Computational Analysis for the Response of an RLC Circuit

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Abstract

RLC circuit is one of the most basic electric circuits that is usually constituted by a first-order system equation. In spite of the fact that it is one of the key concepts of electrical engineering, hardly any scientific experimentation has been done it probably due to the circuit forming the basis of a science. This paper serves as a starting point for undergraduate students who wants to see the experimental outcome of the theoretical findings that they have seen over the course of their curriculum accompanied by computational analyzes. Among others, the algorithm used in this projects portrays a rather fascinating structure which composes a harmony alongside with this prerequisite of electrical sciences.

Computational Analysis for the Response of an RLC Circuit

Charging and discharging of a capacitor that are accompanied by a resistor constitutes the basis of many basic electric circuits which involves second-order system equations. This circuit structure is a key concept that requires the understanding of itself in order for one to delve into the electric-based sciences. With modern-day technology, RLC circuits find their way into in the form of the framework of more complex electrical devices. Hence, the gap between undergraduate students and their theoretical findings should be closed due to the circuit being spread around many science majors. Among aeronautical industry, especially the electronics and control theory seems to have major upward trend over the last decade. Unmanned aerial vehicles and drones (e.g., Fan & Zhang, 2017; Towaha & Omar, 2018) have very complicated hardware designs which proves that the aeronautical industry is starting to rely heavily on a small portion of computer science and electronic engineering. These trends are especially affected by many advantages that modern-day circuits provide, e.g., removal of the human factor, cost-effectiveness and complicated maneuver patterns. Besides the basic hardware of an aircraft, electrical circuits open many more alternative paths that can be taken to construct an aircraft such as measuring techniques and experimentation (see Wu et al., 2018). This is where the importance of electronics come into play: On one hand, comprehension of RLC circuits can indirectly be used for many applications of measurement related to an aircraft's data. On the other hand, this circuits could be the very basis of an electronic component of a typical aircraft. However, hardly any RLC circuits experimentation or analyzation approaches have been proposed outside of a classroom yet.

Therefore, this paper focuses on the transient and steady-state forms of an RLC circuit in order to constitute the basis of future complex structures. After this introduction, the voltage charge and discharge parameter in time domain and certain constants that are related to the system equation model of the circuit are explained. By finding the theoretical equations, the prerequisite for the analyzes are addressed. At the later sections, experimentation methods is explained before the discussion section of the paper which questions the results obtained so far. In addition to analytical solutions that is followed by empirical results, numerical methods with computational applications will be presented before the discussion portion. By identifying and disclosing application scenarios for RLC circuits, this paper serves as a starting point for an undergraduate student's future research in the respective fields.

Theoretical Background

Transient Analysis

The loading and discharge of a capacitor and an inductor where a voltage source keeps feeding the capacitor and inductor with electricity has a transient phase. The name comes from the unsteady nature of voltage until a certain state where the voltage keeps a specific behaviour. RLC circuits tend to show different characteristics under varying voltage sources, i.e., AC or DC. The equation shows inconsistency With varying materials, e.g., voltage source and resistors, due to the paramaters in the equation having different values. The degree of complexity in transient analysis depends on the number of energy storage elements in the circuit. For higher-order circuits, i.e., for circuits with high number of energy storage elements, computer aided analyzes are used. Figure 1 is the visualization of a second-order circuit. For a second-order circuit, the amount of resistors are simply irrevelant beyond one because Thevenin and Norton theorems can transform the resistors into a single entity. Therefore, regardless of how many resistors the circuit contains, the circuit is still said to be second-order.

Second-order circuits are one of the fundamental circuits and therefore need to be explained before further investigating the electrical theorems that will be found in this paper. Visualization of a simple RLC circuit is constitueded in Figure 1 and the following calculations will be based on this figure. Applying Kirchoff's voltage law in this circuit results in

$$v_s(t) - Ri_L - v_C(t) - L\frac{\mathrm{d}i_L(t)}{\mathrm{d}t} = 0$$

At this stage it is clear that on the same line currents must be equal that is $i_R = i_C = c\frac{\mathrm{d}v_C(t)}{\mathrm{d}t}$. Hence, the following equations is obtained:

$$LC\frac{\mathrm{d}^2 v_C(t)}{\mathrm{d}t^2} + RC\frac{\mathrm{d}v_C(t)}{\mathrm{d}t} + v_C(t) = v_S(t) \tag{1}$$

