SOLUTIONS OF DIRECT METHODS

1. List the advantages of the direct methods over iterative methods
   - Direct methods give precise results, iterative methods give approximate values.
   - Direct methods can find the result in one go, iterative methods can get the result after many iterations.

2. Consider the following linear system of equations

$$\begin{aligned} 2x_1 + 3x_2 + x_3 &= 0 \\ 20x_1 + 18x_2 + x_3 &= 3 \\ x_1 + 36_x2 + x_3 &= 34 \end{aligned}$$

   - A = L x U

A= $\begin{bmatrix} 2 & 3 & 1 \\ 20 & 18 & 1 \\ 1 & 36 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ -34 \end{bmatrix}$

$U = \begin{bmatrix} 2 & 3 & 1 \\ 0 & 2 & -9 \\ 1 & 36 & 1 \end{bmatrix}$ $R_2 \leftarrow R_2 - (10)R_1$

$\begin{bmatrix} 2 & 3 & 1 \\ 0 & -12 & -9 \\ 0 & \frac{69}{2} & \frac{1}{2} \end{bmatrix}$ $R_3 \leftarrow R_3 - (\frac{1}{2})R_1$

$\begin{bmatrix} 2 & 3 & 1 \\ 0 & -12 & -9 \\ 0 & 0 & \frac{69}{24} \end{bmatrix}$ $R_3 \leftarrow R_3 - (\frac{69}{24})R_2$

The result of U is: $\begin{bmatrix} 2 & 3 & 1 \\ 0 & -12 & -9 \\ 0 & 0 & \frac{69}{24} \end{bmatrix}$ so according to the U we can find L is : $\begin{bmatrix} 1 & 0 & 0 \\ 10 & 1 & 0 \\ \frac{1}{2} & \frac{69}{24} & 1 \end{bmatrix}$

Ly = b $\rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 10 & 1 & 0 \\ \frac{1}{2} & \frac{69}{24} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ -34 \end{bmatrix}$ Solving the system for y

$$\begin{aligned} y_1 &= 0 \\ y_2 &= 3 \\ y_3 &= -25.375 \end{aligned}$$

Now solve: Ux = y $\rightarrow \begin{bmatrix} 2 & 3 & 1 \\ 0 & -12 & -9 \\ 0 & 0 & \frac{69}{24} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ -25.375 \end{bmatrix}$

$$\begin{aligned} x_1 &= 1 \\ x_2 &= -1 \\ x_3 &= 1 \end{aligned}$$

Code Implementation

Properties of the matrix

- Name = impcol_b
- Dimension = 9
- Number of nonzeros = 50
- Kind = Chemical Process Simulation Problem

Pseudocode

def ULFactorization(A): function takes A_diag_dom matrix which is diagonally dominant
create permutation matrix using np.identity
define B then apply pivotting to A matrix 2 times B=PAP
B_U =B and B_L = np.zeros((n,n),dtype=np.float64)
define L and fill diagonal with ones
find B_U and B_L with using basic LU factorization Gaussian Elimination
Apply transpose funciton to P find P_T
Then we can solve UL factorization is by similarity transformation.
Find A_U = PB_L P_T
Then find A_L = PB_UP_T
In this way, we find the equation A = UL

Result: Since we solved it using direct solution, we see that the exact values and the values of the direct method are the same.

SOLUTIONS OF ITERATIVE METHODS

1. List advantages of the iterative methods over direct methods
   - Iterative methods are more efficient when deal with big data problem, it is more powerful in computational way because it is converge values.

2. Consider the following linear system of equations. Apply Jacobi and Gauss-Seidel Method for 4 iterations

$$\begin{aligned} 2x_1 + 3x_2 + x_3 &= 0 \\ 20x_+ 18x_2 + x_3 &= 3 \\ x_1 + 36x_2 + x_3 &= 34 \end{aligned}$$

Jacobi iterative method

First iteration with $\quad P_0 = (x_1^0, x_2^0, x_3^0) = (0, 0, -3)$

Rearrange the system in above such that the coefficient matrix is strictly diagonally dominant:

$$\begin{aligned} 20x_1 + 3x_2 + x_3 &= 0 \\ x_1 + 36x_2 + x_3 &= -34 \\ 2x_1 + 3x_2 + x_3 &= 3 \end{aligned}$$

(1) These equation can be written in the form:

$$x_1^1 = \frac{3 - x_3^0 - 18x_2^0}{20} = 0.3$$
$$x_2^1 = \frac{-34 - x_1^0 - x_3^0}{36} = -0.861$$
$$x_3^1 = -3x_2^0 - 2x_1^0 = 0$$

(2) Now the new point become: $P_1 = (x_1^1, x_2^1, x_3^1) = (0.3, -0, 861, 0)$

$$x_1^2 = \frac{3 - x_3^1 - 18x_2^1}{20} = 0.925$$
$$x_2^2 = \frac{-34 - x_1^1 - x_3^1}{36} = -0.9527$$
$$x_3^2 = -3x_2^1 - 2x_1^1 = 1.9833333333333334$$

