$

Taylor Polynomials

$$P_1(x) = f(a) + f'(a)(x - a)$$

Definition Let f be a function with derivatives of order k for $k = 1, 2, \cdots, N$ in some interval containing a as an interior point. Then for any integer n from 0 through N, the **Taylor polynomial of order n** generated by f at $x = a$ is the polynomial

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(k)}(a)}{k!}(x-a)^k + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

10.9 Convergence of Taylor Series

Taylor's Theorem If f and its first n derivatives $f', f'', \cdots, f^{(n)}$ are continuous on the closed interval between a and b, and $f^{(n)}$ is differentiable on the open interval between a and b, then there exists a number c between a and b such that

$$f(b) = f(a) + f'(a)(b-a) + \frac{f''(a)}{2!}(b-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(b-a)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}$$

Taylor's Formula If f has derivatives of all orders in an interval I containing a, then for each positive integer n and for each x in I,

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x)$$

where

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} \quad \text{for some } c \text{ between } a \text{ and } x.$$

If $R_n(x) \to 0$ as $n \to \infty$ for all $x \in I$, we say that the Taylor series generated by f at $x = a$ **converges**f on I, and we write

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

Examples

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + R_n(x)$$

where $R_n(x) = \dfrac{e^c}{(n+1)!}x^{n+1}$ for some c between 0 and x.

$$|R_n(x)| \le \frac{|x|^{n+1}}{(n+1)!} \quad \text{when } x \le 0, e^c < 1$$

$$|R_n(x)| < e^x \frac{x^{n+1}}{(n+1)!} \quad \text{when } x > 0, e^c < e^x$$

$$\lim_{n\to\infty} \frac{x^{n+1}}{(n+1)!} = 0 \quad \text{for every } x$$

par $\lim_{n\to\infty} R_n(x) = 0$, and the series converges to e^x for every x. Thus,

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^k}{k!} + \cdots$$

$$e = 1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{n!} + R_n(1)$$

$$R_n(1) = e^c \frac{1}{(n+1)!} < \frac{3}{(n+1)!}$$

Estimating the Remainder

The Remainder Estimation Theorem If there is a positive constant M such that $|f^{(n+1)}(t)| \leq M$ for all t between x and a, inclusive, then the remainder term $R_n(x)$ in Taylor's Theorem satisfies the inequality.

$$|R_n(x)| \leq M \frac{|x-a|^{n+1}}{(n+1)!}$$

If this inequality holds for every n and the other conditions of Taylor's Theorem are satisfied by f, then the series converges to $f(x)$.

A Proof of Taylor's Theorem

10.10 The Binomial Series and Applications of Taylor Series

The Binomial Series for Power and Roots

The Binomial Series
For $-1 < x < 1$,

$$(1+x)^m = 1 + \sum_{k=1}^{\infty} \binom{m}{k} x^k$$

where we define

$$\binom{m}{1=m}, \quad \binom{m}{2} = \frac{m(m-1)}{2!}$$

and

$$\binom{m}{k} = \frac{m(m-1)(m-2)\cdots(m-k+1)}{k!} \quad \text{for } k \geq 3$$

Evaluating Nonelementary Integrals Taylor series can be used to express nonelementary integrals in terms of series. Integrals like $\int \sin x^2 dx$ arise in the study of the diffraction of light.

Example

$$\sin x^2 = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \cdots$$

$$\int \sin x^2 dx = C + \frac{x^3}{3} - \frac{x^7}{7 \cdot 3!} + \frac{x^{11}}{11 \cdot 5!} - \cdots$$

Arctangent

$$\frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7}$$

$$\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}, \quad |x| \le 1$$

when $x = 1$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots + \frac{(-1)^n}{2n+1} + \cdots$$

Evaluating Indeterminate Forms

Euler's Identity

$$e^{i\theta} = 1 + \frac{i\theta}{1!} + \frac{i^2\theta^2}{2!} + \frac{i^3\theta^3}{3!} + \frac{i^4\theta^4}{4!} + \frac{i^5\theta^4}{4!} + \cdots$$

$$= (1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots) + i(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots) = \cos\theta + i\sin\theta$$

Definition For any real number θ, $e^{i\theta} = \cos\theta + i\sin\theta$

$$e^{i\pi} = -1$$

Frequently used Taylor series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad |x| < 1$$

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n, \quad |x| < 1$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad |x| < \infty$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}, \quad |x| < \infty$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}, \quad |x| < \infty$$

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}, \quad -1 < x \le 1$$

$$\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}, \quad |x| \le 1$$

11 Parametric Equations and Polar Coordinates

11.1 Parametrizations of Plane Curves