Equation 1 belongs to a second-order system. In this format a couple of specific concepts can be defined. Natural frequency, denoted as ω_n, damping ratio, denoted as ζ, and DC gain, denoted as K_S, are essential variables that are extremely important and almost always considered to be key aspects of electric and electronic circuits, devices and et cetera. Equation 1 holds these values, therefore an engineer can easily determine the results as in $\omega_n = \sqrt{\frac{1}{LC}}$, $\zeta = \frac{RC}{2}\sqrt{\frac{1}{LC}} = \sqrt{\frac{R^2C}{4L}}$ and $K_S = 1$. Basic differential equation solving methods is going to be applied on Equation 1 in order to in a sense solve the equation. Particular and complementary solutions is going to be investigated in the following calculations. Complementary solution of Equation 1 is in the exponential form

$$y(x) = \alpha \mathrm{e}^{sx} \tag{2}$$

where the variables turned into conventional x domain that is affiliated with a function that is denoted as y. s parameter denotes characteristic polynomial that will give characteristic roots that are s_1 and s_2. In other words Equation 2 can be written as $y(x) = \alpha_1 \mathrm{e}^{s_1 x} + \alpha_2 \mathrm{e}^{s_2 x}$. Equation 2 can be substituted in Equation 1 to give the result of

$$\frac{1}{\omega_n^2} s^2 \alpha \mathrm{e}^{sx} + \frac{2\zeta}{\omega_n} s \alpha \mathrm{e}^{sx} + \alpha \mathrm{e}^{sx} = 0 \tag{3}$$

In order to Equation 3 to be equal to zero, $\frac{s^2}{\omega_n^2} + \frac{2\zeta}{\omega_n} s + 1$ has to be equal to zero. From this point on, roots of the equation can be determined. This will give results to

$$s_{1,2} = -\zeta\omega_n \pm \frac{1}{2}\sqrt{(2\zeta\omega_n)^2 - 4\omega_n^2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} \tag{4}$$

Equation 4 interprets that three different scenarios are possible. The equation can have real and distinct roots, i.e., overdamped ($\zeta > 1$), that would bring about $s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$. Other than that, the equation can have real roots but the roots have the possibility to be repeated, i.e., critically damped ($\zeta = 1$). So the previous statement can be shortened to $s_{1,2} = -\omega_n$. Moreover, the equation can have complex roots, i.e., underdamped ($\zeta < 1$), which will be defined as $-\zeta\omega_n \pm j\omega_n\sqrt{1 - \zeta^2}$. As of now, only the complementary solution is defined and particular solution is absent. In order to access a valid function both complementary and particular solution must be obtained. Particular solution is easily obtainable compared to the complementary solution. Particular solution of Equation 1 is found by setting the equation to a certain parameter. Which parameter that should be chosen is beyond the scope of this paper and will not be discussed. In order to get a broad sense of understanding regarding first-order system equations differential equations related books (e.g., Edwards et al., 2018) can be studied.

$$\frac{1}{\omega_n^2}\frac{\mathrm{d}^2 y(x)}{\mathrm{d}x^2} + \frac{2\zeta}{\omega_n}\frac{\mathrm{d}y(x)}{\mathrm{d}x} + y(x) = K_S y(x)$$

. Certain parameters are disregarded as they no longer serve purpose for $x = 0$ because due to the constant excitation derivative terms are set to zero. Hence particular solution becomes

$$y(x) = y(\infty) \tag{5}$$

This results is oddly similar to the steady-state particular solution of an RC Circuit, nevertheless that system is not the main scope of this paper. By combining Equation 2 and Equation 5 , the solution, i.e., $y(x)$, can be found. For three different scenarios three different solutions can be obtained.

$$y(x) = \begin{cases} \text{overdamped solution,} & \alpha_1 \mathrm{e}^{(-\zeta\omega_n + \omega_n\sqrt{\zeta^2-1})x} + \dots \\ & \dots \alpha_2 \mathrm{e}^{(-\zeta\omega_n - \omega_n\sqrt{\zeta^2-1})x} + y(\infty) \\ \text{critically damped solution,} & \alpha_1 \mathrm{e}^{-\zeta\omega_n x} + \alpha_2 x \mathrm{e}^{-\zeta\omega_n x} + y(\infty) \\ \text{underdamped solution,} & \alpha_1 \mathrm{e}^{(-\zeta\omega_n + j\omega_n\sqrt{1-\zeta^2})x} + \dots \\ & \dots \alpha_2 \mathrm{e}^{(-\zeta\omega_n - j\omega_n\sqrt{1-\zeta^2})x} + y(\infty) \end{cases}$$

Method

Equipment

Equipment list for this experiment consists of a breadboard, capacitors, resistors, a DC power supply accompanied by a computer and an Arduino brand single-board micro-controller that converts analog signal to digital to be processed by means of a computer program that runs and compiles Arduino related binaries such as a MATLAB/Simulink package that provides back-end support for Arduino products. Arduino used in the experiment is run under maximum 5 V.

