IE221 – Probability

Team Work 3 (TW3) Technical Report

Experimental Verification of

Strong Law of Large Numbers (SLLN),
Central Limit Theorem (CLT)
and **Monte Carlo Estimation of π**

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1. Introduction

This report presents an experimental study of two fundamental results in probability theory: the **Strong Law of Large Numbers (SLLN)** and the **Central Limit Theorem (CLT)**. In addition, a practical application of SLLN is demonstrated through the **Monte Carlo estimation of the constant π**.

The main objective of this project is to observe how these theoretical results emerge in practice when the number of observations increases. All simulations are performed using computer-generated random variables, and the results are analyzed using graphical representations. The effect of sample size and convergence behavior are the main focus of this study.

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2. Strong Law of Large Numbers (SLLN)

2.1 Theoretical Background

The Strong Law of Large Numbers states that for a sequence of independent and identically distributed (i.i.d.) random variables with finite expected value μ, the sample mean converges almost surely to μ:

$P(\lim_{n\to\infty} (S_n / n) = \mu) = 1$

This type of convergence is called **almost sure convergence**, meaning that with probability one, the sample path eventually stabilizes at the expected value.

In this study, random variables are generated from the **uniform distribution U[0,1]**. For this distribution:

Expected value:
$\mu = 0.5$

Variance:
$\sigma^2 = 1 / 12$

2.2 Simulation Method

To experimentally verify the SLLN, the following steps are performed:

- Random observations are generated from the U[0,1] distribution.
- For each number of observations n, the cumulative mean is calculated as:

$\bar{X}_n = (X_1 + X_2 + ... + X_n) / n$

- The simulation is carried out up to $n = 10{,}000$.
- A reference line corresponding to $\mu = 0.5$ is added to the graph.

2.3 Results and Discussion

The resulting graph shows that the cumulative mean is highly volatile for small values of n. As the number of observations increases, these fluctuations decrease and the mean stabilizes around the value 0.5.

This behavior clearly confirms the **almost sure convergence** predicted by the Strong Law of Large Numbers. The simulation results are fully consistent with the theoretical expectations.

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3. Central Limit Theorem (CLT)

3.1 Theoretical Background

The Central Limit Theorem states that the standardized sum of i.i.d. random variables converges in distribution to the standard normal distribution:

$Z_n = (S_n - n\mu) / (\sigma \sqrt{n}) \rightarrow N(0,1)$

Unlike SLLN, CLT concerns the **distribution of sums**, not individual sample paths.

3.2 Simulation Method

The CLT is verified using the following approach:

- Distribution: U[0,1]
- Sample sizes: n = 2, 5, 10, 30, 50
- For each value of n:
    - m = 1000 independent experiments are performed.
    - In each experiment, n uniform random variables are summed.
    - The sums are standardized using the formula above.
- Histograms and Normal Q–Q plots are generated for the standardized values.

3.3 Histogram Analysis

The histograms show that for small sample sizes (n = 2 and n = 5), the distribution deviates significantly from the normal distribution. As the sample size increases (n = 30 and n = 50), the histograms gradually approach the bell-shaped curve of the standard normal distribution.

These results demonstrate the validity of the Central Limit Theorem in practice.

3.4 Q–Q Plot Analysis

The Q–Q plots indicate noticeable deviations from linearity for small values of n. However, as n increases, the points align more closely along a straight line.

This observation supports the conclusion that the standardized sums converge to a normal distribution as predicted by CLT.

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4. Monte Carlo Estimation of π

4.1 Theoretical Background

The Monte Carlo method is a direct application of the Strong Law of Large Numbers. Random points are generated inside the unit square $[0,1] \times [0,1]$. The proportion of points that fall inside the quarter unit circle is used to estimate π.

The estimator is given by:

$\hat{\pi} = 4 \times$ (Number of points inside the circle) / (Total number of points)

4.2 Simulation Method

- Random points (x, y) are generated from $U[0,1] \times U[0,1]$.
- For each point, the condition $x^2 + y^2 \leq 1$ is checked.
- The number of points is increased gradually.
- The true value of π is shown as a reference line in the graph.

4.3 Results and Discussion

At the beginning of the simulation, the π estimate exhibits large fluctuations. As the number of points increases, the estimate stabilizes around the value 3.14159.

This result clearly illustrates how the Monte Carlo method converges to the true value of π as a consequence of the Strong Law of Large Numbers.

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5. Conclusion

In this project:

- The Strong Law of Large Numbers is verified through the almost sure convergence of the sample mean.
- The Central Limit Theorem is experimentally demonstrated by the convergence of standardized sums to the normal distribution.
- The Monte Carlo method successfully estimates π as a practical application of SLLN.

All results are consistent with theoretical expectations and clearly demonstrate the role of large sample sizes in probabilistic convergence.