(3) Now the new point become: $P_2 = (x_1^2, x_2^2, x_3^2) = (0.3, -0, 861, 0)$ is used in the new approximation P_2

$$x_1^3 = \frac{3 - x_3^2 - 18x_2^2}{20} = 0.9083333333333332$$
$$x_2^3 = \frac{-34 - x_1^2 - x_3^2}{36} = -1.0252314814814814$$
$$x_3^3 = -3x_2^2 - 2x_1^2 = 1.0083333333333333$$

(4) Now the new point become: $P_2 = (x_1^3, x_2^3, x_3^3) = (0.9083333333333332, -1.0252314814814814, 1.0083333333333333)$ is used in the new approximation P_3

$$x_1^4 = \frac{3 - x_3^3 - 18x_2^3}{20} = 1.0222916666666666$$
$$x_2^4 = \frac{-34 - x_1^3 - x_3^3}{36} = -0.9976851851851851$$
$$x_3^4 = -3x_2^3 - 2x_1^3 = 1.2590277777777779$$

n	x_1	x_2	x_3
0	0	0	-3
1	0.3	-0.8611111111111112	0
2	0.925	-0.9527777777777777	1.9833333333333334
4	0.9083333333333332	-1.0252314814814814	1.0083333333333333
5	1.0222916666666666	-0.9976851851851851	1.2590277777777779

Gauss-Sedel iterative method

We solve same equation so the equation can be written in the form:

$$x_1 = \frac{3 - x_3 - 18x_2}{20}$$
$$x_2 = \frac{-34 - x_1 - x_3}{36}$$
$$x_3 = -3x_2 - 2x_1 = 0$$

This suggest the following Gauss-Siedel iterative process

$$x_1^{n+1} = \frac{3 - x_3^n - 18x_2^n}{20}$$
$$x_2^{n+1} = \frac{-34 - x_1^{n+1} - x_3^n}{36}$$
$$x_3^{n+1}1 = -3x_2^{n+1} - 2x_1^{n+1} = 0$$

We start with initial guess $P_0 = (x_1^0, x_2^0, x_3^0) = (0, 0, -3)$ Substitute $x_2^0 = 0 and x_3^0 = -3$ in the first equation and have

$$x_1^1 = \frac{3 - x_3^0 - 18x_2^0}{20} = 0.3$$

Then subsitute the new value $x_1 = 0.3$ and $x_3^0 = -3$ into the second equation to obtain:

$$x_2^1 = \frac{-34 - (0.3) - x_3^0}{36} = -0.8694444444444444$$

Finally, subsitute the new values $x_1^1 = 0.3$ and $x_2^1 = -0.8694444444444444$ in the third equation and get

$$x_3^1 = -2x_1^1 - 3x_2^1 = 2.008333333333333$$

Now, we have the point $P_1 = (x_1^1, x_2^1, x_3^1) = (0.3, -0.8694444444444444, 2.008333333333333)$ is used to find the next approximation P_2.

$$x_1^2 = \frac{3 - x_3^1 - 18x_2^1}{20} = 0.8320833333333333$$
$$x_2^2 = \frac{-34 - x_1^2 - x_3^1}{36} = -1.0233449074074072$$
$$x_3^2 = -2x_1^2 - 3x_2^2 = 1.4058680555555552$$

Now, we have the point $P_2 = (x_1^2, x_2^2, x_3^2) = (0.8320833333333333, -1.0233449074074072, 1.4058680555555552)$ is used to find the next approximation P_2.

$$x_1^3 = \frac{3 - x_3^2 - 18x_2^2}{20} = 1.0007170138888888$$
$$x_2^3 = \frac{-34 - x_1^3 - x_3^2}{36} = -1.0112940297067903$$
$$x_3^3 = -2x_1^3 - 3x_2^3 = 1.0324480613425933$$

Now, we have the point $P_3 = (x_1^3, x_2^3, x_3^3) = (1.0007170138888888, -1.0112940297067903, 1.0324480613425933)$ is used to find the next approximation P_3.

$$x_1^4 = \frac{3 - x_3^3 - 18x_2^3}{20} = 1.0085422236689816$$
$$x_2^4 = \frac{-34 - x_1^4 - x_3^3}{36} = -1.0011386190280993$$
$$x_3^4 = -2x_1^4 - 3x_2^4 = 0.9863314097463349$$

n	x_1	x_2	x_3
0	0	0	-3
1	0.3	-0.8694444444444444	2.008333333333333
2	0.8320833333333333	-1.0233449074074072	1.4058680555555552
4	1.0007170138888888	-1.0112940297067903	1.0324480613425933
5	1.0085422236689816	-1.0011386190280993	0.9863314097463349

Compare the performance of the Jacobi and Gauss-Seidel Methods and which method has the better performance:

Gauss-Seidel is better performance because it is calculate every x values in each operation then takes it and use for other x operation not like compute all x values like Jacob so this provide us better performance when data and number of iterations get bigger.