Design

Firstly, real circuit, i.e., Figure 1, is going to be modified because the arduino device is run under low electric potential difference. Therefore, the circuit in Figure 1b should be obtained by connecting two different resistors as in the mentioned figure. This allows us to measure the capacitor voltage by correlating it to the electric potential difference between the two externally added resistors. One of these resistors have a resistance value of 10 times greater than the other one's. If the resistors that added extra doesn't have a significant amount of resistance, then the circuit would fail to work as intended because the resistors would dominate the circuit due to the Kirchhoff's Law of current behaviour. Then, an RLC circuit is formed on a breadboard with the components resistance, capacitance and a DC power supply. Short jumper cables are used in order to provide the connection on the breadboard. Red wire starts the positive electric flow and black wire ends in negative the negative pole. Also, black wire is responsible for the grounding. Arduino device and MATLAB with proper plug-ins for Arduino must be connected to the circuit as well. Arduino device is selected to be 10 bit. That type of device would represent 5 V as 1023 and no potential difference, i.e., 0 V, as 0. Voltage source is varied such as it is taken by means of setting it to an arbitrary electrical potential difference as it is taken like the following directives for the equipment during a phase of the experiment: the power source is 7 V, the resistor is $4.62 \times 10^{3}\ \Omega$, the inductor is 840×10^{-6} H and the capacitor is 2.2×10^{-3} F. Voltage source is calculated in a similar manner by introducing external circuit elements on the left side of the Figure 1a as in the right side of Figure 1b.

Procedure

Analog signal is transformed to the digital equivalent and analysed by means of MATLAB (Simulink). When the power supply is turned on, the capacitance starts to get charged. The voltage value is observed on a computer by virtue of an inspection of the scope window in MATLAB that provides more information on the amplitude of the capacitance wave than a regular multi-meter. Then power supply is turned off and the capacitance discharges while the same value characteristic as in the previous step are observed on the the said computer. In this experiment, a computer is used to monitor the potential difference, and thus, indirectly, the charge on a capacitor as well for throughout the experiment. Forthwith, next experimentation follows the previously stated procedure policy. Therefore, this data gathering technique constitutes a basis for the experiment procedure.

Computation. Second order differential equation can be solved by following the 4th order Runge-Kutta method provided by the SciPy library of Python. The code for the numerical calculation can be inspected in Appendix A.

Results

Before presenting the experimental results, one should always bear in mind the theoretical evaluations. From the parameters that are gathered from the previous sections, Equation 5 can be calculated. The design of the circuits denotes that $\omega_n = 735.6124$, $\zeta = 3738.38$. This denotes that the system is over-damped. Overall, the quality factor of the system is found to be 0.0001337 According to the theoretical analyzes, the settling time is 47.0 s at 6.9313 V. Final voltage of the capacitance is therefore guaranteed to be 7 V. The settling time can be extended by increasing R or C values. Numerical evaluations indicates that it would take 47.2435 s -settling time- to reach 6.9329 V. However, by checking the data of the experiment settling time of experimentation is seems to be 40 s for a voltage of 6.7155 V.

Discussion

Short jumper cables shows a resistance that we ignored to the electric flow. This would alter the accuracy of our results. Hence, one would expect the analytical solution to be different of the one obtained from experimentation. Also, noise from signal processing is valid just like in almost every other system. The noise arose from the communication could be hindered with a filter, however the nondifferentiable state of the system prevented the usage of filters for a PID controller. A filter would render the blue line, capacitor voltage, smoother than it is obtained by virtue of the experiment. Two external resistors that we introduced to the RLC circuit also affects the system. As mentioned before, if these were to be of low resistance, electric would try to go through the resistors instead of the capacitor as intended. Numerical solution coincides with the analytical one. This refers that the approximation is highly accurate and precise. Experimentation on the other hand is rather low than expected. This gives an inaccurate settling time for the voltage source. Settling times of analytical and numerical solutions are the same as expected, however experimentation lags by 7 s. One would realise that the settling times for both

RC and RLC circuits for the capacitor charging is the same. This makes sense considering an inductor acts like a short circuit under DC voltage.

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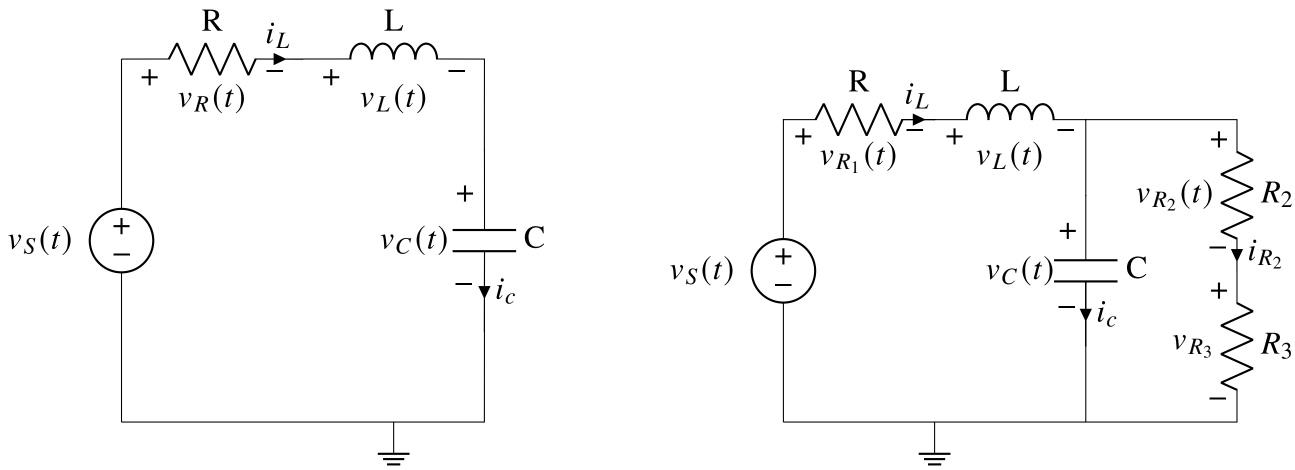

(a) Standard RLC circuit in series.

(b) Modified RLC circuit in series.

Figure 1. Simplified circuit design

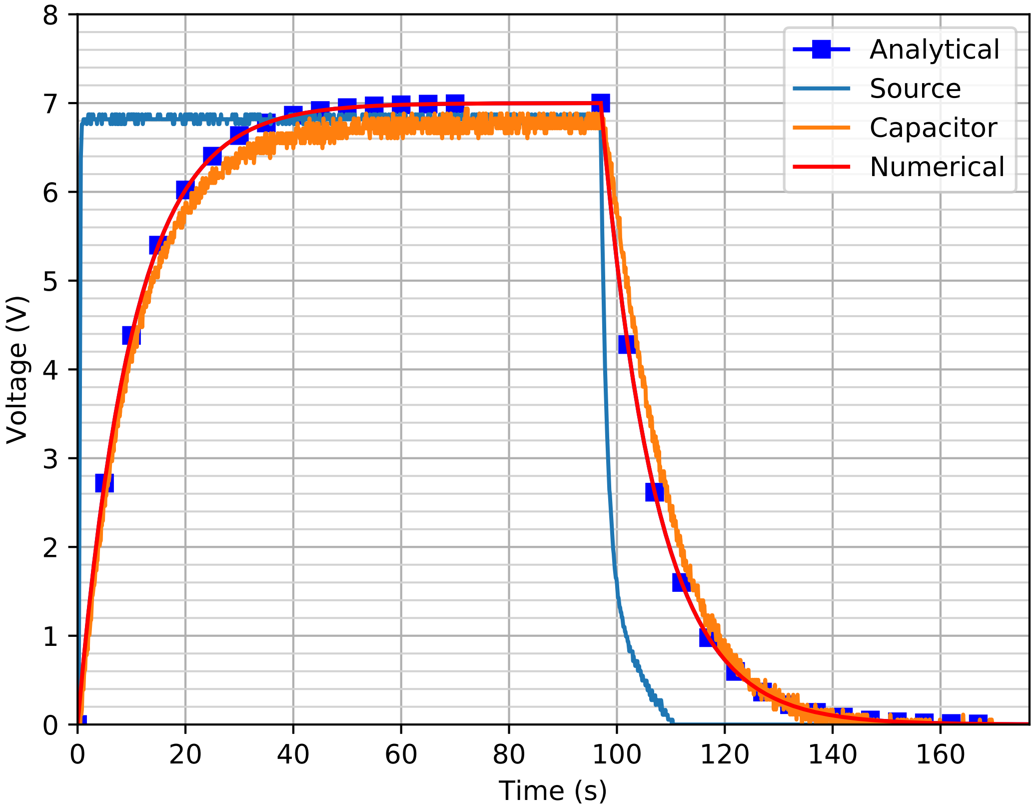

Figure 2. Results

Appendix

Numerical Solution

```
#!/usr/bin/env python
import h5py
import numpy as np
import matplotlib.pyplot as plt
import matplotlib
from scipy.integrate import solve_ivp, odeint

R=4620
Vs=7
L=840/10**6
C=2.2/10**3

f3 = h5py.File('data_Grp1_RLC_exp1.mat','r') #RLC
q = f2["ans"]
q = np.array(y)

x=np.arange(0,97,.5)
x_minor=np.concatenate((x,x+80))
fig2 = plt.figure()
ax = fig2.add_subplot(111)
om=np.sqrt(1/L/C)
zeta=R*C*om/2
s1=(-zeta*om+om*np.sqrt(zeta**2-1))
s2=(-zeta*om-om*np.sqrt(zeta**2-1))
a_1=7/((s1/s2)-1)
a_3=-a_1
```

```
a_2=7/((s2/s1)-1)
a_4=-a_2
V_c=a_1*np.exp((-zeta*om+om*np.sqrt(zeta**2-1))*x)+a_2*np.exp((-zeta*om-om*np.sqrt(zeta**2-1))*x)+7
V_c2=a_3*np.exp((-zeta*om+om*np.sqrt(zeta**2-1))*x)+a_4*np.exp((-zeta*om-om*np.sqrt(zeta**2-1))*x)
plt.xlim(0, x[-1]+80)
plt.ylim(0, (Vs+1))
plt.grid()
for minor in np.linspace(0, (Vs+1), (Vs+1)*5+1):
    if minor not in range(0, (Vs+1)):
        plt.plot(x_minor, minor*np.ones(len(x_minor)), color='lightgray', linewidth=.75)

plt.xlabel("Time (s)")
plt.ylabel("Voltage (V)")
line2,=ax.plot(x, V_c, color='blue', linestyle='-', marker='s', markevery=[index for index in range(0, 150, 10)])
ax.plot(x+97, V_c2, color='blue', linestyle='-', marker='s', markevery=[index for index in range(0, 150, 10)])
qq = q[135:,0]-q[135:,0][0]
line_v,=ax.plot(qq,q[135:,1])
line_c,=ax.plot(qq,q[135:,2])
def func_rlc(y, x):
    return [y[1], (7-y[0])/L/C - y[1]*R/L]
def func_rlc2(y, x):
    return [y[1], -y[0]/L/C - y[1]*R/L]
xs = np.linspace(0, 97, 97*2)
Us = odeint(func_rlc, [0, 0], np.linspace(0, 97, 97*2))
```

```python
Us2 = odeint(func_rlc2, [7, 0], np.linspace(0, 97, 97*2))
ys = Us[:,0]
ys2 = Us2[:,0]
line_rk,=ax.plot(xs,ys, 'r')
ax.plot(xs+97, ys2,'r')
# ax.plot(sol2_rc.t+80, sol2_rc.y[0], color='red')
ax.legend((line2,line_v,line_c,line_rk),("Analytical","Source","Capacitor","Numerica
plt.savefig('rcl.pdf')
plt.close()
for index in range(len(v_t)):
    if v_t[index] - Vs*.99 >= 0:
        print("Settling time of theory is", x[index], "seconds and V(t_s) =", np.round(v_t[index],4), 'V')
        break
for index in range(len(ys)):
    if ys[index] - Vs*.99 >= 0:
        print("Settling time of approximation is", xs[index], "seconds and V(t_s) =", np.round(ys[index],4), 'V')
        break
for index in range(len(q)):
    if q[138+index,2] - q[138+index,1]*.99 >= 0:
        print("Settling time of experimentation is", qq[138+index]-qq[138], "seconds and V(t_s) =", np.round(q[138+index,2],4), 'V')
        break
```